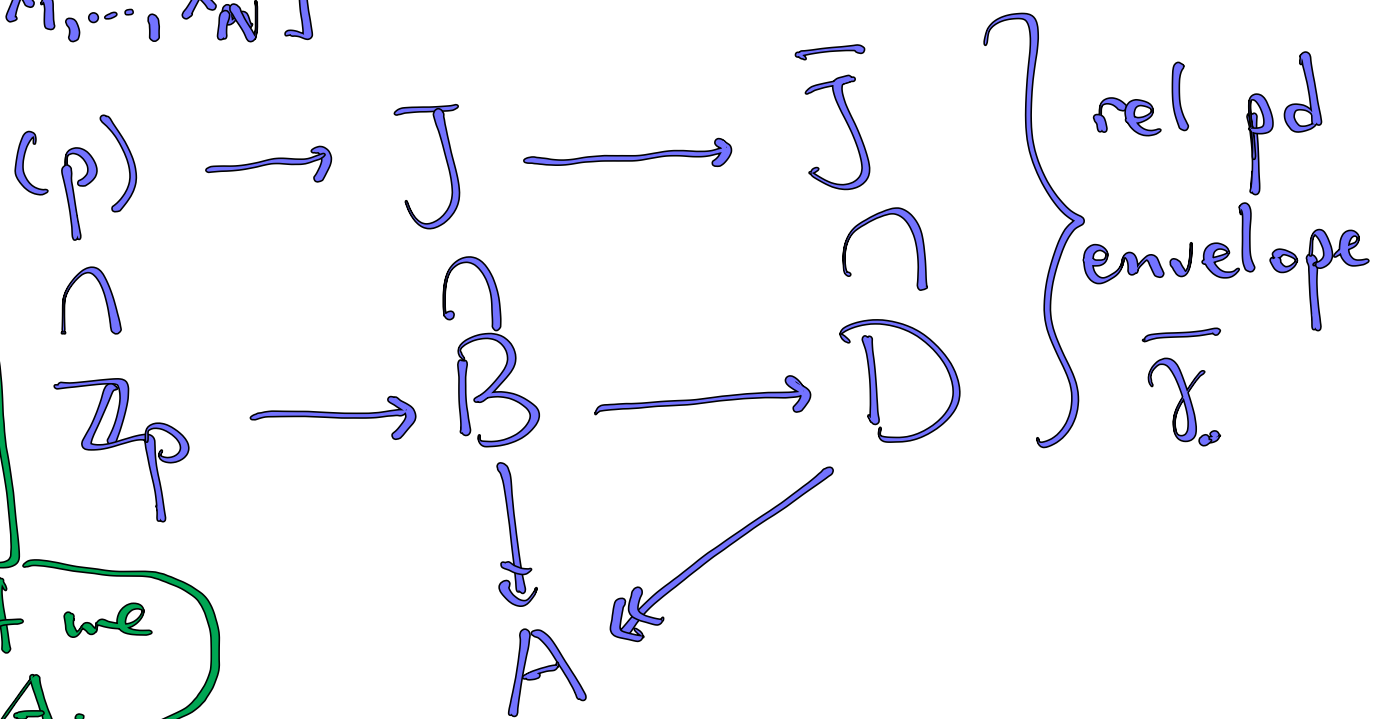


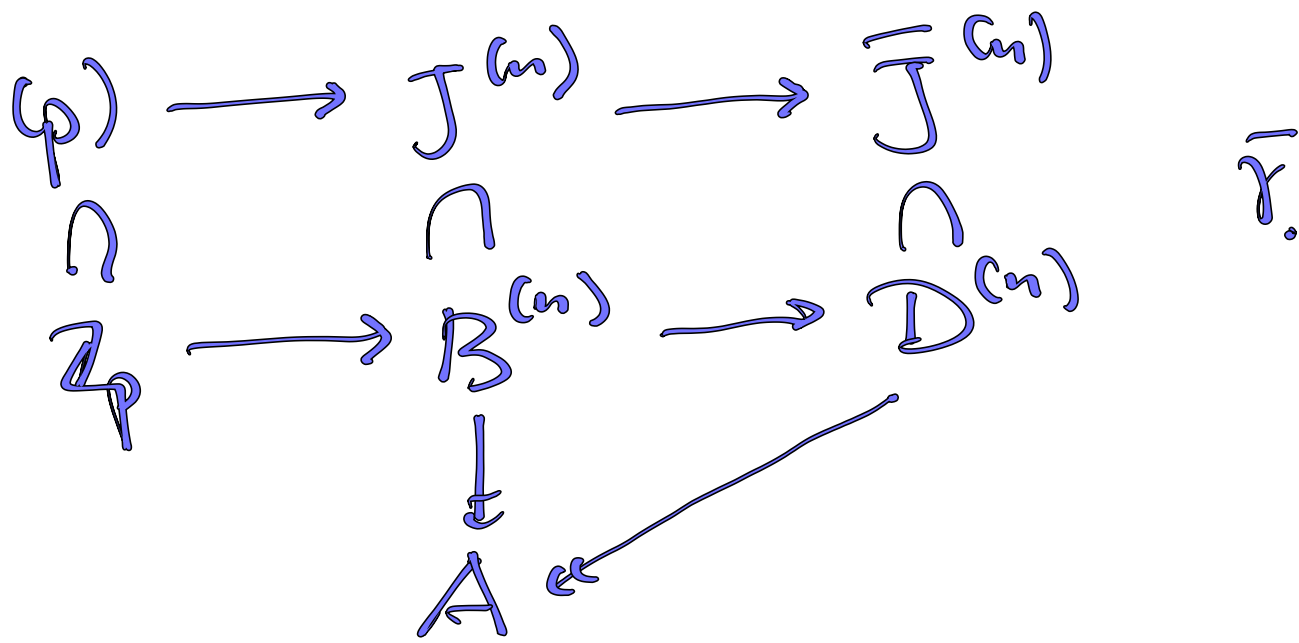
Recall: $A/\mathbb{F}_p$ fn gen

$$B = \mathbb{Z}_p[x_1, \dots, x_N]$$

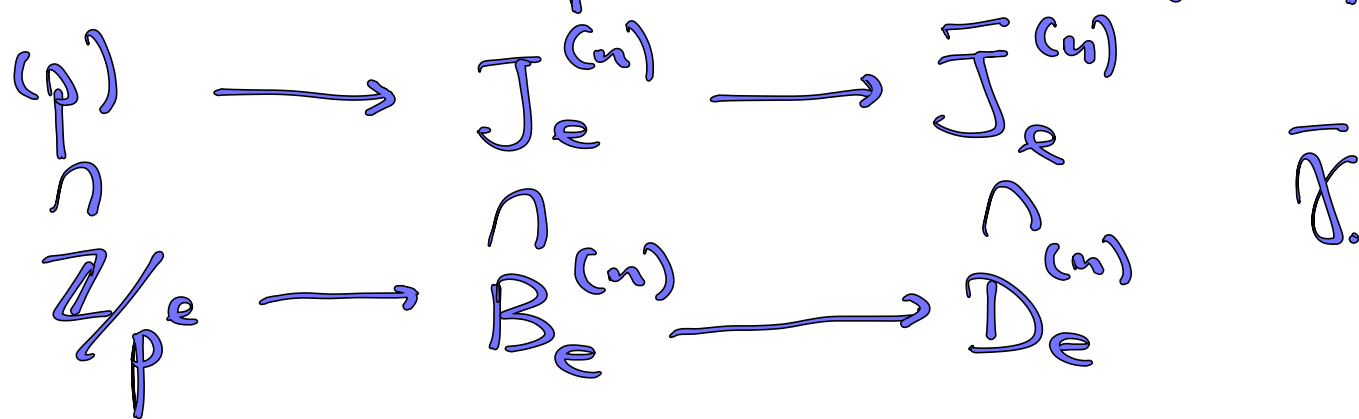
Warning:
in Stack
project, etc.:
often $A = \mathbb{Z}_p$
and C is what we
call A .



$$B^{(n)} = B \otimes_{\mathbb{Z}_p} \dots \otimes_{\mathbb{Z}_p} B \quad n \text{ times}$$



Same but over $\mathbb{Z}/p^e$ instead of $\mathbb{Z}_p$:



Completion : Two sorts of completion.

1) Most common: just p -adic completion.

Note: how does this remember $\bar{J}$ and therefore A ? Somehow it does: e.g. if $x \in \bar{J}$ then $x^n \rightarrow 0$ since $x^n = n! \cdot \bar{f}_n(x) \in p^{v_p(n!)} D$

So, for example: can see the one-cocycle for $A = \mathbb{F}_p[x, x^{-1}]$. Indeed:

$$B = \mathbb{Z}_p[x, y]$$

$$\begin{aligned} x^{-1} &= y(xy)^{-1} = y(1 + (xy - 1))^{-1} = \\ &= y \cdot \sum_{k=0}^{\infty} (xy - 1)^k \end{aligned}$$

converges in the p -adic completion.

2. Complete wrt filtration

$J[e]$ = ideal gen. by

$$\delta_{k_1}(x_1) \cdots \delta_{k_\ell}(x_\ell), \quad k_1 + \cdots + k_\ell \geq e$$

Completion 1 leads to the crystalline site; completion 2 to the nilpotent crystalline site
 (appeared in: Berthelot, as well as
 1; Mazur-Messing; Grothendieck,
 Dix Exposés (?!))
 But also

0. When $\text{char}(k) = 0$:

$$A/k; \quad B \twoheadrightarrow A; \quad B = k[x_1, \dots, x_n];$$

$$\begin{array}{c} \uparrow \\ J \end{array}$$

$$B^{(n)} = B \underset{k}{\otimes} \dots \underset{k}{\otimes} B \quad n \text{ times};$$

$$\hat{B}^{(n)} = J^{(n)}\text{-adic completion where}$$

$$J^{(n)} = \ker(B^{(n)} \rightarrow A)$$

Example

$$A = \mathbb{F}_p, B = \mathbb{Z}_p[x]$$

$$x \mapsto 0:$$

The completion 1 is $\mathbb{Z}_p\langle x \rangle^\wedge$
 $= \left\{ \sum_{k=0}^{\infty} a_k \gamma_k(x) \text{ (or } \sum a_k x^{[k]} \right\}:$
 $v_p(a_k) \rightarrow \infty \text{ as } k \rightarrow \infty \}.$

The completion 2 is $\mathbb{Z}_p\langle\langle x \rangle\rangle =$
 $= \left\{ \sum_{k=0}^{\infty} a_k x^{[k]} \right\}.$

Differentials. For a pd ring

$A, I, \gamma:$ (over k)

$\Omega_{A,I}^1$ = the usual, + additional relation

$$d\gamma_n(x) = \gamma_{n-1}(x)dx, x \in I$$

Recall: $\Omega_{A,I}^1 \cong I_\Delta / I_\Delta^2$ where
 $I_\Delta = \ker(A \otimes A \rightarrow A)$

Proof Mutually inverse maps

$$\Omega_A^1 \xrightarrow{\quad} I_\Delta / I_\Delta^2$$

$$a \mapsto a \otimes 1$$

$$da \mapsto a \otimes 1 - 1 \otimes a$$

$$a_0 da_1 \mapsto a_0 \otimes a_1$$

Lemma $\Omega_{D, \bar{J}}^1 \cong I_\Delta / I_\Delta^{[2]}$

where $I_\Delta = \ker (D^{(2)} \rightarrow D)$

Proof We have two structure maps induced by $B \rightarrow 1 \otimes B$
 $B \otimes 1$

$$b \mapsto d_0 b, d_1 b$$

or simply $1 \otimes b, b \otimes 1$.

$$D \oplus \Omega_{D/k}^1 \xleftarrow{\quad} b_0 \otimes b_1$$

$$b_0 b_1 + b_0 d b_1$$

$\rightarrow$ pd algebra; by universality:

$$D \oplus \Omega_{D, \bar{J}}^1 \xleftarrow{U} D \otimes D$$

$$\Omega_{D, \bar{J}}^1 \xleftarrow{I_\Delta} I_\Delta$$

easy to see: $I_\Delta^{(2)}$ vanishes;
the two maps are mutually inverse.

→ (Note: For any pd algebra B, J and any B -module M :

$B \oplus M$ is a pd algebra;

$$\gamma_n(b+m) = \gamma_n(b) + \gamma_{n-1}(b)m$$

Also used: I_Δ is stable under divided power ops γ_n . Simply because $D^{(2)} \rightarrow D$ is a pd morphism.

Also:

$$dh \mapsto d_0 b - d_1 b$$

$$a \mapsto a$$

is a well-defined map

$$\Sigma_{D, \bar{J}}^1 \rightarrow I_{\Delta} / I_{\Delta}^{[2]}$$

(have to check:

$$\gamma_n(x) - \gamma_{n-1}(x) dx \mapsto 0$$

But

$$\gamma_n(d_0(x)) = \gamma_n(\underbrace{d_0 x - d_1(x)} + d_1 x) =$$

$$= \sum_{k=0}^n \gamma_k(d_0 x - d_1 x) \gamma_{n-k}(d_1 x)$$

$$= \gamma_0(d_0 x - d_1 x) \cdot \gamma_n(d_1 x) +$$

$$+ \gamma_1(d_0 x - d_1 x) \cdot \gamma_{n-1}(d_1 x) + \dots$$

$$= \gamma_n(d_1 x) + (d_0 x - d_1 x) \gamma_{n-1}(d_1 x)$$

$$d_0 \gamma_n(x) - d_1 \gamma_n(x) \equiv (d_0 x - d_1 x) \cdot d_1 \gamma_{n-1}(x) \pmod{I_{\Delta}^{[2]}} \quad \square$$

Lemma. Let $D, \bar{J}, \bar{\delta}$ be
a (relative) pd envelope of
 B, J . Then

$$\Omega'_{D, \bar{J}} \cong \Omega'_{B, k} \otimes_B D$$

Proof Same idea: pd derivations
 $D \rightarrow M$ (any D -module) are same
as usual derivations $B \rightarrow M$, b/c:

- 1) a derivation θ is a morphism
of algs $B \rightarrow B \oplus M$ $b \mapsto b + \theta(b)$
- 2) a pd derivation is a pd morphism
of algs $D \rightarrow D \oplus M$ (as above).

By universal property of the pd
envelope, we get the proof.

Also:

$$D^{(n)} = D \langle x_1^{(j)} - x_1^{(0)}, \dots, x_N^{(j)} - x_N^{(0)} \rangle \mid_{j=1, \dots, n-1}$$

(actually

$$B = \mathbb{Z}_p[x_1, \dots, x_N];$$

$$B^{(n)} = \mathbb{Z}_p[x_1^{(0)}, \dots, x_N^{(0)}; \dots; x_1^{(n-1)}, \dots, x_N^{(n-1)}]$$

$$= B[x_1^{(j)} - x_1^{(0)}; \dots; x_N^{(j)} - x_N^{(0)}] \mid_{j=1, \dots, n-1}$$

Completion:

1. The p -adic completion
of $D^{(n)}$

2. $D \langle x_k^{(j)} - x_k^{(0)} \rangle \mid_{k=1, \dots, N}$
 $j=1, \dots, n-1$

De Rham complexes:

$$\Omega_{D^{(n)}, \bar{J}} = D^{(n)} \left\{ dx_1, \dots, dx_N; \right. \\ \left. d(x_k^{(j)} - x_k^{(l)}) \right\}_{\substack{k=1, \dots, N \\ j=1, \dots, n-1}}$$

Completion: same
as for $D^{(n)}$.