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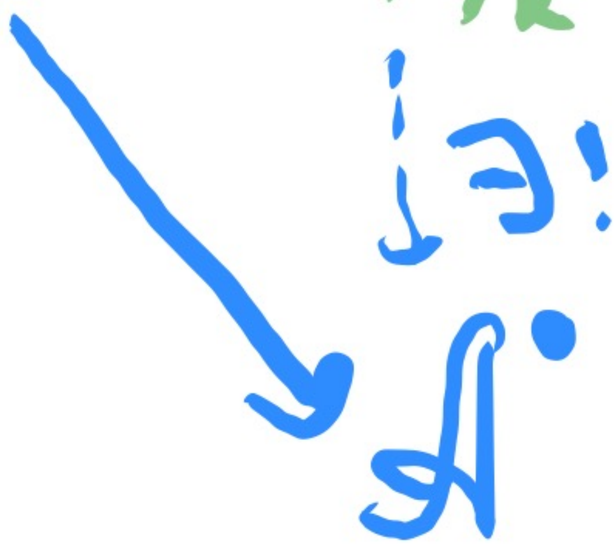

Can drop
commutativity
everywhere;

$(\Omega^{NC,\bullet}_{A/k}, d)$

$\downarrow$ when A comm

$\Omega^\bullet_{A/k}$

char(k) = p : $\boxed{F(b_1,b_2) = \frac{1}{p}\left((b_1+b_2)^p - b_1^p - b_2^p\right)}$

$$\Omega^\cdot_{A/k} \xrightarrow{C^{-1}} \mathcal{Z}^\cdot_{A/k} \Big/ B^\cdot_{A/k}$$

$$\mathcal{Z}^\cdot_{A/k} = \ker(d) \qquad B^\cdot_{A/k} = \operatorname{coker}(d)$$

(morphism of algebras

k acts — k acts via $\mathrm{Frob}(a) = a^p$

$a \longmapsto a^p$ — linear ✓

$db \longmapsto b^{p-1}db$ — linear mod $B^\cdot_{A/k}$

$\boxed{F_1(b_1,b_2) = \sum_{1}^{p-1} \frac{1}{p}\binom{p}{j} b_1^j b_2^{p-j}}$

$(b_1+b_2)^{p-1} d(b_1+b_2) - b_1^{p-1}db_1 - b_2^{p-1}db_2$

$= dF_1(b_1,b_2)$ in $\Omega^\cdot_{\mathbb{Z}[b_1,b_2]/\mathbb{Z}}$

$$\Omega^{\cdot}_{A/k} \xrightarrow{C^{-1}} H^{\cdot}_{dR}(A/k)$$

$$\| $$

$$Z^{\cdot}_{A/k} / B^{\cdot}_{A/k}$$

no chance for $\simeq$ in gen.

ex. $A = k[\varepsilon]/(\varepsilon^2)$ $p > 2$

$H^{\cdot}_{dR} \simeq k$ in degree 0

But: Isom if $k[x_1, \dots, x_n]$

$\boxed{\text{Frob} : k \xrightarrow{\sim} k}$ k perfect field

$A = k[X]$ X smooth algebraic variety / k

Proof $\Omega^{\cdot}_{A/k} \longrightarrow H^{\cdot}_{dR}(A/k)$

is a morphism over

$k[V] = A^p \left(\in \ker d\right)$ $\quad da^p = pa^{p-1} = 0$

linear span of p^{th} powrs

Smooth $\Rightarrow$ isom at all

Completions at all points

$$\Omega^{\cdot}_{A/k} \longrightarrow Z^{\cdot}/B^{\cdot}$$

$$a\omega \longmapsto a^p \cdot C^{-1}(\omega)$$

$F(a) \cdot C^{-1}(\omega) = a \cdot C^{-1}(\omega)$

How does an $\mathbb{F}_p$-algebra grow into a $\mathbb{Z}_p$-algebra?

With vectors

Motivation: Teichmüller repres.

$$\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^{n+1}\mathbb{Z}$$

$\downarrow$

$$\mathbb{F}_p = \mathbb{Z}/p \qquad a^p = a, \quad \forall a \in \mathbb{F}_p$$

$\forall a \quad \exists! [a] \in \mathbb{Z}_p$:

$$[a] \longmapsto a \quad \text{and}$$

$$[a]^p = [a] \qquad a=0: \checkmark$$

$a \neq 0$: $\underline{[a]^{p-1} = 1}$ $[a] \mapsto a$

Last time: Cartier

A comm $\mathbb{V}$ over k $\quad \mathrm{char}(k) = p > 0$

$\Omega^{\cdot}_{A/k}$ = the algebra of (Kähler) differentials

Gen. by a, da $\quad a \in A$ (k-linear in a)

$a \cdot b = ab$ in A;

$da \cdot b = b \cdot da$; $\quad (da)^2 = 0$

$d1 = 0$

$d(1 \cdot 1) = 2 \cdot 1 \cdot d1$

$d1 = 2d1$

or: $1 \in A$ is the unit in Ω

$a \mapsto da \mapsto 0$

$d : \Omega^{\cdot} \to \Omega^{\cdot+1}$

$d^2 = 0$

$d(\omega_1 \omega_2) = d\omega_1 \cdot \omega_2 + (-1)^{|\omega_1|} \omega_1 d\omega_2$

Any element of $\mathbb{Z}_p$, uniquely:

$$[a_0] + p[a_1] + p^2[a_2] + \dots$$

$a_j \in \mathbb{F}_p \quad j = 0, 1, 2, \dots$

How do addition/mult. work!

$[a] + [b]$ – how to write as

mod p^2:

$$[a] + [b] = [a_0] + p[a_1] + p^2 \cdots$$

$a_0 + b_0$

$$[a] + [b] \neq [a+b]$$

$$[a][b] = [ab]$$

b/c mod p

$x^{p-1} = 1$

$$[a+b] = [a] + [b] + p[c] + p^2 \_$$

$c = \mathcal{F}_1(a,b) \bmod p$

$$[a+b]^p = [a+b] \quad \text{- know. So.}$$

$$\checkmark \left([a] + [b] + p[c] + \dots \right)^p =$$

$$= [a] + [b] + p[c] + \dots$$

$$([a]+[b])^p + \sum \binom{p}{\quad} \cdot p[c] + \_$$

$$= [a] + [b] + p[c] + p^2 \_$$

$$[a]^p + [b]^p + p\mathcal{F}_1(a,b)$$

$$\boxed{\mathcal{F}_1(a,b) = \frac{1}{p}\left((a+b)^p - a^p - b^p\right)}$$

$$[a] + [b] = [a+b] + p[c] + \dots$$

where $\boxed{c = F_1(a,b)}$

$$= \frac{1}{p}\left((a+b)^p - a^p - b^p\right)$$

$$\left([a_0] + p[a_1] + \dots\right) + \left([b_0] + p[b_1] + \dots\right) =$$

$$= \left([c_0] + p[c_1] + \dots\right)$$

WHERE:

FORMALLY

1) $c_0 = a_0 + b_0$

2) $c_0^p + pc_1 = (a_0^p + pa_1) + (b_0^p + pb_1)$

Design + and ×

on formal symbols

$[a_0]+p[a_1]+p^2[a_2]+\dots$

::

$(a_0, a_1, a_2, a_3, \dots)$ $a_j \in A$

1)

or ×

$(a_0, a_1, \dots) + (b_0, b_1, b_2, \dots) =$

$= (c_0, c_1, c_2, \dots)$

e.g. $A = F_p$

$c_n = \text{polynomial}_n(a_0, a_1, \dots, a_n; b_0, b_1, \dots, b_n)$

$c_0 = a_0 + b_0$

(or $a_0 b_0$)

If we define

$W_n(a_0, a_1, \ldots, a_n)$ as follows:

$$W_0(a_0) = a_0$$

$$W_1(a_0, a_1) = a_0^p + p a_1$$

$$W_2(a_0, a_1, a_2) = a_0^{p^2} + p a_1^p + p^2 a_2$$

- - -

$$W_n(a_0, \ldots, a_n) = a_0^{p^n} + p a_1^{p^{n-1}} + p^2 a_2^{p^{n-2}} + \ldots + p^n a_n$$

REQUIRE:

If $(c_0, \ldots) = (a_0, \ldots) \overset{\times}{+} (b_0, \ldots)$

THEN $W_n(c_0, \ldots, c_n) = W_n(a_0, \ldots, a_n) + W_n(b_0, \ldots, b_n)$

Solve $x^{p-1} = 1$ $\quad x \mapsto a$

By iterations: given

$$x_n^{p-1} = 1 + p^n b$$

$$x_{n+1} = x_n + p^n \epsilon$$

$$(x_n + p^n \epsilon)^{p-1} = 1 + p^{n+1} b' \quad ?$$

$$x_n^{p-1} + (p-1) x_n^{p-2} \cdot p^n \epsilon + \dots (p^n)^2 + \dots$$

$$\| \qquad \text{prime to } p!$$

$$1 + p^n b + \dots$$

$$x_n^{p-2} \epsilon \equiv b \pmod{p}$$

solve for ϵ

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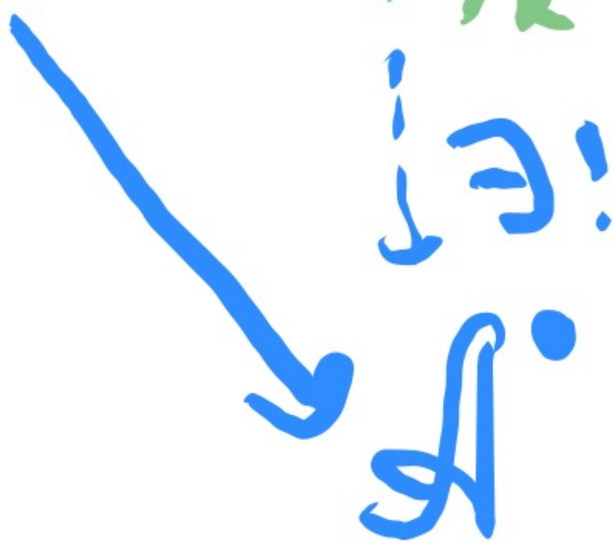

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$$(a_0, a_1, a_2, \ldots) + (b_0, b_1, b_2, \ldots) =$$

$$=: (c_0, c_1, c_2, \ldots)$$

Some cong. mod p^2

$$c_0 = a_0 + b_0$$

$$c_0^p + p c_1 = a_0^p + p a_1 + b_0^p + p b_1$$

$$c_0^{p^2} + p c_1^p + p^2 c_2 =$$

$$= (a_0^{p^2} + \ldots) + (b_0^{p^2} + \ldots)$$

$\exists!$ solution

Solve for: $c_0, c_1, c_2, \ldots$

Solve WHERE? in

$$A = \mathbb{Z}[a_0, a_1, a_2, \ldots; b_0, b_1, b_2, \ldots]$$