PUTNAM TRAINING EXERCISE
POLYNOMIALS
(ANSWERS)

November 3rd, 2015

1. Prove that $\sqrt[3]{2+\sqrt{5}} - \sqrt[3]{-2+\sqrt{5}}$ is rational.

- *Answer:* Let $\alpha = \sqrt[3]{2+\sqrt{5}} - \sqrt[3]{-2+\sqrt{5}}$. By raising to the third power, expanding and simplifying we get that α verifies the following polynomial equation:

$$\alpha^3 + 3\alpha - 4 = 0\,.$$

We have $x^3+3x-4 = (x-1)(x^2+x+4)$. The second factor has no real roots, hence x^3+3x-4 has only one real root equal to 1, i.e., $\alpha = 1$.

2. If $a, b, c > 0$, is it possible that each of the polynomials $P(x) = ax^2+bx+c$, $Q(x) = cx^2+ax+b$, $R(x) = bx^2+cx+a$ has two real roots?

- *Answer:* The answer is No. If $P(x)$ has two real roots we would have $b^2 > 4ac$. Analogously for $R(x)$ and $Q(x)$ we should have $a^2 > 4cb$, and $c^2 > 4ab$ respectively. Multiplying the inequalities we get $a^2b^2c^2 > 64a^2b^2c^2$, which is impossible.

3. Suppose that α, β, and γ are real numbers such that

$$\begin{aligned} \alpha+\beta+\gamma &= 2\,,\\ \alpha^2+\beta^2+\gamma^2 &= 14\,,\\ \alpha^3+\beta^3+\gamma^3 &= 17\,. \end{aligned}$$

Find $\alpha\beta\gamma$.

- *Answer:* Writing the given sums of powers as functions of the elementary symmetric polynomials of α, β, γ, we have

$$\begin{aligned} \alpha+\beta+\gamma &= s\,,\\ \alpha^2+\beta^2+\gamma^2 &= s^2-2q\,,\\ \alpha^3+\beta^3+\gamma^3 &= s^3-3qs+3p\,, \end{aligned}$$

where $s = \alpha+\beta+\gamma$, $q = \alpha\beta+\beta\gamma+\alpha\gamma$, $p = \alpha\beta\gamma$.

So we have $s = 2$, and from the second given equation get $q = -5$. Finally from the third equation we get $p = -7$. So, this is the answer, $\alpha\beta\gamma = -7$.

4. Let $n \geq 2$. Write $(1 + x + \cdots + x^n)^2 - x^n$ as the product of two (non-constant) polynomials.

- *Answer:* Letting $A_{n-1} = 1 + x + \cdots + x^{n-1}$, we have

$$\begin{aligned}(1 + x + \cdots + x^n)^2 - x^n &= (A_{n-1} + x^n)^2 - x^n \\ &= A_{n-1}^2 + 2A_{n-1}x^n + x^{2n} - x^n \\ &= A_{n-1}^2 + 2A_{n-1}x^n + (x^n - 1)x^n \\ &= A_{n-1}^2 + 2A_{n-1}x^n + A_{n-1}(x-1)x^n \\ &= A_{n-1}(A_{n-1} + 2x^n + (x-1)x^n) \\ &= A_{n-1}(A_{n-1} + x^n + x^{n+1}) \\ &= (1 + x + \cdots + x^{n-1})(1 + x + \cdots + x^{n+1})\,.\end{aligned}$$

5. Is it possible to write $f(x) = x^{105} - 9$ as the product of two polynomials of degree less than 105 with integer coefficients?

- *Answer:* The answer is no. Proof by contradiction. Assume $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ have integral coefficients and degree less than 105. Assume $g(x)$ has degree k $(0 < k < 105)$, and let $\alpha_1, \ldots, \alpha_k$ the (complex) roots of $g(x)$. For each $j = 1, \ldots, k$ we have $\alpha_j^{105} = 9$, hence $|\alpha_j| = \sqrt[105]{9}$, and $|\alpha_1\alpha_2 \cdots \alpha_k| = (\sqrt[105]{9})^k =$ the absolute value of the constant term of $g(x)$ (an integer), say $(\sqrt[105]{9})^k = a$. Then $a^{105} = 9^k = 3^{2k}$, which is impossible, because an even power of 3 cannot equal an odd power of an integer.

6. Determine all polynomials such that $P(0) = 0$ and $P(x^2 + 1) = P(x)^2 + 1$.

- *Answer:* The answer is $P(x) = x$.

In order to prove this we show that $P(x)$ equals x for infinitely many values of x. In fact, let a_n the sequence $0, 1, 2, 5, 26, 677, \ldots$, defined recursively $a_0 = 0$, and $a_{n+1} = a_n^2 + 1$ for $n \geq 0$. We prove by induction that $P(a_n) = a_n$ for every $n = 0, 1, 2, \ldots$. In the basis case, $n = 0$, we have $P(0) = 0$. For the induction step assume $n \geq 1$, $P(a_n) = a_n$. Then $P(a_{n+1}) = P(a_n^2 + 1) = P(a_n)^2 + 1 = a_n^2 + 1 = a_{n+1}$.

Since in fact $P(x)$ coincides with x for infinitely many values of x, we must have $P(x) = x$ identically.

7. Let n be an even positive integer, and let $p(x)$ be an nth-degree polynomial such that $p(-k) = p(k)$ for $k = 1, 2, \ldots, n$. Prove that there is a polynomial $q(x)$ such that $p(x) = q(x^2)$.

- *Answer:* Let $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$. The $(n-1)$th-degree polynomial $p(x) - p(-x) = 2(a_1x + a_3x^3 + \cdots + a_{n-1}x^{n-1})$ vanishes at n different points, hence, it must be identically null, i.e., $a_1 = a_3 = \cdots = a_{n-1} = 0$. Hence $p(x) = a_0 + a_2x^2 + a_4x^4 + \cdots + a_nx^n$, and $q(x) = a_0 + a_2x + a_4x^2 + \cdots + a_nx^{n/2}$.

8. Prove that there is no non-constant polynomial $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$ with integer coefficients and with the property that $P(0)$, $P(1)$, $P(2)$,..., are all prime numbers.

- *Answer:* By contradiction. We have that $a_0 = P(0)$ must be a prime number. Also, $P(ka_0)$ is a multiple of a_0 for every $k = 0, 1, 2, \ldots$, but if $P(ka_0)$ is prime then $P(ka_0) = a_0$ for every $k \geq 0$. This implies that the polynomial $Q(x) = P(a_0x) - a_0$ has infinitely many roots, so it is identically zero, and $P(a_0x) = a_0$, contradicting the hypothesis that P is non constant.

9. Let $f(x)$ a polynomial with real coefficients, and suppose that $f(x) + f'(x) > 0$ for all x. Prove that $f(x) > 0$ for all x.

- *Answer:* If $f(x_0) = 0$ at some point x_0, then by hypothesis we would have $f'(x_0) > 0$, and f would be (strictly) increasing at x_0. Hence, if the polynomial f becomes zero at some point x_0, then $f(x) > 0$ for every $x > x_0$, and $f(x) < 0$ for every $x < x_0$. Writing $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, we have $f(x) + f'(x) = a_nx^n + (a_{n-1} + na_n)x^{n-1} + \cdots + (a_0 + a_1)$. Given that that $f(x) + f'(x) > 0$ for all x, we deduce that $a_n = \lim_{n\to\infty} \frac{f(x)+f'(x)}{x^n} > 0$. On the other hand n must be even, otherwise $f(x) + f'(x)$ would become negative as $x \to -\infty$. Hence $\lim_{x\to-\infty} f(x) = +\infty$, contradicting $f(x) < 0$ for every $x < x_0$

Remark: The statement is not true for non-polynomial functions in general, e.g., $f(x) = -e^{-2x}$ verifies $f(x) + f'(x) = e^{-2x} > 0$, but $f(x) < 0$ for every x.

10. Determine all polynomials such that $P(0) = 0$ and $P(x^2 + 1) = P(x)^2 + 1$.

- *Answer:* The answer is $P(x) = x$. In order to prove it we show that $P(x)$ equals x for infinitely many values of x. In fact, let a_n the sequence $0, 1, 2, 5, 26, 677, \ldots$, defined recursively $a_0 = 0$, and $a_{n+1} = a_n^2 + 1$ for $n \geq 0$. We prove by induction that $P(a_n) = a_n$ for every $n = 0, 1, 2, \ldots$. In the basis case, $n = 0$, we have $P(0) = 0$. For the induction step assume $n \geq 1$, $P(a_n) = a_n$. Then $P(a_{n+1}) = P(a_n^2 + 1) = P(a_n)^2 + 1 = a_n^2 + 1 = a_{n+1}$. Since in fact $P(x)$ coincides with x for infinitely many values of x, we must have $P(x) = x$ identically.

11. Let $p(x)$ be a polynomial with integer coefficients satisfying that $p(0)$ and $p(1)$ are odd. Show that p has no integer zeros.

- *Answer:* If k is an even integer we have $p(k) \equiv p(0) \pmod 2$, and if it is odd then $p(k) \equiv p(1) \pmod 2$. Since $p(0)$ and $p(1)$ are odd we have $p(k) \equiv 1 \pmod 2$ for every integer k, so $p(k)$ cannot be zero.

12. Prove that the sum

$$\sqrt{1001^2+1} + \sqrt{1002^2+1} + \cdots + \sqrt{2000^2+1}$$

is irrational.

- *Answer:* We prove it by showing that the sum is the root of a monic polynomial with integer coefficients, but not an integer—so by the rational roots theorem it must be irrational.

First we notice that $n < \sqrt{n^2+1} < n + 1/n$, hence the given sum is of the form

$$S = 1001 + \theta_1 + 1002 + \theta_2 + \cdots + 2000 + \theta_{1000}$$

were $0 < \theta_i < 1/1001$, consequently

$$0 < \theta_1 + \theta_2 + \cdots + \theta_{1000} < 1\,,$$

so S is not an integer.

Now we must prove that S is the root of a monic polynomial with integer coefficients. More generally we will prove that a sum of the form

$$\sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$$

where the a_i's are positive integers, is the root of a monic polynomial. This can be proved by induction on n. For $n = 1$, $\sqrt{a_1}$ is the root of the monic polynomial $x^2 - a_1$. Next assume that $y = \sqrt{a_1} + \sqrt{a_2} + \cdots + \sqrt{a_n}$ is a zero of a monic polynomial with integer coefficients $P(x) = x^r + c_{r-1}x^{r-1} + \cdots + c_0$. We will find a monic polynomial with integer coefficients that has $z = y + \sqrt{a_{n+1}}$ as a zero. We have

$$0 = P(y) = P(z - \sqrt{a_{n+1}}) = (z - \sqrt{a_{n+1}})^r + c_{r-1}(z - \sqrt{a_{n+1}})^{r-1} + \cdots + c_0\,.$$

Expanding the parentheses and grouping the terms that contain $\sqrt{a_{n+1}}$:

$$0 = P(z - \sqrt{a_{n+1}}) = z^r + Q(z) + \sqrt{a_{n+1}}\,R(z)\,.$$

Putting radicals on one side and squaring

$$(z^r + Q(z))^2 = a_{n+1}\,(R(z))^2\ ,$$

so

$$T(x) = (x^r + Q(x))^2 - a_{n+1}\,(R(x))^2$$

is a monic polynomial with integer coefficients with z as a root.