<u>Last time</u>: Globalizing the Alexander-Čech-Spanier complex. (sketch)

1. The local complexes:

given A over $\mathbb{F}_p$,

$(\hat{D}^{(\cdot)}, d) \leftarrow$ a <u>choice</u>

2. $\hat{D}^{(\cdot)} \xleftarrow{\varphi} \hat{E}^{(\cdot)}$

morphism of complexes for two choices.

$$\begin{array}{l} J \overset{\ker}{} \\ \downarrow \\ P \simeq \mathbb{Z}_p[x_1, \dots, x_N] \\ \downarrow \\ A \end{array}$$

$\hat{D}$ = pd envelope of J rel to (p)

See more of reminder in Lecture 13-2, marked $\star$

Do this for $\mathcal{P}_N = \mathcal{P}/p^N\mathcal{P}$

$$\begin{tikzcd} J_N^{(n)} \arrow[r] \arrow[d, phantom, "\cap"] & \bar{J}_N^{(n)} \arrow[d] \\ \mathcal{P}_N^{(n)} \arrow[r] \arrow[d, "\pm"] & D_N^{(n)} \arrow[d, "\pm"] \\ A \arrow[r, equal] & A \end{tikzcd} \quad \forall N$$

$$d(a_0 \otimes \dots \otimes a_{n-1}) = \sum_{j=0}^{n-1} (-1)^j a_0 \otimes \dots \otimes 1 \otimes \dots \otimes a_{n-1}$$

extends to $D_N^{(n)} \to D_N^{(n+1)}$

Inverse system of complxs:

$$\begin{tikzcd}
\cdots \arrow[r] & D_N^{(\bullet)} \arrow[r] \arrow[d] & D_N^{(\bullet+1)} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & D_{N-1}^{(\bullet)} \arrow[r] \arrow[d] & D_{N-1}^{(\bullet+1)} \arrow[r] \arrow[d] & {} \\
& \cdots & \cdots &
\end{tikzcd}$$

Take $R\varprojlim$: for $F_0 \leftarrow F_1 \leftarrow F_2 \leftarrow \cdots$

$$\begin{tikzcd}
R^1\varprojlim & F_0 \times & F_1 \times & F_2 \times & \cdots \\
R^0\varprojlim & F_0 \times \arrow[u, equal] & F_1 \times \arrow[u] \arrow[ul] & F_2 \times \arrow[u, equal] \arrow[ul] & \cdots
\end{tikzcd}$$

$$
\begin{tikzcd}
\hat{D}^{(\cdot)} \arrow[d] & \hat{E}^{(\cdot)} \arrow[l, "\exists\varphi"'] \arrow[d] \\
A & B \arrow[l, "\varphi_0"]
\end{tikzcd}
$$

morphism of complexes

morphism of $\mathbb{F}_p$-algs

$$
\begin{tikzcd}
\hat{D}^{(\cdot)} & \hat{E}^{(\cdot)} \arrow[l, bend right, "\varphi"'] \arrow[l, bend left, "\psi"] \arrow[l, phantom, "\Downarrow h"]
\end{tikzcd}
$$

$$[d, h(\varphi, \psi)] = \varphi - \psi$$

Higher homotopies:

$[d, h(\varphi_1, \dots, \varphi_m)]$ of degree $1-m$

$$= \sum \pm h(\varphi_1, \dots, \hat{\varphi}_j, \dots, \varphi_m)$$

And also:

$$\begin{tikzcd}
\hat{D}^{(\cdot)} \arrow[d] & \hat{E}^{(\cdot)} \arrow[l, bend right, "\varphi_1"'] \arrow[l, bend left, "\varphi_m"] \arrow[l, phantom, "\vdots"] \arrow[d] & \hat{F}^{(\cdot)} \arrow[l, bend right, "\psi_1"'] \arrow[l, bend left, "\psi_l"] \arrow[l, phantom, "\vdots"] \arrow[d] \\
A & B \arrow[l] & C \arrow[l]
\end{tikzcd}$$

$$h(\varphi_1, \dots, \varphi_m) \circ h(\psi_1, \dots, \psi_l) =$$

= (explicit formula in terms of a linear combination of $h(\varphi_\cdot \psi_\cdot, \dots, \varphi_\cdot \psi_\cdot)$)

n.b. more later; related closely to brace structures and 2-categories...

This allows to globalize into a twisting cochain: ρ_{α_0} = diff. ~~differential~~

$X = \bigcup_\alpha U_\alpha$ (open cover)

A twisting cochain: complex of

1) $\mathcal{F}_\alpha$ – a sheaves on U_α

2) $\forall \alpha_0, \ldots, \alpha_n$:

$$\rho_{\alpha_0 \ldots \alpha_n} : \mathcal{F}_{\alpha_0}^{1-n+\bullet} \leftarrow \mathcal{F}_{\alpha_n}^{\bullet}$$

on $U_{\alpha_0} \cap \ldots \cap U_{\alpha_n}$

$\rho_{\alpha_0}^2 = 0$

$\rho_{\alpha_0}\rho_{\alpha_0\alpha_1} - \rho_{\alpha_0\alpha_1}\rho_{\alpha_1} = 0$

$\rho_{\alpha_0} : \mathcal{F}_{\alpha_0}^{\bullet} \to \mathcal{F}_{\alpha_0}^{\bullet+1}$

$\rho_{\alpha_0\alpha_1} : \mathcal{F}_{\alpha_0}^{\bullet} \leftarrow \mathcal{F}_{\alpha_1}^{\bullet}$

...

$\rho_{\alpha_0\alpha_2} - \rho_{\alpha_0\alpha_1}\rho_{\alpha_1\alpha_2}$ = homot to zero

$$\sum_{j=1}^{n-1} (-1)^j \left[\rho_{\alpha_0 \ldots \hat{\alpha}_j \ldots \alpha_n} - \rho_{\alpha_0 \ldots \alpha_j}\rho_{\alpha_j \ldots \alpha_n}\right] = 0$$

As if: Category;

objects are U_α;

a morphism from U_{α_0} to U_{α_1} for every α_0, α_1;

An A_∞ functor from this category to complexes.

(the identity is exactly the A_∞ identity).

Shorthand: $\check{\partial}\rho + \rho \smile \rho = 0$

$$(\rho \smile \rho)_{\alpha_0 \ldots \alpha_n} = \sum \pm \rho_{\alpha_0 \ldots \alpha_k} \rho_{\alpha_k \ldots \alpha_n}$$

$$(\check{\partial}\rho)_{\alpha_0 \ldots \alpha_n} = \sum_{1}^{n-1} \pm \rho_{\alpha_0 \ldots \hat{\alpha}_j \ldots \alpha_n}$$

Morphism of two twisting cochains:

φ_{α_0} of deg 0

$\varphi_{\alpha_0\alpha_1}$ of deg -1

...

$\check{\delta}\varphi + \varphi \smile \rho - \tilde{\rho} \smile \varphi = 0$

Claim 1). Given a scheme $X/\mathbb{F}_p$ and a Zariski open cover

$$X = \bigcup_\alpha U_\alpha :$$

$\boxed{\text{Can}}$ write a twisting cochain:

$\mathcal{F}^{\bullet}_{\alpha} = \underbrace{\hat{D}^{(\bullet)}_{(\alpha)}}$ for $A = A_{\alpha} = \mathcal{O}(U_{\alpha})$

(make some choice).

$\mathcal{F}^{\bullet}_{\alpha} \xleftarrow{\rho_{\alpha\beta}} \mathcal{F}^{\bullet}_{\beta}$ on $U_{\alpha} \cap U_{\beta}$;

choose $\hat{D}^{(\bullet)}_{(\alpha)} \xleftarrow{\varphi_{\alpha\beta}} \hat{D}^{\bullet}_{(\beta)}$

extend: e.g. $\rho_{\alpha\beta\gamma} = h(\varphi_{\alpha\beta}\varphi_{\beta\gamma}, \varphi_{\alpha\gamma})$

$$\begin{tikzcd}
\hat{D}^{(\bullet)}_{(\alpha)} & \hat{D}^{(\bullet)}_{(\beta)} \arrow[l, "\varphi_{\alpha\beta}"'] & D^{(\bullet)}_{(\gamma)} \arrow[l, "\varphi_{\beta\gamma}"'] \arrow[ll, bend left, "\varphi_{\alpha\gamma}"]
\end{tikzcd}$$

For a twisting cochain ρ, the Čech complex:

$$\check{C}^\bullet(X,\rho) = \prod_{\alpha_0,\dots,\alpha_n} \Gamma\left(\mathcal{F}^\bullet_{\alpha_n}(U_{\alpha_0}\cap\dots\cap U_{\alpha_n})\right)$$

the differential:

$$\check{\partial} + \rho\cdot = \check{\partial}_\rho$$

$$(\check{\partial}_\rho s)_{(\alpha_0\dots\alpha_n)} = \sum \pm s_{\alpha_0\dots\hat{\alpha}_j\dots\alpha_n} + \sum_{j=0}^{n} \pm \rho_{\alpha_0\dots\alpha_j} s_{\alpha_j\dots\alpha_n}$$

And: a morphism of two twisting cochains produces a morphism of Čech complexes.

Claim 2. Different choices for the twisting cochain ρ lead to homotopically equivalent Čech complexes.

Tw. coch.: Toledo, Tong $\approx$ '85

O'Brien,

☆

Should have been earlier in the file

$$\begin{array}{ccccc} \mathbb{Z}_p & \to & P & \to & D \\ \cup & & \cup & & \cup \\ (p) & \to & J & \to & \bar{J} \end{array}$$

relative pd envelope = div. powers

$$\begin{array}{ccccc} \mathbb{Z}_p & \to & P^{(n)} & \to & D^{(n)} \\ \cup & & \cup & & \cup \\ (p) & \to & J^{(n)} & \to & \bar{J}^{(n)} \end{array}$$

$$P^{(n)} = \underbrace{P \otimes \cdots \otimes P}_{n \text{ times}} \to A$$

$J^{(n)} =$ kernel

$n \geq 0$ $\quad D = D^{(0)}$

And complete.

Meaning: $\longrightarrow$

De Rham cohomology.

$$\begin{tikzcd} \hat{D} \arrow[d, two heads] & \\ A & P \arrow[l, two heads] \arrow[ul, bend right] & J \arrow[l, hook'] \end{tikzcd}$$

$\left(= \hat{D}^{(0)}\right)$

$\hat{D}$: pd envelope + + completion.

$$\Omega^0_{\hat{D}/\mathbb{Z}_p} \xrightarrow{d} \Omega^1_{\hat{D}/\mathbb{Z}_p} \xrightarrow{d} \cdots$$

(completed) De Rham complex

(details next time).

Claim: computes same as above.

Alexander-Čech vs De Rham comparison (details on Friday).

The double complex:

$$\begin{tikzcd}
\hat{D}^{(2)} \arrow[r] & \Omega^1_{\hat{D}^{(2)}}/ \arrow[r] & {} \\
\hat{D}^{(1)} \arrow[r] \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & \Omega^1_{\hat{D}^{(1)}}/ \arrow[r] \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] & \Omega^2_{\hat{D}^{(1)}} \arrow[u, shift left=2] \arrow[u] \arrow[u, shift right=2] \\
\hat{D}^{(0)} \arrow[r] \arrow[u, shift left, "d_0"] \arrow[u, shift right, "d_1"'] & \Omega^1_{\hat{D}^{(0)}}/\mathbb{Z} \arrow[r] \arrow[u, shift left] \arrow[u, shift right] & \Omega^2_{\hat{D}^{(0)}} \arrow[u, shift left] \arrow[u, shift right]
\end{tikzcd}$$

ACYCLIC

ACYCLIC

1). (Poincaré lemma): all (cohomology of) horizontal complexes SAME.

<u>Next</u>: 1) Crystals; DeRham

2) The above has a lot of NC flavor; apart from the completion, can do everything for NC algebras.

<u>Can</u> we get something interesting noncomm. <u>exactly</u> like this?

Not sure. Close

2 to this: Hoch/cyclic

3 back to ~~Cartier~~, etc.

"Proof" of Poincaré lemma:

$$\hat{D}^{(n)} = \mathbb{Z}_p[\underbrace{x_1^{(1)}, \dots, x_N^{(1)}; x_1^{(2)}, \dots, x_N^{(2)}}_{P}; \dots; x_1^{(n)}, \dots, x_N^{(n)}]$$

this is P_J

$P^{(n)}$, and then:

add divided powers

of: 1) $\ker = J$ in the first position

2) $x_j^{(k)} - x_j^{(k+1)}$

"$\hat{D}^{(n)}$ differs from $\hat{D}^{(0)}$ by adding free pd variables".