Topics in NC geometry

Last time: NC forms $\Omega^{\bullet}_{A/k} \xrightarrow{d} \Omega^{\bullet+1}_{A/k}$

$f_0, f_1 : A \rightrightarrows B$

$$\Omega^{\bullet}_{A/k} \underset{f_{1*}}{\overset{f_{0*}}{\rightrightarrows}} \Omega^{\bullet}_{B/k}$$

$h(f_0, f_1)$

homotopic

(almost for sure) [higher homotopies, coherent

Very similar to what we have with Alexander-Čech

HERE: $h(\cancel{f_0}, f_1) = \imath_f$: $\cancel{f_0} \to \mathrm{id}$

$a_0 \otimes a_1 \mapsto f(a_1)a_0 - a_0 a_1$

HERE → $h(f_0, f_1)$ when $f_0 = f_1 = \mathrm{id}_A$

$b = \sum (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n$

$h(\mathrm{id}_A, \mathrm{id}_A)$:

$\neq$ last Monday

$$
\begin{tikzcd}
 & a_0 \otimes a_1 \arrow[dl, mapsto] \arrow[dr, mapsto, "\text{the old Alexander-Čech}"] & \\
a_0 a_1 - a_1 a_0 & & a_0 a_1
\end{tikzcd}
$$

$$
\begin{tikzcd}
a_0 \otimes \dots \otimes a_n \arrow[d, mapsto] & a_0 \otimes \dots \otimes a_n \arrow[d, mapsto] \\
1 \otimes a_0 \otimes \dots \otimes a_n & \sum_{j=0}^{n} (-1)^j a_0 \otimes \dots \otimes 1 \otimes \dots \otimes a_n \\
A \otimes \bar{A}^{\otimes n} \arrow[r] & A \otimes \bar{A}^{\otimes n+1} \\
A^{\otimes n+1} \arrow[r] & A^{\otimes n+2}
\end{tikzcd}
$$

commutes

Today: But we KNOW important differentials that start with

$$a_0 \otimes a_1 \overset{-b}{\longmapsto} f(a_1)\cdot a_0 - a_0 a_1$$

$$A \xleftarrow{+b} A \otimes \bar{A} \xleftarrow{b} A \otimes \bar{A} \otimes \bar{A} \xleftarrow{b} \cdots$$

$$f(a_2)a_0 \otimes a_1 - a_0 a_1 \otimes a_2 + a_0 \otimes a_1 a_2 \xleftarrow{+b} a_0 \otimes a_1 \otimes a_2$$

b_f: $\underbrace{b'}_{\sum_{j=0}^{n-1} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots \otimes a_n} a_0 \otimes \ldots \otimes a_n + (-1)^n f(a_n) a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}$

$H_\cdot(A,M) =$ homology of $C_\cdot(A,M)$

if everything k-flat: $\simeq M \overset{\mathbb{L}}{\otimes}_{A\otimes A^{op}} A$

$$H_0(A,M) = M \otimes_{A\otimes A^{op}} A = M/_{[A,M]}$$

$$[A,M] = \langle ma - am \rangle$$

- TRACE flavour

$M/_{[A,M]} \longrightarrow K \overset{df}{=} K$-valued TRACE on M

$C.(A, M)$ is a cplex of k-mods

A-mods if A is commutative.

$$\underset{a_0}{\underline{m}} \otimes a_1 \otimes \cdots \otimes a_n \longmapsto a m \otimes a_1 \otimes \cdots \otimes a_n$$

Obs.: the trace property, 2

$$A \underset{g}{\overset{f}{\rightleftarrows}} B \qquad gf \quad A \otimes_{A\otimes A^{op}} A \xrightarrow{\sim} B \otimes_{B \otimes B^{op}} B \quad fg$$

$a_0 \otimes \dots \otimes a_n \xmapsto{b} \sum_{j=0}^{n} (-1)^j \underbrace{a_0 \otimes \dots \otimes a}_{j=0}$

$C_\bullet(A, M)$

$+ (-1)^n \underline{a_n a_0} \otimes a$

$a_0 \in M$ Hochschild chains of

M - an

$a_j \in \bar{A} = A/k \cdot 1$

Before

$f : A \to A$ $M = {}_f A = A$

$A \underset{g}{\overset{f}{\rightleftarrows}} B$

$A / \langle gf(a_1) \cdot a_0 - a_0 a_1 \rangle$

$\cong$

$B / \langle \underline{fg(b_1)} \cdot b_0 - b_0 \underline{b_1} \rangle$

k-mod spans

well def.

$$\begin{array}{ccc} a & & g(b) \\ \downarrow & & \uparrow \\ f(a) & & b \end{array}$$

rels $\to$ rels

$a - gf(a) \in I$

$f(a) - fgf(a) = b - fg(b)$

mod...

$a = gf(a)$

$a_0 = 1$

$\downarrow$

$f(a)$

$$\mathrm{tr}(M) := M/[A,M] \cong M \otimes_{A\otimes A^{op}} A$$

$$\mathrm{tr}_A(f) := {}_fA/[A, {}_fA] \cong A/\langle a_0 a_1 - f(a_1)a_0\rangle$$

$$\mathrm{tr}_A(\overbrace{M \otimes_B N}^{A\text{-bim}}) \overset{k\text{-mod}}{\simeq} \mathrm{tr}_B(\overbrace{N \otimes_A M}^{B\text{-bim}})$$

$$\boxed{\mathrm{tr}_B(fg) \simeq \mathrm{tr}_A(gf)}$$

$$\mathrm{TR}_A(M) := M \overset{L}{\otimes}_{A\otimes A^{op}} A$$

$$\boxed{\mathrm{TR}_A(f) = {}_fA \overset{L}{\otimes}_{A\otimes A^{op}} A}$$

Claim $\mathrm{TR}_A(gf) \simeq \mathrm{TR}_B(fg)$

$$\begin{tikzcd} A_0 \arrow[loop left, "\mathrm{id}"] \arrow[loop right, "gf"] \end{tikzcd}$$

$$(\mathrm{id} - gf)(a_0) = a_0 1 - gf(a_0) 1 =$$

$$= b_f(1 \otimes a_0)$$

$$\mathrm{id} - (gf)_* = [b_f, \underline{B_f}]$$

$$\begin{tikzcd}
A \arrow[r, bend right, "B_f"'] & A \otimes \bar{A} \arrow[l, "b_f"'] \arrow[r, bend right, "B_f"'] & A \otimes \bar{A}^{\otimes 2} \arrow[l, "b_f"'] \arrow[r, bend right, "B_f"'] & A \otimes \bar{A}^{\otimes 3} \arrow[l]
\end{tikzcd}$$

$$B_f(a_0) = 1 \otimes a_0$$

$$C_\bullet(A, {}_{gf}A) \underset{g_*}{\overset{f_*}{\rightleftarrows}} C_\bullet(B, {}_{fg}B)$$

$$a_0 \otimes \dots \otimes a_n \longmapsto f(a_0) \otimes \dots \otimes f(a_n)$$

$$g(b_0) \otimes \dots \otimes g(b_n) \longleftarrow\!\shortmid\ b_0 \otimes \dots \otimes b_n$$

expect (<u>know</u>):

$$(gf)_* \sim \mathrm{id}_{C_\bullet(A, {}_{gf}A)} \qquad (fg)_* \sim \mathrm{id} \dots$$

nice to have

$B(\omega) = \lim_{f \mapsto id} \Big(f$

"nc de Rham"

$f : A \to A$

"fun

$bB + Bb = id - (id)_*$
$= 0$

$b^2 = (b+B)^2 = B$

$=0$ in $A \otimes \bar{A}^{\otimes}$

$a_0 \otimes \ldots \otimes a_n \mapsto 1 \otimes \ldots \mapsto$

and we find:

$$B_{gf}(a_0) = 1 \otimes a_0$$

(in fact: $a_0 \to 1 \otimes a_0 \to 1 \cdot a_0 - gf(a_0) 1 = a_0 - f(a_0)$)

$$B_{gf}(a_0 \otimes a_1) = 1 \otimes a_0 \otimes a_1 - 1 \otimes gf(a_1) \otimes a_0$$

$$\ldots$$

$$B_{gf}(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} 1 \otimes gf(a_j) \otimes \ldots \otimes gf(a_n) \otimes a_0 \otimes a_1 \otimes \ldots \otimes a_{j-1}$$

$$b_{gf} B_{gf} + B_{gf} b_{gf} = 0 \text{ ; also } B_{gf}^2 = 0$$

When $gf = id$:

$$B = B_{id} = \sum_{j=0}^{n} (-1)^{nj} 1 \otimes a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes \ldots \otimes a_{j-1}$$

(Rinehart '63)

now we have:

$$\begin{tikzcd}
C_\cdot(A,A) \simeq A \otimes \bar{A}^{\otimes \cdot} \arrow[r, "HKR^{nc}"] \arrow[dr, "HKR"'] & \Omega^{\cdot, NC}_{A/k} \simeq A \otimes \bar{A}^{\otimes \cdot} \arrow[d, "proj"] \\
 & \Omega^{\cdot}_{A/k}
\end{tikzcd}$$

(b, B) $\quad$ (ι_Δ, d) $\quad$ $(0, d_{DR})$

when $\mathbb{Q} \subset k$

proj: when A comm