Connections (and ⇝ crystals)

$k \to B$ morphism ~~of algs~~

B alg$/k$

$\Omega^1_{B/k} \simeq I_\Delta / I_\Delta^2$ where

$I_\Delta = \ker(B \otimes_k B \to B)$

B-module

Let M be a B-module:

$$M \xrightarrow[\nabla]{} \Omega^1_{B/k} \otimes_B M$$

k-linear morphism

$$\nabla(bm) = db \cdot m + b \cdot \nabla m$$

Then it extends to

$$\nabla : \Omega^{\bullet}_{B/k} \otimes_B M \to \Omega^{\bullet+1}_{B/k} \otimes_B M$$

$$\nabla(\omega \otimes_B m) = \underbrace{d}\omega \otimes_B m + (-1)^{|\omega|} \omega \otimes \nabla m$$

w.d.: $\omega b \otimes m - \omega \otimes bm \longmapsto 0$

$$0 \leftarrow \begin{cases} d(\omega b) \otimes m - d\omega \otimes bm \\ + \omega b \otimes \nabla m - \omega \otimes \nabla(bm) \end{cases}$$

$= \underbrace{d\omega \cdot b \pm \omega \cdot db}$ $\quad \underbrace{db \cdot m} + b \cdot \nabla m$

b/c $\otimes_B$

∇ is <u>flat</u> if $\nabla^2 = 0$

(enough to check:

$$M \xrightarrow{\nabla^2} \Omega^2_{B/\mathbb{Z}_p} \otimes_B M \quad \text{is } 0;$$

$$\underbrace{\nabla^2(\omega \cdot m) = \omega \cdot \nabla^2 m}_{\text{always.}})$$

<u>Now</u>:

$$\begin{tikzcd}
(p) \arrow[r] \arrow[d, phantom, "\cap"] & J \arrow[r] \arrow[d, phantom, "\cap"] & \bar{J} \arrow[d, phantom, "\cap"', "\text{p.d. env.}"] \\
\mathbb{Z}_p \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & D \arrow[d] \\
\mathbb{F}_p \arrow[r] & A \arrow[r, equal] & A
\end{tikzcd}$$

p.d.

$\Omega^{\bullet}_{D,\bar{J}}$

Now: consider a D-module M.

Assume a (flat) connection which is by definition

$$M \to \Omega^1_{D,\bar J} \otimes_D M$$

$$\nabla(bm) = db \cdot m + b \cdot \nabla m$$

$$b \in D$$

Consequences of that?

Already have:

pd

thickening.

meaning: pd stre on KER

$$\begin{tikzcd}
\bar J \arrow[d, phantom, "\cap"] & & I \arrow[dl] \\
D \arrow[r, "\exists \varphi"] \arrow[d] & C \arrow[dl] & \\
A & &
\end{tikzcd}$$

Any pd thick.

not assuming any freeness

$$
\begin{tikzcd}
J \arrow[r] \arrow[d, hook] & \bar{J} \arrow[d, hook, "\text{p.d.}"] & I \arrow[d, hook, "\text{p.d.}"'] \\
B \arrow[r] \arrow[dr, two heads] \arrow[rr, bend right, "\exists"] & D \arrow[r, dotted, "\exists"] \arrow[d, two heads] & C \arrow[dl] \\
& A &
\end{tikzcd}
$$

$$
\begin{tikzcd}
x_j \arrow[rr, mapsto] \arrow[dr, mapsto] & & \exists c_j \arrow[dl, mapsto] \\
& \mathrm{proj}(x_j) &
\end{tikzcd}
$$

$$
\begin{tikzcd}
M \circlearrowleft D \arrow[rr, "\exists \varphi"] \arrow[dr, two heads] & & C \arrow[dl] \\
& A &
\end{tikzcd}
$$

D-mod

$C \otimes_D M$

a.k.a.

$C \otimes_\varphi M$

Note: $b \in D \xrightarrow{\varphi} C$

$\circlearrowleft$

$M, \nabla \qquad C \otimes_D M, \nabla$

induced connection.

$$\nabla(c \otimes m) = dc \otimes_C m + c \otimes \varphi(\nabla m)$$

$\in \Omega^1_{C,I}$ $\quad \in \Omega^1_{D,J} \otimes_D M$

$\boxed{b_0 \, db_1 \mapsto \varphi(b_0)\, d\varphi(b_1)}$

check:

$\nabla(\varphi(d$

$$\Omega^1_{D/k} \xrightarrow{\varphi} \Omega^1_{C/k}$$

$$\Omega^1_{D,J} \xrightarrow{\varphi} \Omega^1_{C,I}$$

φ:
$$\begin{array}{ccc} J & \longrightarrow & I \\ \cap & & \cap \\ D & \xrightarrow[\varphi]{} & C \end{array}$$

$$\Omega^1_{D,J} \xrightarrow[\varphi]{} \Omega^1_{C,I}$$

ext. to $\Omega^1_{D,J} \otimes_D M \xrightarrow{\varphi} \Omega^1_{C,I} \otimes_C M$

$$\omega \underset{D}{\otimes} m \longmapsto \varphi(\omega) \underset{C}{\otimes} m$$

w.d.: $(b\omega) \otimes m \longmapsto \varphi(b\omega) \otimes m$

$\omega \otimes bm \longmapsto \varphi(\omega) \otimes bm$

$\varphi(b)$

$$\underset{D}{\otimes} = \underset{\varphi}{\otimes}$$

$$\nabla : M \to \Omega^1_{D,J} \otimes_D M$$

$$\nabla(c \otimes_D m) = dc \otimes_D m + c \otimes \underbrace{(\nabla m)}_{\Omega^1_{D,J} \otimes_D M}$$

well-def. (exercise)

$$\Omega^1_{D,J} \otimes_D M \to \Omega^1_{C,I} \otimes_C M$$

$$D \xrightarrow{\varphi} C$$

$$\underset{\text{on } M}{\nabla} \rightsquigarrow \nabla \text{ on } C \otimes_D M$$

$$C \underset{\varphi_1}{\otimes} M \xleftarrow[T_{1,2}]{\sim} C \underset{\varphi_2}{\otimes} M$$

of C-modules

$$\sum_{k \geq 0} \frac{h^k(x)}{k!} \otimes \underline{\nabla_x^k} m \longleftarrow 1 \otimes m$$

D has ~~variables~~ $x_1, \ldots, x_N$
elts

$$B = \mathbb{Z}_p[x_1, \ldots, x_N]$$

$$\boxed{\begin{array}{c} \Omega^1_{D,J} \\ \| \\ \bigoplus^N D \cdot dx_j \\ \| \end{array}}$$

$$\Omega^1_B \otimes_B D$$

$$? \pm h = \varphi_1 - \varphi_2$$

$$\gamma_{k_1}(h(x_1)) \cdots \gamma_{k_N}(h(x_N)) \otimes \nabla_{x_1}^{k_1} \cdots \nabla_{x_N}^{k_N} m$$

$$
\begin{tikzcd}
M \arrow[r] \arrow[ddddr, bend right, "\nabla_{x_i}"'] & \Omega^1_{D,J} \otimes_D M \\
& \bigoplus D \otimes_D M \cdot dx_j \arrow[u, phantom, sloped, "\cong"] \\
& \bigoplus M \cdot dx_j \arrow[u, phantom, sloped, "="] \arrow[dd, "\text{proj onto the } dx_i\text{-compt}"] \\
& \\
& M
\end{tikzcd}
$$

$$C \underset{\varphi_1}{\otimes} M \longleftarrow C \underset{\varphi_2}{\otimes} M$$

where $D \overset{\varphi_1}{\underset{\varphi_2}{\rightrightarrows}} C$

$$\boxed{h(x) = \varphi_2(x) - \varphi_1(x)}$$

$\circlearrowleft$ M

$$\sum_{k \geq 0} \frac{h^k(x)}{k!} \underset{\varphi_1}{\otimes} \nabla_x^k m \longleftarrow\!\shortmid 1 \underset{\varphi_2}{\otimes} \cdot m$$

$$= \sum_{k_1, \dots, k_N} \gamma_{k_1}(x_1) \dots \gamma_{k_N}(x_N) \underset{\varphi_1}{\otimes} \nabla_{x_1}^{k_1} \dots \nabla_{x_N}^{k_N} m$$

well-defined:

$$\sum \frac{h^k(x)}{k!} \otimes \nabla^k(bx) \xleftarrow{T_{12}} 1 \underset{\varphi_2}{\otimes} bm$$

$$\sum_{k_1, k_2 \geq 0} \frac{h^{k_1+k_2}(x)}{(k_1+k_2)!} \cdot \binom{k_1+k_2}{k_1} \underset{\varphi_1}{\otimes} \partial_x^{k_1} b \cdot \nabla_x^{k_2} m$$

$$= \sum \frac{h^{k_1+k_2}(x)}{k_1! k_2!} \varphi_1(\partial_x^{k_1} b) \otimes_{\varphi_1} \nabla_x^{k_2} m =$$

$$\sum \frac{h^{k_2}}{k_2!} \cdot \frac{h^{k_1}(x)}{k_1!} (\partial_x^{k_1} b)(\varphi_1(x)) \otimes_{\varphi_1} \nabla_x^{k_2} m$$

$$= \sum \frac{h^{k_2}(x)}{k_2!}\, b(\varphi_2(x)) \otimes \nabla_x^{k_2} =$$

$$= \varphi_2(b) \cdot T_{12}(1 \otimes_{\varphi_2} m). \qquad \square$$

<u>Claim</u> $T_{12}\, T_{23} = T_{13}$, given

$$\begin{tikzcd} M \circlearrowleft D \arrow[r, bend left, "\varphi_1"] \arrow[r, "\varphi_2"] \arrow[r, bend right, "\varphi_3"'] & C \end{tikzcd}$$

Follows from flatness:

"$\exp(A+B) = \exp(A)\exp(B)$"

<u>and</u>, crucially,

$$\nabla_{x_j} \nabla_{x_i} = \nabla_{x_i} \nabla_{x_j}$$

Restriction :

Involves infinite sum, so:

- If ∇ is ANY flat connection: requires completion 2 as above

Namely: with respect to the filtration $J^{[e]}$, $e \geq 0$.

- If ∇ is LOCALLY NILPOTENT: then the completion 1 is enough, (i.e. the p-adic compl.)

Loc. nilp.: $\forall m \; \forall e \; \exists k$: $\nabla^k m \in (p^e)$

Next: A connection IS a T_{12} for some thickening.

$$\begin{tikzcd}
D \arrow[r, shift left, "\varphi_1"] \arrow[r, shift right] \arrow[dr, shift left, "\varphi_2"] \arrow[dr, shift right, "id"'] & D^{(2)}/I_\Delta^{[2]} \arrow[d] \\
& D
\end{tikzcd}$$

$$\begin{array}{c} D \\ \circlearrowleft \\ M \end{array} \qquad D \underset{\varphi_2}{\overset{\varphi_1}{\rightrightarrows}} D^{(2)}/I_\Delta^{[2]} =: C$$

$$T_{21}: \quad C \otimes_{\varphi_1} M \longrightarrow C \otimes_{\varphi_2} M$$

$$1 \otimes_{\varphi_1} m \longmapsto 1 \otimes_{\varphi_2} m + \nabla m$$

$$\nabla m \in (I_\Delta / I_\Delta^2) \otimes_{\varphi_2} M$$

or:

$$1 \otimes_{\varphi_1} m \longmapsto 1 \otimes_{\varphi_2} m + \sum_j (\varphi_1(x_j) - \varphi_2(x_j)) \otimes_{\varphi_2} \nabla_j m$$

$$\left(= 1 \otimes_{\varphi_2} m + \sum d x_j \otimes_{\varphi_2} \nabla_j m \right)$$

Well-defined morphism:

$$1 \otimes_{\varphi_1} bm \longmapsto 1 \otimes_{\varphi_2} bm + \nabla(bm) = 1 \otimes_{\varphi_2} bm + db \otimes m + b \cdot \nabla m$$

$$= \varphi_2(b) \otimes m + (\varphi_1(b) - \varphi_2(b)) \otimes m + \varphi_1(b) \cdot \nabla m = \varphi_1(b) \cdot T_{21}(1 \otimes m)$$

$\varphi_1(b)$ or $\varphi_2(b)$: mod $I_\Delta^{[2}$ are the same.

Flatness of ∇ is equivalent to:

$$D \overset{\varphi_1}{\underset{\varphi_3}{\rightrightarrows}} D^{(3)} / \text{ideal gen. by } \gamma_{\geq 2}(\varphi_1(x) - \varphi_2(x));\ \gamma_{\geq 2}(\varphi_2(x) - \varphi_3(x)),\ \gamma_{\geq 2}(\varphi_1(x) - \varphi_3(x))$$

$$T_{3,2} T_{2,1} = T_{3,1}$$

Proof

$$1 \otimes_{\varphi_1} m \overset{T_{21}}{\longmapsto} 1 \otimes_{\varphi_2} m + \sum (\varphi_1(x_j) - \varphi_2(x_j)) \otimes_{\varphi_2} \nabla_j m$$

$$\downarrow T_{32}$$

$$1 \otimes_{\varphi_3} m + \sum 1 \otimes (\varphi_2(x_j) - \varphi_3(x_j)) \otimes_{\varphi_3} \nabla_j m +$$

$$+ \sum (\varphi_1(x_j) - \varphi_2(x_j)) \otimes_{\varphi_3} \nabla_j m + \sum (\varphi_1(x_j) - \varphi_2(x_j))(\varphi_2(x_k) - \varphi_3(x_k)) \otimes \nabla_j \nabla_k m$$

$$= 1 \otimes_{\varphi_3} m + \sum (\varphi_1(x_j) - \varphi_3(x_j)) \otimes_{\varphi_3} m$$

iff ∇ is flat. $\square$

<u>Conclusion.</u> a D-module with a flat connection $\longleftrightarrow$ a CRYSTAL, i.e.:

Any p-complete pd thickening $C \rightsquigarrow$ a C-module M_C

Any morphism of pd-thickenings $C_1 \to C_2 \rightsquigarrow$ isomorphism

$$M_{C_2} \xrightarrow{\sim} C_2 \otimes_{C_1} M_{C_1}$$

Such that:

$$M_{C_3} \xrightarrow{\sim} C_3 \otimes_{C_2} M_{C_2} \xrightarrow{\sim} C_3 \otimes_{C_2} C_2 \otimes_{C_1} M_{C_1}$$

commutes $\quad \cong$

$$M_{C_3} \xrightarrow{\sim} C_3 \otimes_{C_1} M_{C_1}$$

To construct M_C:

<u>choose</u> $D \xrightarrow{\varphi} C$ over A; $M_C := C \otimes_{\varphi} M$