

Wrapping up crystalline cohom.

Week 7: NC geom

Week 8: Deligne-Illusie degeneration

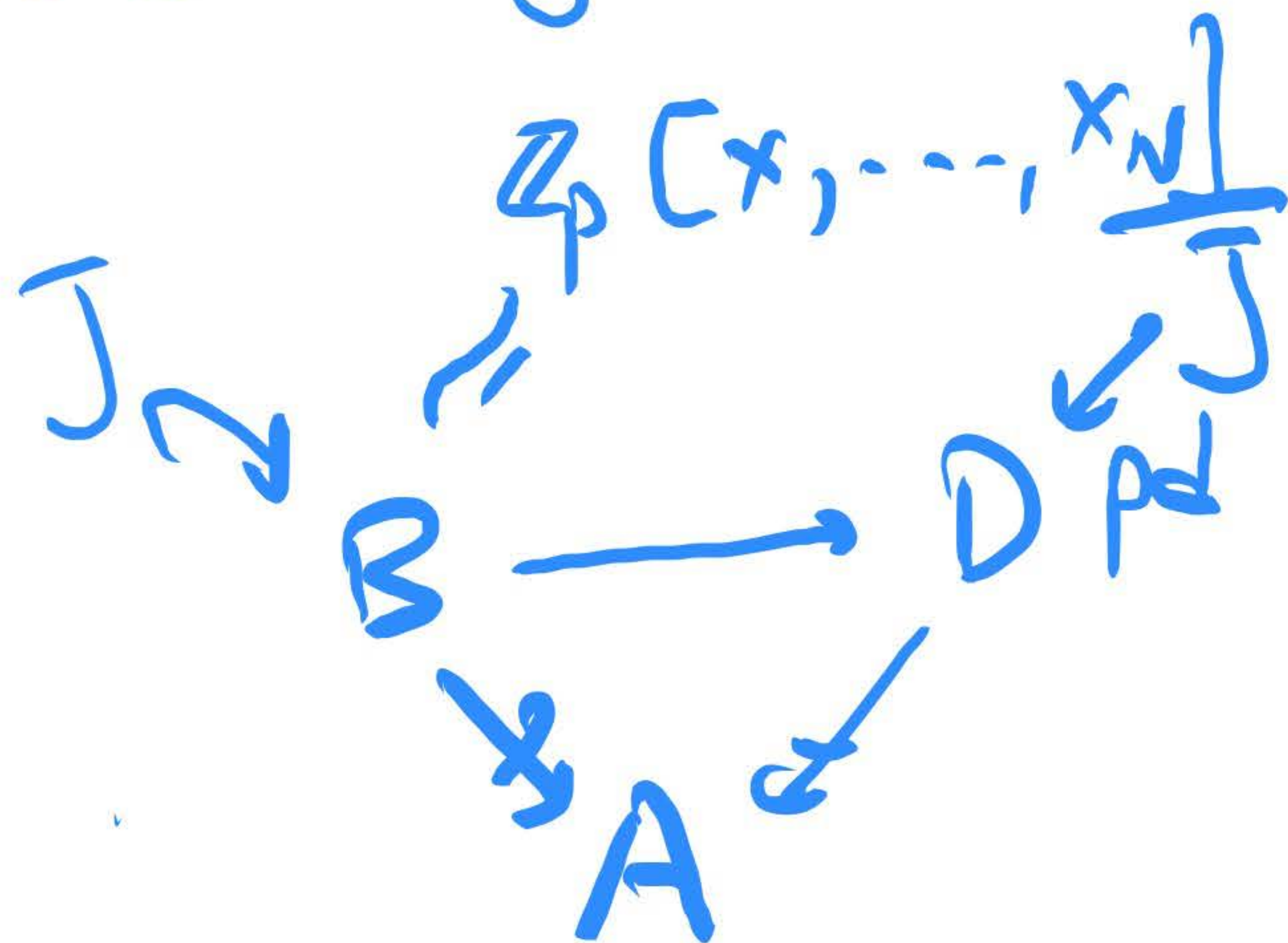
+ ...

Prisms

...

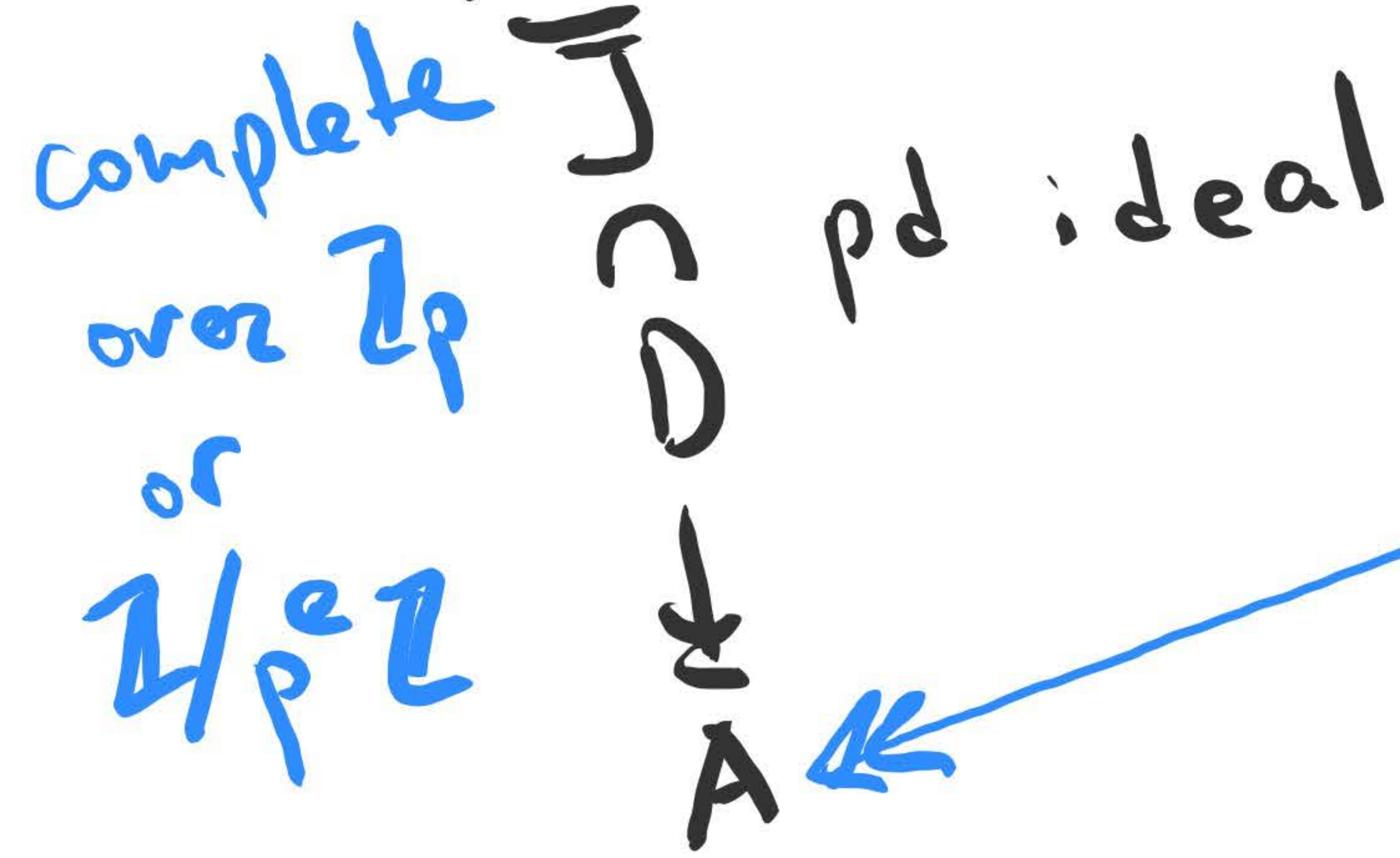
=

A over $\mathbb{F}_p$



Comparison to DRWitt.

kernel
has a
pd structure.



$W(A)$

$\ker (W(A) \rightarrow A)$
consists of
 $V(x), x \in W(A)$

$\chi_n(V(x)) = ?$

$V(x)V(y) = p \cdot V(xy)$

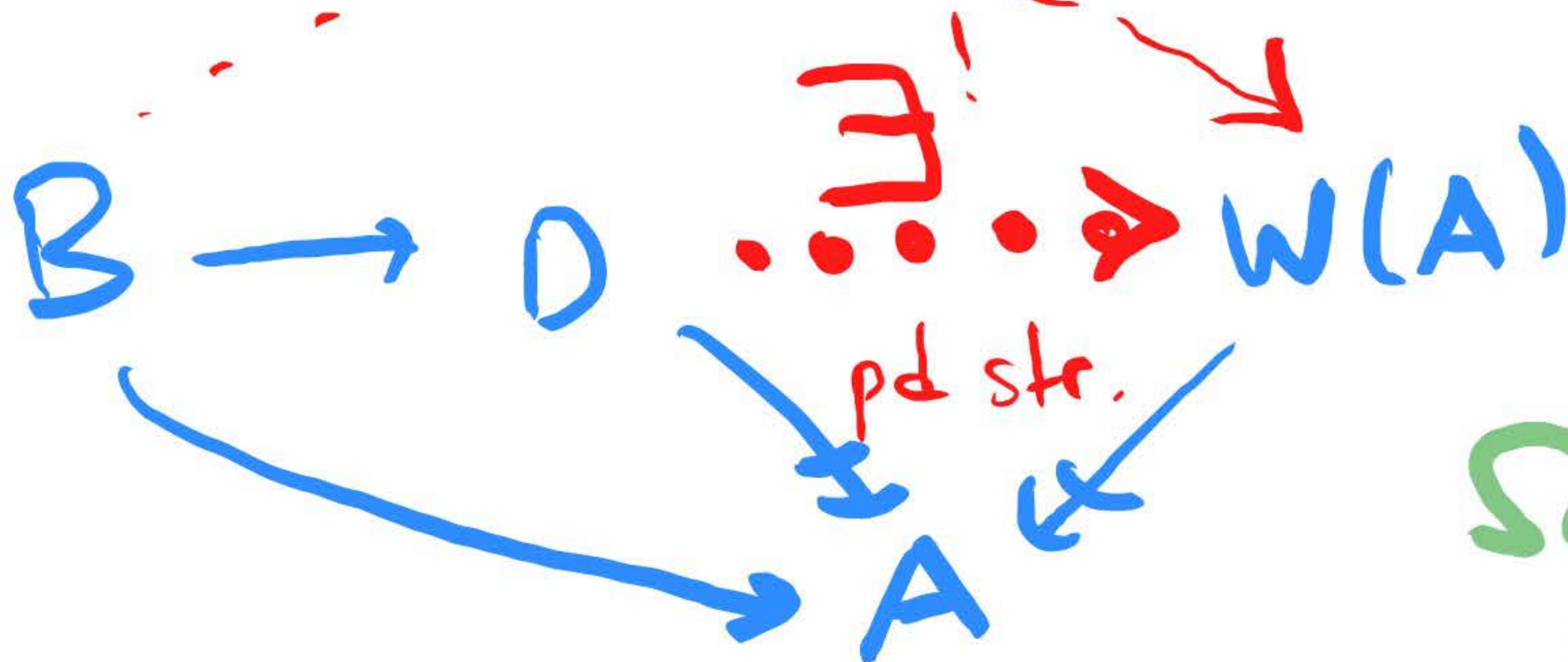
$V(x)^n = p^{n-1} V(x^n) \mid \frac{p^{n-1}}{n!} \in \mathbb{Z}$

$$\chi_n(V(x)) = \frac{p^{n-1}}{n!} V(x^n), \quad n > 0$$

(all axioms are satisfied)



... $\exists$ since B is free alg over $\mathbb{Z}_p$.



$$\Omega_{D/\mathbb{Z}_p}^i \rightarrow \Omega_{W(A)/\mathbb{Z}_p}^i$$

$$\Omega_{D, \bar{\mathbb{Z}}}^i \xrightarrow{\exists!} W\Omega_A^i$$

here

$$d\chi_n(x) - \chi_{n-1} dx_1 \rightarrow 0$$

$$\begin{array}{ccc}
 \Omega_p / \pi_p & \longrightarrow & \Omega_{W(A)} / \pi_p \\
 \downarrow & & \downarrow \\
 \Omega_{D, \bar{J}} & \xrightarrow{\exists!} & W \Omega_A
 \end{array}$$

"divided exponents"
 Cancel out by d_{DR}
 $p^n x_j / p^n$

Next: This is a quasi-isomorphism
 for $A = F_p[x_1, \dots, x_N]$
 $B = \mathbb{Z}_p[\quad]$ ✓

Localization

The Čech-Alexander complex

far from
a complex
of A -mods.

$$D^{(\bullet)} \quad \text{and} \quad \Sigma_{D, \bar{J}}$$

the DR cplx
behave well

under

localization in Zariski topology.

Why

When you take $s \in A$ (over $\mathbb{F}_p$):

$$S \text{ inv.} \iff \tilde{S} \text{ inv.}$$

when you localize in s , you
localize in any $\tilde{s} \mapsto s$ you
lp-adic completeness

$$\tilde{s} \in B \rightarrow D$$

As for localization in $\tilde{S} \in D$:

both $\Sigma_{p, \bar{j}}$ and

$D^{(\cdot)}$

behave well.

the differential is NOT a
morphism of mgs, but a
 $\Sigma \pm d_j$ of those. They descend
to localization.

d_{PR}
compat.
w/ localiz.

$$\frac{d(s^{-1}) = -s^{-2}}{--s^{-2}}$$

BTW: Everywhere above, could replace

$B = \mathbb{Z}_p[x_1, \dots, x_n]$ by a SMOOTH

algebra over $\mathbb{Z}_p$.

Crystalline cohomology

Assume:

X a scheme

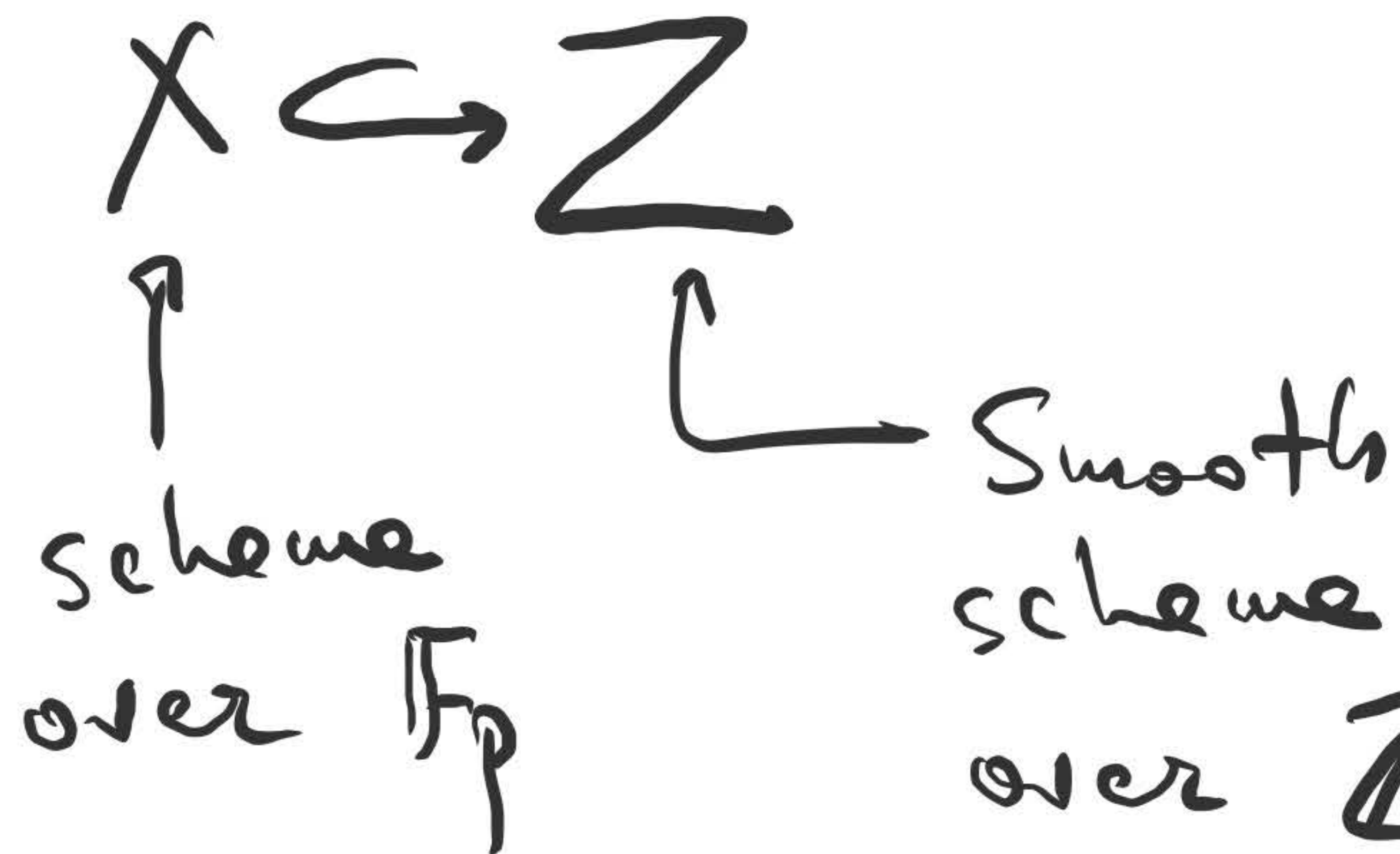
over $\mathbb{F}_p$

$X \hookrightarrow Z$

SMOOTH
scheme over $\mathbb{Z}_p$

e.g.
(quasi)
 X a projective

scheme over
 $Z = \mathbb{P}^n(\mathbb{Z}_p)$ $\mathbb{F}_p$.



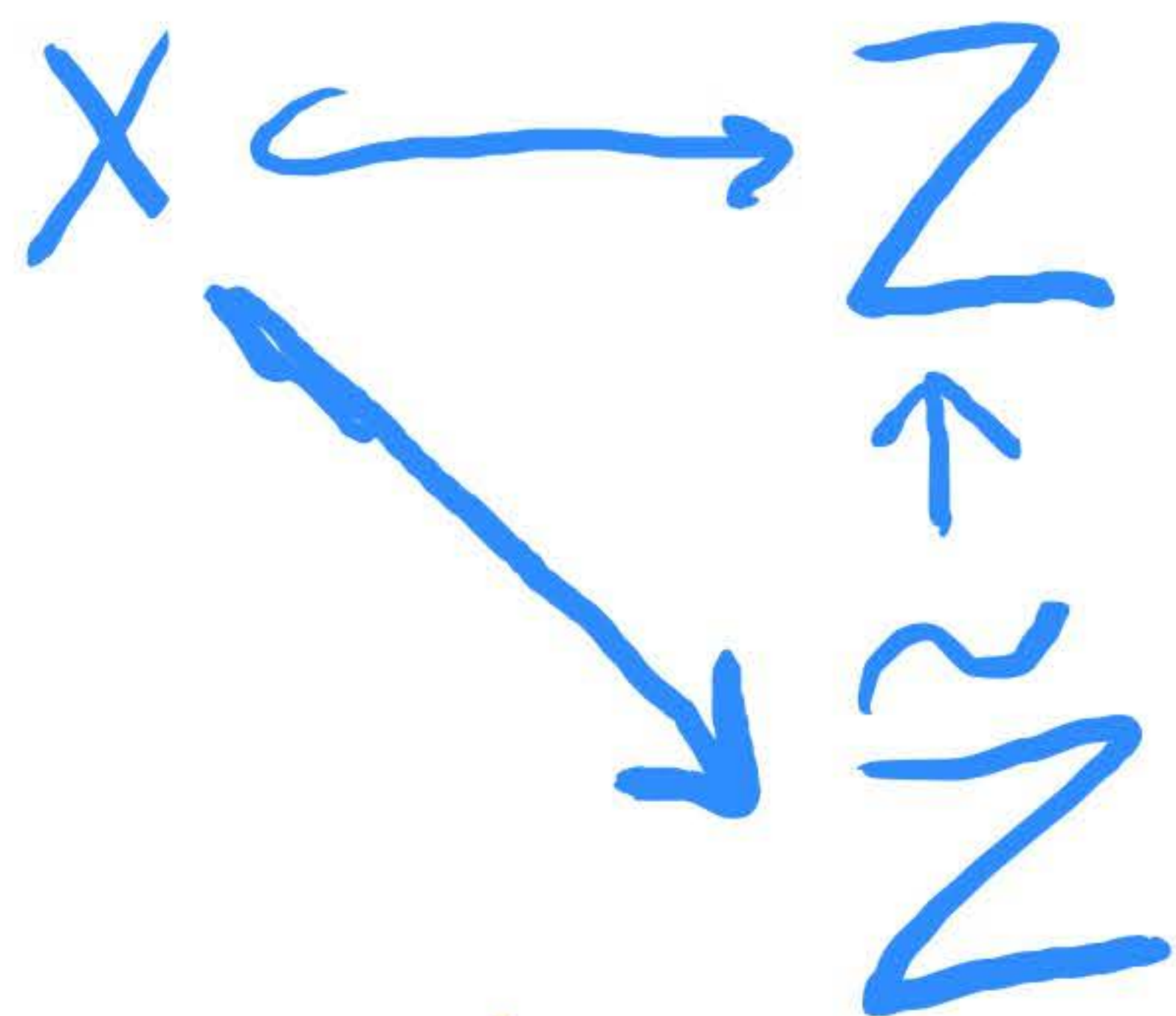
e.s. $\mathbb{P}^n(\mathbb{Z}_p)$

$\mathcal{I}_X$ is the defining
ideal of X
in Z

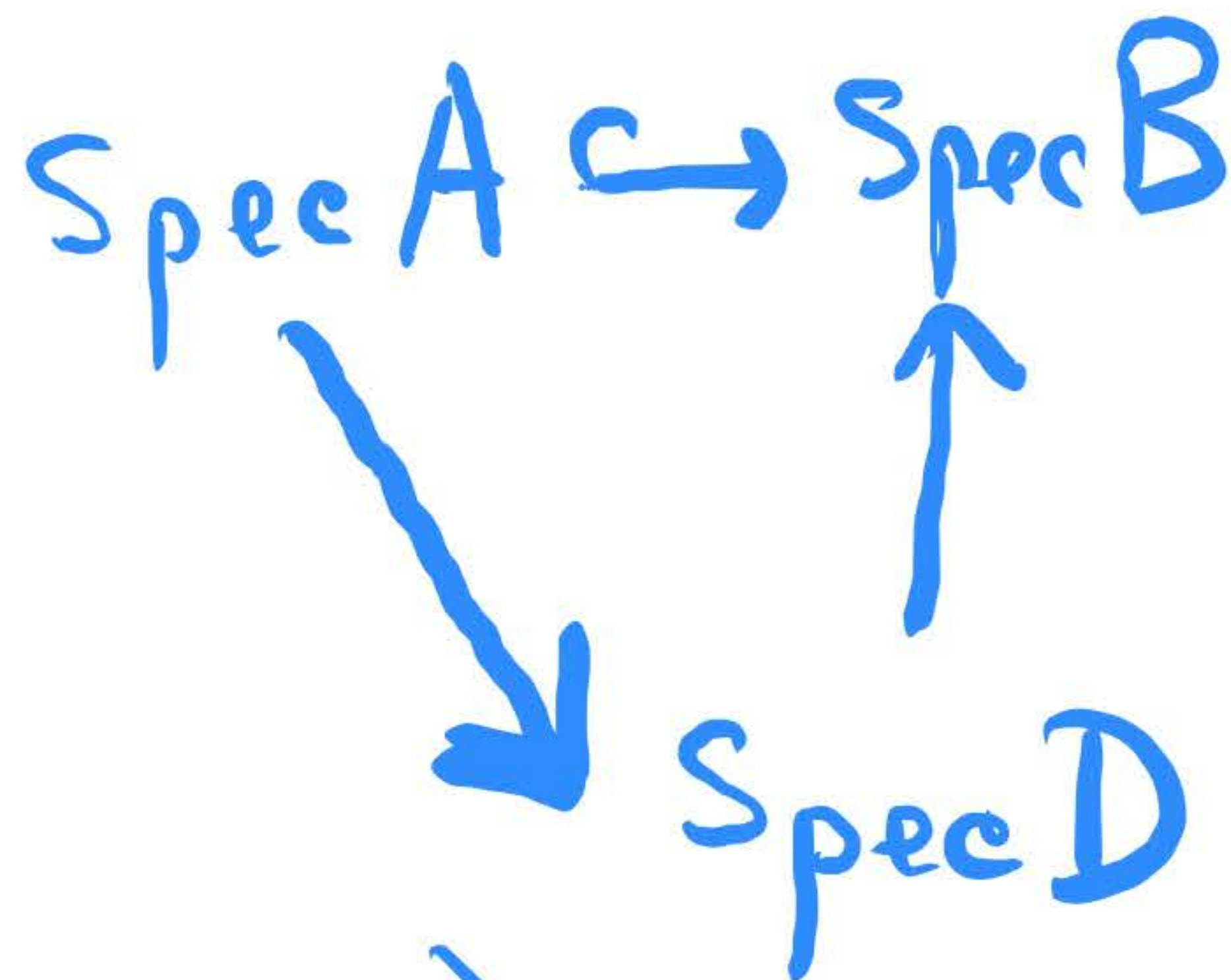
its pd
envelope.

locally:

$$\begin{array}{l}
 \mathcal{O}_Z = B \leftarrow D = \mathcal{O}_Z \\
 \pm \\
 \mathcal{O}_X = A
 \end{array}$$



LocALLY:



$$O_{\tilde{Z}}^{(\bullet)}$$

$$D^{(\bullet)} \text{ or }$$

in pd
sense

$$\Omega_{\tilde{Z}}$$

(pre?) sheaves

$$\Omega_{D, \bar{J}}$$

on X in Zariski topology.

Crystalline site :

Site: a category ("opens")
with extra structure: fibered products,
...

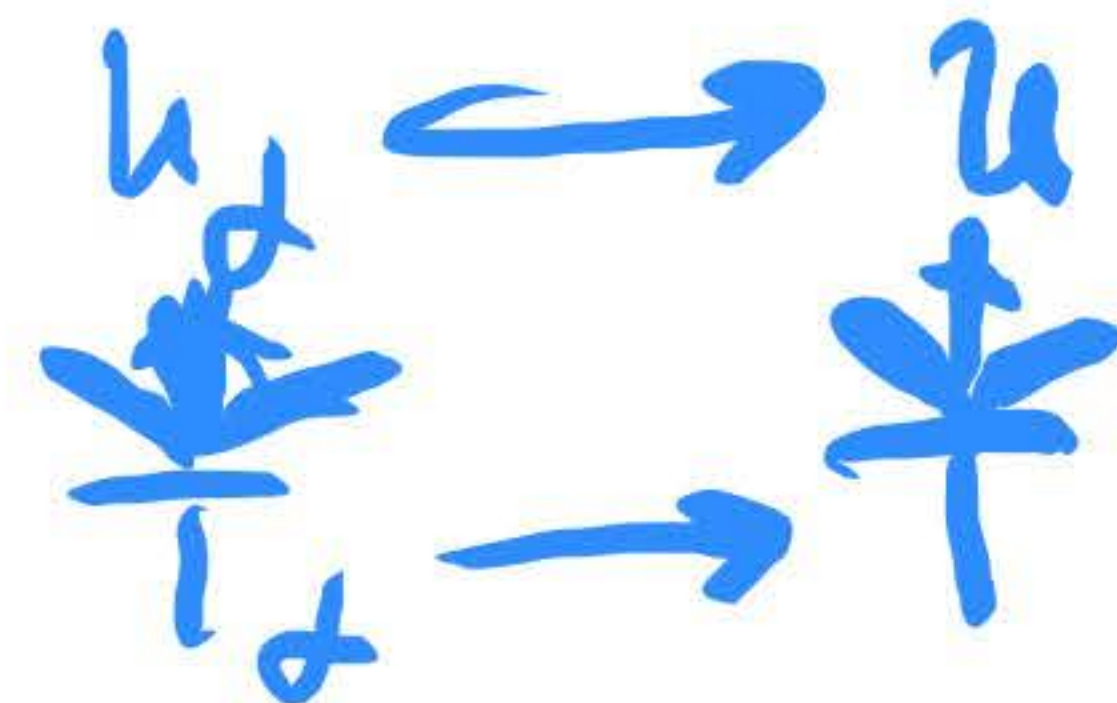
Coverings

$$\begin{array}{c} \mathcal{U}_\alpha \rightarrow \mathcal{U} \\ \alpha \in I \end{array}$$

satisfying
axioms...

Sheaves on a site; cohom of sheaves;

Our site: given $X/\mathbb{F}_p$;



$$V \hookrightarrow X$$

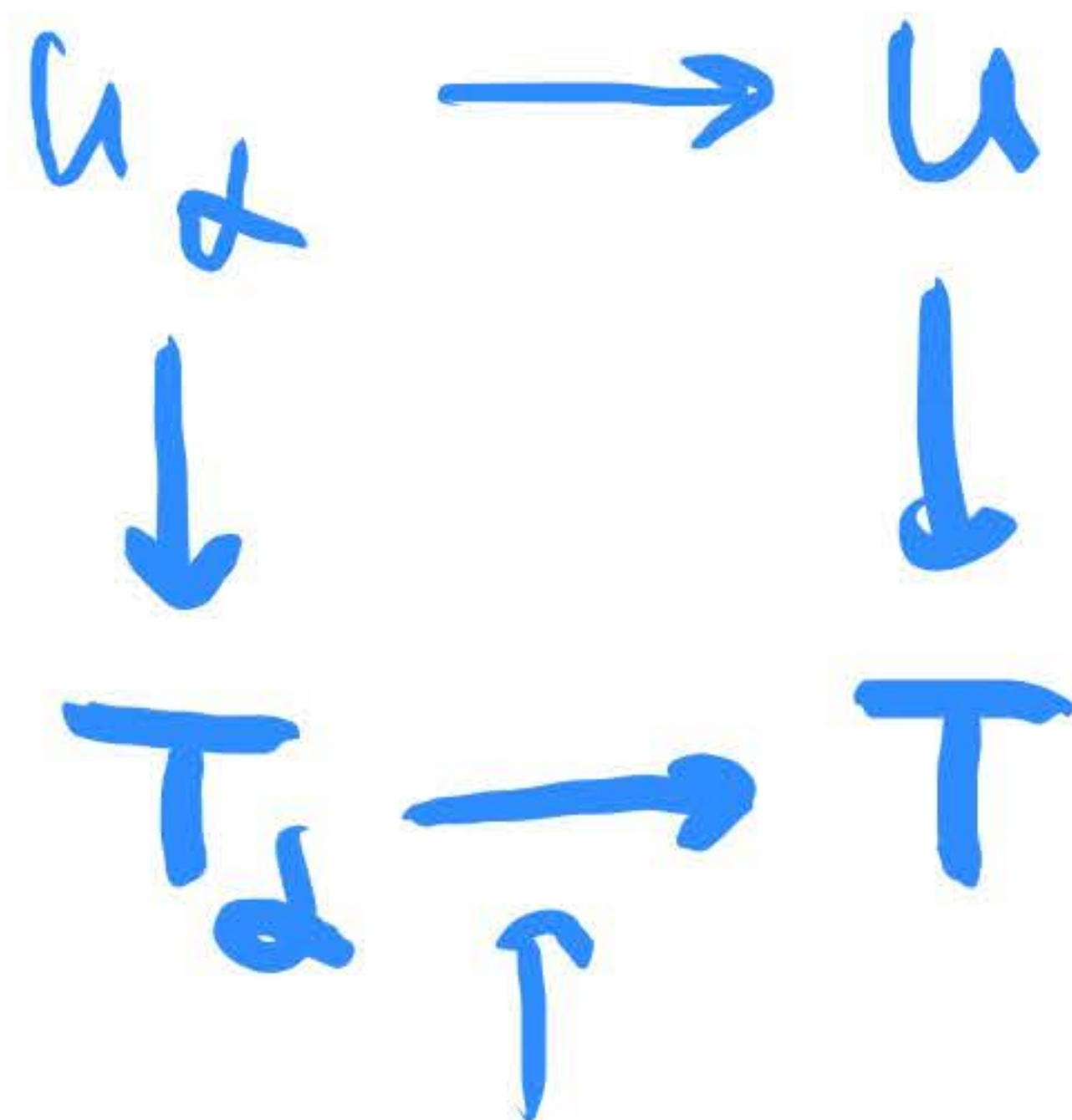
pd thickening
(scheme over

Zariski open

↓

T

Covers!



these form
a cover in
Zariski topology.

or: complete over $\mathbb{Z}_p$

Then When $X \hookrightarrow Z$
 $\uparrow$
 Smooth over $\mathbb{Z}_p$

the above

$R\Gamma(\mathcal{O}_Z^{(n)} \text{ or } \Omega_Z^n)$ computes

the cohomology of the sheaf

$$\mathcal{O} : \begin{pmatrix} n \\ 1 \\ 1 \end{pmatrix} \rightsquigarrow \mathcal{O}_T.$$