Let $A/\mathbb{F}_p$ (fin. gen.)

$J \hookrightarrow P \twoheadrightarrow A$ $\qquad P \simeq \mathbb{Z}_p[x_1, \dots, x_N]$

free (polyn.) over $\mathbb{Z}_p$

$(p) \subset \mathbb{Z}/p^n$ still pd structure

$$\left.\begin{array}{ccccc} \mathbb{Z}_p & \longrightarrow & P & \longrightarrow & D \\ \cup & & \cup & & \cup \\ (p) & \longrightarrow & J & \longrightarrow & \bar{J} \end{array}\right\}$$ pd envelope rel to $(p) \subset \mathbb{Z}_p$

<u>Complete</u>: 1) Do this for $P_n = \mathbb{Z}/p^n[x_1, \dots, x_N]$

$$P_{n+1} \to P_n \to \_\_\_$$

$$D_{n+1} \to D_n \to \cdots$$

projective system of algebras

To be safe: completion is $\mathbb{R}\varprojlim$

Recall: Inverse system

$$\cdots \longrightarrow R_n \to R_{n-1} \to \cdots$$

$\mathbb{R}\varprojlim$: complex of length 2:

$$\begin{tikzcd}
C^1: & R_0 \times & R_1 \times & R_2 \times & \cdots \\
C^0: \arrow[u] & R_0 \times \arrow[u, "="] & R_1 \times \arrow[u, "="] \arrow[ul, "f_1"] & R_2 \times \arrow[u, "="] \arrow[ul, "f_2"] & \cdots \arrow[ul, "f_3"]
\end{tikzcd}$$

$$P^{(n)} = \underbrace{P \otimes \cdots \otimes P}_{(n\ \text{times})} \xrightarrow{\text{mult}} P \to A$$

$$\begin{tikzcd} P^{(n)} \arrow[r] & A \\ J^{(n)} \arrow[u, phantom, "\cup"] & \end{tikzcd}$$

$J^{(n)}$ = kernel

$$\begin{tikzcd} \mathbb{Z}_p \arrow[r, hook] & P^{(n)} \arrow[r] & D^{(n)} \\ p\mathbb{Z}_p \arrow[u, phantom, "\cup"] \arrow[r, hook] & J^{(n)} \arrow[u, phantom, "\cup"] \arrow[r] & \overline{J}^{(n)} \arrow[u, phantom, "\cup"] \end{tikzcd}$$

pd envelope

And complete: $\widehat{D}^{(n)}$ as above.

Why extends to D? b/c only add divided powers of: $x_0 \otimes a_1 \otimes \dots \otimes a_n$

$x_0 \in J$

Together generate $\ker(p^{\otimes(n)})$ or just $\boxed{x_0} \otimes 1 \otimes \dots \otimes 1$

generate $\ker(\text{mult})$ $p \otimes \cdot \to p$

$\dots a \otimes \dots \otimes 1 \otimes \dots$

$\dots \dots 1 \otimes \dots \otimes a \otimes \dots$

$d_j(a_0 \otimes \dots \otimes a_n) = a_0 \otimes \dots \otimes 1 \otimes \dots \otimes a_n$

<u>Next</u>: Given

$$\begin{tikzcd} P \arrow[rr, "\varphi"] \arrow[dr] & & Q \arrow[dl] \\ & A & \end{tikzcd}$$

morphism over A (of rings)

preserves ideal $\Rightarrow$

$\Rightarrow$ extends to pd envelopes.

Induces $\varphi : D_P^{(\bullet)} \longrightarrow D_Q^{(\bullet)}$ morphism of complexes.

$$\begin{tikzcd} P \arrow[rr, bend left, "\varphi"] \arrow[rr, "\psi"'] \arrow[dr] & & Q \arrow[dl] \\ & A & \end{tikzcd} \qquad \begin{tikzcd} D_P^{(\cdot)} \arrow[r, bend left, "\varphi"] \arrow[r, bend right, "\psi"'] \arrow[r, phantom, "2"] & D_Q^{(\cdot)} \end{tikzcd}$$

$$\begin{array}{ccc} \mathcal{D}_P^{(\cdot)} & \rightrightarrows & \mathcal{D}_Q^{(\cdot)} \\ \cup & & \\ P^{(\cdot)} & \underset{\psi}{\overset{\varphi}{\rightrightarrows}} & Q^{(\cdot)} \end{array}$$

$$\boxed{[d, h(\varphi,\psi)] = \varphi - \psi} \quad (\pm)$$

$h(\varphi,\psi)$: $a_0 \otimes \ldots \otimes a_n$

$$\downarrow$$

$$\sum_{j=0}^{n-1} (-1)^j \varphi(a_0) \otimes \ldots \otimes \underbrace{\varphi(a_j)\psi(a_{j+1})}_{} \otimes \ldots \otimes \psi(a_n)$$

$$\varphi(a_0) \otimes \ldots \otimes \varphi(a_n) \xleftarrow{\varphi} a_0 \otimes \ldots \otimes a_n \xrightarrow{\psi} \psi(a_0) \otimes \ldots \otimes \psi(a_n)$$

Rmk

$h(id, id) =$
$=$ bar differential $\boxed{b'}$

Those homotopies are <u>coherent</u> in the following sense

$$\begin{tikzcd}
P^{(\cdot)} \arrow[r, bend left, "\varphi"] \arrow[r, "\psi"] \arrow[r, bend right, "\theta"'] \arrow[dr] & Q^{(\cdot)} \arrow[d] \\
& A
\end{tikzcd}$$

$h(\varphi,\psi) - h(\varphi,\theta) + h(\psi,\theta)$

commutes with d

$[d,-]$ $h(\varphi) - h(\psi) + \_\_ = 0$

$= [d, \underbrace{h(\varphi,\psi,\theta)}_{\text{of degree } -2}]$

$h(\varphi_1, \varphi_2, \dots, \varphi_n)$ of deg $1-n$

$h(\varphi_1, \dots, \varphi_n)a$

$$[d, h(\varphi_1, \dots, \varphi_n)] = \sum_{j=1}^{n} \pm h(\varphi_1, \dots, \hat{\varphi}_j, \dots, \varphi_n)$$

$= \sum \varphi_1(a_1) * \dots * \varphi_n(a_n)$

$a = a_1 \otimes \dots \otimes a_n$

$$h(\varphi_1, \varphi_2, \varphi_3)(a_0 \otimes \dots \otimes a_n) =$$

$$= \sum \pm \boxed{\varphi_1(a_0) \otimes \dots \otimes \varphi_1(a_j)} \boxed{\varphi_2(a_{j+1}) \otimes \dots \otimes \varphi_2(a_k)} \cdot$$

$$\boxed{\cdot \varphi_3(a_{k+1}) \otimes \dots \otimes \varphi_3(a_n)}$$

$(a_0 \otimes \dots \otimes a_k) * (b_1 \otimes \dots \otimes b_l)$

$\|$

$a_0 \otimes \dots \otimes \boxed{a_k b_1} \otimes b_2 \otimes \dots \otimes b_l$

Also:

$$\underbrace{\begin{tikzcd} P^{(\cdot)} \arrow[r, bend left, "\varphi_1"] \arrow[r, bend right, "\varphi_2"'] & Q^{(\cdot)} \end{tikzcd}} \begin{tikzcd} {} \arrow[r, "\psi"'] & R^{(\cdot)} \end{tikzcd}$$

$$\begin{tikzcd} P^{(\cdot)} \arrow[rr, bend left, "\psi\varphi_1"] \arrow[rr, bend right, "\psi\varphi_2"'] & & R^{(\cdot)} \end{tikzcd}$$

$$\psi \cdot \underbrace{h(\varphi_1, \varphi_2)} = h(\psi\varphi_1, \psi\varphi_2)$$

Or:

$$\begin{tikzcd} P^{(\cdot)} \arrow[r, bend left, "\varphi_1"] \arrow[r, bend right, "\varphi_2"'] & Q^{(\cdot)} \arrow[r, bend left, "\psi_1"] \arrow[r, bend right, "\psi_2"'] & R^{(\cdot)} \end{tikzcd}$$

$h(\psi_1, \psi_2)\, h(\varphi_1, \varphi_2)$

$\Vert$

linear comb of

$h(\cdot, \cdot)$

But first: Why survives under pd envelopes? Again, we only need to have div powers of: $x_0 \otimes 1 \otimes \dots \otimes 1$

$$1 \otimes \dots \otimes \boxed{a \otimes 1 \otimes \dots} \otimes 1$$
$$\dots \boxed{1 \otimes a \otimes} \dots$$
$$\mathrm{T}$$
$$\varphi(\cdot)\psi(\cdot)$$

(envelope of $P^{\otimes 2}$) $\ni \gamma_n(a \otimes 1 - 1 \otimes a)$

$$\mathrm{I}$$

envelope of P $\ni \gamma_n(\varphi(a) - \psi(a))$

In other words: $h(\varphi,\psi)$ involves: $a\otimes 1 - 1\otimes a$

$$\downarrow$$

$J \ni \varphi(a) - \psi(a)$

$$\ldots \otimes a \otimes \ldots$$
$$\downarrow$$
$$\ldots \otimes \varphi(a) \otimes \ldots$$

(or $\psi(a)$)

and

$$\ldots \otimes \underbrace{a_k \otimes a_{k+1}}\otimes \text{—}$$
$$\downarrow$$
$$\varphi(a_k)\cdot\psi(a_{k+1})$$

And that extends to pd envelopes of ideal generated by $\begin{cases} x_0 \otimes 1 \otimes \text{—} \otimes 1 & x_0 \in J \\ \text{—} \otimes (a\otimes 1 - 1\otimes a)\otimes \text{—} \end{cases}$

<u>So far</u>: defined crystalline complex of any <u>affine</u> scheme / $\mathbb{F}_p$ (<u>quite</u>) canonically up to homotopy.

<u>Enough to glue</u>. $X = \bigcup_\alpha U_\alpha$

a <u>twisted cochain</u>:

a) a complex of sheaves $\mathcal{F}_\alpha^\bullet$ on U_α

b) a morphism of complexes

Same formula as for A_∞ functor

1) $\mathcal{F}^\bullet_\alpha$, $d =: \rho_\alpha$ on $\mathcal{U}_\alpha$

2) $\rho_{\alpha\beta} : \mathcal{F}^\bullet_\alpha \leftarrow \mathcal{F}^\bullet_\beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$

$\rho_\alpha \rho_{\alpha\beta\gamma} + \rho_{\alpha\beta\gamma}\rho_\gamma$

$\|$

$\boxed{\rho_{\alpha\gamma} - \rho_{\alpha\beta}\rho_{\beta\gamma}}$

3) $\rho_{\alpha\beta\gamma} : \mathcal{F}^\bullet_\alpha \leftarrow \mathcal{F}^\bullet_\gamma$ of deg -1 on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \cap \mathcal{U}_\gamma$

$\vdots$

$\rho_{\alpha_0 \ldots \alpha_n} : \mathcal{F}^\bullet_{\alpha_0} \leftarrow \mathcal{F}^\bullet_{\alpha_n}$ of degree $1-n$ on $\mathcal{U}_{\alpha_0 \ldots \alpha_n}$

$$\sum_{j=1}^{n-1} \pm \rho_{\alpha_0 \ldots \hat{\alpha}_j \ldots \alpha_n} + \sum_{j=1}^{n-1} \rho_{\alpha_0 \ldots \alpha_j}\rho_{\alpha_j \ldots \alpha_n} = 0$$

Toledo, Tong '85