Wrapping up crystalline cohom.

Week 7: NC geom

Week 8: Deligne-Illusie degeneration

+ . . .

Prisms

. . .

=

A over $\mathbb{F}_p$;

$$\begin{tikzcd}
 & & \mathbb{Z}_p[x_1,\dots,x_N] \arrow[dl] \arrow[dr] & \\
J \arrow[r] & B \arrow[rr] \arrow[dr, two heads] & & D \text{ pd} \arrow[dl] \\
 & & A &
\end{tikzcd}$$

$\overline{J}$

Comparison to DRWitt.

complete over $\mathbb{Z}_p$ or $\mathbb{Z}/p^e\mathbb{Z}$

$$\begin{array}{l} J \\ \cap \quad \text{pd ideal} \\ D \\ \downarrow \\ A \end{array}$$

kernel has a pd structure.

$W(A)$

ker $(W(A) \to A)$ consists of $V(x)$, $x \in W(A)$

$\gamma_n(V(x)) = ?$

$\boxed{V(x)V(y) = p \cdot V(xy)}$ $V(x)^n = p^{n-1} \cdot V(x^n)$ $\Big| \ \frac{p^{n-1}}{n!} \in \mathbb{Z}$

$$\gamma_n(V(x)) = \frac{p^{n-1}}{n!} V(x^n), \quad n>0$$

(all axioms are satisfied) ✓

$\exists$ since B is free alg over $\mathbb{Z}_p$.

$$\begin{tikzcd}
B \arrow[r] \arrow[dr] \arrow[rr, dashed, bend left, "\exists"] & D \arrow[r, dotted, "\exists!"] \arrow[d, "\text{pd str.}"] & W(A) \arrow[dl] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
\Omega^\bullet_{D/\mathbb{Z}_p} \arrow[r] \arrow[d] & \Omega^\bullet_{W(A)/\mathbb{Z}_p} \arrow[d] \\
\Omega^\bullet_{D,\bar{J}} \arrow[r, dotted, "\exists!"] & W\Omega^\bullet_A
\end{tikzcd}$$

$$d\gamma_n(x) - \gamma_{n-1}\, dx \longmapsto 0$$

here

$$
\begin{tikzcd}
\Omega^{\bullet}_{D/\mathbb{Z}_p} \arrow[r] \arrow[d] & \Omega^{\bullet}_{W(A)/\mathbb{Z}_p} \arrow[d] \\
\Omega^{\bullet}_{D,\bar{J}} \arrow[r, "\exists !"'] & W\Omega^{\bullet}_{A}
\end{tikzcd}
$$

~~"divided exponents"~~

Cancel out by d_{DR}

$p^n x_j^{1/p^n}$

<u>Next</u>: This is a quasi-isomorphism

for $A = \mathbb{F}_p[x_1, \dots, x_N]$

$B = \mathbb{Z}_p[\quad\quad]$

Localization The Čech-Alexander cplex $D^{(\bullet)}$ and the DR cplx $\Omega^{\bullet}_{D,\bar{J}}$: behave well under localization in Zariski topology.

far from a complex of A-mods.

Why? When you take $s \in A$ (over $\mathbb{F}_p$): when you localize in s, you localize in any $\tilde{s} \mapsto s$ $\tilde{s} \in \underline{B} \to D$
Σ

s inv. $\Updownarrow$ $\forall \tilde{s}$ inv. (p-adic completeness)

As for localization in $\tilde{S} \in D$:

both $\Omega^{\bullet}_{p,\bar{J}}$ and

$D^{(\cdot)}$ behave well

the differential is NOT a morphism of rings, but a $\sum \pm d_j$ of those. They descend to localization.

d_{DR} compat. w/ localiz.: $d(s^{-1}) = -s^{-2}$

<u>BTW</u>: Everywhere above, could replace $B = \mathbb{Z}_p[x_1, \ldots, x_d]$ by a SMOOTH algebra over $\mathbb{Z}_p$.

<u>Crystalline cohomology</u>.

e.g. X a <u>(quasi)projective</u> scheme over $\mathbb{F}_p$.

$Z = \mathbb{P}^n(\mathbb{Z}_p)$

<u>Assume</u>: X a scheme over $\mathbb{F}_p$

$X \hookrightarrow Z$ SMOOTH scheme over $\mathbb{Z}_p$

$X \hookrightarrow Z$

scheme over $\mathbb{F}_p$

Smooth scheme over $\mathbb{Z}_p$

e.g. $\mathbb{P}^n(\mathbb{Z}_p)$

$\mathcal{J}_X$ is the defining ideal of X in Z

its pd envelope.

$\underline{\text{locally}}$:

$$\begin{array}{ccccc} \mathcal{O}_Z = B & \leftarrow & D = \mathcal{O}_{\tilde{Z}} \\ \downarrow & & \\ \mathcal{O}_X = A & & \end{array}$$

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & Z \\ & \tilde{Z} \arrow[u] \end{tikzcd}$$

Locally:

$$\begin{tikzcd} \operatorname{Spec} A \arrow[r, hook] \arrow[dr] & \operatorname{Spec} B \\ & \operatorname{Spec} D \arrow[u] \end{tikzcd}$$

$\mathcal{O}^{(\bullet)}_{\tilde{Z}}$ $\qquad$ $D^{(\bullet)}$ or

$\Omega^{\bullet}_{D,\bar{J}}$

in pd sense $\Big]$ $\Omega^{\bullet}_{\tilde{Z}}$ - (pre?) sheaves on X in Zariski topology.

Crystalline site:

Site: a category ("opens")
with extra structure: fibered products, ...

Coverings:

$$U_\alpha \to U \quad \alpha \in I$$

satisfying axioms...

Sheaves on a site; cohom of sheaves;

<u>Our</u> site: given $X/\mathbb{F}_p$;

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook, "\text{Zariski open}"] & X \\ T & \end{tikzcd}$$

pd thickening (scheme over $\mathbb{Z}/p^e\mathbb{Z}$)

" <u>or</u>: complete over $\mathbb{Z}_p$ ".

$$\begin{tikzcd} U_\alpha \arrow[r] \arrow[d, hook] & U \arrow[d, hook] \\ T_\alpha \arrow[r] & T \end{tikzcd}$$

Covers!

$$\begin{tikzcd} U_\alpha \arrow[r] \arrow[d] & U \arrow[d] \\ T_\alpha \arrow[r] & T \end{tikzcd}$$

those form a cover in Zariski topology.

Thm When $X \hookrightarrow Z$:

Z smooth over $\mathbb{Z}_p$

the above

$\mathbb{R}\Gamma(\mathcal{O}^{(\bullet)}_{\tilde{Z}}$ or $\Omega^{\bullet}_{\tilde{Z}})$ computes

the cohomology of the sheaf

$\mathcal{O} : \begin{pmatrix} U \\ \downarrow \\ T \end{pmatrix} \rightsquigarrow \mathcal{O}_T .$