Étale Homotopy Theory

and

Simplicial Schemes

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Cohomology and K-Theory

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The last row will play an important role in this conference.

I will discuss the second row.
Étale Cohomology

Let $X$ be a scheme of finite type over a field $k$ and let $\ell$ be a prime $\neq$ the characteristic of $k$.

The étale cohomology groups

$$H^p_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z})$$

can be defined “topologically” via the Čech construction. Let $\mathcal{U} = \{U_i \to X\}$ be an étale cover of $X$ and set

$$U_{i_0, \ldots, i_p} = U_{i_0} \times_X \cdots \times_X U_{i_p}.$$

Then the étale $p$-chains are given by

$$C^p(\mathcal{U}, \mathbb{Z}/\ell\mathbb{Z}) = \prod H^0(U_{i_0, \ldots, i_p}, \mathbb{Z}/\ell\mathbb{Z}).$$
If we define

\[ H^p(\mathcal{U}, \mathbb{Z}/\ell\mathbb{Z}) = H^p(C^*(\mathcal{U}, \mathbb{Z}/\ell\mathbb{Z})) , \]

then the \( p \)th étale cohomology group of \( X \) is the direct limit

\[ H^p_\text{ét}(X, \mathbb{Z}/\ell\mathbb{Z}) = \lim_{\mathcal{U}} H^p(\mathcal{U}, \mathbb{Z}/\ell\mathbb{Z}) \]

over the directed set of all étale covers \( \mathcal{U} \) of \( X \).

**Key Observation:** The global sections \( H^0(U_{i_0,\ldots,i_p}, \mathbb{Z}/\ell\mathbb{Z}) \) are determined by the set of connected components

\[ \pi_0(U_{i_0,\ldots,i_p}) . \]

As we vary over all \( i_0, \ldots, i_p \), we get a simplicial set.
Simplicial Sets and Schemes

Let $\Delta$ be the category with objects $[n] = \{0, 1, \ldots, n\}$ and morphisms monotone maps $[n] \to [m]$.

A simplicial object in a category $C$ is a contravariant functor

$$X_\bullet : \Delta \to C.$$ 

The maps $[1] \to [0]$ and $[0] \implies [1]$ give

$$X_0 \Leftrightarrow X_1 \cdots$$

$C = \text{Sets}$ gives $\text{SSets}$ and $C = \text{Sch}/k$ gives $\text{SSch}/k$. We also have a connected component functor

$$\pi_0 : \text{SSch}/k \longrightarrow \text{SSets}.$$
Étale Homotopy Theory

Due to Artin and Mazur, using ideas of Verdier and Lubkin. $\mathcal{H}$ is the homotopy category of SSets (ignore base points).

Given a scheme $X$, $X_{\text{ét}}$ is the category with objects étale maps $Y \to X$ and morphisms commutative diagrams

\[
\begin{array}{ccc}
Y & \to & Y' \\
& \searrow & \swarrow \\
& & X
\end{array}
\]

where $Y \to Y'$ is also étale. By the Čech construction, an étale cover $\{U_i \to X\}$ gives a simplicial object $U$. in $SX_{\text{ét}}$. This is an example of a hypercovering.
The étale homotopy type of $X$

$$(X)_{\text{ét}} = \{\pi_0(U)\} \in \text{Pro-} \mathcal{H}$$

given by the connected components of the inverse system of all hypercoverings of $X$.

If $X$ is a simplicial scheme, one also has

$$(X_\cdot)_{\text{ét}} = \{\pi_0(\Delta U_\cdot)\} \in \text{Pro-} \mathcal{H}.$$ 

Furthermore, if $U$ is a hypercovering of $X$, then the natural map

$$(U_\cdot)_{\text{ét}} \to (X)_{\text{ét}}$$

is a weak equivalence in $\text{Pro-} \mathcal{H}$.
Applications

Étale homotopy theory has many applications, including:

- Comparison Theorems
- The Adams Conjecture
- Tubular Neighborhoods
- Poincaré Duality
- Finite Chevalley Groups
- Étale K-Theory
Comparison Theorems

When $X$ is a scheme of finite type over $\mathbb{C}$, the most basic comparison theorem asserts

$$H^p_{\text{ét}}(X, \mathbb{Z}/\ell\mathbb{Z}) \simeq H^p(X(\mathbb{C}), \mathbb{Z}/\ell\mathbb{Z})$$

for any prime $\ell$. This generalizes:

- For $X$ geometrically unibranch over $\mathbb{C}$,
  $$(X)_{\text{ét}} \simeq X(\mathbb{C})^{\wedge}$$
  in $\text{Pro-}\mathcal{H}$ ($^{\wedge}$ is pro-finite completion).

- For $X.$ over $\mathbb{C}$, we have
  $$(X.)_{\text{ét}}^{\wedge} \simeq_{\text{weak}} X.(\mathbb{C})^{\wedge}.$$

- For $f : X \to Y$ smooth and proper,
  $$H^p(\text{fib}(f_{\text{ét}}), \mathbb{Z}/\ell\mathbb{Z}) \simeq H^p_{\text{ét}}(f^{-1}(y), \mathbb{Z}/\ell\mathbb{Z}).$$
Tubular Neighborhoods

Topologically, a tubular neighborhood $T_{X/Y}$ of $Y \subset X$ is easy to picture:

\begin{center}
\begin{tikzpicture}
\draw[fill=gray!30] (0,0) -- (6,0) -- (6,3) -- (0,3) -- cycle;
\draw[fill=gray!50] (1,0.5) rectangle (5,2.5);
\node at (3,1) {$Y$};
\node at (3,3) {$X$};
\end{tikzpicture}
\end{center}

Some nice properties of $T_{X/Y}$:

- $Y \subset T_{X/Y}$ is a homotopy equivalence.
- For $X, Y$ smooth, $\partial T_{X/Y} \to Y$ is a spherical fibration that carries the Thom class. Up to homotopy, this fibration is $T_{X/Y} - Y \hookrightarrow T_{X/Y}$. 
Tubular Neighborhoods in Algebraic Geometry

**Zariski:** A Zariski neighborhood of \( Y \subset X \) is too big. Except in trivial cases, it can’t be a tubular neighborhood.

**Étale:** An étale neighborhood is an étale map \( V \to X \) such that \( V \times_{X} Y \cong Y \). These are also too big:

**Example.** One can prove that the only étale neighborhoods of \( \mathbb{P}^1 \subset \mathbb{P}^2 \) are Zariski neighborhoods of \( \mathbb{P}^1 \) in \( \mathbb{P}^2 \).
**Ringed Space:** Given $Y \subset X$, one can construct:

- its *henselization* $Y \subset X^h_Y \to X$.
- its *formal completion* $Y \subset \hat{X}_Y \to X$.

These are ringed spaces supported on $Y$ with some nice properties. But we can’t remove $Y$ to get a spherical fibration. So these aren’t geometric enough.

**Simplicial:** Let $t_{X/Y}$ be the category of simplical objects $V. \in SX_{\text{ét}}$ such that $V. \times_X Y \to Y$ is a hypercovering. Then:

The *tubular neighborhood* of $Y$ in $X$ is

$$T_{X/Y} = \{V. \mid V. \in t_{X/Y}\}.$$
Here is a glimpse of life before TeX:

In 1974, I paid $3 to have this page typed.
Properties of $T_{X/Y}$

- $(Y)_{\text{ét}} \simeq (T_{X/Y})_{\text{ét}}$ is a homotopy equivalence.
- $H^*_{\text{ét},Y}(X, \mathbb{Z}/\ell \mathbb{Z})$ is isomorphic to $H^*(T_{X/Y}, T_{X/Y} - Y, \mathbb{Z}/\ell \mathbb{Z})$.
- When $Y$ and $X$ are smooth, there is an algebraic exponential map
  \[(N_{X/Y} - Y)_{\text{ét}} \simeq (T_{X/Y} - Y)_{\text{ét}}\]
  where $N_{X/Y}$ is the normal bundle of $Y$ in $X$.
- Friedlander used $T_{X/Y}$ to give a “topological” proof of Poincaré duality for étale cohomology.
Twisted Chevalley Groups

Let $H$ be a “twisted” group of Chevalley, Steinberg, or Suzuki-Rees type. Then there is a simple algebraic group $G$ over $\overline{\mathbb{F}}_p$ such that $H = \text{the fixed point set of}$ an algebraic endomorphism

$$\phi : G \longrightarrow G.$$ 

In 1953, Lang showed that $\Phi(g) = g\phi(g)^{-1}$ is onto. This gives a fibration

$$H \longrightarrow G \overset{\Phi}{\longrightarrow} G.$$

In 1970 Quillen suggested that this would be relevant to étale homotopy theory. Friedlander pursued this in the 1970s. His results compute the $\mathbb{Z}/\ell\mathbb{Z}$ cohomology of $H$ in terms of $H^*(BG, \mathbb{Z}/\ell\mathbb{Z})$ for $\ell \neq p$. 

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Classifying spaces were originally constructed topologically and are not algebraic varieties.

Working simplicially, we have the simplicial scheme $BG$ such that $BG_n$ is the cartesian product

$$G \times_k \cdots \times_k G.$$ \(n\) times

Boundary and degeneracy maps are built from the identity $\text{Spec}(k) \to G$ and multiplication $G \times_k G \to G$.

By the comparison theorem, the étale homotopy type of $BG$ is the same as $BG(\mathbb{C})$, up to pro-finite completion. This brings topology into algebraic geometry.
Étale K-Theory

For a CW complex $T$, ordinary K-theory with coefficients in $\mathbb{Z}/m\mathbb{Z}$ is defined by

$$K^0(T, \mathbb{Z}/m\mathbb{Z}) = [C(m) \wedge T, BU]$$
$$K^1(T, \mathbb{Z}/m\mathbb{Z}) = [\Sigma C(m) \wedge T, BU]$$

where $C(m)$ comes from the cofiber triple

$$S^1 \xrightarrow{m} S^1 \longrightarrow C(m).$$

Using étale homotopy theory, we get the following definition of Friedlander:

The étale K-theory of a scheme $X$ is

$$K^0_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}) = [C(m) \wedge (X)_{\text{ét}}, \#BU]$$
$$K^1_{\text{ét}}(X, \mathbb{Z}/m\mathbb{Z}) = [\Sigma C(m) \wedge (X)_{\text{ét}}, \#BU]$$
Properties

\begin{itemize}
\item $K^\ast_\text{ét}(X, \mathbb{Z}/\ell\mathbb{Z}) \simeq K^\ast(X(\mathbb{C}), \mathbb{Z}/\ell\mathbb{Z})$.
\item $\text{Gal}(\overline{k}/k)$ acts on $K^\ast_\text{ét}(X \times_k \overline{k}, \mathbb{Z}/\ell\mathbb{Z})$.
\item There is a spectral sequence relating $H^\ast_\text{ét}(X, \mathbb{Z}/\ell\mathbb{Z})$ to $K^\ast_\text{ét}(X, \mathbb{Z}/\ell\mathbb{Z})$.
\item The map $K^0_\text{alg}(X) \to K^0(X(\mathbb{C})) \otimes \mathbb{Z}_\ell$ factors through $K^\ast_\text{ét}(X, \mathbb{Z}_\ell)$.
\end{itemize}

A more sophisticated definition of étale K-theory was given in 1985 by Dwyer and Friedlander. Most cohomology theories can be represented by spectra. Just as we brought the topological $BG$ into the category of simplicial schemes, the idea here is to bring spectra into $\text{SSch}/k$. 
First: Given a $k$-algebra $A$,

$$K_A = \text{Sp}(\text{Hom}_g(A, B\overline{G\ell}_*)_k),$$

where “$g$” means scheme-theoretic maps. Then

$$\pi_i(K_A) = \text{Quillen } K\text{-theory of } A.$$

Second: Given a scheme $X$ over $k$,

$$\hat{K}_X^{\text{ét}} = \text{Sp}(\text{Hom}_l(X, B\overline{G\ell}_*)_k),$$

where “$\ell$” means maps between the $\ell$-adic completions of the étale homotopy types. Then

$$\pi_i(\hat{K}_X^{\text{ét}} \wedge \mathcal{M}(\nu)) = \hat{K}_i^{\text{ét}}(X, \mathbb{Z}/\ell^\nu\mathbb{Z}).$$