Homological projective duality

(Alexander Kuznetsov!)

Beilinson

$$\begin{aligned} D(\mathbb{P}^{n-1}) &= \langle \mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1) \rangle \\ &= \langle D(\mathrm{pt}), D(\mathrm{pt}), \dots, D(\mathrm{pt}) \rangle. \end{aligned}$$

The notation $\langle\ \rangle$ means a *semi-orthogonal decomposition:*

- They are semi-orthogonal: $\mathrm{Ext}^*(\mathcal{O}(i), \mathcal{O}(j)) = 0$ for $i > j$,
- $\mathcal{O}(i)$ is *exceptional*: $R\,\mathrm{Hom}(\mathcal{O}(i), \mathcal{O}(i)) = \mathbb{C}.\,\mathrm{id}$. Equivalently,

$$\begin{aligned} D(\mathrm{pt}) &\to D(\mathbb{P}^{n-1}) \\ \mathbb{C} &\mapsto \mathcal{O}(i) \end{aligned}$$

  is an embedding,
- The sheaves $\mathcal{O}, \mathcal{O}(1), \dots, \mathcal{O}(n-1)$ *generate* $D(\mathbb{P}^{n-1})$.

Sketch proof

The first two conditions are simple cohomology computations.

Generation: Since $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-1)$ are exceptional and semi-orthogonal the Gram-Schmidt process shows that any object of $D(\mathbb{P}^{n-1})$ is an extension of a piece in their span and a piece in their orthogonal.

Thus we want to show their orthogonal is zero.

Use the Koszul resolution associated to the section

$$\mathcal{O} \xrightarrow{(x_0, x_1, \ldots, x_{n-1})} \mathcal{O}(1)^{\oplus n}$$

with zero locus one point $\{p\}$. Shows that $\mathcal{O}_p$ is in the span of $\mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(n-1)$ for any point $p \in \mathbb{P}^{n-1}$.

Thus any object F in the orthogonal has

$$F|_p \cong R\operatorname{Hom}(\mathcal{O}_p, F)[n-1] = 0$$

for any $p \in \mathbb{P}^{n-1}$. Therefore $F = 0$ by the Nakayama lemma.

Family version

Fix

- $X = \mathbb{P}(E) \xrightarrow{\pi} B$, a $\mathbb{P}^{n-1}$-bundle, where
- E is a rank n vector bundle over B.

Orlov proved that

$$D(X) = \Big\langle D(B), D(B)(1), \ldots, D(B)(n-1) \Big\rangle.$$

Here the ith copy of $D(B)$ is embedded via $\pi^*(\,\cdot\,) \otimes \mathcal{O}(i)$, where $\mathcal{O}(-1) \hookrightarrow \pi^* E$ is the tautological bundle on X.

Blow ups

Blow up in $Z \subset X$ of codimension-n (both smooth):

$$\begin{array}{ccc} E & \xhookrightarrow{j} & \mathrm{Bl}_Z X \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z & \xhookrightarrow{i} & X \end{array}$$

Orlov also proved

$$D(\mathrm{Bl}_Z X) = \Big\langle D(X),\, D(Z)(-E),\, \dots,\, D(Z)(-(n-1)E) \Big\rangle.$$

Here

- $D(X)$ is included via π^* and $D(Z)$ by $j_* p^*$
- $p\colon E \to Z$ is a $\mathbb{P}^{n-1}$-bundle
- its $\mathcal{O}(1)$ line bundle is $\mathcal{O}(-E)$

so this is an analogue of his projective bundle result.

Universal hyperplane

Set up:

- variety and line bundle $(X, \mathcal{O}_X(1))$
- basepoint free linear system $V \subseteq H^0(\mathcal{O}_X(1))$

Equivalently

- map $f\colon X \to \mathbb{P}(V^*)$ such that
- $\operatorname{im} f$ not contained in any hyperplane

Over the dual projective space $\mathbb{P}(V)$ we have the universal family of hyperplanes $\mathcal{H} \to \mathbb{P}(V)$:

$$\mathcal{H} := \{(x, s) : s(x) = 0\} \subset X \times \mathbb{P}(V).$$

Discriminant locus is classical projective dual $X^\vee \subset \mathbb{P}(V)$ of X.

Given a linear subspace $L \subseteq V$ with annihilator $L^\perp \subseteq V^*$, set

$$X_{L^\perp} := X \times_{\mathbb{P}(V^*)} \mathbb{P}(L^\perp),$$
$$\mathcal{H}_L := \mathcal{H} \times_{\mathbb{P}(V)} \mathbb{P}(L).$$

Projective duality

Baselocus $X_{L^\perp}$ of $L \subseteq H^0(\mathcal{O}_X(1))$ is *contained in every fibre of* $\mathcal{H}_L \to \mathbb{P}(L)$, giving a diagram

$$\begin{array}{ccccc} X_{L^\perp} \times \mathbb{P}(L) & \overset{j}{\hookrightarrow} & \mathcal{H}_L & \overset{\iota}{\hookrightarrow} & X \times \mathbb{P}(L) \\ \downarrow p & & \downarrow \pi & \swarrow \rho & \\ X_{L^\perp} & \overset{i}{\hookrightarrow} & X & & \end{array}$$

Notice that

- π has general fibre $\mathbb{P}^{\ell-2}$ $\quad (\ell := \dim L)$
- over $X_{L^\perp}$ the fibre is $\mathbb{P}(L) = \mathbb{P}^{\ell-1}$

E.g. $\ell = 2$ then $\mathcal{H}_L \to X$ is blow up in codimension-two $X_{L^\perp} \subset X$.

Homological projective duality I

Suppose that $\dim X_{L^\perp} = \dim X - \ell$.

The above diagram gives an inclusion of the derived category of $X_{L^\perp}$ into that of the universal hypersurface $\mathcal{H}_L$ over the linear system:

$$j_* p^* \colon D\big(X_{L^\perp}\big) \hookrightarrow D(\mathcal{H}_L).$$

π^* is also full and faithful. Together these give a semi-orthogonal decomposition

$$D(\mathcal{H}_L) = \Big\langle D\big(X_{L^\perp}\big),\ \pi^* D(X)(0,1),\ \dots\ ,\ \pi^* D(X)(0,\ell-1)\Big\rangle.$$

(i,j) is twist by $\mathcal{O}_X(i) \boxtimes \mathcal{O}_{\mathbb{P}(V)}(j)$ (restricted to $\mathcal{H}_L \subset X \times \mathbb{P}(L)$).

The "interesting part" of the derived category

Want to make $D(\mathcal{H})$ smaller; more comparable to $D(X)$.

Cut it down to its "interesting part" (Kuznetsov).

Standard example: start with $D(\mathbb{P}^{n-1}) = \langle \mathcal{O}, \ldots, \mathcal{O}(n-1) \rangle$.

Restrict to degree-d hypersurface $H \subset X$, then $\mathcal{O}(d), \ldots, \mathcal{O}(n-1)$ remains exceptional semi-orthogonal collection on restriction to H.

Define "interesting part" of $D(H)$ to be its right orthogonal:

$$\begin{aligned} \mathcal{C}_H &:= \left\langle \mathcal{O}_H(d), \mathcal{O}_H(d+1), \ldots, \mathcal{O}_H(n-1) \right\rangle^{\perp} \\ &= \left\{ E \in D(H) \colon R\,\mathrm{Hom}(\mathcal{O}(i), E) = 0 \text{ for } i = d, \ldots, n-1 \right\}. \end{aligned}$$

So

$$D(H) = \left\langle \mathcal{C}_H, \mathcal{O}_H(d), \mathcal{O}_H(d+1), \ldots, \mathcal{O}_H(n-1) \right\rangle.$$

Amazingly, $\mathcal{C}_H$ is always a *fractional Calabi-Yau* category.

Examples

Interesting examples include $(d = 2, n \text{ even})$ and $(d = 3, n = 6)$:

- Even dimensional quadrics. $\mathcal{C}_H$ is generated by two spinor bundles which are exceptional and orthogonal to each other;

$$\mathcal{C}_H \cong D(\mathrm{pt} \sqcup \mathrm{pt}).$$

  Families $\rightsquigarrow$ double covers of linear systems of quadrics.

- Cubic fourfolds. $\mathcal{C}_H$ is the derived category of a K3 surface, noncommutative in general.

2-dimensional CY category $RHom(E, F)^* \cong RHom(F, E)[2]$, deformation of $D(K3)$.

"Explains" the Beauville-Donagi holomorphic symplectic form on the Fano variety $F(H)$ of lines in H. $F(H)$ is a moduli space of objects $\pi_{\mathcal{C}_H}(\mathscr{I}_L) \in \mathcal{C}_H$ so inherits Mukai's symplectic structure.

More later!

Families

Put categories $\mathcal{C}_H \subset D(\mathcal{H})$ together over the $\mathbb{P}\big(H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d)\big)$ family of all Hs. So:

- Set $V = H^0(\mathcal{O}_{\mathbb{P}^{n-1}}(d))$
- For $L \subseteq V$, with universal hypersurface $\mathcal{H}_L \subset \mathbb{P}^{n-1} \times \mathbb{P}(L)$,

$$\mathcal{C}_{\mathcal{H}_L} := \big\langle D(\mathbb{P}(L))(d,0),\ D(\mathbb{P}(L))(d{+}1,0),\ \ldots,\ D(\mathbb{P}(L))(n{-}1,0)\big\rangle^{\perp}$$

so that $D(\mathcal{H}_L)$ is

$$\big\langle \mathcal{C}_{\mathcal{H}_L}, D(\mathbb{P}(L))(d,0),\ D(\mathbb{P}(L))(d+1,0),\ \ldots,\ D(\mathbb{P}(L))(n-1,0)\big\rangle$$

- Putting $L = V$ gives $\mathcal{C}_{\mathcal{H}}$, the **HP dual** of $(\mathbb{P}^{n-1}, \mathcal{O}(d))$.

Soon we will refine HPD I by replacing $D(\mathcal{H}_L)$ with $\mathcal{C}_{\mathcal{H}_L}$.

Lefschetz collections

For general $(X, \mathcal{O}_X(1))$ Kuznetsov replaces the Beilinson decomposition with a (rectangular) Lefschetz decomposition:

- an admissible subcategory $\mathcal{A} \subseteq D(X)$
- generating a semi-orthogonal decomposition

$$D(X) = \langle \mathcal{A}, \mathcal{A}(1), \ldots, \mathcal{A}(i-1) \rangle.$$

(Above example is $(X, \mathcal{O}_X(1)) = (\mathbb{P}^{n-1}, \mathcal{O}_{\mathbb{P}^{n-1}}(d))$ and $\mathcal{A} = \langle \mathcal{O}, \mathcal{O}(1), \ldots, \mathcal{O}(d-1) \rangle$ with $i = n/d$. If $d \nmid n$ can use non-rectangular Lefschetz collection.)

Setting $\mathcal{C}_H := \langle \mathcal{A}(1), \mathcal{A}(2), \ldots, \mathcal{A}(i-1) \rangle^{\perp} \subset D(H)$ (for $H \in |\mathcal{O}_X(1)|$) to be the interesting part of $D(H)$ gives

$$D(H) = \langle \mathcal{C}_H, \mathcal{A}(1), \mathcal{A}(2), \ldots, \mathcal{A}(i-1) \rangle.$$

Lefschetz collections and families

Similarly for $X \to \mathbb{P}(V^*)$ and $L \subseteq V$ we set

$$\mathcal{C}_{\mathcal{H}_L} := \left\langle \mathcal{A}(1) \boxtimes D(\mathbb{P}(L)), \dots, \mathcal{A}(i-1) \boxtimes D(\mathbb{P}(L)) \right\rangle^{\perp}$$

so that $D(\mathcal{H}_L)$ is

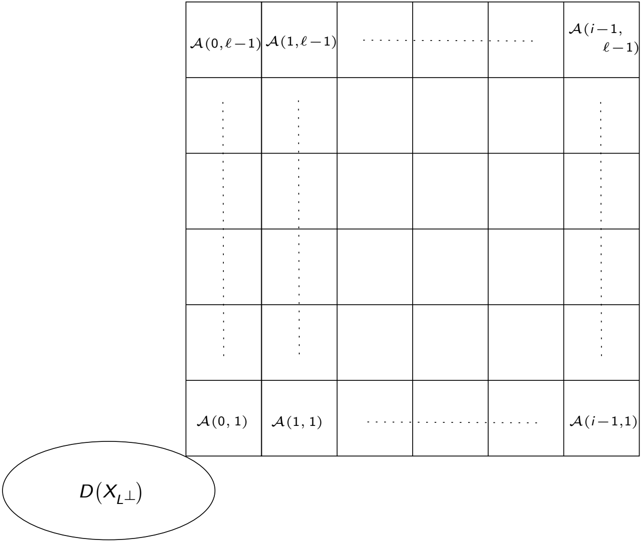
$$\left\langle \mathcal{C}_{\mathcal{H}_L}, D(\mathbb{P}(L))(d,0), D(\mathbb{P}(L))(d+1,0), \dots, D(\mathbb{P}(L))(n-1,0) \right\rangle$$

Putting $L = V$ gives $\mathcal{C}_{\mathcal{H}}$, the **HP dual** of $X \to \mathbb{P}(V)$ with the (rectangular) Lefschetz decomposition $D(X) = \left\langle \mathcal{A}, \dots, \mathcal{A}(i-1) \right\rangle$.

(Kuznetsov asks for $\mathcal{C}_{\mathcal{H}}$ to be *geometric*: i.e. to have a $D(\mathbb{P}(V))$-linear equivalence $\mathcal{C}_{\mathcal{H}} \cong D(Y)$ for some $Y \to \mathbb{P}(V)$.)

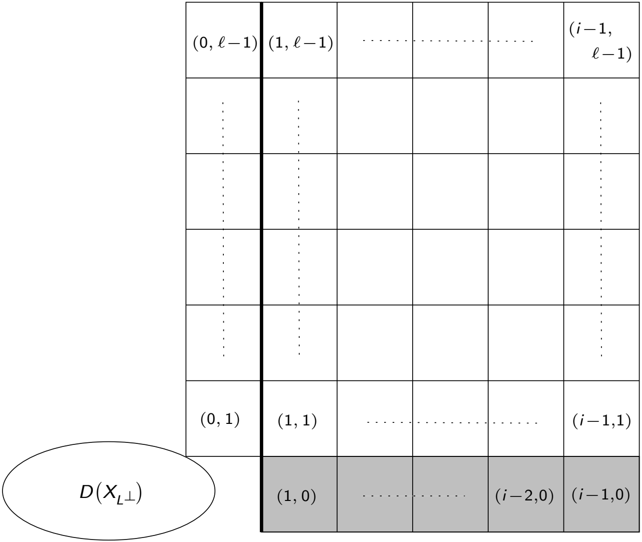
Picturing HPD I

$$D(\mathcal{H}_L) = \left\langle D\left(X_{L^\perp}\right),\ \pi^* D(X)(0,1),\ \dots\ ,\ \pi^* D(X)(0, \ell - 1)\right\rangle$$



Picturing interesting part of $D(\mathcal{H}_L)$

$$\mathcal{C}_{\mathcal{H}_L} = \left\langle \mathcal{A}(1) \boxtimes D(\mathbb{P}(L)), \ldots, \mathcal{A}(i-1) \boxtimes D(\mathbb{P}(L)) \right\rangle^{\perp}$$



HPD II

Projecting ("mutating") the grey boxes into the white boxes in first column we find

Theorem (Kuznetsov)

- *If $\ell > i$ then* $\mathcal{C}_{\mathcal{H}_L} = \big\langle D\big(X_{L^\perp}\big),\ \mathcal{A}(0,1),\ \mathcal{A}(0,2),\ \dots\ ,\ \mathcal{A}(0,\ell-i)\big\rangle$
- *If $\ell = i$ then* $\mathcal{C}_{\mathcal{H}_L} \cong D\big(X_{L^\perp}\big)$
- *If $\ell < i$ then* $D\big(X_{L^\perp}\big) = \big\langle \mathcal{C}_{\mathcal{H}_L},\ \mathcal{A}(1,0),\ \mathcal{A}(2,0),\ \dots\ ,\ \mathcal{A}(i-\ell,0)\big\rangle$

Looks like a duality now! $X \to \mathbb{P}(V^*)$ and its HP dual $\mathcal{C}_{\mathcal{H}}/D(\mathbb{P}(V))$ have similar size and are on the same footing.

Passing to a codimension-ℓ linear section,

- lose the first ℓ copies of $\mathcal{A}(\ \cdot\ )$
- gain (restriction to $X_{L^\perp}$ of) the $\mathbb{P}(L)$ family of categories $\mathcal{C}_H$ as H runs through the hyperplanes containing $X_{L^\perp}$

Example I (with Addington)

Cubic fourfold $X \subset \mathbb{P}^5$. Hodge diamond

$$\begin{matrix} & & 1 & & \\ & & 1 & & \\ 0 & 1 & 21 & 1 & 0 \\ & & 1 & & \\ & & 1 & & \end{matrix}$$

Generically $H^{2,2}(X,\mathbb{Z}) = \langle h^2 \rangle$, but special X have an extra class $T \in H^{2,2}_{\mathrm{prim}}(X,\mathbb{Z})$.

Hassett determined when the integral Hodge structure $\langle h^2, T \rangle^\perp \subset H^4(X,\mathbb{Z})$ is isometric to (Tate twist of) $H^2_{\mathrm{prim}}(S,\mathbb{Z})$ for some polarised K3 surface S.

Kuznetsov: $D(X) \cong \big\langle \mathcal{C}_X, \mathcal{O}, \mathcal{O}(1), \mathcal{O}(2) \big\rangle$ where $\mathcal{C}_X$ is a noncommutative K3 and sometimes really $\cong D(K3)$.

We showed (more-or-less) that X is Hassett $\Longleftrightarrow$ Kuznetsov.

Cubic fourfolds containing a plane

Cubic fourfold $X \subset \mathbb{P}^5$ containing a plane $P \subset X \subset \mathbb{P}^5$.
Defines another plane

$$\mathbb{P}^2 := \left\{\text{3-planes } P \subset \mathbb{P}^3 \subset \mathbb{P}^5\right\}$$

Such a 3-plane intersects X in a singular cubic surface $P \cup Q$.

$\Rightarrow \mathbb{P}^2$ family of quadric surfaces Q; in fact

$$\mathrm{Bl}_P X \to \mathbb{P}^2$$

is a quadric surface fibration, generic fibre $\mathbb{P}^1 \times \mathbb{P}^1$, singular fibres (cone over a conic) over discriminant sextic curve $\subset \mathbb{P}^2$.

$$\begin{array}{c} S := \left\{\text{rulings of fibres}\right\} \\ \downarrow \\ \mathbb{P}^2 \end{array}$$

is the double cover corresponding to the 2 objects in $\mathcal{C}_{\mathrm{fibre}}$.
(Branched over discriminant sextic curve.)

$S \ni s$ parameterises the sheaves $\iota_* \mathscr{I}_L$ on X, where L is any line in the ruling corresponding to the point $s \in S$.

Projecting into $\mathcal{C}_X \subset D(X)$, find S is a moduli space of objects in $\mathcal{C}_X$. Gives

Theorem (Kuznetsov)

Universal object on $S \times X$ gives an equivalence $D(S) \xrightarrow{\sim} \mathcal{C}_X \subset D(X)$.

(Modulo Brauer class issues.)

We use this description and deformation theory to reach all other divisors of special cubic fourfolds (where a different description of $\mathcal{C}_X \cong D(K3)$ holds).

Example II (with Calabrese)

Two ways to get a Calabi-Yau 3-fold from a pencil of cubic 4-folds:

- $X = X_{3,3} \subset \mathbb{P}^5$ is baselocus of the pencil, i.e. the intersection of two cubic fourfolds,
- $Y \to \mathbb{P}^1$ is (noncommutative) K3 fibration associated to the universal cubic fourfold over the pencil $\mathbb{P}^1$.

Choose *special* cubic fourfolds so Y geometric (commutative). Then HPD predicts

$$D(X) = D(Y).$$

Sheaf on $Y \rightsquigarrow$ sheaf on each cubic fourfold fibre $\rightsquigarrow$ sheaf on baselocus X (contained in each fibre)

Derived (auto)equivalences of CY 3-folds

Pencil *special* $\Rightarrow$ X, Y both singular.

For cubic fourfolds containing a plane we were able to resolve crepantly to give *non-birational* Calabi-Yau 3-folds $\widehat{X}$, $\widehat{Y}$ with

$$D(\widehat{X}) \cong D(\widehat{Y}).$$

For cubic fourfolds with an ODP we were able to resolve crepantly to give *birational* Calabi-Yau 3-folds $\widehat{X}$, $\widehat{Y}$ with an autoequivalence

$$D(\widehat{X}) \circlearrowleft .$$

Example III (with Segal)

The Grassmannian of 2-planes in $\mathbb{C}^{2n}$,

$$\mathrm{Gr}(2,2n) \subset \mathbb{P}\big(\Lambda^2\mathbb{C}^{2n}\big)$$

has classical projective dual the Pfaffian variety of degenerate 2-forms (rank $\leqslant 2n-2$) on $\mathbb{C}^{2n}$:

$$\mathrm{Pf}(2n-2,2n) \subset \mathbb{P}\big(\Lambda^2(\mathbb{C}^{2n})^*\big).$$

Pf is a (singular) degree-n hypersurface $\big\{\omega \in \Lambda^2(\mathbb{C}^{2n})^* : \omega^{\wedge n} = 0\big\}$.

Kuznetsov conjectures one can pass to a "categorical crepant resolution" so that $\mathrm{Pf} \to \mathbb{P}^*$ becomes HP dual to $\mathrm{Gr} \to \mathbb{P}$.

Taking $L \subset \Lambda^2\mathbb{C}^{2n}$ such that $\mathbb{P}(L^\perp) \subset \mathbb{P}\big(\Lambda^2(\mathbb{C}^{2n})^*\big)$ misses the singularities of Pf, would get relation

$$D\big(\mathrm{Gr} \cap \mathbb{P}(L)\big) \longleftrightarrow D\big(\mathrm{Pf} \cap \mathbb{P}(L^\perp)\big)$$

Quintic 3-fold

We prove this under some conditions by different methods
("LG models" or matrix factorisations)

Taking $\ell = \dim L = 40$, for instance, gives full and faithful embeddings

$$D(\mathrm{Pf} \cap \mathbb{P}^4) \longrightarrow D(\mathrm{Gr} \cap \mathbb{P}^{39}).$$

LHS is a quintic 3-fold.
Beauville shows the *general* quintic 3-fold is Pfaffian.

RHS is Fano 11-fold, linear section of Gr(2,10).