Just emailed pics "for Lecture 14"

Today: will summarize what needed to go forward; DETAILS are

$B \simeq \mathbb{Z}_p[x_1, \ldots, x_N]$ $\quad J \subset B \twoheadrightarrow A$

kernel

$\downarrow$

A over $\mathbb{F}_p$

notation clashes a bit with Stack Proj ch. 23, 59

$(A = \mathbb{Z}_p)$

pd str.

$$\begin{tikzcd}
(p) \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & J \arrow[r] \arrow[d, phantom, "\cap" rotate=-90] & \bar{J} \arrow[d, phantom, "\cap" rotate=-90] \\
\mathbb{Z}_p \arrow[r] & B \arrow[r] \arrow[d] & D \arrow[dl] \\
& A &
\end{tikzcd}$$

$\bar{\gamma}$, ; rel pd envelope

$$\begin{tikzcd}
\underbrace{B \otimes \cdots \otimes B}_{n \text{ times}} = B^{(n)} \arrow[r] \arrow[d] & D^{(n)}, \bar{J}^{(n)}, \bar{\gamma}. \arrow[dl] \\
A &
\end{tikzcd}$$

$D^{(n)} \approx D\langle x_k^{(j)} - x_k^{(0)} \rangle$

$B = \mathbb{Z}_p[x_1, \ldots, x_N]$

$B^{(n)} = \mathbb{Z}_p[x_k^{(j)}]_{k=1,\ldots,N;\ j=0,\ldots,n-1} = B[x_k^{(j)} - x_k^{(0)}]_{\substack{k=1,\ldots,N \\ j=1,\ldots,n-1}}$

__Forms__ Given B, J p.d. ring

__our__ B, J does not!

should have said: $\boxed{D, \bar{J} \ldots}$

but I want it for ANY pd ring. not just for pd envelopes.

$\Omega^\bullet_{B,J}$

the usual $\Omega^\bullet_{B/k}$ with extra relations:

$$\boxed{d\gamma_n(x) = \gamma_{n-1}(x)dx}_{x \in J}$$

$$\boxed{d\left(\frac{x^n}{n!}\right) = \frac{x^{n-1}}{(n-1)!}dx}$$

<u>But</u>: derivations $B \to M$ (for $\Omega^1_{D,\bar{J}}$)

same as <u>algebra morphisms</u>

$$B \to B \oplus M \;\left(\xrightarrow{\text{proj}} B \right), \qquad B \to B \oplus M \xrightarrow{\text{proj}} B \;=\; \mathrm{id}_B$$

trivial extension of square zero.

$$\boxed{\Omega^\bullet_{D,\bar{J}} \cong \mathrm{Sym}_D\, \Omega^1_{D,\bar{J}}}$$

pd

pd ders = algebra morphisms

$$D \longrightarrow D \oplus M \supset J \oplus M$$

pd structure <u>here</u>.

$\boxed{\text{B AND: pd envelope is D } \underline{\text{universal}} \text{ for pd alg morphisms}}$

So:

$$\Omega^\bullet_{D^{(n)}, \bar{J}^{(n)}} \simeq$$

In particular: $H^\bullet_{DR}$ same for $D^{(n)}$ and $D^{(0)}$, $\forall n$.

$$\simeq D\left\langle x_k^{(j)} - x_k^{(0)} \right\rangle \left\{ dx_k, dx_k^{(j)} - dx_k^{(0)} \right\}$$

$k = 1, \dots, N$

$j = 1, \dots, N-1$

$\Omega^\bullet_{D^{(0)}, \bar{J}^{(0)}}$

$$\Omega^\bullet_D \left\langle \underbrace{x_k^{(j)} - x_k^{(0)}} \right\rangle \left\{ d\underbrace{\left(x_k^{(j)} - x_k^{(0)}\right)} \right\}$$

$B \to D$

$\Omega^\bullet_B, d$ → not a B-mod cplex, but: B, ∇_{flat}

$\Omega^\bullet_{D, J}, d$

$\parallel$

$D \otimes_B (\Omega^\bullet_B, d)$ $\quad d(a \otimes \omega) = da \otimes \omega + \_$

Poincaré lemma:

$$\underbrace{\mathbb{Z}_p \langle y_1, \ldots, y_k \rangle}_{\text{free pd alg}} \{dy_1, \ldots, dy_k\}$$

(And the completions)

is acyclic $(\simeq \mathbb{Z}_p$ in deg $0)$

$k=1$:

$$y^{[n]} \mapsto y^{[n-1]} dy$$

$$\boxed{\Omega^{\bullet}_{D^{(0)}, \bar{J}^{(0)}} \otimes \mathbb{Z}_p \langle \ldots \rangle \simeq \Omega^{\bullet}_{D^{(\omega)}, \bar{J}^{(\omega)}}}$$

Completions

1). Just p-adic completion

$$\hat{D}^{(n)} = \hat{D}^{(0)}\langle x_k^{(j)} - x_k^{(0)} \rangle^{\wedge \text{p-adic}}$$

$$B = \mathbb{Z}_p[x] \to \mathbb{F}_p$$

$$x \mapsto 0$$

$\hat{D}$ = p-adic compl of $\mathbb{Z}_p\langle x \rangle$

$$\sum_{k=0}^{\infty} a_k x^{[k]} : v_p(a_k) \to \infty$$

(crystalline site)

2). Completion wrt filtration $J^{[e]}$, $e \geq 0$.

(nilpotent crystalline site). $\hat{D} = \{ \sum_{k=0}^{\infty} a_k x^{[k]} \}$

in particular:

$$\exp(x) = \sum_{k=0}^{\infty} x^{[k]}$$

Čech – De Rham :

1. All vertical d_j's are BIJECTIONS on horizontal cohomology

2. All vertical cohomologies but the extreme left are ZERO

$$
\begin{tikzcd}
{} & {} & {} & \\
\Omega^0_{D^{(1)},\bar{J}^{(1)}} \arrow[u, "d_0 \,\cdots\, d_2"] \arrow[r] & \Omega^1_{D^{(1)},\bar{J}^{(1)}} \arrow[u] \arrow[r] & \Omega^2_{D^{(1)},\bar{J}^{(1)}} \arrow[u] \arrow[r] & {} \\
\Omega^0_{D^{(0)},\bar{J}} \arrow[u, "d_0 \,\cdots\, d_1"] \arrow[r] & \Omega^1_{D^{(0)},\bar{J}} \arrow[u] \arrow[r] & \Omega^2_{D^{(0)},\bar{J}} \arrow[u] \arrow[r] & \cdots \\
 & \text{ACYCLIC} & \text{ACYCLIC} &
\end{tikzcd}
$$

The first column: . . .

$$
\begin{array}{ccccc}
\vdots & & & & \\
\uparrow d & & & & \\
\Omega_{D^{(2)},\bar{J}^{(2)}} & \simeq & \Omega^{0}_{D^{(2)},\bar{J}^{(2)}} & & \{dx_k^{(0)}; dx_k^{(1)}; dx_k^{(2)}\} \\
\uparrow d \;\uparrow\;\uparrow & & & & \overbrace{\phantom{\{dx_k^{(0)}; dx_k^{(1)}\}}}^{\text{linear span of}} \\
\Omega_{D^{(1)},\bar{J}^{(1)}} & \simeq & \Omega^{0}_{D^{(1)},\bar{J}^{(1)}} & & \{dx_k^{(0)}; dx_k^{(1)}\} \\
\uparrow d \;\uparrow & & & & \overbrace{\phantom{\{dx_k^{(0)}\}}}^{\text{linear span of}} \\
\Omega_{D^{(0)},\bar{J}^{(0)}} & \simeq & \Omega^{0}_{D^{(0)},\bar{J}^{(0)}} & & \{dx_k^{(0)}\} \\
 & & & & k=1,\dots,N
\end{array}
$$

1).

Bhatt–de Jong 2011?

$$\Omega^0(-) \qquad \begin{array}{c} \vdots \\ D^{(2)} \\ d_0 \uparrow \ \uparrow \ \uparrow d_2 \\ D^{(1)} \\ d_0 \uparrow \ \uparrow d_1 \\ D^{(0)} \end{array}$$

just our Čech-Alex. cplx

Cosimplicial $\mathbb{Z}_p$-module

$d_i d_j = \dots$

relations guaranteeing $d^2 = 0$

ACYCLIC

$$\begin{array}{c} D^{(2)} \otimes V^3 \\ \uparrow\uparrow\uparrow \quad \nearrow\nearrow\nearrow \\ D^{(1)} \otimes V^2 \\ \uparrow\uparrow \quad \nearrow\nearrow \\ D^{(0)} \otimes V^1 \end{array}$$

cartesian power $V \times V$

$V := \mathbb{Z}_p$-basis: $\boxed{dx_1, \dots, dx_N}$