Recall: different versions of Hamiltonian action.

① Hamiltonian:

$$\begin{tikzcd} G \arrow[d, phantom, "\circlearrowright"] & G \arrow[d, phantom, "\circlearrowright"] \\ \underset{\omega}{X} \arrow[r, "\Phi"] & \mathfrak{g}^* \end{tikzcd}$$

$$d\omega = 0$$

$v \in \mathfrak{g}$: $\iota_v \omega = \Phi^* dv$ ← linear fn on $\mathfrak{g}^*$

$\Rightarrow \omega \in \Omega^2(X)^G$

② Poisson spaces (Lu):

G Poisson Lie group

$$\begin{tikzcd} G \arrow[d, phantom, "\circlearrowright"] & G \arrow[d, phantom, "\circlearrowright"] \\ \underset{\omega}{X} \arrow[r, "\mu"] & G^* \end{tikzcd}$$

— dressing action

$v \in \mathfrak{g}$: $\iota_v \omega = 2\mu^* \langle r, dx.x^{-1} \rangle$

$\Rightarrow L_v \omega = d\iota_v \omega = 2\mu^* d \langle r, dx.x^{-1} \rangle =$

$$= g_\mu^* \sum \langle v^{(1)}, dx \cdot x^{-1} \rangle \wedge \langle v^{(2)}, dx \cdot x^{-1} \rangle$$

where $\delta v = \sum v^{(1)} \wedge v^{(2)}$

its Lie bialg ⟵ "Formal Poisson-Lie group"

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathrm{BiAlg}_{\mathrm{Lie}_{n+1}} \arrow[r, shift left, "U"] \arrow[d, shift right, "\Omega"'] & \underline{\mathrm{Alg}}(\mathrm{CoAlg}_{\mathbb{P}_n}) \arrow[l, shift left, squiggly] \arrow[d, shift right, "\Omega"'] \\
\mathrm{Alg}_{\mathbb{P}_{n+1}} \arrow[u, shift right, "B"'] & \underline{\mathrm{Alg}}(\mathrm{Alg}_{\mathbb{P}_n}) \arrow[u, shift right, "B"']
\end{tikzcd}$$

issue: how does Ω, B respect $\otimes$?

DETOUR

Also: $B(A,A) = \mathrm{Bar}\, C^{\bullet}(A,A)$

$\cap$

$$\begin{tikzcd}[row sep=huge, column sep=huge]
\mathrm{BiAlg} \arrow[r, "="'] \arrow[d, shift right, "\mathrm{DeQuant}"'] & \underline{\mathrm{Alg}}(\mathrm{CoAlg}) \arrow[d, shift right, "\mathrm{Cobar}"'] \\
\mathrm{BiAlg}_{\mathrm{Lie}} \arrow[u, shift right, "\mathrm{Quant}"'] \arrow[d, shift right] & \underline{\mathrm{Alg}}(\mathrm{Alg}) \arrow[u, shift right, "\mathrm{Bar}"'] \arrow[d, leftrightarrow, "\wr"'] \\
\mathrm{Alg}_{\mathbb{P}_2} \arrow[u, shift right] & \mathrm{Alg}_{\mathbb{E}_2}
\end{tikzcd}$$

in what sense ?

$\mathbb{E}_1 \times \mathbb{E}_1 = \mathbb{E}_2$ Dunn-Lurie

$\Rightarrow C^{\bullet}(A,A)$ Gerst_{∞} alg

$\Rightarrow C^{\bullet}(A,A)$ $\mathbb{E}_2$ alg

Recall: A assoc alg; $C^\bullet(A,A) = \mathrm{Hom}(A^{\otimes\bullet}, A)$

$$(\varphi \cup \psi)(a_1, \ldots, a_{n+m}) = \varphi(a_1, \ldots, a_n)\psi(a_{n+1}, \ldots, a_{m+n})$$

$$\varphi\{\psi_1, \ldots, \psi_k\}(a_1, \ldots, a_N) = \sum \pm \varphi(\_\ \varphi_1(\_)\ \_ \ldots \varphi_k(\_)\_)$$

where $\_\_$ is $(a_1, \ldots, a_N)$

$[\varphi, \psi] = \varphi\{\psi\} \mp \psi\{\varphi\}$ Gerst. bracket

$$m(a_1, a_2) = (-1)^{|a_1|} a_1 a_2$$

$$\delta = [m, -]$$

$B(A,A) = \mathrm{Bar}\, C^\bullet(A,A) = \mathrm{Tens}\, C^\bullet(A,A)[1]$

coproduct cofree;

$(\varphi_1 | \ldots | \varphi_m) \bullet (\psi_1 | \ldots | \psi_n)$:

1) shuffle them: $\sum \pm (\_\ |\varphi_i|\ \_\ |\psi_j|\ \_)$

2) For any fragment

$$|\varphi_i|\psi_{k+1}| \ldots |\psi_{k+l}|:$$

replace it with $\varphi_i\{\psi_{k+1}, \ldots, \psi_{k+l}\}$.
(OR NOT).

e.g. $(\varphi)\bullet(\psi) = (\varphi|\psi) \pm (\psi|\varphi) \pm (\varphi\{\psi\})$

Fact (Gerst.-Voronov, Getzler-Jones '94).

This is a bialgebra, in fact a Hopf alg.

Why $B(A,A)$?

$\mathcal{B}(A_1, A_2)$ dg <u>cocategory</u>

objects: $f : A_1 \to A_2$

$$\mathcal{B}(A_1, A_2) = \mathrm{Bar}\, \underbrace{C^\bullet(A_1, A_2)}_{\text{dg category}}$$

$$\downarrow$$

objects: $f : A_1 \to A_2$

morphisms $C^\bullet(A_1, A_2)(f,g)$:

$$C^\bullet(A_1, \underbrace{{}_f {A_2}_g}_{A_1\text{- bimodule via } f,g})$$

Fact: $\mathcal{B}(A_1, A_2) \otimes \mathcal{B}(A_2, A_3) \xrightarrow{\bullet} \mathcal{B}(A_1, A_3)$

dg functors ; associative.

(in addition:

$$\mathrm{Bar}(A_1) \otimes \mathcal{B}(A_1, A_2) \longrightarrow \mathrm{Bar}(A_2))$$

So: Algebras form a category in cocategories.

Should Cobar make it into a category in categories, i.e. a 2-category?

Issue: how does Cobar (and Bar) behave under $\otimes$?

For algebras:

$$\mathrm{Bar}(A_1) \otimes \mathrm{Bar}(A_2) \longrightarrow \mathrm{Bar}(A_1 \otimes A_2)$$

shuffle

strictly associative.

$$\begin{tikzcd} \mathrm{Cobar}(B_1 \otimes B_2) \arrow[d, "\mathrm{cosh.}"'] \\ \mathrm{Cob}(B_1) \otimes \mathrm{Cob}(B \end{tikzcd}$$

Way to use this:

$$\mathcal{C}(A_1, \dots, A_{n+1}) = \text{Cobar}\left(\underline{B(A_1, A_2) \otimes \dots \otimes B(A_n, A_{n+1})}\right)$$

Two types of morphisms:

I:

$$\begin{tikzcd}
\mathcal{C}(A_1, \dots, A_{n+1}) \arrow[d] \\
\mathcal{C}(A_1, A_{i_1}, \dots, A_{i_k}, A_{n+1})
\end{tikzcd}$$

induced by $\bullet$ product

II

$$\begin{tikzcd}
\mathcal{C}(A_1, \dots, A_{n+1}) \arrow[d] \\
\mathcal{C}(A_1, \dots, A_k) \otimes \mathcal{C}(A_k, \dots, A_{n+1})
\end{tikzcd}$$

$$2 \leq k \leq n$$

induced by $\underline{\text{co}}$shuffle.

I, II agree with themselves and with each others; II is weak equ.

This is Leinster's definition of a 2-category. It is Segal-like but not quite.

ex.

$$\begin{tikzcd}
& \mathcal{C}(A_1, A_2, A_3) \arrow[r, "I"] \arrow[d] & \mathcal{C}(A_1, A_3) \\
& \mathcal{C}(A_1, A_2) \otimes \mathcal{C}(A_2, A_3) \arrow[dl, "proj"'] \arrow[dr, "proj"] & \\
\mathcal{C}(A_1, A_2) & & \mathcal{C}(A_2, A_3)
\end{tikzcd}$$

If those where allowed:

would get a Segal simplicial structure

$$\begin{tikzcd}
& \mathcal{C}(A_1, A_2) \\
\mathcal{C}(A_1, A_2, A_3) \arrow[ur, "d_2"] \arrow[r, "d_1"] \arrow[dr, "d_0"'] & \mathcal{C}(A_1, A_3) \\
& \mathcal{C}(A_2, A_3)
\end{tikzcd}$$

Perhaps passing to Nerve$_{dg}$ ∘∘∘

END DETOUR

① How do we define $\mathrm{Poly}(\mathfrak{X}, n)$ for a derived stack?

② What could it be for $[X/G]$? (and how do we naturally arrive at Lie bialgebras / Poisson-Lie group/spaces?

① Big issue: forms on $\mathfrak{X}$ much more straightforward. Reason: forms pull back; standard dfns work. (i.e.: forms $\Omega^{\cdot}_{A/k}$ are functorial in A)

$$\Omega^{\cdot}_{\mathfrak{X}/k} = \lim_{T \to \mathfrak{X}} \Omega^{\cdot}_{T}$$

where $T = \mathrm{Spec}(A)$...

But: $\mathrm{Poly}(n, A) = \mathrm{Sym}_A(\mathrm{Der}(A,A)[-n])$

$= \mathrm{Sym}_A(\mathrm{Hom}(\Omega^1_A, A)[-n])$

is NOT functorial in A.

Answer (CPTTV): formal geometry à la Gelfand-Fuks. (Had been used in def. quant. for decades; turns out works in this vast generality).

(LATER).

② Let us try to find our way towards $\mathrm{Poly}(n, \mathfrak{X})$ where $\mathfrak{X} = [X/G]$.

Recall: $T_{[X/G]} = (\overset{0}{\mathfrak{g} \otimes \mathcal{O}_X} \to \overset{1}{T_X})$

(we are not sure what this is; its dual

$T^*_{[X/G]}$ was $C^\bullet_{alg}(G; \underset{-1}{T^*_X} \to \underset{0}{\mathfrak{g}^* \otimes \mathcal{O}_X}))$

Anyway, start with

$$\left[\operatorname{Sym}(\overset{0}{\mathfrak{g}}) \otimes \underset{\operatorname{Sym}_{\mathcal{O}_X} T_X[-1]}{\underset{\parallel}{\Lambda^\bullet \overset{1}{T_X}}}\right]^G$$

$$\partial(X^n \cdot \alpha) = n X^{n-1} \cdot \operatorname{vect}_X \wedge \alpha$$

Try to impose $\{,\}$ of degree -1:

$$\{X\alpha, Y\beta\} = XY\{\alpha,\beta\}$$

Fact: yes, ∂ is a derivation of $\{,\}$ (and of the product).

Proof $\{\partial(X_1\dots X_m\alpha), Y_1 — Y_m\beta\}$

$$= \sum \pm \dots \hat{X}_i — \cdot\, \mathrm{vect}_{X_i} \wedge \{\alpha,\beta\}$$

$$\dots \hat{X}_i — \quad \alpha\cdot \mathrm{vect}_{X_i}(\beta)$$

$$= \sum \pm — \hat{X}_i \dots \cdot\, \mathrm{vect}_{X_i} \wedge \{\alpha,\beta\}$$

(★) $$+ \sum \pm \cancel{— \hat{X}_i} — \hat{Y}_j — [X_i, Y_j]\cdot \alpha\cdot\beta$$

$\partial\{X_1\dots X_m\alpha, Y_1\dots Y_m\beta\}$

$$+ \sum \pm \big(— \underline{\hat{X}_i} — \alpha\big)\cdot \underbrace{X_i(Y_1\dots Y_m\beta)}_{\substack{\| \\ \partial^{CE}(Y_1\dots Y_m\beta)(X_i)}}$$

$$\{X_1 — X_m\alpha, \partial(Y_1\dots Y_m\beta)\} =$$

$$= \sum \pm —— \hat{Y}_j —— \mathrm{vect}_{Y_j} \wedge \{\alpha,\beta\}$$

(★) $\cancel{——}$

$$+ \sum \pm Y_j(X_1\dots X_m\alpha)\cdot\big(— \hat{Y}_j — \beta\big)$$

We see: $\{,\}$ well-defined on

$$[\mathrm{Sym}(\mathfrak{g}) \otimes \Lambda^\bullet T_X]^G$$

There is a natural way to extend it to $C^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}) \otimes \Lambda^\bullet T_X), \partial^{CE}$:

$$\|$$

$$\mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes \Big(\mathrm{Sym}(\mathfrak{g}) \otimes \underbrace{\mathrm{Sym}_{\mathcal{O}_X} T_X[-1]}_{\{,\}_{Sch}}\Big)$$

$$\{,\}$$

Shifted version:

$$C^\bullet(\mathfrak{g}, \mathrm{Sym}(\mathfrak{g}[-n]) \otimes \mathrm{Sym}_{\mathcal{O}_X} T_X[-1-n])$$

$$\|$$

$$\mathrm{Sym}(\mathfrak{g}^*[-1]) \otimes \Big[\mathrm{Sym}\,\mathfrak{g}[-n] \otimes \overset{\{,\}}{\mathrm{Sym}_{\mathcal{O}_X} T_X[-1-n]}\Big]$$

$$\{,\}$$

of degree $-1-n$

For $n=1$ and $X=pt$:

$$C^\bullet(\mathfrak{g}, \mathrm{Sym}\,\mathfrak{g}[-1]) = \mathrm{Sym}(\underbrace{\mathfrak{g}^*[-1] \oplus \mathfrak{g}[-1]}_{\{,\} \text{ of degree } -2})$$

$$\partial^{CE}_{\mathfrak{g}}$$

working candidate/approximation for $\mathrm{Poly}(BG, 1)$.

<u>Claim</u>: a MC element of this dgla sitting in $C^1(\mathfrak{g}, \Lambda^2\mathfrak{g})$

$$\Updownarrow$$

turning $\mathfrak{g}, [,]$ into a Lie bialgebra.

$$(\delta \in C^1(\mathfrak{g}, \Lambda^2\mathfrak{g}); \ \partial^{CE}\delta = 0 ; \ [\delta, \delta] = 0 - \text{coJacobi})$$

<u>Roughly</u>: Lie cobracket compatible with given $[,] \rightsquigarrow$ 1-shifted Poiss. on $\hat{BG}$.

Obvious next step:

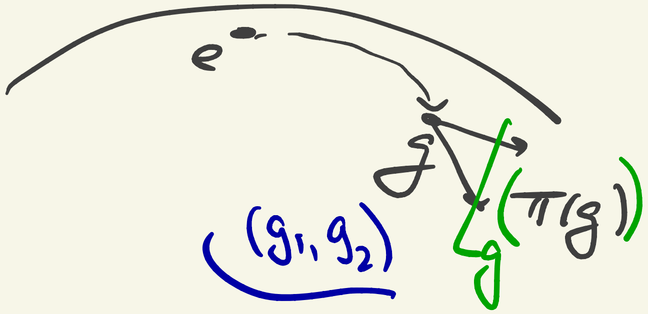
Do this with

$$C^\bullet\left(G, \mathrm{Sym}\,\mathfrak{g}[-n] \otimes \mathrm{Sym}_{\mathcal{O}_X} T_X[-1-n]\right)$$

Schouten-like bracket

$\{ , \}_{Sch}$

Big hint: a Poisson-Lie structure on G is a cocycle in $Z^1(G, \wedge^2\mathfrak{g})$ subject to a condition.



$$\Delta(\{a,b\}) = \{a,b\}(g_1 \cdot g_2) =$$

$$(a\otimes b)\left(g_1 g_2 \cdot \overset{\pi(g_1 g_2)}{\searrow}\right)$$

$$\{a^{(1)}, b^{(1)}\} \cdot a^{(2)} b^{(2)} (g_1, g_2) =$$

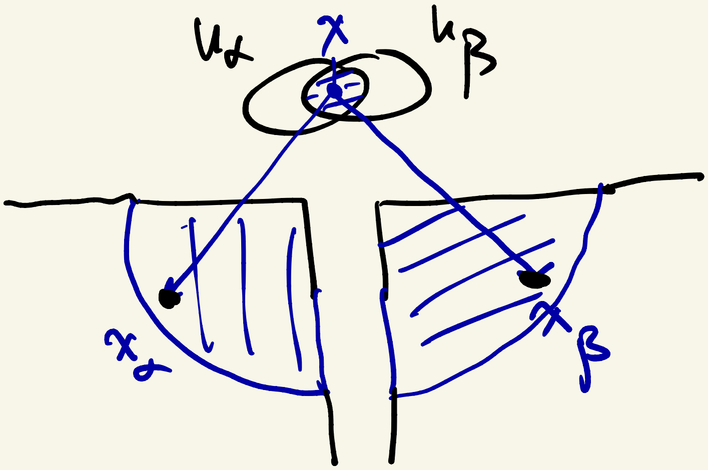
$$= (a\otimes b)\left(g_1 \cdot \overset{\pi(g_1)}{\searrow}\; g_2\right) = (a\otimes b)\left(g_1 g_2 \cdot \mathrm{Ad}_{g_2}^{-1}\pi(g_1)\right)$$

On Formal Geometry

① Classic (Gelfand-Fuks-Kazhdan(-Grothendieck)

M smooth; C^∞ or complex analytic.

The jet bundle on M:



$g_{\alpha\beta}(x_\beta) = x_\alpha$

Bundle of algebras: fiber $\hat{\mathcal{O}} = \mathbb{C}[[\underset{\substack{\parallel \\ (x_1,\ldots,x_n)}}{\hat{x}}]]$

$$x_\alpha + \hat{x}_\alpha = g_{\alpha\beta}(x_\beta + \hat{x}_\beta)$$

In other words: $G_{\alpha\beta} : \mathbb{C}[[\hat{x}_\alpha]] \xrightarrow{\sim} \mathbb{C}[[\hat{x}_\beta]]$

$$\hat{x}_\alpha \longmapsto g_{\alpha\beta}(x_\beta + \hat{x}_\beta) - x_\alpha$$

$$= g'_{\alpha\beta}(x_\beta) \cdot \hat{x}_\beta + \ldots$$

Get a bundle of algebras $\hat{\mathcal{O}}_M$; filtration by powers of $\mathfrak{m} \subset \hat{\mathcal{O}}$;

$$\mathrm{gr}^k \hat{\mathcal{O}}_M = \mathrm{Sym}^k T^*_M$$

$$\nabla_{can} = d_{DR} - \frac{\partial}{\partial \hat{x}_\alpha} \cdot dx_\alpha \quad \text{on} \quad \mathcal{U}_\alpha;$$

well-defined flat connection.

$$\mathcal{O}_M \xrightarrow{\sim} \hat{\mathcal{O}}^\bullet_M = \Omega_M(\hat{\mathcal{O}}_M) = \hat{\mathcal{O}}\text{- valued forms};$$

quis of dga

Creason: locally, $\hat{\mathcal{O}}^\bullet_M = C^\infty(x)[[\cancel{\hat{x}}]]\{\cancel{dx}\}$

$\frac{\partial}{\partial \hat{x}} dx$ – "leading term" in ∇_{can}

$$\begin{array}{ccc} f & \longmapsto & f(x_\alpha + \hat{x}_\alpha) \\ \cap & & \shortparallel \\ \mathcal{O}_M & & \sum \frac{1}{n!} f^{(n)}(x_\alpha) \cdot \hat{x}_\alpha^{\,n} \end{array}$$

well-defined.

No holonomy: $\exp\left((y-x)\frac{\partial}{\partial \hat{x}}\right) : \hat{x} \mapsto \hat{x} + y - x$

not well-defined on $\mathbb{C}[[\hat{x}]]$;

only defined if $y-x$ is itself a *formal* var.

Meaning: $\hat{O}_M$ lifts to a sheaf of modules

on

$$\begin{tikzcd} M & M \hat{\times}_{\Delta} M \arrow[l, shift right, "d_0"'] \arrow[l, shift left, "d_1"] & \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

$\hat{O}_M$ on M; $d_0^* \hat{O}_M \xleftarrow{\sim} d_1^* \hat{O}_M$ on $M \hat{\times}_{\Delta} M$;

$$\begin{tikzcd} d_0^* d_0^* \hat{O}_M & & d_2^* d_1^* \hat{O}_M \arrow[ll, shift right=2, "\sim"'] \arrow[ll, shift left=2, "\sim"] \arrow[ll, phantom, "\parallel"] \end{tikzcd}$$

on $M \hat{\times}_{\Delta} M \hat{\times}_{\Delta} M$.

Now: natural geometric structures like multivectors or forms produce modules over $\hat{O}_M$. For polyvectors, fiber: $\hat{O}\{\frac{\partial}{\partial \hat{x}_1}, \dots, \frac{\partial}{\partial \hat{x}_n}\}$

With this in mind: algebraic case.

1) Recall the standard Čech-DeRham

2) Bhatt: apply 1) in a derived setting.

3) Apply to

$$\begin{tikzcd} X \arrow[d] & X \hat{\times}_{\Delta} X \arrow[l, shift right] \arrow[l, shift left] & \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \\ \text{---} \quad X_{dR} & & \end{tikzcd}$$

Important part: interprete completion

$\left\{\begin{array}{l} \text{in terms of } X_{dR} \\ \text{as a DeRham cplx} \end{array}\right.$

Reminder: "classical" Čech-DeRham

$$\begin{tikzcd} & P \arrow[d] \\ A \arrow[ur] \arrow[r] & B \end{tikzcd}$$

P polynomial over A.

$J_B = \ker(P \twoheadrightarrow B)$

Goal: express I-adic compl. in terms of derived DeRham.
will use to study X_{dR}

$$P \rightrightarrows P \otimes_A P \overset{\longrightarrow}{\rightrightarrows} P \otimes_A P \otimes_A P \cdots$$

$$\partial_j (a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots \otimes 1 \otimes a_j \otimes \ldots \otimes a_n$$

Čech complex. $A \to (P \rightrightarrows \ldots)$ is a

(homotopy: if $\varepsilon : P \to A$ (actually any section of $A \to P$), then
quis

$$\varepsilon(a_0) a_1 \otimes \ldots \otimes a_n \longleftarrow\!\shortmid \; a_0 \otimes \ldots \otimes a_n$$

Now: $J_B^{(n)} = \ker(P^{\otimes n} \longrightarrow B)$

$$P(n+1) = \widehat{P}^{\otimes_A n} \quad (J_B^{(n)}\text{-adic})$$

No longer acyclic b/c ε does not extend to completions.

Fact: up to homotopy, does not depend on choice of $P \twoheadrightarrow A$.

Pf 1)
$$\begin{tikzcd} P \arrow[r, shift left] \arrow[dr] & Q \arrow[l, shift left] \arrow[d] \\ & A \end{tikzcd}$$
2) $\forall$
$$\begin{tikzcd} P \arrow[rr, bend left, "\phi"] \arrow[rr, "\psi"'] \arrow[dr] & & P \arrow[dl] \\ & A & \end{tikzcd}$$
ϕ_*, ψ_* extend to completions; are homotopic.

Homotopy:

$h(\phi,\psi)$

$$\sum \phi(a_0)\otimes\dots\otimes\underbrace{\phi(a_i)\psi(a_{i+1})}\otimes\psi(a_{i+2})\otimes\dots\otimes\psi(a_n) \longleftarrow\!\shortmid\ a_0\otimes\dots\otimes a_n$$

(and similarly higher homotopies $h(\phi_1,\dots,\phi_k)$)

The Čech-DeRham complex: $\left(\Omega^k_{P(n)}\ \text{completed}\ J_B^{(n)}\text{-adically}\right)$

$$\begin{tikzcd}
\vdots & & & \infty & \prod^{\infty} \\
\hat\Omega^2_{P(0)} \arrow[u, "d"] \arrow[r, shift left] \arrow[r, shift right] & \hat\Omega^2_{P(1)} \arrow[u] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \hat\Omega^2_{P(2)} & & \\
\hat\Omega^1_{P(0)} \arrow[u, "d"] \arrow[r, shift left] \arrow[r, shift right] & \hat\Omega^1_{P(1)} \arrow[u] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & {} & & \\
\hat\Omega^0_{P(0)} \arrow[u, "d"] \arrow[r, shift left] \arrow[r, shift right] & \hat\Omega^0_{P(1)} \arrow[u] \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & {} & &
\end{tikzcd}$$

① ALL ARE ~ 0

② All vertical complexes are same up to hom. eq.

Pf $J_B^{(n)}$ differs from J_B by adding formal free variables $x_i^{(0)}-x_i^{(1)},\dots,x_i^{(n-1)}-x_i^{(n)}$ where $x_i | i\in I$ - free generators of P/A.

The horizontal complex $\Omega^1_{P(*)/A}$ acquires a tensor factor

$$\begin{tikzcd}
 & \sum A\,dx_i^{(0)} \arrow[r] & \sum A\,dx_i^{(0)} & \\
\sum A\cdot dx_i^{(0)} \arrow[ur] \arrow[dr] & \oplus \arrow[r] & \sum A\,dx_i^{(1)} & \cdots \\
 & \sum A\cdot dx_i^{(1)} \arrow[r] & \sum A\,dx_i^{(2)} &
\end{tikzcd}$$

which is ~ 0.

So $\hat{\Omega}^1_{P(*)/A}$, and by Künneth $\hat{\Omega}^k_{P(*)/A}$ are contractible to zero.

②: $\hat{\Omega}^\bullet_{P(n)/A}$ differs from $\hat{\Omega}^1_{P(1)/A}$ by tensoring by the DeRham complex of $A[x_i^{(0)}-x_i^{(1)},\dots,x_i^{(n-1)}-x_i^{(n)}]$ over A (completed). ■

<u>We get:</u>

$$\begin{tikzcd}
\hat{\Omega}^\bullet_{P/A} & \cdots \arrow[l, "\sim"', "\text{hom. eq.}"] \arrow[r, "\sim"] & P(*)
\end{tikzcd}$$

$$d_{DR} \qquad\qquad \hat{\partial} = d_0 - d_1 + \dots$$

__Next__: replace

$$\begin{tikzcd} & P \arrow[d, two heads, "\text{polynomial}"] \\ A \arrow[ur] \arrow[r] & B \end{tikzcd}$$

by

$$\begin{tikzcd} & P_* \arrow[d, "\wr"] \\ A \arrow[ur] \arrow[r] & B \end{tikzcd}$$

dg (or simplicial) resolution.

__Idea__: $J_B = \ker(P \to B)$

$J_{B,*} = \ker(P_* \to B)$

$J_{P,*} = \ker(P_* \to P_0)$

1) The above works without change if you complete P_* $J_{B,*}$-adically; get the same answer. (extended indep. of P).

2). Can replace $J_{B,*}$ by $J_{P,*}$.

3). When complete wrt $J_{P,*}$: nothing changes (b/c $J_{P,*}^{m+1}$ does not intersect $P_{\leq m}$). Will get:

completed wrt Hodge filtration

$$\left(\hat{\Omega}^\bullet_{\hat{P}}, d_{DR}\right) \simeq \Omega_{P_*}, \partial_P + d_{DR}$$

One version: FT'85. Bhatt '12: simplicial case.

Completion as Čech or DeRham

Let $A \to \underset{\substack{\shortparallel \\ B}}{A/I}$ where A is Noetherian.

$$\begin{tikzcd} & P_\bullet \arrow[d, "\wr"] \\ A \arrow[ur] \arrow[r] & A/I \end{tikzcd}$$

$P_\bullet(*), \check{\partial}$ Čech (now double) complex

$$\check{C}ech(A\to B)\left\{\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \infty \\ P_1(0) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & P_1(1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & {} \\ P_0(0) \arrow[r, shift left] \arrow[r, shift right] \arrow[uurr, dash] & P_0(1) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & {} \end{tikzcd}\right.$$

((Derived) Čech

(no need to complete J_B adically)

and

$$dR^\cdot_{B/A}\left\{\begin{tikzcd} \vdots \arrow[d] & \vdots \arrow[d] & \infty \\ \Omega^0_{P_1/A} \arrow[r] \arrow[d] & \Omega^1_{P_1/A} \arrow[r] \arrow[d] & {} \\ \Omega^0_{P_0/A} \arrow[r] \arrow[uurr, dash] & \Omega^1_{P_0/A} \arrow[r] & {} \end{tikzcd}\right.$$

(derived DR)

Thm Both compute the I-adic completion of A (Bhatt '12).

More generally, for M Noetherian A-module,

$$M \overset{\mathbb{L}}{\underset{A}{\otimes}} \check{C}ech\,(A \to A/I) \simeq \hat{M} \quad (I\text{-adic}).$$

(same for $dR^{\cdot}$ ...)

Direct proof for Čech; then do Čech-deRham comparison.

=

We now have our new interpretation of I-adic completion. Now back to formal geometry.

=

Recall: For X a smooth scheme,

$$\bigstar \qquad X_{dR} \simeq X \leftleftarrows X \overset{\wedge}{\underset{\Delta}{\times}} X \;\substack{\longleftarrow\\ \longleftarrow\\ \longleftarrow}\; \cdots$$

In general, we will define for a derived stack:

$$X_{dR}(A) = X(A^{red}) \qquad A^{red} = \left(A^{0}/\partial A^{-1}\right)^{red}$$

for a usual ring, $B^{red} = B/\{nilp\}$

Proof of ★: say, for $X = \mathbb{A}^n$:

$$(A^{red})^n \simeq \mathbb{A}^n \leftleftarrows \mathbb{A}^n \times_{(A^{red})^n} \mathbb{A}^n \Lleftarrow \cdots \qquad \text{true}$$

X$_{dR}$ and completion: for $f: X \to Y$

$$\begin{tikzcd}
\hat{Y}_f \arrow[r] \arrow[d] & X_{dR} \arrow[d] \\
Y \arrow[r] & Y_{dR}
\end{tikzcd}$$

<u>Example</u> $S \twoheadleftarrow R \hookleftarrow J$

$$\begin{tikzcd}
? \arrow[r] \arrow[d] & X_{dR}(A) \arrow[d] \\
Y(A) \arrow[r] & Y_{dR}(A)
\end{tikzcd} \qquad \begin{tikzcd}
? \arrow[r] \arrow[d] & \{S \to A^{red}\} \arrow[d] \\
\{R \to A\} \arrow[r] & \{R \to A^{red}\}
\end{tikzcd}$$

i.e. $\begin{matrix} R \to A \\ \cup \quad \cup \\ J \to \ker(A \to A^{red}) \end{matrix}$ s.t.

in Noetherian case: (continuous) morphisms $\hat{R} \longrightarrow A.$ (J-adic)

$$p : X \longrightarrow X_{dR}$$

General definitions via formal geometry will be given in terms of the $\mathcal{O}_{X_{dR}}$-algebra $p_*\mathcal{O}_X$; and they, in turn, will be expressed in terms of $dR^\bullet(A \twoheadrightarrow B)$ studied above.

<u>But first</u>: $p_*\mathcal{O}_X$ and $\mathcal{O}_{X_{dR}}$: what to expect?

<u>Smooth scheme X</u>:

$\mathcal{O}_{X_{dR}}$-modules $= \mathcal{O}$-modules $/ \; (X \leftleftarrows X \hat{\times}_{\Delta} X \Lleftarrow \cdots$

$\parallel$ (over $\mathbb{Q}$)

$\mathcal{O}$-modules on X w/flat conn.

$\parallel$

dg modules$/ \left(\Omega^\bullet_X, d_{DR}\right)$

$$\begin{tikzcd} \Omega^\bullet_X \arrow[r, two heads] & \mathcal{O}_X \\ \Omega^{>0} \arrow[r] & 0 \end{tikzcd}$$

$$\text{"Spec } \Omega^\bullet_X\text{"} \xleftarrow{\;p\;} \text{"Spec } \mathcal{O}_X\text{"}$$

(<u>awkward</u>: $\Omega^\bullet_X \notin cdga^{\leq 0}$).

$p_*\mathcal{O}_X$: $\mathcal{O}_X$ on which $\Omega^{>0}$ acts by 0.

semifree resolution:

$$k[x_1,\dots,x_n][\underset{0}{\hat{x}_1},\dots,\underset{0}{\hat{x}_n}]\{\underset{1}{dx_1},\dots,\underset{1}{dx_n}\};\quad \partial = d_{DR} - \sum \frac{\partial}{\partial \hat{x}_i}\cdot dx_i$$

very close to $\hat{\mathcal{O}}_X$.

So: general (enough) derived stack

X:

$$A_X \longrightarrow B_X$$

cdga; A_X in terms of $\mathcal{O}_{X_{dR}}$,

B_X in terms of $p_*\mathcal{O}_X$

$$\mathrm{Poly}(X,n) = \mathrm{Sym}_{B_X}\left(\mathrm{Der}_{A_X}(B_X)[-1-n]\right)$$

n-shifted Poisson structures on X

$\Updownarrow$

MC elements of $\mathrm{Poly}(X,n)$.

Ex: X affine smooth scheme

$\mathcal{A}_X = \Omega^{\bullet}_X \qquad \mathcal{B}_X = \Omega^{\bullet}_X(\hat{\mathcal{O}}_X)$

$\hat{\mathcal{O}}_X$ – valued forms

↑

bundle of jets

$\mathrm{Poly}(X,n) = \Omega^{\bullet}_X(\widehat{\mathrm{Poly}}(n))$

$\parallel$

forms on X with coeffs in jets of polyvectors

What to expect when $X = BG$?

$\mathcal{O}_{BG} = \mathcal{O}(G^{\bullet})^{\urcorner\,\text{power}}$ cosimplicial comm alg

↑

(resolution of).

$\mathcal{O}_{(BG)_{dR}} = \Omega^{\bullet}(G^{\bullet})$ – cosimplicial cdga

Could take for $\mathcal{O}_{BG}$:

$$\Omega^\bullet_{G^\bullet}(\hat{\mathcal{O}}_{G^\bullet})$$

$$\Omega^\bullet_{G^\bullet} \longrightarrow \Omega^\bullet_{G^\bullet}(\hat{\mathcal{O}}_{G^\bullet})$$

take $\lim_{\Delta}$ and use <u>strict</u> cdga model for that (i.e deRham-Sullivan forms). This is our

$$A_{BG} \longrightarrow B_{BG}.$$

Again, roughly:

we know

$$\Omega^1_{BG}[-1] \simeq C^\bullet_{alg}(G, \mathfrak{g}^*[2])$$

$$\Omega^\bullet_{BG} \simeq C^\bullet_{alg}(G, \mathrm{Sym}\, \mathfrak{g}^*[-2])$$

$$\mathcal{O}_{BG} \simeq C^\bullet_{alg}(G)$$

Resolution of $p^*\mathcal{O}_{BG}$ over $\Omega^\bullet_{BG}$:

$$B_X \simeq C^\bullet_{alg}(G, \mathrm{Sym}\,\mathfrak{g}^*[-2] \otimes \mathrm{Sym}\,\mathfrak{g}^*[-1])$$

new term in ∂

$\mathrm{Poly}_{A_X}(B_X)$:

$$C^\bullet_{alg}(G, \cancel{\mathrm{Sym}\,\mathfrak{g}^*[-2]} \otimes \cancel{\mathrm{Sym}\,\mathfrak{g}^*[-1]} \otimes \overbrace{\mathrm{Sym}\,\mathfrak{g}[-n]}^{\text{the shifted polyvectors}})$$

new term in ∂

$$\cong$$

$$C^\bullet_{alg}(G, \mathrm{Sym}\,\mathfrak{g}[-n])$$

By transfer of structure:

L_∞ structure of degree $-1-n$.

ITS MC ELEMENTS = shifted Poiss. strs (on BG)

3) Perhaps (K.-Takeda-V.?) construct a Pre-CY structure for a right CY structure.