Over $\mathbb{R}$, etc.:

$$\frac{x^{n+1}}{n+1} \xleftarrow{\int} x^n dx$$

$$H^0_{dR} = \frac{\ker(d)}{\operatorname{im}(d)} \simeq R \qquad d \circ d = 0$$

$$0 \xrightarrow{d} \Omega^0 \xrightarrow{d} \Omega^1 \longrightarrow 0$$

$$H^1_{dR} = \frac{\ker(d)}{\operatorname{im}(d)} = 0$$

So: if $d\omega = 0$,

$\omega = d\alpha$ unless

$\omega = \text{const} \in \Omega^0$

over $\mathbb{F}_p$:

$$\begin{aligned}
x^p &\longmapsto p x^{p-1} dx = 0 \\
&\vdots \\
x^2 &\longmapsto 2x\,dx \\
x &\longmapsto dx \\
1 &\xrightarrow[d]{} 0
\end{aligned}$$

$d x^p = 0$ and, of course,

$x^p \neq d(\dots)$ $\quad 0 \to \Omega^0 \to \Omega^1 \to 0$

$$? \cancel{\longmapsto} \overset{\omega}{x^p} \longmapsto 0$$

$x^{2p}, \dots$ $\quad \ker(d : \Omega^0 \to \Omega^1)$

$= F_p[x^p]$

$d(f_1 f_2) =$
$= df_1 \cdot f_2 + f_1 \cdot df_2$

$$\Omega^0 \xrightarrow{d} \Omega^1$$

$$\mathbb{F}_p[x] \qquad \mathbb{F}_p[x]\,dx$$

$$?? \ \cancel{\longmapsto} \ \boxed{x^{p-1}dx}$$

$$x^2 \longmapsto 2x\,dx$$

$$x \longmapsto 1\,dx$$

$$\frac{x^{n+1}}{n+1} \longleftarrow\!\dashv x^n dx$$

works unless

$$p \mid n+1$$

$$\underbrace{x^{p-1}dx,\ x^{2p-1}dx,\ x^{3p-1}dx \ldots}_{\text{basis of } \Omega^1/d\Omega^0}$$

not in $d\Omega^0$

$$\Omega^0 \xrightarrow{\quad d \quad} \Omega^1$$

$$\ker(d) = \mathbb{F}_p[x^p]\cdot 1 \qquad \operatorname{coker}(d) = \mathbb{F}_p[x^p]\cdot x^{p-1}dx$$

both free rank one

$\mathbb{F}_p[x^p]$-modules

$$\big\downarrow \wr$$

$$\mathbb{F}_p[x]$$

$$\boxed{\begin{aligned} H^0_{dR} &\cong \Omega^0 \\ H^1_{dR} &\cong \Omega^1 \end{aligned}}$$

will generalize into Cartier isomorph

But what if we do want "usual" situation with Poincaré lemma, de Rham, etc.?

("topological" invariants of varieties over $\mathbb{F}_p$, $\overline{\mathbb{F}}_p$, ...)

?

BUT HOW NATURAL IS THIS ?

$$
\begin{tikzcd}
\mathbb{F}_p[x] \\
\mathbb{Z}_p[x] \arrow[u, two heads] \arrow[r] & \mathbb{Q}_p[x] \\
\mathbb{Z}[x] \arrow[u, hook]
\end{tikzcd}
$$

$$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/\mathbb{Z}p^{n+1}$$

$$\mathbb{Z}_p[x] \xrightarrow{d} \mathbb{Z}_p[x]dx$$

$\ker = \mathbb{Z}_p$

$$\mathbb{Q}_p[x] \xrightarrow[d]{} \mathbb{Q}_p[x]dx$$

$\ker = \mathbb{Q}_p$ $\operatorname{coker} = 0$

$$\mathbb{Z}_p \twoheadrightarrow \mathbb{F}_p$$

"grows naturally out of $\mathbb{F}_p$" by some invar. procedure

$$\mathbb{Z}_p[x] \twoheadrightarrow \mathbb{F}_p[x]$$

no, not that naturally, though in some sense canonically.

$W(\mathbb{F}_p[x])$ ← is completely natural / canonical

$W(A)$ = Witt vectors

1. Differentials; Cartier (iso) morphism.

$k \to A$ k-algebra A commutative

unital commutative

MAY CHANGE LATER

!

$\Omega^{\bullet}_{A/k}$

Kähler differentials:

Generators: $a, a \in A$

$da, a \in A$

both k-linear in a

$d(a+b) = da + db$

k-algebra, graded

Relations: $a \cdot b$ as in A

$d(ab) = da \cdot b + a \cdot db$ $\quad (da)^2 = \boxed{0}$

Generators: $\underbrace{a,\ da}_{k\text{-linear in } a}$ $a \in A$

$a \cdot b$ as in A

$d(ab) = da \cdot b + a \cdot db \quad a, b \in A$

$da \cdot db + db \cdot da = 0 \Leftarrow (da)^2 = 0$

$\boxed{\begin{array}{c} a \cdot db = \\ = db \cdot a \end{array}}$ for now?

Ex $A = k[x]$

$\Omega^{\bullet}_{A/k}$: generators x, dx

relations: $x\,dx = dx \cdot x$

$(dx)^2 = 0$

basis: $x^n \quad x^n dx$

$n \geq 0$

Pf. $k[x] + k[x] \cdot dx$

$$\downarrow \uparrow$$

$$\Omega^*_{k[x]/k}$$

morphism of algs

$x \mapsto x$, $dx \mapsto dx$

Observe: $\Omega^\bullet$ is an algebra by definition;

It is graded (by # of factors da);

$$\boxed{d^2 = 0}$$

$$\omega_1 \cdot \omega_2 = (-1)^{|\omega_1||\omega_2|} \omega_2 \cdot \omega_1$$

$d : \Omega^\bullet \to \Omega^{\bullet+1}$ $\quad d: a \mapsto da$

$da \mapsto 0$

$$d(\omega_1 \omega_2) = d\omega_1 \cdot \omega_2 + (-1)^{|\omega_1|} \omega_1 d\omega_2$$

Now: $k = \mathbb{F}_p$ $\quad A$ $\quad \mathbb{F}_p$-alg

$\Omega^\bullet_{A/\mathbb{F}_p}$ still an alg over $\mathbb{F}_p$

($\mathbb{Z}_p$ comes later)

Recall from $A = \mathbb{F}_p[x]$:

$$x \longmapsto \underline{x^p} \qquad dx \longmapsto \underset{?}{\underline{x^{p-1}dx}}$$

$$F : A \to A \qquad F(a) = a^p$$

LINEAR $\quad (a+b)^p = \sum_{j=0}^{p} \binom{p}{j} a^j b^{p-j}$

but $\binom{p}{j} = \dfrac{p(p-1)\cdots}{j!}$ $\quad \boxed{0<j<p}$

$\boxed{(a+b)^p = a^p + b^p}$

$\mathrm{char}(k) = p$ $\quad F : k \to k$

$\qquad\qquad\qquad\qquad \alpha \mapsto \alpha^p$

$$F(a) = a^p$$

$$F(\alpha a) = \alpha^p F(a) = F(\alpha)F(a)$$

__Not__ k-linear, rather

semi-linear

$$\begin{tikzcd} A \arrow[r, "F"] & A \\ k \text{ acts} \arrow[u] & k \text{ acts via } F \arrow[u] \end{tikzcd}$$

__Extend to:__

$$\Omega^\bullet_{A/k} \longrightarrow \Omega^\bullet_{A/k}$$

$$a \longmapsto a^p$$

$$db \longmapsto b^{p-1}db$$

Differential (and integral) calculus in characteristic p.

Calculus (single-variable) over $\mathbb{F}$

$$\mathbb{F}_p[x] \xrightarrow{d} \mathbb{F}_p[x]\,dx$$

$$x^n \longmapsto nx^{n-1}\,dx$$

$$f\,dx \longmapsto$$

integral calculus would be "inv

$(ab)^p = a^p b^p$ ✓

$\boxed{\mathbb{F}_p}$

$d(ab) - da \cdot b - a \cdot db$

$F(\alpha) = \alpha$

$\downarrow$

$(ab)^{p-1} \underbrace{d(ab)}_{\parallel} - a^{p-1} da \cdot b^p - a^p \cdot b^{p-1} db$

$da \cdot b + a \cdot db$ $= 0$ ✓

$(db)^2 \mapsto b^{p-1} db \cdot b^{p-1} db = 0$ ✓

$(b_1 + b_2)^{p-1} d(b_1 + b_2) - b_1^{p-1} db_1 - b_2^{p-1} db_2$

$p = 2$:

$(b_1 + b_2) d(b_1 + b_2) - db_1 \cdot b_1 - db_2 \cdot b_2$

$= b_1 \cdot db_2 + b_2 db_1 = \boxed{d(b_1 b_2)}$

NOTE:

$$d(a^p) = 0$$

$$d(b^{p-1}\,db) = 0$$

So what we are constructing:

$$\Omega^\circ_{A/k} \longrightarrow \mathcal{Z}^\circ_{A/k} \big/ \mathcal{B}^\circ_{A/k}$$

$\mathcal{Z}^\circ_{A/k} = \ker(d)$, $\quad \mathcal{B}^\circ_{A/k} = \operatorname{im}(d)$

$$d(b_1+b_2) - db_1 - db_2 = 0 \longmapsto \dots \neq 0$$

$$\dots = d(\dots)$$

$A/k \qquad \operatorname{char}(k) = p$

$F : A \to A \qquad a \longmapsto a^p$

Now $\quad F \,(= C^{-1}) : \Omega^1_{A/k}$

$$a \longmapsto a^p$$

$$db \longmapsto b^{p-1}\, db$$

and require it to be a morphism of (graded) algebras??

linearity in a : ✓

linearity in b : ??

$$d(b_1 + b_2) \longmapsto \cancel{db}\ b_1^{p-1}\, db_1 + b_2^{p-1}\, db_2$$
$$\neq (b_1 + b_2)^{p-1}\, d(b_1 + b_2)$$

$$\underbrace{(b_1+b_2)^{p-1}\,d(b_1+b_2) - b_1^{p-1}\,db_1 - b_2^{p-1}\,db_2}_{dF_1(b_1,b_2)} =$$

$$= (\text{formally}) =$$

Say: in $\Omega^{\bullet}_{\mathbb{Q}[b_1,b_2]/\mathbb{Q}}$

$$\frac{1}{p}\,d(b_1+b_2)^p - \frac{1}{p}\,db_1^p - \frac{1}{p}\,db_2^p$$

$$= d\left[\underbrace{\frac{1}{p}\left((b_1+b_2)^p - b_1^p - b_2^p\right)}_{\text{makes perfect sense as a polynomial in } \mathbb{Z}[b_1,b_2]}\right]$$

$$F_1(b_1,b_2) := \sum_{j=1}^{p-1} \underline{\frac{1}{p}\binom{p}{j}}\, b_1^j\, b_2^{p-j}$$