Topics in noncommutite geometry

A a (nc) algebra$/k$ **Diff forms:**

Generators a, da $a \in A$

k-linear in $a \in A$

Relations: $a \cdot b$ as in A

$\boxed{d1 = 0}$ $d(ab) = da \cdot b + a \cdot db$

and no commuting relations!

$\Omega^{\bullet, NC}_{A/k}$ noncommutative forms

1) It is a graded algebra:

$|a|=0, \quad |da|=1 \qquad |\omega_1\omega_2| = |\omega_1| + |\omega_2|$

$d: \Omega^{\bullet}_{A/k} \to \Omega^{\bullet+1}_{A/k} \qquad d^2=0$

$d(\omega_1\omega_2) = d\omega_1 \cdot \omega_2 + (-1)^{|\omega_1|}\omega_1 \cdot d\omega_2$

$d: a \mapsto da \mapsto 0$

relations $\to$ relations, so well-defined.

$d^2=0 \qquad (\Omega^{\bullet}_{A/k}, d)$ DGA

<u>Universal</u> dga over k: $(\mathcal{A}^\bullet, d)$ together with $A \to \mathcal{A}^\bullet$ as (graded) algebras.

$$\begin{tikzcd} A \arrow[r] \arrow[dr] & \mathcal{A}^\bullet \\ & \Omega^\bullet_A \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

$\exists!$ morphism of dgas

<u>Lemma</u> The map $a_0 \otimes a_1 \otimes \ldots \otimes a_n \mapsto a_0\, da_1 \ldots da_n$

$\bar{A} = A/_{k \cdot 1}$ $\qquad A \otimes \bar{A}^{\otimes n} \xrightarrow{\sim} \Omega^n_{A/k}$

is an isomorphism.

$$(A\otimes\bar{A}^{\otimes n})\otimes(A\otimes\bar{A}^{\otimes m})\to A\otimes\bar{A}^{\otimes(n+m)}$$

check: associative. $a_0\cdot(1\otimes a_1)\cdots(1\otimes a_n)$

$d: A\otimes\bar{A}^{\otimes n}\to A\otimes\bar{A}^{\otimes n+1}$ In the algebra $A\otimes\bar{A}^{\otimes\bullet}$

$a_0\,da_1\ldots da_n\mapsto 1\cdot da_0\,da_1\ldots da_n$

check: do get a dga structure.

$$\Omega^\bullet_{A/k}\underset{\longleftarrow}{\xrightarrow{\exists!}}A\otimes\bar{A}^{\otimes\bullet}$$

$a_0\,da_1\ldots da_n\longmapsfrom a_0\otimes\cdots\otimes a_n$

mutually inverse on a and on da ✓

So: $H^\bullet(\Omega^\bullet_{A/k}, d) \simeq k\cdot 1$

Quick rmk: $\Omega^\bullet_{A/k}, d$ <u>does</u> remind a bit of $D^{(\bullet)}, d$ (Čech-Alexander)

$$\simeq\ (\Omega^\bullet_{D,\bar J}, d_{DR})$$

A is k-projective $\simeq k \oplus \bar A$ as k-mods

$$\begin{tikzcd}[column sep=small]
\Omega^0 & \Omega^1 & \Omega^2 & \\
k & 1\cdot d\bar A & 1\cdot d\bar A\cdot d\bar A & {} \\
\bar A \arrow[ur, "\sim"] & d\bar A\cdot d\bar A \arrow[ur, "\sim"] & d\bar A\cdot d\bar A\cdot d\bar A \arrow[ur, "\sim"] &
\end{tikzcd}$$

$$A\otimes \bar{A}^{\otimes n} \xrightarrow{\sim} \Omega^n_{A/k}$$

$$a_0 \otimes \dots \otimes a_n \longmapsto a_0\, da_1 \dots da_n$$

$$\Omega^0_{A/k} = A = k\cdot 1 \oplus A/k$$

$$\Omega^1_{A/k} \cong A\otimes \bar{A} = k\otimes \bar{A} + \bar{A}\otimes \bar{A}$$

$1\cdot da$ $\qquad$ $a_0\, da_1$

In fact: nice homotopy exists.
(Natural, given by formulas).
[Is there a way of seeing that without computing? Probably...

Just look for this homotopy.

$$\boxed{\begin{array}{c} a_0 a_1 - f(a_1) a_0 \\ \uparrow \\ a_0 \otimes a_1 \end{array}} \qquad \begin{array}{ccc} A & \xrightarrow{f} & A \\ \uparrow & & \uparrow \\ \Omega^1_{A/k} & & \Omega^1_{A/k} \end{array}$$

$a - f(a) = ?(da)$

$?(a_0 da_1) = ??$

$?(1 da_1) = a_1 - fa_1$

$$\underbrace{?\,(a_0 \cdot da_1)}_{\underline{\text{any}}\ \text{elt} \in \Omega^1_{A/k}} \in A$$

$$\begin{tikzcd} a_1 \in A \arrow[d, "d"] \\ da_1 = 1 \cdot da_1 \arrow[u, bend right, "?"'] \end{tikzcd}$$

$$?\,(1 \cdot da_1) = a_1 - f(a_1)$$

<u>Observe:</u>

$$?\,(a_0 \cdot da_1) = \underline{a_0 a_1 - f(a_1) \cdot a_0} \quad \text{works.}$$

note: <u>trace</u> motive appears!

$$
\begin{tikzcd}
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] \arrow[l, bend left, "\text{homot.}"] & \Omega^2 \arrow[l, bend left, "?"]
\end{tikzcd}
$$

$$\text{homot.} \cdot d(a_0\, da_1) + ?\big(d(a_0\, da_1)\big) =$$

$$= a_0\, da_1 - f(a_0)\, df(a_1)$$

Find that, and procede by induction.

BTW:

Answer:

$$a_0\, da_1 \dots da_n \overset{\sim\Delta}{\longmapsto} \sum_{j=1}^{n} a_0\, da_1 \dots da_{j-1} \overbrace{\boxed{a_j \otimes 1 - 1 \otimes a_j}}^{\text{was } = da_j \text{ in the Alexander-Čech story!}} da_{j+1} \dots da_n$$

The Ginzburg–Schedler differential:

$$\iota_{id}: (a_0 da_1 \ldots da_n) \mapsto \sum \pm a_0 da_1 \ldots da_{j-1} \cdot (a_j \otimes 1 - 1 \otimes a_j) \cdot da_{j+1} \ldots da_n$$

$$\boxed{\begin{aligned} L_D(a_0 da_1 \ldots da_n) &= \sum a_0 da_1 \ldots d Da_j \, da_{j+1} \ldots \\ \iota_D(\ ) &= \sum \pm a_0 da_1 \ldots a_j \, da_{j+1} \ldots \end{aligned}}$$

NOTE: Why ι? As in usual calc. but also

$D \in Der_k(A)$, $L_D : \Omega^\bullet_{A/k} \circlearrowleft$

$\iota_D : \Omega^\bullet \to \Omega^{\bullet - 1}$

$\underline{D} \in \mathrm{Der}(A)$

$L_D : \Omega^{\cdot}_{A/k} \circlearrowleft$

$\iota_D : \Omega^{\cdot}_{A/k} \to \Omega^{\cdot-1}_{A/k}$

Der sending:

$$\boxed{\begin{aligned} a &\longmapsto 0 \\ da &\longmapsto Da \end{aligned}}$$

deg -1

The derivation of degree 0

sending:

$a \mapsto a$

$da \mapsto d(Da)$

ι_Δ not like that;

$\Delta : A \to A \otimes A$

So: (notation)

$$\rightleftarrows \Omega^{\bullet}_{A/k} \underset{\iota_f}{\overset{d}{\rightleftarrows}} \Omega^{\bullet+1}_{A/} \rightleftarrows$$

$$[d, \iota_f] = \mathrm{id} - f_*$$

What if NOT? $\iota^2_{\mathrm{id}} : \Omega^{\bullet} \to \Omega^{\bullet-2}$ morphism of complexes must be 0 on cohom.

In particular: when $f = \mathrm{id}$

$$[d, \iota_f] = 0.$$

also observe: $\iota^2_{\mathrm{id}} = 0$.