A - comm alg / $\mathbb{F}_p$ (fin. gen.) $\quad P \twoheadrightarrow A$

$$P \cong \mathbb{Z}_p[x_1, \dots, x_N]$$

$$\underset{\text{kernel}}{J} \subset P \twoheadrightarrow A$$

Consider the (relative) pd envelope of J:

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r] & P \arrow[r] & D \\ (p) \arrow[u, phantom, "\cup"] \arrow[r] & J \arrow[u, phantom, "\cup"] \arrow[r] & \bar{J} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

(<u>Recall</u>: $\frac{p^n}{n!} \in \mathbb{Z}$; on any pB for a $\mathbb{Z}_p$-algebra B, $\gamma_n(pb) = \frac{p^n}{n!} \cdot b^n$ is a well-defined pd structure)

<u>Then</u> take <u>completion</u>.

What exactly <u>is</u> completion?

<u>One</u> way: in fact we have a projective system of

$$\begin{tikzcd}
\mathcal{P}_k = \mathcal{P}/(p^k) \arrow[r, two heads] \arrow[ddr] & A \\
J_k \arrow[u, phantom, "\subseteq" rotate=90] \arrow[dr] & \\
& \bar{J}_k \subset D_k \quad \bar{\gamma}_k
\end{tikzcd}$$

relative pd completion.

Get a projective system

$$\begin{tikzcd}
\arrow[r] & D_k \arrow[r] & D_{k-1} \arrow[r] & \cdots \\
\arrow[r] & \bar{J}_k \arrow[u, phantom, "\cup" rotate=180] \arrow[r] & \bar{J}_{k-1} \arrow[u, phantom, "\cup" rotate=180] \arrow[r] & \cdots
\end{tikzcd}$$

Next: $D^{(n)}$ = relative pd envelope of

$$\begin{gathered}
J^{(n)} \\
\cap \\
\mathcal{P}^{(n)} = \mathcal{P} \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} \mathcal{P} \\
\downarrow \\
A
\end{gathered}$$

multiplication followed by $\mathcal{P} \to A$

Again, actually have a projective system

$$\begin{tikzcd}
\cdots \arrow[r] & D_k^{(n)} \arrow[r] & D_{k-1}^{(n)} \arrow[r] & \cdots \\
\cdots \arrow[r] & \bar{J}_k^{(n)} \arrow[u, phantom, "\cup" rotate=180] \arrow[r] & \bar{J}_{k-1}^{(n)} \arrow[r] & \cdots
\end{tikzcd}$$

The Alexander (Čech) - Spanier complex:

$$\hat{D} \to \dots \to \hat{D}^{(n)} \to \hat{D}^{(n+1)} \to \dots$$

$$a_0 \otimes \dots \otimes a_n \mapsto \sum_{j=0}^{n+1} (-1)^j a_0 \otimes \dots \otimes 1 \otimes \dots$$

<u>Fact</u>: the homotopy class of this complex does not depend on P.

<u>Proof</u>

$$\begin{tikzcd} P \arrow[rr, "\exists\varphi"] \arrow[dr] & & Q \arrow[dl] \\ & A & \end{tikzcd}$$

φ - morphism of algebras over A

$$\begin{tikzcd} \hat{D}_P \arrow[rr, dashed, bend left, "\exists\varphi"] \arrow[dr] & & \hat{D}_Q \arrow[ll, dashed, bend left, "\psi"] \arrow[dl] \\ & P \arrow[r, shift left, "\varphi"] \arrow[d] & Q \arrow[l, shift left, "\psi"] \arrow[dl] \\ & A & \end{tikzcd}$$

φ, ψ exist by universal property extend to completions.

Same for $\hat{D}^{(n)}_P, \hat{D}^{(n)}_Q$.

We claim $\psi\varphi \sim id$, $\varphi\psi \sim id$. Enough to construct a homotopy between any $\varphi, \psi : \hat{D}^{(\cdot)} \circlearrowleft$, morphisms of $\hat{D}^{(\cdot)}$ induced by $\varphi, \psi : \hat{D} \circlearrowleft$ over $\mathring{A}$.

Explicitly:

$$h(\varphi,\psi)(a_0 \otimes \ldots \otimes a_n) =$$

$$= \sum_{j=0}^{n-1} (-1)^j \varphi(a_0) \otimes \ldots \otimes \varphi(a_j)\psi(a_{j+1}) \otimes \ldots \otimes \psi(a_n)$$

<u>In fact:</u> $a_0 \xmapsto{d} a_0 \otimes 1 - 1 \otimes a_0 \xmapsto{h} \varphi(a_0) - \psi(a_0)$ ✓

$$a_0 \otimes a_1 \xmapsto{d} 1 \otimes a_0 \otimes a_1 - a_0 \otimes 1 \otimes a_1 + a_0 \otimes a_1 \otimes 1$$

$$\downarrow h$$

$$\boxed{\psi(a_0) \otimes \psi(a_1)} - \cancel{\varphi(a_0) \otimes \psi(a_1)} + \varphi(a_0)\psi(a_1) \otimes$$
$$- 1 \otimes \varphi(a_0)\psi(a_1) + \cancel{\varphi(a_0) \otimes \psi(a_1)} - \boxed{\varphi(a_0) \otimes \varphi(a_1)}$$

h

$$\varphi(a_0)\psi(a_1) \xmapsto{d} 1 \otimes \varphi(a_0)\psi(a_1) - \varphi(a_0)\psi(a_1) \otimes 1 \ \checkmark$$

etc.

<u>Next</u>: 1) why is this homotopy <u>coherent</u> (and therefore we can glue)

2) Why do they extend to completions.

3) Comparison to De Rham.