Crystalline cohomology

Divided powers. A divided power (pd) structure on an ideal $I \subset A$ (all commutative):

$$I \ni x \mapsto \gamma_n(x) \in A \,;$$

$$\gamma_0(x) = 1;\ \gamma_1(x) = x;\ \gamma_n(x) \in I;$$

$$\gamma_n(x)\gamma_m(x) = \frac{(m+n)!}{m!\,n!}\gamma_{n+m}(x) \quad [\Rightarrow x^n = n!\,\gamma_n(x)$$

$$\gamma_n(ax) = a^n \gamma_n(x),\quad a \in A,\ x \in I$$

$$\gamma_n(x+y) = \sum_{j=0}^{n} \gamma_j(x)\gamma_{n-j}(y)$$

$$\gamma_n(\gamma_m(x)) = \frac{(mn)!}{m!^n\,n!} \cdot \gamma_{mn}(x)$$

If no torsion: unique pd structure iff $I^n \subset n!A$

a pd ring is (A, I, γ)

a morphism of pd rings:

$$\begin{array}{ccc} (A, & I, & \gamma) \\ \downarrow & \downarrow & \| \\ (B, & J, & \delta) \end{array}$$

all limits exist: just take ring-theoretic (= set-theoretical) limits; all operations are defined componentwise.

e.g. $\prod (A_j, J_j, \gamma_j) = \prod A_j \supset \prod J_j$;

$$\gamma_n((x_j)) = (\gamma_n(x_j))$$

Colimits also exist, but they do not coincide with the ring-theoretical ones:

example

$$\begin{array}{ccc} (A, I) & \longrightarrow & (B', J') \\ \downarrow & & \downarrow \\ (B, J) & \longrightarrow & B \otimes_A B', \; B \otimes_A J' + J \otimes_A B' \end{array}$$

as rings, but:

may need to factor out something. Ex.

$$\begin{array}{ccc} (\mathbb{Z}, 0; \gamma = 0) & \longrightarrow & (\mathbb{Z}/4, (2); \gamma_2(2) = 0) \\ \downarrow & & \\ (\mathbb{Z}/4, (2); \gamma_2(2) = 2) & & \end{array}$$

(the two pd structures on $(\mathbb{Z}/4, (2))$, i.e. the two solutions of $2\gamma_2(2) = 2^2 = 0$)

$\mathbb{Z}/4 \otimes_{\mathbb{Z}} \mathbb{Z}/4 = \mathbb{Z}/4$; to get a pd-structure, have to factor out 2.

Answer: the colim is $(\mathbb{Z}/2, 0; \gamma = 0)$

<u>pd envelopes</u> Given $J \subset P$, there is a universal pd algebra $(\bar{J} \subset D, \bar{\gamma})$

$$\begin{array}{ccc} J & \subset & P \\ \downarrow & & \downarrow \\ \bar{J} & \subset & D \end{array}$$

(just introduce new elements $\gamma_n(x)$ and factor out relations).

<u>Note</u>: even if J has a pd structure we will get something new.

<u>Example</u> For a $\mathbb{Z}_{(p)}$-algebra R, pR has a pd structure.

<u>Proof</u>: p^n is divisible by $n!$

since $v_p(n!) = [\frac{n}{p}] + [\frac{n}{p^2}] + \ldots$

For any p, it is > 0.

(For $p > 2$: $v_p(n!) \to \infty$ as $n \to \infty$).

<u>So</u>: since $\mathbb{Z}_{(p)}$ has no torsion, unique pd structure on (p).

Extends to any pR:

$\gamma_n(pr) = r^n \cdot \gamma_n(p)$. Why well-def., even if there is torsion?

$p(r-r') = 0 \Rightarrow \gamma_n(p) \cdot (r^n - r'^n) = \ldots \cdot p \cdot (r-r') \cdot \ldots = 0.$

But, if we formally add $\gamma_n(x)$ to any $x \in (p) \subset \mathbb{Z}_p$, we get something new.

What if we want to keep <u>old</u> γ_n that already exist? <u>Relative</u> version:

$$\begin{tikzcd}
A, I, \gamma \arrow[d] \arrow[drr, dashed] & & \\
B, J \arrow[rr, bend right] & & (D, \bar{J}, \bar{\gamma})
\end{tikzcd}$$

pd-algebra

algebra + ideal

$\longleftarrow$ universal such.

Meaning:

$$\begin{tikzcd}
A, I, \gamma \arrow[d] \arrow[rr, bend left] & & C, K, \delta \\
B, J \arrow[urr] \arrow[rr, bend right] & & D, \bar{J}, \bar{\gamma} \arrow[u, dotted, "\exists !"']
\end{tikzcd}$$

$\forall$ pd algebra over (A, I, γ)

(Do the universal case;
then factor out $\gamma_n^{\text{new}}(x) - \gamma_n(x)$).

Our main example will be:

$$A \text{ over } \mathbb{F}_p; \quad \begin{array}{c} P \to A \\ \wr\wr \\ \mathbb{Z}_p[x_1,\dots,x_n] \end{array}$$

$$\mathbb{Z}_p,(p) \longrightarrow (P, J) \qquad J = \ker \text{ of } P \to A$$

$D, \bar{J}, \bar{\gamma}$
is the pd envelope

$$\begin{tikzcd} \mathbb{Z}_p,(p),\gamma_n \arrow[r] \arrow[dr] & P, J \arrow[d] \\ & D, \bar{J}, \bar{\gamma} \end{tikzcd}$$

pd envelope of $\ker(P \to A)$
relatively to $\mathbb{Z}_p \to P$.

<u>Example</u> $A = \mathbb{F}_p[x]$ $\quad P = \mathbb{Z}_p[x,y]$

$J = (p,y)$ $\quad D = \mathbb{Z}_p[x]\langle y \rangle$

Here we use:

$$R\langle y\rangle = \bigoplus_{n\geq 0} R\, y^{[n]}$$

$$y^{[n]}\, y^{[m]} = y^{[n+m]} \cdot \frac{(m+n)!}{m!\, n!}$$

Similarly $R\langle x_1, \dots, x_n\rangle$

Motivation: Alexander-(Čech)-Spanier cohomology.

$X \subset M$

└ smooth, e.g. $\mathbb{R}^N$

$$\mathbb{C} \xrightarrow{d_0} C(M) \underset{d_1}{\overset{d_0}{\rightrightarrows}} C(M \times M) \rightrightarrows \cdots$$

$$d_j f(x_0, \dots, x_n) = f(x_0, \dots, \hat{x}_j, \dots, x_n)$$

$$d = \sum_{j=0}^{n} (-1)^j d_j \qquad d^2 = 0$$

And contracting homotopy

$$\mathbb{C} \xleftarrow{h} C(M) \xleftarrow{h} C(M \times M) \xleftarrow{h} \cdots$$

$$hf(x_0, \dots, x_{n-1}) = f(x_0, \dots, x_{n-1}, x_*)$$

some fixed based point

Now: localize near $X \subset M^n$ diag

$$C^n = \operatorname{germs}_X C(M^n) = \lim_{\substack{X \subset U \subset M^n \\ \text{open}}} C(U)$$

get a complex.

This is our motivation; we will replace $C(M) \twoheadrightarrow C(X)$ by

$$\hat{D} \twoheadrightarrow A$$

completed pd envelope of $P \twoheadrightarrow A$.