More on cyclic, Hochschild, ...

From last time:

A certain dga constructed from $C^\bullet(A,A)$ and $\mathfrak{g}_A^\bullet = C^{\bullet+1}(A,A)$

(A_∞) acts on $CC_\bullet^-(A) \simeq C_\bullet(A)[[u]]$

$b+uB$

(tb def.)

"Proof" 1). $\mathrm{Bar}(A) = \bigoplus_{n=1}^{\infty} A[1]^{\otimes n}$

$(a_1|\ldots|a_n) \mapsto \sum (a_1|\ldots|a_j)\otimes(a_{j+1}|\ldots|a_n)$

$\sum \pm (a_1|\ldots|\overline{a_i \ldots a_{j+1}}|\ldots|a_n)$

dg coalgebra

$\mathfrak{g}_A^{\cdot}$ = coder. $\mathrm{Bar}(A)$ $\Longrightarrow$ $\big|$ yes, a dgla

$k \geq 0$

$\mathrm{Bar}^+ \to \mathrm{Bar}$

$\varphi \in C^k(A, A)$:

$\varphi : (a_1 | \dots | a_n) \mapsto \sum \pm (a_1 | \dots | \varphi(a_{j+1}, \dots, a_{j+k}) | a_{j+k+1} | \dots | a_n)$

$\pm = (-1)^{j(k-1)}$

$\varphi = \lambda_\varphi$

$\mathrm{Bar}^+ = k + \mathrm{Bar}_+ = \bigoplus_{n \geq 0}$

$b' = \partial_{bar} = \lambda_m$ $\quad m(a_1, a_2) = a_1 a_2$

b' = differential

$U(\mathfrak{g}_A^{\cdot})$ dg Hopf

$$U(\mathfrak{g}^{\cdot}_A) \otimes \underbrace{\mathrm{Bar}^{+}(A)}\ \dot{\to}\ \underbrace{\mathrm{Bar}^{+}(A)}_{\text{dg coalg}}$$

by composing coderivations

$\mathfrak{g}^{\cdot}_A = \mathrm{coder}\ \mathrm{B\dot{a}r}^{+}(A)$

$A[1] \otimes 0$

MORPHISM OF DG COALGEBRAS

$\underline{1} \in \mathrm{Bar}_0 = k$

$$\mathrm{Bar}^{+}_{+}(A) \longrightarrow \mathrm{Bar}_{+}(A) = \text{cofree coalg of } A[1]$$

$$\downarrow \text{proj}$$

$$\underline{A}[1]$$

determines a coderivation uniquely

dually:

$$\begin{tikzcd} T(V) \arrow[r, "D"] & T(V) = \bigoplus_{n\geq 0} T^n(V) \\ V \arrow[u] \arrow[ur] & \end{tikzcd}$$

derivation

$(= T_+$ in my notation$)$

uniquely determines

dual to:

$D(1)=0$

$$\begin{tikzcd} \underline{Bar_*(A)} \arrow[d] & Bar_*^+(A) \arrow[l] \arrow[dl] \\ A[1] & \end{tikzcd}$$

will land

D

$\bigoplus_{n\geq 0} A[1]^{\otimes n}$

uniquely determ coder

$$U(\mathfrak{g}_A) \otimes Bar(A) \to Bar(A')$$

MORPHISM OF DG COALGEBRAS

$$\begin{tikzcd} C^0_{\mathrm{I}} & C^1_{\mathrm{II}} \arrow[l] & C^2_{\mathrm{II}} \arrow[l] \\ & C^0_{\mathrm{II}} \arrow[u] & C^1_{\mathrm{II}} \arrow[l] \arrow[u] \\ & & C^0_{\mathrm{II}} \arrow[u] \end{tikzcd}$$

$\star$ ↓ enlarge it later

APPLY $CC^\bullet_{\mathrm{II}}$

cyclic co chain complex (dual to complex of cyclic chains for algebras

still $\bigoplus$ NOT $\prod$

bicomplex

$$\begin{tikzcd} & \arrow[d, "b"] \\ C_1 \arrow[r, "B"] \arrow[d] & C_2 \arrow[d, "b"] \\ C_0 \arrow[r, "B"] & C_1 \arrow[d, "b"] \\ & C_0 \end{tikzcd}$$

$$X = \bigoplus \left(A \otimes \bar{A}^{\otimes \underline{n}}\right)^m \leftarrow \text{grading of } A^\bullet$$

$$
\begin{tikzcd}
V & U\otimes\bar{U} \arrow[l, "B"'] & U\otimes\bar{U}^{\otimes 2} \arrow[l, "B"'] & V\otimes\bar{V}^{\otimes 3} \arrow[l, "B"'] \arrow[u] \\
& V \arrow[u] & V\otimes\bar{V} \arrow[l, "B"'] \arrow[u] & V\otimes\bar{V}^{\otimes 2} \arrow[l, "B"'] \arrow[u] \\
& & V \arrow[u] & V\otimes\bar{V} \arrow[l, "B"'] \arrow[u] \\
& & & V \arrow[u, "b"']
\end{tikzcd}
$$

diagonals formed using $\oplus$

$CC^{\bullet}_{\mathrm{I}}(\bar{V})$

$$
\begin{tikzcd}
A\otimes\bar{A} \arrow[d, "{[\,,\,]}"] \\
A
\end{tikzcd}
\qquad
\begin{tikzcd}
\sum x^{(1)}\otimes x^{(2)} - x^{(2)}\otimes x^{(1)} \\
x \arrow[u, mapsto]
\end{tikzcd}
$$

2. $\boxed{CC^{\bullet}_{\mathbb{I}}(\mathrm{Bar}(A)) \simeq CC^{-}_{\bullet}(A)}$

Cunta - Quillen; J. Jones; ...

FT

$$\left[\begin{array}{l} U(\mathfrak{g}_A) \otimes \mathrm{Bar}(A) \to \mathrm{Bar}(A) \\ U(\mathfrak{g}_A) \otimes U(\mathfrak{g}_A) \to U(\mathfrak{g}_A) \end{array}\right.$$

agree

$CC^{\bullet}_{\mathbb{I}}(-)$

$\Downarrow$

$CC^{\bullet}_{\mathbb{I}}(U(\mathfrak{g}_A))$ is an A_∞ algebra

$CC^{\bullet}_{\mathbb{I}}(\mathrm{Bar}(A))$ is an A_∞ module

Finally: $CC^{\cdot}_{\mathrm{II}}(U(\mathfrak{g})) \simeq U(\mathfrak{g}) \ltimes \mathrm{Cobar}\,\overline{U}\,[[u]]$, $\delta + b + uB$ as last time.

$\mathfrak{g}$

cocomm Hopf alg U

Can be enlarged (in NOTES?) much bigger Hopf algebra

one is deformation of the other; requires some argument about no obstruction.

Cyclic objects

The category Λ (and Λ^{op}).

Λ^{op}: objects: $[n]$, $n \geq 0$

$\Lambda^{op}([n],[m])$: natural operations

$$X^{n+1} \longrightarrow X^{m+1}$$

for any unital monoid $X \ni 1$ OF THE FORM:

$$(a_0, a_1, a_2) \longmapsto (1, a_1 a_2, 1, a_0)$$

$$X^3 \longrightarrow X^4$$

$$\cancel{(a_0, a_1) \longmapsto (a_0^2, a_1)}$$

Can use: multiplication and 1

THE cyclic order of the a_j's SAME as on left

Any a_j EXACTLY ONCE!

$\boxed{\Lambda \simeq \Lambda^{op}}$

Let X be algebra with a trace.

$tr: X \to k$

$tr(a_0 a_1 - a_1 a_0) = 0$

$$[n] \quad a \in X^{\otimes n+1}$$

$$\lambda \downarrow \uparrow \lambda^+$$

$$[m] \quad b \in X^{\otimes m+1}$$

$tr(a_0 b_0 a_1 b_1 \ldots a_m b_m) = \langle a, b \rangle$,

$a = (a_0, \ldots, a_m)$, $b = (b_0, \ldots, b_m)$

$\langle \lambda(a), b \rangle = \langle a, \lambda^+ b \rangle$

$\boxed{\Lambda \simeq \Lambda^{op}}$

Let X be algebra with a trace

$tr : X \to k$

$tr(a_0 a_1 - a_1 a_0) = 0$

$$\begin{array}{ll} [n] & a \in X^{\otimes n+1} \\ \lambda \downarrow \uparrow \lambda^+ & \\ [m] & b \in X^{\otimes m+1} \end{array}$$

$tr(a_0 b_0 a_1 b_1 \dots a_m b_m) = \langle a, b \rangle$,

$a = (a_0, \dots, a_m)$, $b = (b_0, \dots, b_m)$

$\langle \lambda(a), b \rangle = \langle a, \lambda^+ b \rangle$

1) Δ^{op} is THE simplicial category.

$$(a_0,\dots,a_n) \xmapsto{d_0} a_0a_1,\dots,a_n$$

$$(a_0,\dots,a_n) \xmapsto{d_1} a_0, a_1a_2,\dots,a_n$$

$$\vdots$$

$$(a_0,\dots,a_n) \xmapsto{d_{n-1}} a_0,\dots,a_{n-2},a_{n-1}a_n$$

$$(a_0,\dots,a_n) \xmapsto{d_n} a_na_0, a_1,\dots,a_{n-1}$$

Generate Δ^{op}

$j \geq 0$

$$(a_0,\dots,a_n) \xmapsto{s_j} (a_0,\dots,a_j,1,a_{j+1},\dots,a_n)$$

missing from Δ^{op}:

$\longrightarrow (1, a_0,\dots,a_n) \notin \Delta^{op}$ "s_{-1}"

$=$

$\tau : (a_0,\dots,a_n) \mapsto (a_n, a_0,\dots,a_{n-1})$

$$a = a_0, a_1, a_2, a_3 \qquad X^{\otimes 4} \qquad [3]$$

$$\lambda \downarrow \qquad \downarrow \qquad \downarrow \lambda$$

$$a_1 a_2, 1, a_3 a_0 \qquad X^{\otimes 3} \qquad [2]$$

(S^1 and its Poincaré duality

$$b = (b_0, b_1, b_2)$$

$$\langle \lambda a, b \rangle = tr(a_1 a_2 \cdot \underline{b_0} \cdot 1 \cdot \underline{b_1} \cdot a_3 a_0 \cdot \underline{b_2})$$

$$\lambda^+(b_0, b_1, b_2) = (b_2, 1, b_0 b_1, 1)$$

$$= tr(a \cdot \lambda^+(b)) = tr(a_0 ? \cdot a_1 ? \cdot a_2 ? \cdot a_3 ?)$$

$$tr(a_0 \underline{b_2} \cdot a_1 \cdot \underline{1} \cdot a_2 \underline{b_0 b_1} \cdot a_3 \cdot \underline{1})$$