Recall: $A/\mathbb{F}_p$ fin gen

$B = \mathbb{Z}_p[x_1, \dots, x_N]$

$$\begin{tikzcd}
(p) \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & J \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & \bar{J} \arrow[d, phantom, "\cap" rotate=90] \\
\mathbb{Z}_p \arrow[r] & B \arrow[r] \arrow[d, two heads] & D \arrow[dl, two heads] \\
& A &
\end{tikzcd}$$

rel pd envelope $\bar{\gamma}_\cdot$

Warning: in Stack project, etc.: often $A = \mathbb{Z}_p$ and C is what we call A.

$B^{(n)} = B \otimes_{\mathbb{Z}_p} \cdots \otimes_{\mathbb{Z}_p} B$ n times

$$\begin{tikzcd}
(p) \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & J^{(n)} \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & \bar{J}^{(n)} \arrow[d, phantom, "\cap" rotate=90] \\
\mathbb{Z}_p \arrow[r] & B^{(n)} \arrow[r] \arrow[d, two heads] & D^{(n)} \arrow[dl, two heads] \\
& A &
\end{tikzcd}$$

$\bar{\gamma}_\cdot$

Same but over $\mathbb{Z}/p^e$ instead of $\mathbb{Z}_p$:

$$\begin{tikzcd}
(p) \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & J_e^{(n)} \arrow[r] \arrow[d, phantom, "\cap" rotate=90] & \bar{J}_e^{(n)} \arrow[d, phantom, "\cap" rotate=90] \\
\mathbb{Z}/p^e \arrow[r] & B_e^{(n)} \arrow[r] & D_e^{(n)}
\end{tikzcd}$$

$\bar{\gamma}_\cdot$

Completion : Two sorts of completion.

1) Most common: just p-adic completion.

Note: how does this remember $\bar{J}$ and therefore A? Somehow it does: e.g. if $x \in \bar{J}$ then $x^n \to 0$ since $x^n = n!\, \bar{\gamma}_n(x) \in p^{v_p(n!)} D$

So, for example: can see the one-cocycle for $A = \mathbb{F}_p[x, x^{-1}]$. Indeed:

$B = \mathbb{Z}_p[x, y]$

$$x^{-1} = y(xy)^{-1} = y(1 + (xy-1))^{-1} =$$
$$= y \cdot \sum_{k=0}^{\infty} (xy-1)^k$$

converges in the p-adic completion.

2. Complete wrt filtration

$J^{[e]}$ = ideal gen. by

$$\gamma_{k_1}(x_1) \dots \gamma_{k_\ell}(x_\ell), \quad k_1 + \dots + k_\ell \geq e$$

Completion 1 leads to the crystalline site; completion 2 to the nilpotent crystalline site (appeared in: Berthelot, as well as 1; Mazur-Messing; Grothendieck, Dix Exposés (?))

But also

0. When $\mathrm{char}(k) = 0$:

$$A/k; \quad \begin{array}{c} B \twoheadrightarrow A \\ \cup \\ J \end{array} ; \quad B = k[x_1, \ldots, x_N];$$

$B^{(n)} = B \otimes_k \ldots \otimes_k B$ n times;

$\hat{B}^{(n)} = J^{(n)}$-adic completion where

$J^{(n)} = \ker(B^{(n)} \to A)$

<u>Example</u> $A = \mathbb{F}_p$, $B = \mathbb{Z}_p[x]$

$x \mapsto 0$:

The completion 1 is $\mathbb{Z}_p\langle x\rangle^{\wedge}$

$$= \left\{ \sum_{k=0}^{\infty} a_k \gamma_k(x) \ \left(\text{or } \sum a_k x^{[k]}\right) : v_p(a_k) \to \infty \text{ as } k \to \infty \right\}.$$

The completion 2 is $\mathbb{Z}_p\langle\langle x\rangle\rangle =$

$$= \left\{ \sum_{k\geq 0}^{\infty} a_k x^{[k]} \right\}.$$

<u>Differentials.</u> For a pd ring A, I, γ: (over k)

$\Omega^1_{A,I}$ = the usual, + additional relation

$$d\gamma_n(x) = \gamma_{n-1}(x)\, dx, \quad x \in I$$

Recall: $\Omega^1_{A,I} \cong I_\Delta / I_\Delta^2$ where

$$I_\Delta = \ker(A \otimes A \to A)$$

Proof Mutually inverse maps

$$\Omega^1_A \rightleftarrows I_\Delta / I_\Delta^2$$

$$a \longmapsto a \otimes 1$$

$$da \longmapsto a\otimes 1 - 1 \otimes a$$

$$a_0 da_1 \longleftarrow\!\shortmid\; a_0 \otimes a_1$$

Lemma $\Omega^1_{D,\bar{J}} \simeq I_\Delta / I_\Delta^{[2]}$

where $I_\Delta = \ker (D^{(2)} \to D)$

Proof We have two structure maps induced by $B \to 1 \otimes B$, $B \otimes 1$

$b \mapsto d_0 b, d_1 b$

or simply $1 \otimes b$, $b \otimes 1$.

$$D \otimes \Omega^1_{D/k} \longleftarrow b_0 \otimes b_1$$

$b_0 b_1 + b_0 db_1$

$\longmapsto$ pd algebra; by universality:

$$\begin{array}{ccc} D \oplus \Omega^1_{D,\bar{J}} & \longleftarrow & D \otimes D \\ \cup & & \cup \\ \Omega^1_{D,\bar{J}} & \longleftarrow & I_\Delta \end{array}$$

easy to see: $I_\Delta^{[2]}$ vanishes; the two maps are mutually inverse.

(Note: For any pd algebra B, J and any B-module M: $B \oplus M$ is a pd algebra;

$$\gamma_n(b+m) = \gamma_n(b) + \gamma_{n-1}(b)m)$$

Also used: I_Δ is stable under divided power ops γ_n. Simply because $D^{(2)} \to D$ is a pd morphism.

Also:

$$db \longmapsto d_0 b - d_1 b$$

$$a \longmapsto a$$

is a well-defined map

$$\Omega^1_{D,\bar{J}} \longrightarrow I_\Delta / I_\Delta^{[2]}$$

(have to check:

$$\gamma_n(x) - \gamma_{n-1}(x)\,dx \longmapsto 0$$

But

$$\gamma_n(d_0(x)) = \gamma_n(\underbrace{d_0 x - d_1(x)} + d_1 x) =$$

$$= \sum_{k=0}^{n} \gamma_k(d_0 x - d_1 x)\,\gamma_{n-k}(d_1 x)$$

$$= \gamma_0(d_0 x - d_1 x) \cdot \gamma_n(d_1 x) +$$

$$+ \gamma_1(d_0 x - d_1 x) \cdot \gamma_{n-1}(d_1 x) + \underset{I_\Delta^{[2]}}{\underset{\cap}{\ldots}}$$

$$= \gamma_n(d_1 x) + (d_0 x - d_1 x)\,\gamma_{n-1}(d_1 x)$$

$$d_0\gamma_n(x) - d_1\gamma_n(x) \equiv (d_0 x - d_1 x) \cdot d_1\gamma_{n-1}(x) \mod I_\Delta^{[2]}$$

$\square$

Lemma. Let $D, \bar{J}, \bar{\gamma}$ be a (relative) pd envelope of B, J. Then

$$\Omega^1_{D,\bar{J}} \simeq \Omega^1_{B,k} \otimes_B D$$

Proof Same idea: pd derivations $D \to M$ (any D-module) are same as usual derivations $B \to M$, b/c:

1) a derivation θ is a morphism of algs $B \to B \oplus M$ $\quad b \mapsto b + \theta(b)$

2) a pd derivation is a pd morphism of algs $D \to D \oplus M$ (as above).

By universal property of the pd envelope, we get the proof.

Also:

$$D^{(n)} = D\langle x_1^{(j)} - x_1^{(0)}, \ldots, x_N^{(j)} - x_N^{(0)} \rangle \Big|_{j=1,\ldots,n-1}$$

(actually

$$B = \mathbb{Z}_p[x_1, \ldots, x_N];$$

$$B^{(n)} = \mathbb{Z}_p[x_1^{(0)}, \ldots, x_N^{(0)}; \ldots; x_1^{(n-1)}, \ldots, x_N^{(n-1)}]$$

$$= B[x_1^{(j)} - x_1^{(0)}; \ldots; x_N^{(j)} - x_N^{(0)}] \Big|_{j=1,\ldots,n-1}$$

Completion:

1. The p-adic completion of $D^{(n)}$

2. $D\langle\langle x_k^{(j)} - x_k^{(0)} \rangle\rangle \Big|_{\substack{k=1,\ldots,N \\ j=1,\ldots,n-1}}$

De Rham complexes:

$$\Omega^{\cdot}_{D^{(n)}, \bar{J}} = D^{(n)} \left\{ dx_1, \ldots, dx_N ; \; d\left(x_k^{(j)} - x_k^{(0)}\right) \right\}_{\substack{k=1,\ldots,N \\ j=1,\ldots,n-1}}$$

Completion: same as ~~for~~ $D^{(n)}$.