Operations on Hochschild and cyclic complexes

$C^\cdot(A, M)$ $\quad M$ $\quad A$-bimodule

$C^n(A,M) = \mathrm{Hom}_k(\bar{A}^{\otimes n}, M)$ $\quad n \geq 0$

(normalized) cochains

$\bar{A} = A/k\cdot 1$

$\delta: C^n \to C^{n+1}$ $\quad \delta = [m, \cdot]$

$[\,,\,]: C^{n+1}(A,A) \otimes C^{m+1}(A,A) \to C^{n+m+1}(A,A)$

$\cap$

$C^{n+1}_{full}(A,A) = \mathrm{Hom}_k(A^{\otimes n+1}, A)$ $\left[\begin{matrix}\text{not}\\ \bar{A}\end{matrix}\right]$

$m(a,b) = ab$

$\cap$

C^2_{full}

Last time: Start with $\overline{U}(\mathfrak{g}_A^\cdot)$

$\mathfrak{g}_A^\cdot = C^{\cdot+1}(A,A), \delta, [,]$ dgLieA

$\overline{U}(\mathfrak{g}_A^\cdot) = \ker(\varepsilon : U \to k)$

$\Delta x = \sum x^{(1)} \otimes x^{(2)}$

degree $= |x| - 1$

$(x), x \in \overline{U}(\mathfrak{g}_A^\cdot)$

$(x) \xrightarrow{\delta} (\delta x)$

$(x) \xmapsto{b} \sum \pm (x^{(1)})(x^{(2)})$

$uB \mapsto u \cdot x$

dga: generators $U(\mathfrak{g}_A^\cdot)$

- dg subalgebra

u (formal)

relations:

$[X, (x)] = ([X, x])$

$X \in \mathfrak{g}^\cdot$, $(x) \in \overline{U}$, $([X,x]) \in \overline{U}$

$(x) \mapsto (\delta + b + uB)(x)$

Actually, from any (dg) Lie algebra $\mathfrak{g}$, and from any cocommutative Hopf alg U

$\rightsquigarrow$ dg algebra as above.

<u>Theorem</u> $CC_\bullet^-(A)$ has a natural A_∞ module structure over the dga above.

<u>Compare</u>: $\mathfrak{g} = \Lambda^{\bullet+1} T_X$; $\Omega^\bullet_X$

$\pi \in \mathfrak{g}$

$\iota_\pi : \Omega^\bullet \to \Omega^{\bullet - |\pi|}$ $\quad L_\pi = [d, \iota_\pi] : \Omega^\bullet \to \Omega^{\bullet - |\pi| + 1}$

$[L_\pi, L_{\pi'}] = L_{[\pi,\pi']}$

$[L_\pi, \iota_{\pi'}] = (-1)^{|\pi|-1} \iota_{[\pi,\pi']}$ $\qquad [d, \iota_\pi] = L_\pi$

$[\iota_\pi, \iota_{\pi'}] = 0$

odd

$\mathfrak{g}[\varepsilon][u]$, $0 + u\frac{\partial}{\partial \varepsilon}$ $\qquad \varepsilon^2 = 0$

$X + \varepsilon Y \mapsto uY$ $\qquad$ differential on $\mathfrak{g}$

Generators: $\mathfrak{g}$ dg Lie subalg

εX $(=(x))$, $x \in \mathfrak{g}$ $\qquad [X, \varepsilon Y] = \pm \varepsilon [X, Y]$

$[\varepsilon X, \varepsilon Y] = 0$

Univ enveloping alg of that:

(dg) Subalgebra $U(g^\cdot)$ (gen. by L_x, $x \in g$)

Generators: (x) $x \in g$

(anti) ~~commute~~ w/one another

Commuta-tors of $(x), (y)$ will be not zero but boundaries wrt b.

$$[X, (x)] = ([X, x])$$

resol. of Sym(eg)

$$(x) \in Sym_k(\underline{g})$$

$$b(x) = \sum \underline{\pm} (x^{(1)})(x^{(2)})$$

B

OUR ALGEBRA:

$\mathcal{U}(\mathfrak{g})$ instead of $\mathrm{Sym}(\mathfrak{g})$; "QUANTUM VERSION"

Assume: $\mathfrak{g}$ = free $\widehat{gr}$ Lie algebra with one (odd) generator m

$|m| = 1$.

invert 2

$\mathcal{U}(\mathfrak{g}) = k\langle m\rangle$ – the free assoc algebra

$\boxed{m^2 =: R}$

$[m, m] \neq m \cdot m - m \cdot m = 0$

$[m,m] = 2m$

<u>Claim</u>: in $U(\mathfrak{g})$ (obviously) $m^2 \neq 0$

but in $U(\mathfrak{g}) \ltimes \text{Cobar}\, \bar{U}(\mathfrak{g})[[u]]$ "R"

$\delta + b + uB$

can "flatten" m:

$$\mu = m + \sum C_{k,l} R^k (R^l)$$

$\boxed{\mu^2 = 0}$ in some completion of

if[1] we are allowed infinite series where "degree" of R grows

"small"

+ if[2]: small denoms allowed

if[3] u^{-1} allowed !

Once again, looking for:

$$\mu_F := \mu = m + \sum_{k,l} c_{k,l}\, R^k\big(\underbrace{R^l}_{\text{let}}\big)$$

odd

even

even or odd...

? correct in a sec.

will correct in pictures after

$$F(x,y) = \sum_{k,l} c_{k,l}\, y^k x^l$$

suggestive

in light of $\underline{W}(A)$

$$\mu_F^2 = 0 \Longleftarrow F(x,y) = \sum_{n=1}^{\infty} \frac{x(x-y)\cdots(x-(n-1)y)}{n!\, u^n}$$

Corollary: Say, A is an algebra$/\mathbb{F}_p$

lift the prod to <u>some</u> binary product.

$$\begin{array}{c} \tilde{A} \\ \downarrow \\ A \end{array} \qquad \begin{array}{l} -\ \text{free } \mathbb{Z}\text{-module;} \\ \\ A = \tilde{A}/p\tilde{A}. \end{array}$$

$m_0 \in C^2_{full}(A,A)$ -

$m_0^2 = 0$

$m^2 = R$ p-adically small

$m_0^2 = 0$

$m^2 = p\gamma \qquad \gamma \in \mathfrak{g}^2_{\tilde{A}}.$

$m \in \mathfrak{g}^1_{\tilde{A}} \qquad R \in \mathfrak{g}^2_{\tilde{A}}$

Given $A/\mathbb{F}_p$ algebra

IF $A = \tilde{A}/p$

$\tilde{A}$ algebra$/\mathbb{Z}_p$:

$$\boxed{CC^{per}_\bullet(\tilde{A})}$$

a) "canonical in A", indep. of $\tilde{A}$

generalizes

b) even if $\tilde{A}$ is only assoc mod p: CAN construct differen

We do obtain a differential on

$$CC^{per}_{!}(\tilde{A})^{\wedge}_{p\text{-adic}}$$

$$\begin{array}{c} \tilde{A} \\ \downarrow \\ A = \tilde{A}/p\tilde{A} \end{array}$$

defines a differen-

Prove: any two liftings produce isomorphic complexes, etc.

$\tilde{A}/\mathbb{Z}_p$ = non-associative lifting of $A/\mathbb{F}_p$

Next time: Another crystalline-like invariant of $A/\mathbb{F}_p$:

$J \subset P$ free $\boxed{\text{noncommutative NOW}}$

$$P \twoheadrightarrow A$$

completion

Cortiñas–Cuntz–Meyer

Mukherjee

Next: NC Witt vectors (Kaledin)