Lecture 1

Derived forms and — Shifted sympl.

Derived forms and — (NC) CY categories

+ : filtered derived category.

A - a commutative ring.

For now: algebra / $k \supset \mathbb{Q}$

<u>A resolution:</u>

$$\begin{tikzcd} & R_\bullet \arrow[d, two heads, "\wr"] \\ k \arrow[ur] \arrow[r] & A \end{tikzcd}$$

epi, quiso of dga

$R_\bullet$ - (comm.) dga concentrated in homological degree ≥ 0 (= cohom ≤ 0)

(More generally: A could be any dga $A_\bullet$ in homol deg ≥ 0).

$R_\bullet$ is <u>semifree</u> if it is $k[t_j \mid_{j \in J}]$ (t_j homog elements) as a <u>graded</u> algebra.

<u>Facts</u>: ① a semifree resolution of $A_\bullet$ always exists

②
$$\begin{tikzcd} R_\bullet \arrow[r, dashed, "\exists"] \arrow[d, "\wr"'] & Q_\bullet \arrow[d, "\wr"] \\ A_\bullet \arrow[r, "f"] & B_\bullet \end{tikzcd}$$

unique/ homotopy in the fol. sense:

③ $\forall$
$$\begin{tikzcd} R_\bullet \arrow[r, shift left, "\tilde{f}_0"] \arrow[r, shift right, "\tilde{f}_1"'] \arrow[d] & Q_\bullet \arrow[d] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

$\tilde{f}_0$ homotopic to $\tilde{f}_1$:

$$R_\bullet \xrightarrow{\tilde{f}} Q_\bullet[t, dt] \overset{ev_0}{\underset{ev_1}{\rightrightarrows}} Q_\bullet$$

$|t| = 0$

$|dt| = 1$

$ev_0 \cdot f = f_0$, $ev_1 \cdot f = f_1$.

Rmk: for two complexes $A^\bullet, B^\bullet$:

$$A^\bullet \longrightarrow B^\bullet \otimes C^\bullet(\Delta^1) \overset{ev_0}{\underset{ev_1}{\rightrightarrows}} B^\bullet$$

$$\| $$

$$k \cdot e_0 + k \cdot e_1 + k \cdot \varepsilon$$

$$\partial e_0 = \varepsilon = -\partial e_1$$

is the same as two morphisms $A^\bullet \overset{f_0}{\underset{f_1}{\rightrightarrows}} B$ and a homotopy h between them:

$$a \longmapsto f_0(a) e_0 + f_1(a) e_1 + h(a) \varepsilon$$

Note also: $C^\bullet(\Delta^1)$ is a dga (with $\cup$ product) but not commutative. If we talk about NC dga (as we will), we could define homotopy as

$$R_\cdot \longrightarrow Q_\cdot \otimes C^\cdot(\Delta^1) \rightrightarrows Q_\cdot$$

(morphisms of dga). This would allow us to stay with dga even over $\mathbb{Z}$. For commutative dga there is no good model for $C^\bullet(\Delta^1)$, so over $\mathbb{Z}$ we have to go to simplicial rings.

Derived functors

$$F : Alg_k \to \mathcal{C}$$

Apply F to (any) $R_\bullet \xrightarrow{\sim} A$

(needs to extend to dga ...)

Example $\quad F(A) = \Omega^1_{A/k}$

$$\begin{aligned} &\quad \| \\ &A\text{-module gen. by } da,\ a \in A \\ &k\text{-linear in } a;\ d1 = 0 \\ &d(ab) - da.b - a.db = 0 \end{aligned}$$

Extend to dga $(R_\bullet, \partial)$:

$\Omega^1_{R_\bullet/k}$ $\quad R_\bullet$-module gen. by

$dr,\ r \in R;\ |dr| = (r);$

k-linear in r; $d1 = 0$;

$$d(rs) = dr\cdot s + (-1)^{|r|} r\cdot ds$$

Action of ∂:

$$\partial(r.ds) = \partial r.\,ds + (-1)^{|r|+1} r.\,d(\partial s)$$

(have to check that this preserves relations).

<u>Fact</u>: If $R_\bullet = k[t_j \mid j \in J]$ then

$$\Omega^1_{R_\bullet/k} \simeq \bigoplus_{j\in J} R_\bullet \cdot dt_j$$

<u>Cor</u>. $\Omega^1_{R_\bullet/k} \xrightarrow{\sim} A \otimes_{R_\bullet} \Omega^1_{R_\bullet/k}$

$$\simeq \bigoplus_{j\in J} A\cdot dt_j$$

(b/c it is free $R_\bullet$-mod)

Ex. 1 $\quad P = k[x_1, \dots, x_n]$

$$A = P/(f)$$

$$R_\bullet = P[\xi] \qquad \partial \xi = f$$

$$\begin{array}{ccc} A \cdot d\xi & \longrightarrow & \bigoplus_j A \cdot dx_j \\ \| & & \| \\ (P/f)\, d\xi & \longrightarrow & \Omega^1_{P/k} / f \cdot \Omega^1 \\ d\xi & \longmapsto & df \end{array}$$

$$H_0 = \operatorname{coker} = \Omega^1_{P/k} / \langle f, df \rangle$$

$$\simeq \Omega^1_{A/k}$$

$H_1 = \ker$

$\underline{\text{ex}}$. $A = k[x]/(x^2)$

$$A \cdot d\xi \longrightarrow A \cdot dx$$

$$x \cdot d\xi \longmapsto x \cdot dx^2 = 2x^2 \cdot dx = 0$$

$$\cap$$

$$\ker = H_1\left(A \otimes_{R_\bullet} \Omega^1_{R_\bullet}\right)$$

Another ex.:

$$A = k[x,y]/(x^2, xy, y^2)$$

Lecture 2

More generally than last time:

$A \to B$ dga in deg ≤ 0 (cohom)

$$\begin{tikzcd} & R \arrow[d, two heads, "\wr"] \\ A \arrow[ur, hook] \arrow[r] & B \end{tikzcd}$$

epi, quiso of dga

$R = A[t_j \mid j \in J]$

exists, unique up to homotopy

$$\mathbb{L}_{B/A} := B \otimes_{R_\bullet} \Omega^1_{R_\bullet/A}$$

$$\left(\simeq \bigoplus_{j \in J} B \cdot dt_j\right)$$

Higher forms

gr ~~COMMUTATIVE~~

$\Omega^\bullet_{A/k}$ = Algebra / k gen. by

da, $|da| = |a| + 1$ (cohom.),

rel : da k-linear in a,

$d1 = 0$,

$d(ab) = da \cdot b + (-1)^{|a|} a \cdot db$

Differential: $d : a \mapsto da \mapsto 0$

If $R = k[t_j \mid j \in J]$:

$$\Omega^\bullet_{R/k} = k[t_j, dt_j \mid j \in J]$$

$$\simeq \operatorname{Sym}\left(\Omega^1_{R/k}[-1]\right)$$

$$\bigoplus \left(\Lambda^p_R \Omega_{R/k}\right)[-p] \simeq \bigoplus \operatorname{Sym}^p\left(\Omega^1_{R/k}[-1]\right)$$

(will actually need a different grading convention sometimes).

But first: NONCOMM VERSION

A is now an $\underline{\text{assoc}}$ alg

$$\begin{tikzcd} & R_\bullet \arrow[d, "\sim"] \\ k \arrow[ur] \arrow[r] & A \end{tikzcd}$$

(or dga in cohom deg ≤ 0)

as graded alg, $R_\bullet = \underbrace{k\langle t_j \mid_{j\in J}\rangle}_{\text{free assoc}}$

Homotopy btw 2 morphisms of dga:

$$R_\bullet \longrightarrow Q_\bullet \otimes \underbrace{C^\bullet(\Delta^1)}_{\text{or } \Omega^\bullet(\Delta^1)} \rightrightarrows Q$$

NC 1-forms: $\Omega^{1,NC}_{A/k}$

A-bimod gen. by da

k-lin in a; $d1=0$

$$d(ab) = da.b + (-1)^{|a|} a.db$$

L. $A = k\langle t_j \mid j \in J\rangle$:

$$\Omega^{1,NC}_{A/k} \simeq \bigoplus_{j\in J} A\cdot dt_j \cdot A$$

(free bimodule)

Also: $\Omega^{\bullet,NC}_{A/k} = \mathrm{Alg}/k$

gen. by a, da (lin in a),

$d1=0$

$|da| = |a|+1$

$d(ab) =$ (same as above)

$$\Omega^{\bullet,NC}_{A/k} = \bigoplus \Omega^{p,NC}_{A/k}$$

by # of da's

$$d : \Omega^{p,NC}_{A/k} \to \Omega^{p+1,NC}_{A/k}$$

if $(A^\bullet, \partial)$ dga, then

$(\Omega^{\bullet,NC}_{A/k}, d+\partial)$ again a dga

Also:

$$\Omega^{1,q}_{A/k} = \Omega^{1,NC}_{A/k} \Big/ \left[A, \Omega^{1,NC}_{A/k}\right]$$

If $A = k\langle t_j \rangle$

$$\Omega^{1,q}_{A/k} \simeq \bigoplus_j A \cdot dt_j$$

Short De Rham complex

$$\xrightarrow{b} A \xrightarrow{d} \Omega^{1,q}_{A/k} \xrightarrow{b} A \xrightarrow{d}$$

$$a \longmapsto da$$

$$a\,db \longmapsto [a,b]$$

$$b \circ d = d \circ b = 0$$

The bar complex (= bar resolution of A as an A-A bimodule):

$$\mathcal{B}_n(A) = A \otimes A^{\otimes n} \otimes A \qquad n \geq 0$$

$$b'(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{j=0}^{n} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_{n+1}$$

$$\Omega^{1,NC}_{A/k} \xrightarrow{\sim} \operatorname{coker}\left(\mathcal{B}_2(A) \xrightarrow{b'} \mathcal{B}_1(A)\right)$$

$$a_0 \cdot da_1 \cdot a_2 \longmapsto a_0 \otimes a_1 \otimes a_2$$

$$C_n(A) = \mathcal{B}_n(A) / [A, \mathcal{B}_n(A)] = \mathcal{B}_n(A) \otimes_{A \otimes A^{op}} A$$

$$\cong A \otimes A^{\otimes n}$$

$$b(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \ldots \otimes a_j \cdot a_{j+1} \otimes \ldots \otimes a_n + (-1)^n a_n a_0 \otimes a_1 \otimes \ldots$$

$$\Omega^{1,nc}_{A/k} \simeq \operatorname{coker}\left(C_2(A) \xrightarrow{b} C_1(A)\right)$$

$\bar{A} = A/k.1$
(modification from last time)

Lecture 3

$$\mathcal{B}_n(A) = A \otimes A^{\otimes n} \otimes A, \quad n \geq 0$$

$$b' : \mathcal{B}_n \to \mathcal{B}_{n-1}$$

$\mathcal{B}_\bullet(A)$ free bimodule resolution of A-A bimod A (we assume A k-flat).

$$C_\bullet(A) = \mathcal{B}_\bullet(A) \otimes_{A \otimes A^{op}} A \qquad b = b' \otimes_{A \otimes A^{op}} A$$

$$C_n(A) = A \otimes \bar{A}^{\otimes n}$$

$(C_\bullet(A), b)$ the Hochschild complex

Its homology: $HH_\bullet(A)$

$$B : C_\bullet(A) \to C_{\bullet+1}(A)$$

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} 1 \otimes a_j \otimes \dots \otimes a_0 \otimes \dots \otimes a_{j-1}$$

(signs change in graded case).

$$B^2 = bB + Bb = b^2 = 0$$

Digression: How do we know B exists?

For any bimodule ${}_A M_A$:

$$TR_A(M) := M/[A, M]$$

For any ${}_A M_B$ and ${}_B N_A$:

$$TR_A(M \otimes_B N) \xrightarrow{\sim} TR_B(N \otimes_A M)$$

$$m \otimes n \longmapsto n \otimes m$$

$$\mathbb{L}$$

Should be true for $\overset{\mathbb{L}}{\otimes}$.

Example: $f : A \to B$ ${}_f B = \mathrm{graph}(f)$

A-B bimod $\simeq B$;

A acts via $f(a)$

$$A \xrightarrow{f} B \xrightarrow{g} A$$

$${}_f B \otimes_B {}_g A \simeq {}_{gf} A$$

$${}_g A \otimes_A {}_f B \simeq {}_{fg} B$$

Should have:

$$C_\bullet(A, {}_{gf}A) \overset{?}{\xrightarrow{\sim}} C_\bullet(B, {}_{fg}B)$$

(Note: $C_\bullet(A, M) = M \otimes \bar{A}^{\otimes \bullet}$

formula same as above but

$a_0 \in M$.)

$$a_0 \otimes \dots \otimes a_n \longmapsto f(a_0) \otimes \dots \otimes f(a_n)$$

$$g(b_0) \otimes \dots \otimes g(b_n) \longleftarrow b_0 \otimes \dots \otimes b_n$$

Should be homotopy inverse.

Say, $g = id$;

$$f_* : C_\bullet(A, {}_fA) \to C_\bullet(A, {}_fA)$$

$$a_0 \otimes \dots \otimes a_n \mapsto f(a_0) \otimes \dots \otimes f(a_n)$$

should be homotopic to zero.

In fact, the homotopy is:

$$B_f(a_0\otimes\dots\otimes a_n) = \sum \pm 1\otimes f(a_j)\otimes\dots\otimes f(a_n)\otimes a_0\otimes a_1\otimes\dots\otimes a_{j-1}$$

<u>In other words:</u>

$$id - f_* = [b, B_f]$$

$$\Downarrow$$

$$[b, B_{id}] = 0.$$

has the air of

$$\text{"}\lim_{f\to id}\frac{(id-f_*)}{\dots}\text{"}$$

and indeed is a nc DeRham differential.

Hochschild complex of a dga R:

$$\bigoplus_{n\geq 0} A \otimes \bar{A}[1]^{\otimes n}$$ (cohom grading; all ≤ 0)

$b + \partial$

of degree 1.

We write $C_n = A \otimes \bar{A}[1]^{\otimes n}$

(as opposed to total grading.

Now, B is of degree -1.

$C_\bullet(A)$ is a dg module over dga $k[\varepsilon]$ $\varepsilon^2 = 0$, $|\varepsilon| = -1$

ε acts by B

OR BY DFN: MIXED COMPLEX

Complex of the same format:

$$\bigoplus_{n\geq 0} \Omega^n_{R/k}[n]$$

if (R,∂) <u>commutative</u> dga.

Also $k[\varepsilon]$-dg mod; ε acts by d. $(= d_{DR})$.

Comparison btwn the two:

HKR

when R an alg (in deg 0): both sit in degree $-n$.

$$\star \quad \overbrace{R\otimes\bar{R}^{\otimes n}[n] \xrightarrow{HKR} \Omega^n_{R/k}[n]}$$

$$a_0\otimes\ldots\otimes a_n \longmapsto \frac{1}{n!} a_0\, da_1 \ldots da_n$$

Intertwines b with 0, B with d.

Lemma for a free graded algebra,

$$
\begin{tikzcd}
\arrow[r] & C_2(R) \arrow[r, "b"] & C_1(R) \arrow[r, "b"] \arrow[d] & C_0(R) \arrow[d, "="] \\
& 0 \arrow[r] & \Omega^{1,\natural}_{R/k} \arrow[r, "b"] & R
\end{tikzcd}
$$

is a quiso.

Pf $R = k\langle V \rangle$; the free resolution

$$R \otimes V \otimes R \to R \otimes R \to R \to 0$$

$$r_1 \otimes v \otimes r_2 \longmapsto r_1 v \otimes r_2 - r_1 \otimes v r_2$$

Apply $\otimes_{R \otimes R^{op}} R$ to it; get the bottom line. ■

Cor. $CC^-_\bullet(R) := C_\bullet(R)[[u]],\ \partial + b + uB$ (negative cyclic complex of a semifree dga)

quiso.

$$\Omega^{1,\natural}_{R/k} \to R \to \Omega^{1,\natural}_{R/k} \to R \to \cdots$$

The left hand side of HKR ($\star$)

is an example of a graded mixed cplx:

Mixed complex $E = \bigoplus_{p \in \mathbb{Z}} E(p)$

($\bigoplus$ of complexes with differential ∂)

$$
\begin{tikzcd}
E(0) \arrow[r, "\varepsilon"', "(-1)"] \arrow[d, equal, dotted] & E(1) \arrow[r, "\varepsilon", "(-1)"'] \arrow[d, equal] & E(2). \arrow[d, equal] \\
\left(\Omega^0_{R/k}\right)^{-2} \arrow[r, "\varepsilon"] \arrow[d, "\partial"] & \left(\Omega^1_{R/k}\right)^{-2} \arrow[r, "\varepsilon = d_{DR}"] \arrow[d, "\partial"] & \left(\Omega^2_{R/k}\right)^{-2} \arrow[d, "\partial"] \\
\left(\Omega^0_{R/k}\right)^{-1} \arrow[r, "\varepsilon"'] \arrow[d, "\partial"] & \left(\Omega^1_{R/k}\right)^{-1} \arrow[r, "\varepsilon"'] \arrow[d, "\partial"] & \left(\Omega^2_{R/k}\right)^{-1} \arrow[d, "\partial"] \\
\left(\Omega^0_{R/k}\right)^{0} \arrow[r, "\varepsilon"'] & \left(\Omega^1_{R/k}\right)^{0} \arrow[r, "\varepsilon"'] & \left(\Omega^2_{R/k}\right)^{0}
\end{tikzcd}
$$

deg $=-2$

deg $=-1$

deg 0

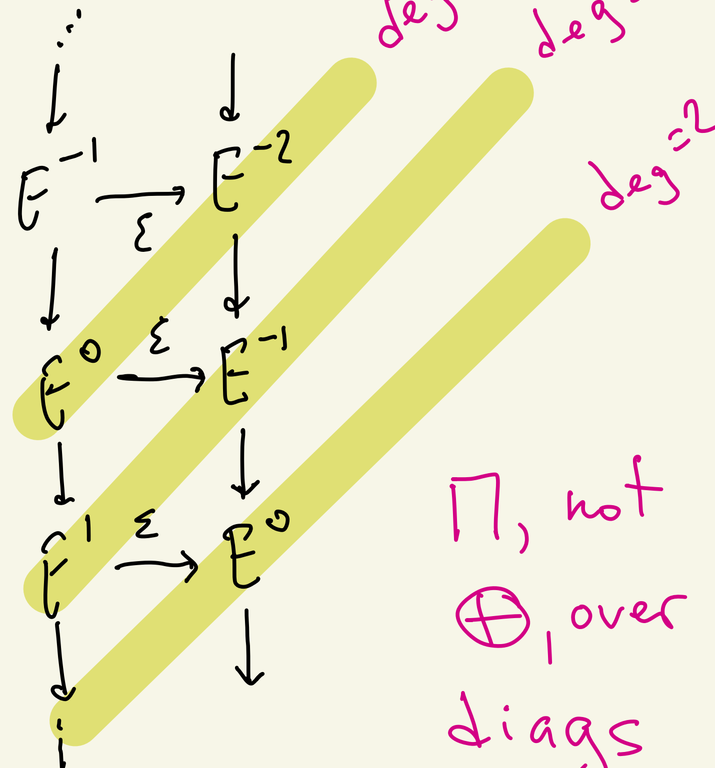
Not a double cplx; $|\partial| = +1$

$|\varepsilon| = -1$

How to turn this into a complex?

For a mixed complex:

given $\cdots \longrightarrow E^{\bullet} \xrightarrow{\partial} E^{\bullet+1} \longrightarrow$ with ε going back.



NC(E): neg cyclic

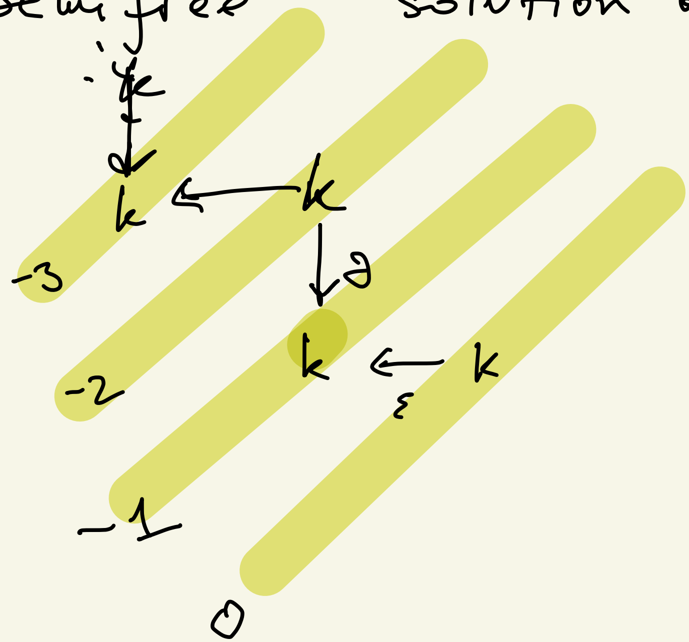
$NC(E) = E[[u]], \partial + u\varepsilon$, $|u| = +2$

$\prod$, not $\bigoplus$, over diags

Fact:

$$NC(E) \simeq \mathbb{R}\mathrm{Hom}_{k[\epsilon]}(k, E)$$

Pf Semifree resolution of k:



$$\| $$

$$\bigoplus_{j\geq 0} k[\epsilon]\cdot a_j \qquad |a_j| = -2j$$

$$\partial : a_j \longmapsto \epsilon a_{j-1}$$

For graded mixed complexes

$E = \bigoplus E(p)$:

$$NC(E)(p)^n = \prod_{j \geq 0} E(p+j)^{n+p-2j}$$

$NC(E)(p)^{-1}$

$NC(E)(p)^{0}$

$NC(E)(p)^{1}$

$$\begin{tikzcd}
{} \arrow[d] & & \\
E(p)^{-1} \arrow[r] \arrow[d] & E(p+1)^{-2} \arrow[r] \arrow[d] & E(p+2)^{-3} \\
E(p)^{0} \arrow[r] \arrow[d] & E(p+1)^{-1} \arrow[r] \arrow[d] & E(p+2)^{-2} \arrow[d] \\
{} & E(p+1)^{0} \arrow[r] \arrow[d] & E(p+2)^{-1} \arrow[d] \\
 & {} & {}
\end{tikzcd}$$

$$NC(E)^w = \bigoplus_p NC(E)(p)$$

$$\mathbb{A}^p(A,n) = \left|\Lambda^p \mathbb{L}_{A/k}[n]\right|$$

$$\mathbb{A}^{p,cl}(A,n) = \left|NC^w(A/k)[n-p](p)\right|$$

What is what here? A - dga in deg ≤ 0

1) $NC^w(A/k) = NC^w(DR(A/k))$

2) For a complex $E^\cdot$, $|E^\cdot|$ is the Dold-Kan of $\tau_{\leq 0}E^\cdot$

$$0 \leq n \longmapsto \mathrm{Hom}_{\text{complxs}}(C_\bullet(\Delta^n), -)$$

$\mathbb{A}^p(A,n)$:

$$\begin{tikzcd}
 & {} \arrow[d, "\partial"'] \\
-1 & (\Omega^p_{A/k})^{n-1} \arrow[d, "\partial"'] \\
0 & (\Omega^p_{A/k})^{n} \arrow[d, "\partial"'] \\
 & (\Omega^p_{A/k})^{n+1} \arrow[d] \\
 & {}
\end{tikzcd}$$

(1.1 of this)

$\mathcal{A}^{p,cl}(A,n)$:

$$\begin{tikzcd}
 & & {} \arrow[d] \\
\left(\Omega^{p}_{A/k}\right)^{n-1} \arrow[r] \arrow[d, "\partial"'] & \left(\Omega^{p+1}_{A/k}\right)^{n-1} \arrow[r] \arrow[d, "\partial"] & \left(\Omega^{p+2}_{A/k}\right)^{n-1} \arrow[d] \\
\left(\Omega^{p}_{A/k}\right)^{n} \arrow[r, "\partial"'] & \left(\Omega^{p+1}_{A/k}\right)^{n} \arrow[r] & {}
\end{tikzcd}$$

deg = 0

deg = −1

(1.1 of this).

Lecture 4

Plan:

A – semi-free comm dga in $\deg \leq 0$.

$\mathcal{A}^{2,cl}(A, n)$: $\quad |\cdot|$ of the complex $\quad \prod^{\infty}$

$$
\begin{tikzcd}
 & u\left(\Omega^3_{A/k}\right)^{n-1} \arrow[r] \arrow[d, "\partial"] & {} \arrow[d] \\
\left(\Omega^2_{A/k}\right)^{n} \arrow[r, "ud"] \arrow[d, "\partial"] & u\left(\Omega^3_{A/k}\right)^{n} \arrow[r] \arrow[d] & u^2 \arrow[d] \\
\left(\Omega^2_{A/k}\right)^{n-1} \arrow[r, "ud"] & u\left(\Omega^3_{A/k}\right)^{n-1} \arrow[r] & {} \\
 & & \cdots
\end{tikzcd}
$$

$\deg = 0$

$\deg = -1$

Cocycle of degree 0:

$$\omega = \omega_2 + \omega_3 + \ldots$$

$$\omega_k \in \left(\Omega^k_{A/k}\right)^{n-k}$$

$$\underbrace{(\partial + d)\omega = 0}_{(\omega \text{ closed})}$$

ω nondegenerate:

$$\omega_2 : T_{A/k} := \mathrm{Der}(A/k) \longrightarrow \Omega^1_{A/k}$$
$$X \longmapsto \iota_X \omega_2$$

is a quasi-isomorphism.

A nondegenerate closed 2-form of degree n is by def. a shifted symplectic structure of degree n.

What happens if we do the above in a noncommutative setting?

Roughly (will develop later):

$$\underset{(?)}{\omega_2} : \Omega^{1,4}_{R/k} \longrightarrow \mathrm{Der}(R/k)$$

But for a commutative A and a noncomm semi-free res. $R \xrightarrow{\sim} A$:

$\Omega^{1,q}_{R/k}$ computes $\simeq$ all $HH_\bullet(A)$

$\mathrm{Der}(R/k)$ computes $\simeq$ all $HH^\bullet(A)$
(later).

HKR: $HH_\bullet(A) \xrightarrow{\sim} \Omega^\bullet_{A/k}$

$HH^\bullet(A) \overset{\sim}{\leftrightarrows} \Lambda^\bullet T_{A/k}$

(not just 1-forms/vectors, but all forms/multivectors).

quis $\{\text{multivectors}\}_{\mathcal{X}} \xrightarrow{\sim} \{\text{vectors}\}_{\mathcal{X}}$

$\updownarrow$

CY structures on $\mathcal{X}$

More generally,

quis $HH^\bullet(A) \longleftrightarrow HH_\bullet(A)$

$\updownarrow$

CY structure on A.

Finally:

a CY strre on a dg category A

$\wr$

"NC version of a shifted symplectic form"

$$\omega_2 : \Omega^{1,4}_{R/k} \xrightarrow{\sim} \mathrm{Der}(R/k)$$

where $R \xrightarrow{\sim} A$

$\wr$

Comm version of a shifted symplectic form on the commutative dga $\mathcal{O}(\mathrm{Rep}_d(R))$ (derived representation scheme).

Plus: Basics of theory of shifted symplectic manifolds (Lagrangian, ...)

How to glue shifted symplectic structures? (Derived) stacks ...
(rather sketchily)

Quantization of shifted s.c.
Poisson —

Lecture 5

From last time:

a couple of calculations/examples.

Ex. 1 $HH_\bullet(k[t]/(t^2))$

$$HH_\bullet\big(k[t]/(t^2)\big)$$

$$\overset{\parallel}{A}$$

$$R \xrightarrow{\sim} A$$

$$\parallel$$

$$k[t,\xi],\ t^2\frac{\partial}{\partial\xi}$$

(1) Direct calculation of $HH_\bullet$:

$$C_n(A) = A \otimes \bar{A}^{\otimes n}$$

$$\parallel$$

$$\langle 1\otimes t^n,\ t\otimes t^n\rangle$$

$$t\otimes t^n \overset{b}{\longmapsto} 0$$

$$1\otimes t \longmapsto 0 \qquad 1\otimes t\otimes t \longmapsto 2t\otimes t$$

$$1\otimes t^3 \longmapsto 0 \qquad \ldots$$

$$\begin{array}{cccc} 1\otimes t^{\otimes 3} & 1\otimes t^{\otimes 2} & 1\otimes t & 1 \\ & t\otimes t^{\otimes 2} \quad 2 \longrightarrow & t\otimes t & t \end{array}$$

$$HH_j(A) \simeq k \quad \text{if } j>0;$$

$$HH_0(A) \simeq A \simeq k^2$$

(2) $$\left(\Omega^{\bullet}_{R/k}, \partial\right) \xrightarrow{\sim} \left(\Omega^{\bullet}_{R/k} \otimes_R A, \partial\right)$$

$$A\cdot 1 \qquad A\cdot dt$$

$$A\cdot (d\xi)^n \qquad A\cdot dt\cdot (d\xi)^n$$

$$d\xi \longmapsto dt^2 = 2t\,dt$$

$$\begin{tikzcd} A & A\,dt \\ A\,d\xi \arrow[ur, "t"] & A\,d\xi\,dt \\ A(d\xi)^2 \arrow[ur, "t"] & \\ {} \arrow[ur, "t"] & \end{tikzcd}$$

ker:
$(d\xi)^n t$
degree $2n$
(homol)

Coker:
$(d\xi)^n dt$
degree
$2n+1$

How to glue? Stacks.

General intro sources:

Fantechi, Stacks for everybody

Toën, ... global overview... '06

Calaque, Three lectures...

Idea #1: (affine) scheme $S \longmapsto$

$\{$Geom structures parametrized by $S\}/\text{iso}$

$\vdots$

<u>Groupoid</u>: objs = $\{$Geom str. param by $S\}$

mor = isos

<u>Ex 1</u> $S \longmapsto$ grpd: objs = $\{$v.bdles of rank r on $S\}$

mor = $\{$isomorphisms$\}$

<u>Ex. 1'</u>. Given G: objs = $\{G\text{-torsors}\}$, ...

Ex. 2 Given a scheme X:

objects = morphisms $X \to S$

morphisms = $\{id\}$ (discrete grpd)

These geom structures <u>should</u>:

① Pull back ② Glue local-to-glob

ex.

$$\begin{tikzcd} f^*E \arrow[r] \arrow[d] & E \arrow[d, "\text{v.b. or } G\text{-torsors}"] \\ T \arrow[r, "f"'] & S \end{tikzcd} \qquad \begin{tikzcd} & X \\ T \arrow[ur] \arrow[r, "f"'] & S \arrow[u] \end{tikzcd}$$

To formalize ①, ②: look at

$$\mathcal{S} = \{\text{Aff schemes}\}$$

as a <u>SITE</u>

<u>A stack</u>: Category fibered in groupoids over $\mathcal{S}$.

- objects glue (i.e. every descent datum is effective)
- Morphisms glue (i.e. form a sheaf)

Recall: a site is a category with a Grothendieck topology, i.e.

1) fiber products $S_i \times_S S_j$ exist;

$$\begin{tikzcd} & S_i \times_S S_j \arrow[dl] \arrow[dr] & \\ S_i \arrow[dr] & & S_j \arrow[dl] \\ & S & \end{tikzcd}$$

2) Class of covers $\{S_i \to S\}_{i \in I}$

Axioms: ...

(generalizing: $U_i \hookrightarrow U$ open)

embeddings...

Generalization #1: Instead of groupoids, ∞-groupoids i.e. Kan simplicial sets (for an actual groupoid Γ, $B\Gamma = $ $= \mathrm{Nerve}(\Gamma)$ is a Kan set).

An ∞-stack is a (homotopy) sheaf on the site $\mathcal{S}$.

The sheaf condition:

$$\mathfrak{X}(S) \xrightarrow[\text{w.e.}]{\sim} \underset{\Delta}{\mathrm{holim}}\ \mathfrak{X}\left(S_{i_0} \underset{S}{\times} \cdots \underset{S}{\times} S_{i_n}\right)$$

(gluing objects and gluing morphisms are both incorporated).

Generalization #2: Derived stacks

Now objects of our $\mathcal{S}$ are derived affine schemes, i.e. (opposite to) $\mathrm{cdga}_{\leq 0}$. MORPHISMS : DFN HOMOT INV

Black box 1 (for now):

dAff ($= \mathrm{cdga}_{\leq 0}$) form a SITE.

(includes: choice of topology)

Toën–Vezzosi, HAG II

(Homotopy) sheaf of Kan sets ($= \infty$-grpds) on that site $=$

$=$ a derived stack.

(Shift of P.O.V.): any construction for $\mathrm{cdg}_{\leq 0}$ that is homotopy invariant + pullsback + glues is a derived stack.

Example $A \longmapsto \mathbb{L}_{A/k} = \Omega^1_{\tilde{A}/k} \otimes_{\tilde{A}} A$

where $\tilde{A} \xrightarrow[\sim]{} A$ a semi-free resolution.

Black box 2: It glues well (descent)

$$\mathbb{L}_{\mathfrak{X}/k} = \operatorname{holim}_{A \to \mathfrak{X}} \mathbb{L}_{A/k}$$

<u>Grey box 3</u> $\mathfrak{X} = X$ (an actual scheme):

$$\mathbb{L}_{X/k} = \text{the usual one}$$

$\mathfrak{X} = \operatorname{hocolim}_{\Delta^{op}} X_\bullet$ (simplicial scheme):

$$\mathbb{L}_{\mathfrak{X}/k} = \operatorname{holim}_{\Delta} \mathbb{L}_{X^n/k}$$

(Usual stack $\Rightarrow$ ∞-stack $\Rightarrow$ derived stack)

$S \mapsto \text{grp } \Gamma(S) \longmapsto \text{Nerve } \Gamma(S)$

$\mathfrak{X}(A) = \mathfrak{X}(S)$

$S = \operatorname{Spec} H^0(A)$

<u>Example</u> G – algebraic grp

$$BG = \operatorname{hocolim}_{\Delta^{op}} B_\bullet G$$

$$\mathbb{L}_{BG/k} = \operatorname{holim}_{\Delta} \mathbb{L}_{G^n/k}$$

$$\| $$

$$\mathbb{L}_{\mathcal{O}(G^n)/k}$$

(G smooth)

$$\| $$

$$\Omega^1_{\mathcal{O}(G^n)/k}$$

More generally: $G \circlearrowright X$

$[X/G]$: $S \mapsto$ action groupoid $\begin{array}{c} G(S)\times X(S) \\ \downarrow\downarrow \\ X(S) \end{array}$

grey box 4: stackification of

or the Kan set $\cdots \substack{\rightarrow\\\rightarrow\\\rightarrow} G(S)\times X(S) \rightrightarrows X(S)$

(sheafification of).

$$\mathbb{L}_{BG/k} \simeq \mathfrak{g}^*[-1]$$

$$\mathbb{L}_{[X/G]/k} \simeq \left(\underset{0}{\Omega^1_X} \to \underset{1}{\mathfrak{g}^*\otimes \mathcal{O}_X}\right)$$

(In what sense and why?)

Cosimplicial commutative ring

$$C^\bullet_{alg}(G, \mathcal{O}_X): \quad \mathcal{O}(X) \underset{d_1}{\overset{d_0}{\rightrightarrows}} \mathcal{O}(X\times G) \underset{d_2}{\overset{d_0}{\substack{\rightarrow\\\rightarrow\\\rightarrow}}} \mathcal{O}(X\times G^2)$$

cosimplicial module:

$$\Omega^1_X \underset{d_1}{\overset{d_0}{\rightrightarrows}} \Omega^1_{X\times G} \substack{\rightarrow\\\rightarrow\\\rightarrow} \Omega^1_{X\times G^2} \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots$$

$$\Omega^1_{G^n\times X} \simeq \Omega^1_X \otimes \mathcal{O}(G^n) \oplus \mathcal{O}_X \otimes \Omega^1_{G^n}$$

$$\Omega^1_{G^n} \simeq \mathfrak{g}^{*\oplus n} \otimes \mathcal{O}_{G^n}$$

$l_i \in \mathfrak{g}$; right-invar vect field

identification: $\underline{\lambda} = (\lambda_1, \dots, \lambda_n) \in \mathfrak{g}^{*\oplus n}$

acts on $l_1 \cdot g_1, \dots, l_n \cdot g_n$ by $\lambda_1(l_1) + \dots + \lambda_n(l_n)$

<u>Example</u> $\quad d_1 : \Omega^1_X \to \mathcal{O}_X \otimes \mathcal{O}_G \otimes \mathfrak{g}^*$

$\underbrace{\quad 1 \quad}_{\text{(component of 1)}}$

The form $\omega \longmapsto$

the form that puts in corresp. to the right-inv vect. field $l \cdot g_1$, $g_1 \in G$

the function $\langle \omega, \underbrace{l \cdot g_1 \cdot x}_{\text{tangent vector @ } g_1 x} \rangle$

$$\| $$

$$g_1^*(\iota_l \omega)$$

$$(d_0\underline{\lambda})(g_1,\dots,g_{n+1}) = (0,\lambda_1,\dots,\lambda_n)(g_2,\dots,g_{n+1})$$

$$(d_j\underline{\lambda})(g_1,\dots,g_{n+1}) = (\lambda_1,\dots,\lambda_j,\, g_j^*\lambda_j,\dots,\lambda_n)(g_1,\dots,g_jg_{j+1},\dots)$$

$$(d_{n+1}\underline{\lambda})(g_1,\dots,g_{n+1}) = g_{n+1}^*\,\underline{\lambda}(g_1,\dots,g_n) \qquad 1\le j\le n$$

$g^*\lambda = (Ad_g)^*(\lambda)$ $\uparrow$ action on this as a fn of X

$$(d_0\omega)(g_1,\dots,g_{n+1}) = \omega(g_2,\dots,g_{n+1})$$

$$(d_j\omega)(g_1,\dots,g_{n+1}) = \omega(\dots,g_jg_{j+1},\dots)$$

$$(d_{n+1}\omega)(g_1,\dots,g_{n+1}) = g_{n+1}^*\,\omega(g_1,\dots,g_n) +$$

$$+\, g_{n+1}^*\,\iota\omega(g_1,\dots,g_n)$$

After a change

$$\omega(g_1,\dots,g_n) \longmapsto (g_1\cdots g_n)^*\,\omega(g_1,\dots,g_n)$$

and

$$(\lambda_1,\dots,\lambda_n)(g_1,\dots,g_n) \mapsto (g_1\cdots g_n)^*\left(\lambda_1, g_1^*\lambda_2,\dots,(g_1\cdots g_{n-1})^*\lambda_n\right)$$

where: g^* is the pullback by g of a $((\mathfrak{g}^*\otimes n)$-valued) function/~~form~~ on X; $g^*\lambda = Ad_g^*(\lambda)$

we get:

$$(d_0 \underline{\lambda})(g_1, \dots, g_{n+1}) = g_1^{-1*}(0, g_1^{-1*}\lambda_1, \dots, g_1^{-1*}\lambda_n)(g_2, \dots, g_{n+1})$$

$$(d_j \underline{\lambda})(g_1, \dots, g_{n+1}) = (\lambda_1, \dots, \lambda_j, \lambda_j, \dots, \lambda_n)(\dots, g_j g_{j+1}, \dots)$$

$$(d_{n+1} \underline{\lambda})(g_1, \dots, g_{n+1}) = (\lambda_1, \dots, \lambda_n, 0)(g_1, \dots, g_n)$$

(for a $\mathfrak{g}^{*\oplus n}$-valued function $\underline{\lambda}$ on $G^n \times X$

$$(d_0 \omega)(g_1, \dots, g_{n+1}) = g_1^{-1*}\omega(g_2, \dots, g_{n+1})$$

$$(d_j \omega)(g_1, \dots, g_{n+1}) = \omega(g_1, \dots, g_j g_{j+1}, \dots, g_{n+1})$$

$$(d_{n+1}\omega)(g_1, \dots, g_{n+1}) = \omega(g_1, \dots, g_n)$$ + cross term:

$$(\underset{1}{0}, \underset{2}{0}, \dots, \underset{n}{0}, \underset{n+1}{\imath . \omega(g_1, \dots, g_n)})$$

Here, for $\omega \in \Omega^1_X$, $\imath\omega \in \mathfrak{g}^* \otimes \mathcal{O}_X$

$l \in \mathfrak{g}$: $\imath\omega(l) = \imath_l \omega$

where $\imath_l$ is contraction by the right invariant field.

$$\imath : \Omega^1_X \longrightarrow \mathfrak{g}^* \otimes \mathcal{O}_X$$

Claim: the cochain complex assoc. to the above cosimplicial v.s. is $\simeq$ to:

$$C^\bullet_{alg}\big(G, \underbrace{\Omega^1_X \xrightarrow{\sim} \mathfrak{g}^* \otimes \mathcal{O}_X}_{\substack{0 \qquad\qquad 1 \\ \text{G-equivariant complex}}}\big)$$

Follows from: for any ab group V, (*)

e.g. $V = \mathfrak{g}^*$,

$$\operatorname{holim}_\Delta \left(0 \rightrightarrows V \substack{\rightarrow\\\rightarrow\\\rightarrow} V \oplus V \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right)$$

$\cong$ (the associated complex)

$\simeq V[-1]$

More precisely:

$\mathfrak{g}^{*\oplus \bullet} \otimes \mathcal{O}(G^\bullet \times X)$ is a bisimplicial k-module.

$$\underbrace{\underbrace{d_0 \downarrow \cdots \downarrow d_{m+1}}_{\text{as in } (*)} \quad \mathfrak{g}^{*\oplus m} \otimes \mathcal{O}(G^n \times X) \quad \underbrace{d_0 \downarrow \cdots \downarrow d_{n+1}}_{C^\bullet(G, (*))}}$$

$$d^{(1)} = \sum_0^{m+1} \pm d_j \qquad d^{(2)} = \sum_0^{n+1} \pm d_j$$

$$C^n_{alg}(G, \mathfrak{g}^{*\oplus n} \otimes \mathcal{O}_X) \xrightarrow{EZ} \bigoplus_{p+q=n} C^p_{alg}(G, \mathfrak{g}^{*\oplus q} \otimes \mathcal{O}_X)$$

$$\downarrow \text{proj}$$

$$C^{n-1}_{alg}(G, \mathfrak{g}^* \otimes \mathcal{O}_X)$$

The Cartan model for equivariant forms/fields and $\Omega^\bullet_{[X/G]/k}$.

$$\Omega^\bullet_{[X/G]/k} = C^\bullet_{alg}\left(G, \mathrm{Sym}^\bullet_{\mathcal{O}_X}\left(\left(\Omega^1_X \to \mathfrak{g}^* \otimes \mathcal{O}_X\right)[-1]\right)\right)$$

$$= C^\bullet_{alg}\left(G, \Omega^\bullet_X \otimes \mathrm{Sym}\, \mathfrak{g}^*[-2]\right)$$

with two differentials ι and d.

$$\left(\Omega^\bullet_X \otimes \mathrm{Sym}\, \mathfrak{g}^*[-2]\right)^G \longrightarrow \Omega^\bullet_{[X/G]/k}$$

On the left, we have Borel's equivariant forms.

Dually,

$$\mathbb{T}_{[X/G]/k} \simeq C^\bullet_{alg}\left(G, \mathrm{Sym}_{\mathcal{O}_X}\left(T_X \leftarrow \mathfrak{g}\otimes\mathcal{O}_X\right)[-1]\right)$$

$$\wr|$$

$$C^\bullet_{alg}\left(G, \Lambda^\bullet_{\mathcal{O}_X} T_X \otimes S(\mathfrak{g}[-2])\right)$$

with the differential induced by $\mathfrak{g} \ni l \longmapsto \xi_l \wedge \cdot$. To that, equivariant multivectors

$$\left(\Lambda^\bullet_{\mathcal{O}_X} T_X \otimes S(\mathfrak{g}[-2])\right)^G$$

map.

Examples of shifted symplectic structures

(M, ω) of degree 0

$T^*[n]X \qquad \omega = d(\sum \xi^i dx_i)$

G reductive group ; of degree 2 on BG (induced by the Killing form)

$$\omega \in S^2(\mathfrak{g}^*)^G$$

of degree one on $[\mathfrak{g}^*/G]$:

$d\omega = 0$

$\iota_\omega = 0$

$$\omega = \sum dx_i \otimes x^i \in [\Omega^1_{\mathfrak{g}^*} \otimes S^1(\mathfrak{g}^*)]^G$$

basis of $\mathfrak{g}$ $= \{$lin fns on $\mathfrak{g}^*\}$ $\qquad$ dual basis of $\mathfrak{g}^*$

On derived critical locus:

$S(x_1, \dots, x_n)$ of degree -1.

$$A = k[x_1, \dots, x_n; \xi_1, \dots, \xi_n]$$

$$\partial : x_j \mapsto 0 \qquad \xi_j \mapsto \frac{\partial S}{\partial x_j}$$

$$|\xi_j| = -1$$

$$(\partial + d)\left(\sum d\xi_j \cdot dx_j\right) = 0$$

$$\parallel$$

$$\sum \frac{\partial^2 S}{\partial x_i \partial x_j} \cdot dx_i\, dx_j = 0$$

Lagrangian structures

$L \xrightarrow{f} X$ (dg schemes; more generally, derived stacks...)
ω

An isotropic structure for f is a homotopy btw $f^*\omega$ and 0 in $\Omega^{2,cl}_{L/k}$.

Such a homotopy defines a homotopy between 0 and

$$\begin{tikzcd} f^*T_X \arrow[r, "\omega"] & T^*_L \\ T_L \arrow[u, "f_*"] \arrow[ur, bend left] & \end{tikzcd}$$

and zero,

and therefore a morphism of complexes of $\mathcal{O}_L$ - modules

$$\begin{tikzcd} f^* T_X \arrow[r, "\omega"] & T^*_L[n] \\ T_L \arrow[u] \arrow[ur] & \end{tikzcd}$$

An isotropic structure is <u>Lagrangian</u> if this morphism is a quisom.

<u>Example</u> $X \circlearrowleft G$ Hamiltonian action.

$$\boxed{\begin{array}{l} H(\ell) \overset{df}{=} \sum a^i H_i \\ \text{for } \ell = \sum a^i \ell_i \end{array}}$$

$x_i \longmapsto H_{x_i}$ (or H_i)

$\{H_i, H_j\} = c_{ij}^k H_k$

$\ell \in \mathfrak{g}$: X_ℓ - corr. vect. field; $L_{X_\ell} = \{H_\ell, -\}$

The shifted symplectic structure of degree $\underline{1}$ on $[\mathfrak{g}^*/G]$:

$$\omega_{taut} \in \left[\Omega^1_{\mathfrak{g}^*} \otimes \mathrm{Sym}^1(\mathfrak{g}^*)\right]^G$$

$$\shortparallel$$

$$\sum d l_i \otimes l^i \quad (l_i \in \mathfrak{g} \text{ as linear functions on } \mathfrak{g}^*)$$

(Nondegenerate closed 2-form of degree 1 on $[\mathfrak{g}^*/G]$).

<u>Claim:</u> $\mu : [X/G] \longrightarrow [\mathfrak{g}^*/G]$

$$\mu(x) = \sum H_i(x) \cdot l^i \in \mathfrak{g}^*$$

is a Lagrangian structure for ω_{taut}.

<u>Indeed:</u>

$$l \mapsto l^p \otimes \{H_p, H(l)\} \qquad \omega_{taut}$$

$$\begin{tikzcd}
\mu^* T_{[\mathfrak{g}/G]} & \mathfrak{g}\otimes\mathcal{O}_X \arrow[r] & \mathfrak{g}^*\otimes\mathcal{O}_X \\
T_{[X/G]} & \mathfrak{g}\otimes\mathcal{O}_X \arrow[u, "="] \arrow[r] & T_X \arrow[u]
\end{tikzcd}$$

$$l^k \otimes L_v H_k \leftarrow\!\shortmid v$$

$$l \mapsto X_l$$

$=$

$$l \mapsto \sum l^p \otimes \{H(l), H_p\}$$

$$\begin{tikzcd}
\mathfrak{g}\otimes\mathcal{O}_X \arrow[r] \arrow[d, "dH(l)"'] & \mathfrak{g}^*\otimes\mathcal{O}_X \arrow[d, "="] & \mu^* T_{[\mathfrak{g}^*/G]} \\
T^*_X \arrow[r] & \mathfrak{g}^*\otimes\mathcal{O}_X & T^*_{[X/G]}
\end{tikzcd}$$

$$l \mapsto l^p \otimes \{H_p, H(l)\}$$

$$\alpha \mapsto l^k \otimes \iota_{X_{l_k}}\alpha$$

$$l \longmapsto l^p \otimes \{H(l), H_p\}$$

$$l^k \otimes L_v H_k$$

$$\begin{tikzcd}
 & \mathfrak{g}\otimes\mathcal{O}_X \arrow[r] \arrow[d, "dH(l)"'] & \mathfrak{g}^*\otimes\mathcal{O}_X \arrow[d, "="] \\
\mathfrak{g}\otimes\mathcal{O}_X \arrow[r] & T_X \ni v \longmapsto \iota_v\omega \in T^*_X \arrow[r] & \mathfrak{g}^*\otimes\mathcal{O}_X
\end{tikzcd}$$

$=$

$$l \longmapsto X_l$$

(the vect. field)

$$\alpha \longmapsto l^k \otimes \iota_{X_{l_k}}\alpha$$

isomorphic

We see:

$$T_{[X/G]} \xrightarrow{\sim} \boxed{\begin{matrix}\mu^* T^*_{[\mathfrak{g}^*/G]} \\ \downarrow \\ T^*_{[X/G]}\end{matrix}}$$

Quasi-Hamiltonian actions

For a reductive group G:

Symplectic structure of deg 1 on $[G/G^{ad}]$. Assume $G = GL_n$.

$$\omega_3 \in \Omega^3_G;\quad \omega_3 = \frac{1}{6}\mathrm{tr}(g^{-1}dg)^3$$

$$\omega_1 \in \Omega^1_G \otimes \mathrm{Sym}^1(\mathfrak{g}^*);$$

$$\omega_1(v) = \frac{1}{2}\mathrm{tr}(v(g^{-1}dg + dg\cdot g^{-1})$$

$$(d+\iota)(\omega_1 + \omega_3) = 0.$$

An isotropic structure:

$$\begin{array}{ccc} X & \xrightarrow{\mu} & G \\ \circlearrowleft & & \circlearrowleft \\ G & & G^{ad} \end{array}$$

equivariant;

$\omega \in (\Omega^2_X)^G$;

$$\iota_v\omega = \frac{1}{2}\mathrm{tr}(v\cdot(d\mu\cdot\mu^{-1} + \mu^{-1}d\mu$$

$$d\omega = \frac{1}{6}\mathrm{tr}(\mu^{-1}d\mu)^3$$

$$\begin{array}{c} \mu^* T_{G/G^{ad}} \\ \uparrow \\ T_{X/G} \end{array}
\qquad
\begin{tikzcd}
\mathfrak{g}\otimes\mathcal{O}_X \arrow[r] & \mathfrak{g}\otimes\mathcal{O}_X & \mathfrak{g}^*\otimes\mathcal{O}_X \arrow[r] \arrow[d, "\mu^\#"] & \mathfrak{g}^*\otimes\mathcal{O}_X \arrow[d, "="] \\
\mathfrak{g}\otimes\mathcal{O}_X \arrow[r] \arrow[u, "="] & T_X \arrow[u, "\mu_\#"'] \arrow[r, "\omega"] & T_X^* \arrow[r] & \mathfrak{g}^*\otimes\mathcal{O}_X
\end{tikzcd}
\qquad
\begin{array}{c} \mu^* T^*_{G/G^{ad}} \\ \downarrow = \\ T^*_{X/G} \end{array}$$

$v \mapsto v - Ad_\mu v$

T_G

$\cong$

contractible subcomplex

$$v^j_i\left((dg.g^{-1})^i_j + (\mu^{-1}.(dg.g^{-1}).\mu)^j_i\right)$$

$$\begin{tikzcd}
 & & \mathfrak{g}^*\otimes\mathcal{O}_X \arrow[d] \\
\mathfrak{g}\otimes\mathcal{O}_X \arrow[r] & T_X \arrow[r, "\omega"] & T_X^*
\end{tikzcd}
\qquad
\begin{array}{c} (dg.g^{-1})^i_j \\ \downarrow \\ (d\mu.\mu^{-1})^i_j \end{array}$$

$\mathfrak{g} \ni v$

$v \mapsto X_v$ – the vector field of v on X

<u>Nondegeneracy</u>: Note that

quasi-Hamiltonian $\Rightarrow$ $X_v \in \ker(\omega)\big|_{T_x}$ if:

$$v + Ad_{\mu(x)} v = 0$$

(Lagrangian str.)

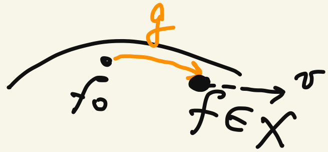
Fact: $\{ X_v \in T_x X \mid v + Ad_{\mu(x)} v = 0 \} = \ker(\omega) \Rightarrow$

Example X = conjugacy class of $f_0 \in G$.

$$\begin{aligned} G &\longrightarrow X \\ g &\longmapsto g f_0 g^{-1} \end{aligned}$$

$v \in \mathfrak{g}$

The 2-form ω on X:

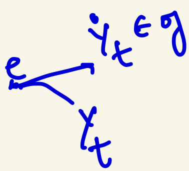


$v = \dot{x}_t$ $\quad e \to x_t$

$X_v \sim (x_t f x_t^{-1})^{\cdot} \in T_f X$

$$\omega(X_v, X_w) = \frac{1}{2}\mathrm{tr}\left(w \cdot \mathrm{Ad}_f(v) - v \cdot \mathrm{Ad}_f(w)\right)$$

The pullback of ω to G under $g \mapsto g f_0 g^{-1}$:



$\gamma_t \cdot g$

$\downarrow$

$\gamma_t \boxed{g \cdot f_0 \cdot g^{-1}} \cdot \gamma_t^{-1} \quad \mapsto \frac{1}{2}\mathrm{tr}\left(\dot{\gamma}_t \cdot g f_0 g^{-1}(\dot{\eta}_t) g f_0 g^{-1}\right.$

$\eta_t \boxed{\phantom{g \cdot f_0 \cdot g^{-1}}} \eta_t^{-1} \quad - \ldots$

$$\omega\big|_G = \mathrm{tr}\left[dg \cdot \cancel{g^{-1}} \cancel{g} \cdot f_0 \cdot g^{-1} \cdot dg \cdot \cancel{g^{-1}} \cancel{g} \cdot f_0^{-1} \cdot g^{-1}\right]$$

$$= \frac{1}{2}\mathrm{tr}\left(g^{-1}dg \cdot \mathrm{Ad}_{f_0}(g^{-1}dg)\right)$$

$$\omega|_G = \tfrac{1}{2}\mathrm{tr}\left[g^{-1}\cdot dg \cdot \mathrm{Ad}_{f_0}(g^{-1}dg)\right]$$

$$\iota_{R_v}\,\omega|_G = \tfrac{1}{2}\mathrm{tr}\left[g^{-1}vg\cdot \mathrm{Ad}_{f_0}(g^{-1}dg) - g^{-1}dg\cdot \mathrm{Ad}_{f_0}(g^{-1}vg)\right]$$

$\uparrow$

right inv vector field of $v \in \mathfrak{g}$ $\quad \|$

$$\tfrac{1}{2}\mathrm{tr}\left[v\cdot \mathrm{Ad}_{gf_0g^{-1}}\underbrace{(dg\cdot g^{-1})} - \underbrace{\cancel{g^{-1}}dg}\cdot f_0 g^{-1}\cdot v\cdot \underbrace{g f_0^{-1} g^{-1}}\right]$$

$$= \mathrm{tr}\left[v\cdot\left(\mathrm{Ad}_{gf_0g^{-1}}(dg\cdot g^{-1}) - \mathrm{Ad}_{gf_0^{-1}g^{-1}}(dg\cdot g^{-1})\right)\right]$$

$=$

$$\tfrac{1}{2}(\mu^{-1}d\mu + d\mu\cdot\mu^{-1}) -$$

$$= \tfrac{1}{2}\left(gf_0^{-1}g^{-1}\cdot d(gf_0g^{-1})\right) + \tfrac{1}{2}d(gf_0g^{-1})\cdot gf_0^{-1}g^{-1}$$

$$= \tfrac{1}{2}\left(gf_0^{-1}g^{-1}\cdot dg\cdot f_0g^{-1}\right) - \tfrac{1}{2}\cancel{(dg\cdot g^{-1})} +$$

$$+\tfrac{1}{2}\cancel{dg\cdot g^{-1}} - \tfrac{1}{2}gf_0g^{-1}\cdot dg\cdot f_0^{-1}g^{-1}$$

$$= \tfrac{1}{2}\left(\mathrm{Ad}_{gf_0^{-1}g^{-1}}(dg\cdot g^{-1}) - \mathrm{Ad}_{gf_0g^{-1}}(dg\cdot g^{-1})\right) \quad \checkmark$$

Next: $d(\omega|_G)$ vs $\frac{1}{6} tr(\mu^{-1}d\mu)^3|_G$

$$d\,tr\left[\frac{1}{2}(g^{-1}dg)\cdot Ad_{f_0}(g^{-1}dg)\right] = -\frac{1}{2}tr\left[(g^{-1}dg)^2\cdot Ad_{f_0}(g^{-1}dg)\right]$$

$$+\frac{1}{2}tr\left[g^{-1}dg\cdot Ad_{f_0}(g^{-1}dg)^2\right]$$

$$\mu^{-1}d\mu = g f_0^{-1} g^{-1}\cdot d(g f_0 g^{-1}) = Ad_{g f_0^{-1} g^{-1}}(dg\cdot g^{-1}) - dg\cdot g^{-1}$$

$$= Ad_g\left[Ad_{f_0^{-1}}(g^{-1}dg) - g^{-1}dg\right]$$

$$tr(\mu^{-1}d\mu)^3 = tr\,\cancel{Ad_g}\left[Ad_{f_0^{-1}}(g^{-1}dg) - g^{-1}dg\right]^3 =$$

$$= \cancel{Ad_g}\,tr\left[\cancel{Ad_{f_0^{-1}}(g^{-1}dg)^3} - 3Ad_{f_0^{-1}}(g^{-1}dg)^2\cdot(g^{-1}dg)\right.$$

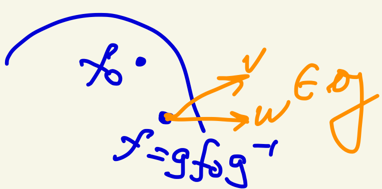
$$\left. + 3Ad_{f_0^{-1}}(g^{-1}dg)\cdot(g^{-1}dg)^2 - \cancel{(g^{-1}dg)^3}\right]$$

Conclusion so far: $X = Ad_G(f_0)$ $\quad f_0\in G$

$\mu: X \hookrightarrow G$ $\qquad \omega$ on X:

$\omega(w,v) = \frac{1}{2}tr(w\cdot Ad_f(v) - v\cdot Ad_f(w))$

is quasi-Hamiltonian.



Ex. The double of G.

$D(G) = G \times G$

$G \times G$ action: $(a,b) \mapsto (g_1 a g_2^{-1}, g_2 b g_1^{-1})$

$\mu : D(G) \to G \times G \qquad (a,b) \mapsto (ab, a^{-1}b^{-1})$

$\omega = \mathrm{tr}\left(a^{-1}da \cdot db \cdot b^{-1} - da \cdot a^{-1} \cdot b^{-1} db\right)$

The quasi-Hamiltonian conditions:

$$\frac{1}{2}\left[d(ab)\cdot(ab)^{-1} + (ab)^{-1}\cdot d(ab)\right] +$$

$$+ \frac{1}{2}\left[d(a^{-1}b^{-1})\cdot ba + ba \cdot d(a^{-1}b^{-1})\right]$$

$$= \frac{1}{2}\left[da \cdot a^{-1} + Ad_a(db \cdot b^{-1}) + Ad_{b^{-1}}(a^{-1}da)\right.$$

$$+ b^{-1}db - a^{-1}da - a^{-1}\cdot b^{-1}db \cdot a +$$

$$\left. + Ad_{b^{-1}}(a^{-1}da)\right]$$

quasi-Hamiltonian vs exponential

$$\exp_s(A) = e^{sA}$$

$$\exp_s^*(dg.g^{-1})(\delta A) = \int_0^s e^{tA}\,\delta A\, e^{-tA}\,dt$$

$$A \in \mathfrak{g} \longrightarrow \delta A$$

$$\frac{\partial}{\partial s}\exp_s^*(dg.g^{-1})(\delta B) = e^{sA}\,\delta B\, e^{-sA}$$

$$\varpi = \frac{1}{2}\int_0^1 \exp_s^*(dg.g^{-1}) \cdot \frac{\partial}{\partial s}\exp_s^*(dg.g^{-1})\,ds$$

$$\varpi(\delta_1 A, \delta_2 A) = \frac{1}{2}\operatorname{tr}\int_0^1 ds \int_0^s e^{tA}\,\delta_1 A\, e^{(s-t)A}\,\delta_2 A\, e^{-sA}\,dt$$

$$-\delta_2 A \qquad \delta_1 A$$

$$= \frac{1}{2}\operatorname{tr}\iint_{0 \le t \le s \le 1} \delta_1 A \cdot e^{(s-t)A} \cdot \delta_2 A\, e^{(t-s)A}\,ds\,dt$$

$$-\delta_2 A \qquad \delta_1 A$$

$v \in \mathfrak{g}$: $\quad \delta A = [v, A]$ (the ad action of $\mathfrak{g}$)

$$\iota_v \varpi(\delta A) =$$

$$= \frac{1}{2} \operatorname{tr} \iint_{0 \le t \le s \le 1} [v, A] e^{(s-t)A} \cdot \delta A \cdot e^{(t-s)A} \, dt \, ds -$$

$$- \frac{1}{2} \operatorname{tr} \iint \delta A \cdot e^{(s-t)A} [v, A] e^{(t-s)A} \, dt \, ds$$

$$= \frac{1}{2} \operatorname{tr} \int_0^1 ds \cdot \delta A \left(\int_0^s e^{(t-s)A} [v, A] e^{(s-t)A} \, dt - \int_0^s e^{(s-t)A} [v, A] e^{(t-s)A} \, dt \right)$$

$$\frac{1}{2} \operatorname{tr} \int_0^1 \left(+ e^{-sA} \cdot [v, e^{sA}] - [v, e^{sA}] e^{-sA} \right) \delta A \, ds$$

$$= - \operatorname{tr}(v \delta A) + \frac{1}{2} \left(\exp_s^* (dg \cdot g^{-1} + g^{-1} dg) \right)$$

$$\iota_v \omega = -d\,\underbrace{\big((v,\cdot)\big)}_{\text{lin fn on } \mathfrak{g}} + \frac{1}{2}\Big(v, \exp_s^*\big(dg\cdot g^{-1} + g^{-1}dg\big)\Big)$$

Recall:

$$\varpi(\delta_1 A, \delta_2 A) = \frac{1}{2}\operatorname{tr} \iint_{0\le t\le s\le 1} \delta_1 A\cdot e^{tA}\cdot \delta_2 A\cdot e^{-tA}\cdot dt\,ds$$

$d\varpi$ will include terms:

$$\frac{1}{2}\operatorname{tr} \iiint_{t_1, t_2 \ge 0 \dots} \delta A_i \cdot e^{t_1 A}\cdot \delta A_j \cdot e^{t_2 A}\cdot \delta A_k\, e^{-(t_1+t_2)A}$$

and

$$\frac{1}{2}\operatorname{tr} \iiint_{t_1, t_2 \ge 0 \dots} \delta A_i \cdot e^{(t_1+t_2)A}\cdot \delta A_j \cdot e^{-t_1 A}\cdot \delta A_k\, e^{-t_2 A}$$

$$\operatorname{tr} \exp^*(dg\cdot g^{-1}) = \operatorname{tr}\Big(\int_0^1\!\!\int_0^1\!\!\int_0^1 e^{x_1 A}\,\delta_1 A\, e^{-x_1 A}\cdot e^{x_2 A}\cdot \delta_2 A\cdot e^{-x_2 A}\cdot \delta_3 A\cdot e^{-x_3 A}\, dx_1\, dx_2\, dx_3$$

$$= \iiint \delta A_1 \cdot e^{(x_1-x_2)A} \cdot \delta A_2 \cdot e^{(x_2-x_3)A} \cdot \delta A_3 \cdot e^{(x_3-x_1)A}$$
$$+ \text{(alt)}$$

two of $x_i - x_j \geq 0$; one ≤ 0.

Comparing the terms, we conclude:

$$d\varpi = -\exp^*\left(\frac{1}{6}\operatorname{tr}(dg\cdot g^{-1})^3\right)$$

<u>and</u>

$$\iota_v \varpi = -d(v,\cdot) + \langle v, \exp_*\left(\tfrac{1}{2}(g^{-1}dg + dg\cdot g^{-1}\right)$$

$v \in \mathfrak{g}$

<u>Corollary</u> (M,σ) $\quad M \xrightarrow{\Phi} \mathfrak{g}^* \underset{\langle,\rangle}{\cong} \mathfrak{g}$

Hamiltonian action of a <u>reductive</u> Lie group.

$$\begin{tikzcd} M \arrow[r, "\Phi"] \arrow[rr, bend right, "\mu"'] & \mathfrak{g} \arrow[r, "\exp"] & G \end{tikzcd}$$

$\omega = \Phi^*\varpi - \sigma$

becomes a <u>quasi</u> Hamiltonian action of G.

<u>Poisson-Lie groups</u> $G, \{,\}$ Poisson;

Lie group; $e: * \to G$; $G \to G$; $G \times G \to G$

$g \mapsto g^{-1}$

Poisson morphisms. Come from deformations of

$$\Delta = \Delta_0 + \hbar \Delta_1 + \hbar^2 \Delta_2 + \cdots \Bigg\} \text{ Hopf algebra}$$

$$a * b = ab + \sum_{n=1}^{\infty} \hbar^n P_n(a,b) \cdots$$

$\boxed{\text{commut cocomm Hopf al}}$

$$\Delta(ab) = \Delta a \cdot \Delta b$$

$$a P_1(b,c) - P_1(ab,c) + P_1(a,bc) - P_1(a,b)c = 0$$

$\Rightarrow \{a,b\} = P_1(a,b) - P_1(b,a)$ is a biderivation;

associativity: $\hbar^2$ term, antisym:

$$\{a,\{b,c\}\} + \circlearrowleft = 0$$

$$\Delta_0(P_1(a,b)) + \Delta_1(ab) = \Delta_0(a)\Delta_1(b) + \Delta_1(a)\Delta_0(b) + (P_1 \otimes 1 + 1 \otimes P_1)(\Delta_0 a, \Delta_0 b)$$

Skew-symmetrize in a, b:

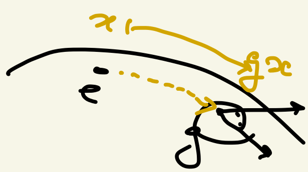
$\Delta_0 : A \to A \otimes A$ is a morphism of Poisson algebras.

$A = \mathbb{C}[G]$ (appropriate class of functions on a Lie group: formal, algebraic, ...) G – a Poisson-Lie group: $G, \{,\}$; $e \mapsto g$, $g \mapsto g^{-1}$, mult – Poisson morphisms

Poisson Lie group $\rightsquigarrow$ Lie bialgebra:

$G \qquad \mathfrak{g} = Lie(G)$

Poisson bracket:



identifies $\Lambda^2 T_g G$ with $\Lambda^2 \mathfrak{g}$

The Poisson-Lie condition:

$$\eta(g_1 g_2) = \eta(g_2) + Ad_{g_2}^{-1} \eta(g_1)$$

Cor: $\eta : \mathfrak{g} \to \mathfrak{g}^*$ skew-self-adjoint)

Differentiate at $g_1 = g_2 = e$:

$$\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g} ;$$

$$\delta([x,y]) = \mathrm{ad}_x \delta(y) - \mathrm{ad}_y \delta(x)$$

Alternatively:

$$\mathfrak{m}_e = \ker(A \xrightarrow{\eta} k) \quad \text{counit} \atop {= ev_{g=e}}$$

$$\mathfrak{g}^* = \mathfrak{m}_e / \mathfrak{m}_e^2 ; \quad \mathfrak{g} = \mathrm{Der}(A, k)$$

(where A acts via η)

$\{a,b\} \in \mathfrak{m}_e$

(e.g. b/c η is a Poisson morphism)

$\Rightarrow \{ , \}$ descends to

$$\Lambda^2 \mathfrak{m}_e/\mathfrak{m}_e^2 \longrightarrow \mathfrak{m}_e/\mathfrak{m}_e^2$$

$$\Lambda^2 \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$$

Lie alg structure

Fact: $X \in \mathfrak{g}$, Xa the left-inv. vector field of $\mathfrak{g}$, $a \in A$:

$$X\{a,b\} = \{Xa, b\} + \{a, Xb\} + \delta(X)(a,b)$$

(where $\delta(X) \in \mathfrak{g} \wedge \mathfrak{g}$ viewed as a left-invariant bivector on G).

Pf $\Delta_0 a := \sum a^{(1)} \otimes a^{(2)}$

$$Xa = \sum a^{(1)} \cdot Xa^{(2)}(e)$$

again, $X\_(e) \in Der(A, k)$

$$X(\{a,b\}) = \sum \{a,b\}^{(1)} \cdot X(\{a,b\}^{(2)})(e)$$

$$= \sum \{a^{(1)}, b^{(1)}\} \cdot X(a^{(2)} b^{(2)})(e) +$$

$$+ \sum a^{(1)} b^{(1)} \cdot X(\{a^{(2)}, b^{(2)}\})(e)$$

$$= \sum \{a^{(1)}, b^{(1)}\} X(a^{(2)})(e) \cdot b^{(2)}(e) +$$

$$+ \sum \{a^{(1)}, b^{(1)}\} \cdot a^{(2)}(e) \cdot Xb^{(2)}(e) +$$

$$+ \sum a^{(1)} b^{(1)} \cdot X(\{a^{(2)}, b^{(2)}\})(e) =$$

$= \{Xa, b\} + \{a, Xb\} + \delta(X)(a \wedge b)$

<u>Remark</u> Same formula for a Poisson action $G \times X \to X$ on a Poisson variety X:

$$\begin{array}{ccc} B & \longrightarrow & B \otimes A \\ \| & & \| \\ \mathcal{O}(X) & & \mathcal{O}(G) \end{array}$$

$a, b \in \mathcal{O}(X)$

$\delta(X) \in \Lambda^2 T_X$ assoc to the action of $\mathfrak{g}$

Manin triple:

$\mathfrak{g} \oplus \mathfrak{g}^*$ Lie alg structure

$\langle\,,\,\rangle$ btw $\mathfrak{g}, \mathfrak{g}^*$ defines an invariant form; $\mathfrak{g}, \mathfrak{g}^*$ - subalgs.

Same as a Lie bialg strucure on $\mathfrak{g}$ (or $\mathfrak{g}^*$).

<u>Example</u> $\mathfrak{g} = \overset{\mathrm{Lie}(K)}{\mathfrak{k}}$ compact Lie group

$$G^{\mathbb{C}} = KAN$$

<u>Example</u>:

$G^{\mathbb{C}} = SL(n, \mathbb{C})$ $G = K = SU(n)$

$A = \{\mathrm{diag}(a_1, \dots, a_n) \,|\, a_i \in \mathbb{R}\}$

$N = N_+ =$ upper triang

$$\mathfrak{g}^{\mathbb{C}} = \underset{\substack{\| \\ \mathfrak{g}}}{\mathfrak{k}} \oplus \underset{\substack{\cong \\ \mathfrak{g}^*}}{\underline{\mathfrak{a} \oplus \mathfrak{n}}}$$

identify via Im(Killing form)

Ex. $\mathfrak{su}(2)$: $E_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $E_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ $E_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$\mathfrak{k}(=\mathfrak{g})$	$\mathfrak{a}$	$\mathfrak{n}$
iE_0	E_0	
$E_+ - E_-$		E_+
$iE_+ + iE_-$		iE_+

$$\mathfrak{g}^* \simeq \mathfrak{a} + \mathfrak{n}$$

$$2(iE_0)^\vee \longleftrightarrow E_0$$

$$(iE_+ + iE_-)^\vee \longleftrightarrow E_+$$

$$-(E_+ - E_-)^\vee \longleftrightarrow iE_+$$

$$\delta : \mathfrak{g}^{\mathbb{C}} \to \Lambda^2 \mathfrak{g}^{\mathbb{C}}$$

$$E_0 \longmapsto 0$$

$$E_\pm \longmapsto iE_0 \wedge E_\pm$$

$$[(iE_+ + iE_-)^\vee, (E_+ - E_-)^\vee] = 0 \; ; \; \left\{ \begin{array}{l} [2iE_0^\vee, (iE_+ + iE_-)^\vee] = 2(\_)^\vee \checkmark \\ [2iE_0^\vee, (E_+ - E_-)^\vee] = 2(E_+E)^\vee \checkmark \end{array} \right.$$

$$\delta : \begin{array}{l} E_+ - E_- \longmapsto iE_0 \wedge (E_+ - E_-) \\ iE_+ + iE_- \longmapsto iE_0 \wedge (iE_+ + iE_-) \end{array}$$

Another formula for δ:

$$\delta(x) = \mathrm{ad}_x\left(-\tfrac{i}{2} E_+ \wedge E_-\right)$$

<u>Rmk</u> More generally: look for Lie bialgebra where $\delta(x) = \mathrm{ad}_x(R)$ $\quad R \in \Lambda^2 \mathfrak{g}$

Get a quadratic condition on R equiv. to the co-Jacobi identity for δ.

In this case, easy to write the Poisson structure on G:

$$\pi(g) = l_g R - r_g R$$

(left - right shift of R to $g \in G$).

<u>Ex.</u> $G = GL_2$ or SL_2. $\quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$

$$g = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix}$$

$$\left.\begin{aligned} \wedge\; & l_g(E_{12}) = t_{11}\frac{\partial}{\partial t_{12}} + t_{21}\frac{\partial}{\partial t_{22}} \\ & l_g(E_{21}) = t_{12}\frac{\partial}{\partial t_{11}} + t_{22}\frac{\partial}{\partial t_{21}} \end{aligned}\right] - \wedge \quad \begin{aligned} & r_g(E_{12}) = t_{21}\frac{\partial}{\partial t_{11}} + t_{22}\frac{\partial}{\partial t_{12}} \\ & r_g(E_{21}) = t_{11}\frac{\partial}{\partial t_{21}} + t_{12}\frac{\partial}{\partial t_{22}} \end{aligned}$$

Up to a constant multiple:

$$\{t_{11}, t_{22}\} = 2 t_{12} t_{21} \qquad \{t_{12}, t_{21}\} = 0$$

(cancels out)

$$\{t_{11}, t_{12}\} = t_{11} t_{12} \qquad \{t_{12}, t_{22}\} = t_{12} t_{22}$$

$$\{t_{11}, t_{21}\} = t_{11} t_{21} \qquad \{t_{21}, t_{22}\} = t_{21} t_{22}$$

Appendix Manin's description of $GL_q(n)$ as Aut (quantum space + Koszul dual).

$n=2$: q plane Koszul dual

$k_q[\mathbb{A}^2]$: $x_2 x_1 = q x_1 x_2$ $\qquad$ $\xi_1 \xi_2 + q \xi_2 \xi_1 = 0$

$\xi_1^2 = 0 = \xi_2^2$

Define relations on t_{ij} so that:

$$x_1 \longrightarrow t_{11} x_1 + t_{12} x_2 \qquad \xi_1 \to t_{11} \xi_1 + t_{21} \xi_2$$

$$x_2 \longrightarrow t_{21} x_1 + t_{22} x_2 \qquad \xi_2 \to t_{12} \xi_1 + t_{22} \xi_2$$

and for dual

would be morphisms

$$k_q(\mathbb{A}^2) \to k_q(\mathbb{A}^2) \otimes k_q(G)$$

$$(t_{21}x_1 + t_{22}x_2)(t_{11}x_1 + t_{12}x_2) =$$

$$= q(t_{11}x_1 + t_{12}x_2)(t_{21}x_1 + t_{22}x_2)$$

$$t_{21}t_{11}x_1^2 + t_{22}t_{11}x_2x_1 + t_{21}t_{12}x_1x_2 +$$

$$+ t_{22}t_{12}x_2^2 =$$

$$= q\,t_{11}t_{21}x_1^2 + q\,t_{12}t_{21}x_2x_1 + q\,t_{11}t_{22}x_1x_2 +$$

$$+ q\,t_{12}t_{22}x_2^2$$

$$t_{21}t_{11} = q\,t_{11}t_{21}; \quad t_{22}t_{12} = q\,t_{12}t_{22};$$

$$q\,t_{22}t_{11} + t_{21}t_{12} = q^2 t_{12}t_{21} + q\,t_{11}t_{22}$$

$$q[t_{22}, t_{11}] = q^2 t_{21}t_{12} - t_{21}t_{12}$$

$$(t_{11}\xi_1 + t_{21}\xi_2)(t_{12}\xi_1 + t_{22}\xi_2)$$
$$= -q(t_{12}\xi_1 + t_{22}\xi_2)(t_{11}\xi_1 + t_{21}\xi_2)$$

$$t_{11}t_{22}\xi_1\xi_2 + t_{21}t_{12}\xi_2\xi_1$$
$$= -q\,t_{12}t_{21}\xi_1\xi_2 - q\,t_{22}t_{11}\xi_2\xi_1$$

$$-q\,t_{11}t_{22} + t_{21}t_{12}$$
$$= q^2 t_{12}t_{21} - q\,t_{22}t_{11}$$

$$q(t_{22}t_{11} - t_{11}t_{22}) = q^2 t_{12}t_{21} - t_{21}t_{12}$$
$$= q^2 t_{21}t_{12} - t_{21}t_{12}$$

$$\Rightarrow [t_{12}, t_{21}] = 0$$

$$[t_{22}, t_{11}] = \frac{q^2-1}{q}\cdot t_{12}t_{21}$$

also: $(t_{11}\xi_1 + t_{21}\xi_2)^2 = 0$

$$(t_{12}\xi_1 + t_{22}\xi_2)^2 = 0$$

We recover $k_q[GL_2]$:

$$t_{12}t_{11} = q\, t_{11}t_{12} \qquad t_{21}t_{11} = q\, t_{11}t_{21}$$

$$t_{22}t_{21} = q\, t_{21}t_{22} \qquad t_{22}t_{12} = q\, t_{12}t_{22}$$

$$t_{12}t_{21} = t_{12}t_{21}$$

$$t_{22}t_{11} - t_{11}t_{22} = (q-q^{-1})\,t_{12}t_{21}$$

<u>quantum determinant</u>:

$$\xi_1\xi_2 \longmapsto (t_{11}\xi_1 + t_{21}\xi_2)(t_{12}\xi_1 + t_{22}\xi_2)$$

$$= t_{11}t_{22}\,\xi_1\xi_2 + t_{21}t_{12}\,\xi_2\xi_1$$

$$= (t_{11}t_{22} - q\, t_{21}t_{12})\,\xi_1\xi_2$$

central

$$\det_q T = t_{11}t_{22} - q\, t_{12}t_{21}$$

subj. rels

central $\quad k_q[GL_2] = k_q[t_{ij}]_{\det_q T}$

1. The general context of reduction (as Lagrangian intersection), via Safronov.

2. Hamiltonian, q Ham., Lie bialgebra reduction

3. Lie bialgebras, Poisson Lie groups...

<u>Reduction as Lagrangian intersection.</u>

<u>Lagrangian correspondence:</u>

$$\begin{tikzcd} & L \arrow[dl] \arrow[dr] & \\ \underset{\omega_X}{X} & & \underset{\omega_Y}{Y} \end{tikzcd}$$

two shifted symplectic structures of degree n.

$\boxed{\text{Lagrangian structure } L \longrightarrow X \times \overline{Y}}$

$$\overline{Y} = (Y, -\omega_Y)$$

$$L_1 \times_Y L_2 = \begin{tikzcd} & & L \arrow[dl] \arrow[dr] & & \\ & L_1 \arrow[dl] \arrow[dr] & & L_2 \arrow[dl] \arrow[dr] & \\ \underset{\omega_X}{X} & & \underset{\omega_Y}{Y} & & \underset{\omega_Z}{Z} \end{tikzcd}$$

$\boxed{\text{a bit of a grey box}}$

will specify

<u>Claim</u>:

Is a Lagr. str. in $X \times \overline{Z}$

$$h = \pi_1^* h_1 + \pi_2^* h_2 \in$$
$$\in \Omega^{2,cl}(L)$$

$$\begin{tikzcd}
 & & L \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dll, bend right] \arrow[drr, bend left] & & \\
 & L_1 \arrow[dl, "p_X"'] \arrow[dr, "p_Y^1"] & & L_2 \arrow[dl, "p_Y^2"'] \arrow[dr, "p_Z"] & \\
X & & Y & & Z
\end{tikzcd}$$

$\omega_X \qquad \omega_Y \qquad \omega_Z$

$(\partial+d)h_1 = p_X^*\omega_X - p_Y^*\omega_Y$

$(\partial+d)h_2 = \underbrace{p_Y^*\omega_Y - p_Z^*\omega_Z}$

$(\partial+d)h =$

$= \pi^* p_X^*\omega_X - \pi^* p_Z^*\omega_Z$

$\underline{\pi_1^*(\partial+d)h_1} = \pi_1^* p_X^*\omega_X - \cancel{\pi_1^* p_Y^*\omega_Y} \qquad \pi_2^*(\partial+d)h_2 = \cancel{\pi_2^* p_Y^*\omega} -$

$- \pi_2^* p_Z^*\omega_Z$

+

Why nondegenerate?

$$L \xrightarrow{f} \mathfrak{X}$$

$(d+\partial)h = f^*\omega \qquad \omega$

$$\begin{tikzcd}
f^*T_{\mathfrak{X}} \arrow[r, "\omega"] & f^*T^*_{\mathfrak{X}} \arrow[d] \\
TL \arrow[u] \arrow[ur, "\text{quis}"] \arrow[r, dashed, "h"'] & T^*L
\end{tikzcd}$$

$$
\begin{array}{c} L_1 \\ \downarrow \\ *_1 \end{array}
\qquad
T_L = \text{this minus } \left(\pi^* p_Y^* T_Y\right)^{diag}
\Leftarrow
\left\{
\begin{array}{l}
\pi_1^* T_{L_1} \to \boxed{\pi_1^* p_X^* T_X^* \oplus \pi_1^* p_Y^{1*} T_Y^* \to \pi_1^* T_{L_1}^*[n]} \\
\oplus \\
\pi_2^* T_{L_2} \to \boxed{\pi_2^* p_Y^{2*} T_Y^* \oplus \pi_2^* p_Z^* T_Z^* \to \pi_2^* T_{L_2}^*[n]}
\end{array}
\right.
$$

$$
\begin{tikzcd}
 & & L \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
 & L_1 \arrow[dl, "p_X"'] \arrow[d, "p_Y^1"] & & L_2 \arrow[dll, "p_Y"'] \arrow[dr, "p_Z"] & \\
X \times & Y & & & Z
\end{tikzcd}
$$

this minus $\left(\pi^* p_Y^* T_Y^*\right)^{diag}$

T_L ?

$$
\begin{tikzcd}
 & L \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
L_1 \arrow[dr, "p_Y^1"'] & & L_2 \arrow[dl, "p_Y^2"] \\
 & Y &
\end{tikzcd}
$$

They are derived stacks. But locally: they are derived Aff schemes, i.e. quasi-free dga$^{\leq 0}$.

$$
\begin{tikzcd}
 & C = A[x_1, \ldots, y_1, \ldots] & & B_1 \otimes_A B_2 = C & \\
A[x_1, \ldots] \arrow[ur] & & A[y_1, \ldots] \arrow[ul] & B_1 \arrow[u, "\pi_1"] & B_2 \arrow[ul] \\
 & k[u_1, u_2, \ldots] = \partial_A \arrow[ul] \arrow[ur] & & A \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$$
C = k[u_1, \ldots, x_1, \ldots, y_1, \ldots] \qquad T_L^* = \Omega^1_{C/k} = \oplus C \cdot \begin{cases} du_i \\ dx_i \\ dy_i \end{cases}
$$

$$
\pi_1^* T_{L_1}^* = \oplus C \cdot \begin{cases} du_i \\ dx_i \end{cases} \qquad \pi_2^* T_{L_2}^* = \oplus C \cdot \begin{cases} du_i \\ dy_i \end{cases}
$$

$$
\boxed{\Big(\underbrace{\pi_1^* p_Y^{1*} = \pi_2^* p_Y^{2*}}\Big) T_Y^*}^{\,diag} \to \boxed{\pi_1^* T_{L_1}^* \oplus \pi_2^* T_{L_2}^*} \to T_L^*
$$

$$
\underset{\oplus C \cdot du_i}{\shortparallel} \qquad \underset{\oplus C \cdot du_i, dx_i}{\shortparallel} \qquad \underset{\oplus C \cdot du_i, dy_i}{\shortparallel}
$$

$$
\begin{tikzcd}
 & & & (\pi^* p_Y^* T_Y^*)^{diag} \arrow[d] & \\
T_L \arrow[d] \arrow[rr, bend left] & & \begin{matrix} \pi_1^* p_X^* T_X \oplus \cancel{\pi^* p_Y^* T_Y} \\ \cancel{\pi^* p_Y^* T_Y^*} \oplus \pi_2^* p_Z^* T_Z \end{matrix} \arrow[r] \arrow[rr, bend right] & \begin{matrix} \pi_1^* T_{L_1}^*[n] \\ \pi_2^* T_{L_2}^*[n] \end{matrix} \arrow[d] & \\
\begin{matrix} \pi_1^* T_{L_1} \\ \oplus \\ \pi_2^* T_{L_2} \end{matrix} \arrow[rru] \arrow[d] & & & T_L^* & \\
(\pi^* p_Y^* T_Y)^{diag} & & & & \\
T_Y^* \arrow[u, "\cong \omega"'] & & & &
\end{tikzcd}
$$

2 copies of $\pi^* p_Y^* T_Y^*$

"Groth group" plausible; cancel out indeed.

Examples

$$\begin{tikzcd}
 & & L \arrow[dl] \arrow[dr] & & \\
 & L_1 \arrow[dl] \arrow[dr] & & L_2 \arrow[dl] \arrow[dr] & \\
* & & Y & & *
\end{tikzcd}$$

ω n-shifted

Lagr. strres
L_1 on Y
L_2 on $\bar{Y}$

$\Downarrow$

$L_1 \times_Y L_2$

Lagr. struct. on $*$

= $(n-1)$-shifted sympl. form.

①. Safronov's context for reduction:

① a shifted sympl. X $(n=1)$

② $M/G \longrightarrow X$

Lagrangian structure

③ "background" Lagrangian structure on X.

$$\begin{tikzcd}
M_{red} = [M/G] \times_X L \arrow[r] \arrow[d] & L \arrow[d] \\
{[M/G]} \arrow[r, "\mu"] & X
\end{tikzcd}$$

"background"

2. e.g.

$$\begin{tikzcd} {[M/G]} \times_{[\mathfrak{g}^*/G]} [\mathcal{O}/G] \arrow[r] \arrow[d] & {[\mathcal{O}/G]} = L \arrow[d] \\ {[M/G]} \arrow[r, "\Phi"'] & {[\mathfrak{g}^*/G]} \end{tikzcd}$$

(or any coadjoint orbit of G)

gen. constr. in derived stacks; its value on "local" $A \in \mathrm{cdga}^{\leq 0}$: just do the fiber prod, perhaps "sheafify"...

How about:

$$\left[(M \times_{\mathfrak{g}^*} \mathcal{O}) / G \right] ?$$

$$\begin{tikzcd} & \mathcal{O}(\mathcal{O}) \arrow[ur] & & \mathcal{O}(M) \otimes^{\mathbb{L}}_{\mathrm{Sym}(\mathfrak{k})} k & k \arrow[l] \\ \mathcal{O}(M) & \mathcal{O}(\mathfrak{g}^*) \arrow[l] \arrow[u] & & \mathcal{O}(M) \arrow[u] & \mathrm{Sym}(\mathfrak{g}) \arrow[l] \arrow[u] \end{tikzcd}$$

$\mathcal{O}(M) \ni H_1, \dots, H_n$

$\mathrm{Sym}(\mathfrak{g}) = k[H_1, \dots, H_n]$

$$= \mathcal{O}[M][\overbrace{\xi_1, \dots, \xi_n}^{\mathfrak{g}}] \circlearrowleft G \qquad \{H_i, H_j\} = c_{ij}^k H_k$$

$$\partial \xi_j = H_j$$

Δ^{op}-dga: $\quad \mathcal{O}(G^{\times \bullet}) \otimes \mathcal{O}(M)[\xi_1, \dots, \xi_n] = C^\bullet_{alg}(G, \ \ )$

2. G reductive;

$$
\begin{tikzcd}
 & {[*/G^{\cancel{ad}}]} \arrow[d] \\
{[M/G]} \arrow[r, "\mu"'] & {[G/G^{ad}]}
\end{tikzcd}
$$

or could be any conj class

qHam reduction.

3. Closely: (Lu) Poisson-Lie reduction

Integration of a Lie bialgebra structure to formal functions on the group:

1) Given a Lie algebra $\mathfrak{g}$:

$\mathrm{Tens}(\mathfrak{g})$ = cofree coalg of $\mathfrak{g}$

1) Shuffle product:

$(x_1 \dots x_n)(y_1 \dots y_m) = \sum (z_1 \dots z_{n+m})$

$z_k = x_i$ or y_j; orders of $x_1, \dots, x_n$ and $y_1, \dots, y_m$ preserved

2). $\{(x_1 \dots x_n), (y_1 \dots y_m)\}$:

write the shuffle product.

In all terms and all positions: take one pair of neighbors $x_i y_j$

and replace by $[x_i, y_j]$.

Get $\mathrm{Tens}^n \otimes \mathrm{Tens}^m \to \mathrm{Tens}^{n+m-1}(\mathfrak{g})$

<u>Fact</u>: $\mathrm{Tens}(\mathfrak{g})$ is a Poisson algebra; Δ cofree a Poisson morphism.

<u>Now dually</u>:

$T(\mathfrak{g}^*)$ free assoc alg + coPoisson coAlg

Assume now: $\mathfrak{g}^*$ <u>also</u> a Lie algebra.

$$T(\mathfrak{g}^*) \twoheadrightarrow U(\mathfrak{g}^*)$$

<u>Fact</u>: if $\mathfrak{g}$ is a Lie bialgebra then the coPoisson coalg str. is ok on

<u>Now</u>: $\mathfrak{g}$ - Lie bialgebra

$U(\mathfrak{g}^*)$: 1) assoc algebra

2) coPoisson coalgebra

$$m: U \otimes U \longrightarrow U$$

morphism of coPoisson coalgebras

$$U^* = U(\mathfrak{g}^*)^*:$$

1) assoc coalgebra

Poisson algebra

$$\Delta: U^* \longrightarrow U^* \otimes U^*$$

is a Poisson morphism.

Get a Poisson-Lie structure on a formal nbhd of e in G^*.

Another way to look at this:

$$\begin{tikzcd} \mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \arrow[r, "\tilde{U}"] \arrow[d, shift left, "\Omega"] & \mathrm{Alg}(\mathrm{CoAlg}_{\mathbb{P}_n}) \\ \mathrm{Alg}_{\mathbb{P}_{n+1}} \arrow[u, shift left, "B"] & \end{tikzcd}$$

$$\begin{tikzcd} \text{Lie bialg where } \delta \text{ is of deg } 1-n \arrow[r, "\tilde{U}"] \arrow[d, shift left, "\Omega"] & \mathrm{Alg}\left(\begin{matrix}\text{Poisson}\\ \text{coalgs}\\ \text{where}\end{matrix}\right) \\ \text{Poisson algebras where } \{,\} \text{ is of deg } -n \arrow[u, shift left, "B"] & \{,\}^\vee \text{ deg } 1-n \end{tikzcd}$$

(e.g. Gerst. algs are $\mathbb{P}_2$ algs)

$$\text{Lie bialg } \mathcal{L} \xmapsto{\ U\ } U(\mathcal{L})$$

$$\Omega \downarrow$$

$S(\mathcal{L}[-n])$

<u>bracket</u>: ind. by $[\,,\,]_{\mathcal{L}}$

<u>differential</u>: ind. by $\delta_{\mathcal{L}}$.

Poiss. cobracket: induced by $\delta_{\mathcal{L}}$

$U \otimes U \to U$: product in $U(\mathcal{L})$ $[\,,\,]_{\mathcal{L}}$

$$\mathcal{L} = (\mathrm{CoLie}(A[1]))[n-1]$$

$$B \uparrow$$

$(A, \{\,,\,\})$

differential: ind. by m_A

bracket: induced by $\{\,,$

And another pair of functors

$$\mathrm{Alg}_{\mathbb{P}_n} \underset{\Omega}{\overset{\mathcal{B}}{\rightleftarrows}} \mathrm{CoAlg}_{\mathbb{P}_n}$$

(Koszul duality).

Operad $\mathbb{P}_{n+1}$		Koszul dual $\mathbb{P}_{n+1}^{!}$
m	$\lvert m \rvert = 1$	$\lvert m^\vee \rvert = 1$
br	$\lvert br \rvert = -n$	$\lvert br^\vee \rvert = 1+n$

$B^\bullet$ a $\mathbb{P}_{n+1}^{!}$-alg:

$\Updownarrow$

$C^\bullet = B^{\bullet-1-n}$ is a $\mathbb{P}_n$-alg.

$\mathbb{P}_n$- alg A $\qquad$ $\mathbb{P}_n$-coalg BA

$$BA = \mathrm{CoCom}\big((\mathrm{CoLie}\, A[d])[e]\big).$$

differential:

$$\mathrm{CoCom}^1\big(\mathrm{CoLie}^2(A[d])[e]\big)$$
$$\downarrow \text{① ind. by } m_A$$
$$(A[d])[e]$$
$$\uparrow \text{② ind. by } br_A$$
$$\mathrm{CoCom}^2\big(\mathrm{CoLie}^1(A[d])[e]\big)$$

d, e
"
?

① $(A[d] \otimes A[d])[e] \xrightarrow{m} A.[d][e]$

must be of deg 1: $\boxed{d=1}$

② $A[1][e] \otimes A[1][e] \xrightarrow{|br_A| = -n} A[1][e]$

must be of deg 1; $\boxed{e=n}$

$BA = \mathrm{CoCom}((\mathrm{CoLie}\, A[1])[n])$

diffl induced by br_A, m_A.

Cobracket of degree $+n$.

<u>Dually</u>: $\Omega C = \mathrm{Com}(\mathrm{Lie}(C[-1])[-n])$

diffl induced by $br^\vee_A, m^\vee_A$

Bracket of degree $-n$

<u>To summarize:</u>

$$\begin{tikzcd}
\mathrm{BiAlg}_{\mathrm{Lie}_{n-1}} \arrow[r, "U"] \arrow[d, shift left, "\Omega"] & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{CoAlg}_{\mathbb{P}_n}) \arrow[d, shift left, "\Omega"] \\
\mathrm{Alg}_{\mathbb{P}_{n+1}} \arrow[u, shift left, "B"] \arrow[r, bend left] \arrow[ur, phantom, "\text{(Safronov)}"] & \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{P}_n}) \arrow[u, shift left, "B"]
\end{tikzcd}$$

which leads to: what are $\mathbb{E}_n$-algs in Sym Mon (∞)-cats? B, Ω vs Sym Mon struct.?