More on cyclic objects

Towards: cyclotomic theory

$$\Lambda^{op}$$

Objects: $[n]$, $n \geq 0$

$\Lambda^{op}([n],[m]) =$

$=$ natural transformations

Product of # of a's $=1$

$$X^{n+1} \longrightarrow X^{m+1}$$

each a_j enters 1 time

X – any unital monoid

$(a_0, \ldots, a_n) \longmapsto (a_{j_0+1} \cdots a_{j_1}, a_{j_1+1}, \ldots, \ldots a \ldots a)$

$$\begin{tikzcd}
X^{n+1} \arrow[r] \arrow[d, "\text{mult}"'] & X^{m+1} \arrow[d, "\text{mult.}"] \\
X \arrow[r, equal] & X
\end{tikzcd} \qquad \Lambda^{op}$$

$$(a_0, a_1, a_2, a_3) \longmapsto (1, a_1 a_2, 1, a_3, a_0)$$

the cyclic order of a_i's should to be same. (required)

If required to commute:

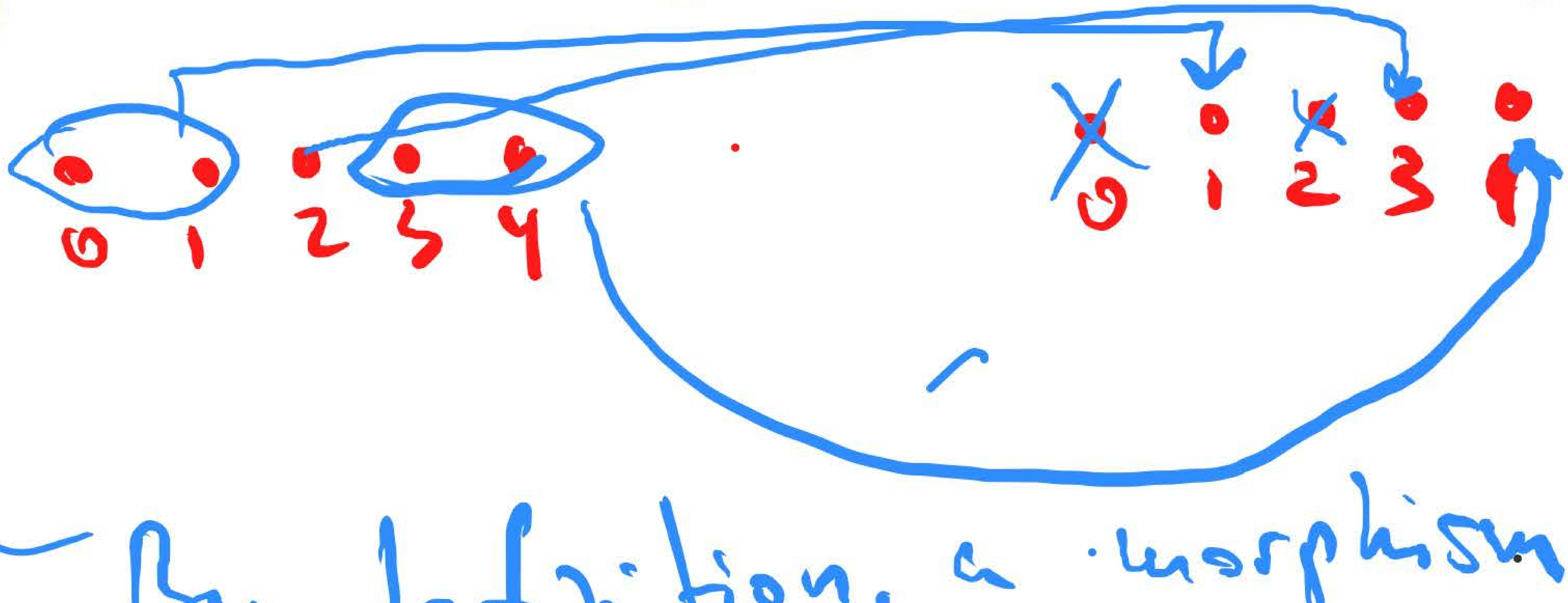
Get the category Δ'.

Operation in Δ':

$$(a_0, \dots, a_4) \mapsto (1, a_0a_1, 1, a_2, a_3a_4)$$

a non-decreasing map

$$\{0,1,2,3,4\} \longrightarrow \{0, \dots, 4\}$$



By definition, a morphism in the category Δ.

We call it $\Delta' \subset \Lambda^{op}$

$$\Delta^1 \subset \Lambda^{op} \quad \simeq \Delta \text{ on the nose}$$

$$\cup$$

$$\Delta^{op}$$

$$\simeq$$

$\{$operations s.t. a_0 lands on the place $0\}$.

$$(a_0, a_1, a_2, a_3, a_4) \longmapsto (1, a_0 a_1, a_2 a_3 a_4)$$

$\Delta^1 \not\ni \ \notin \Delta^{op}$, $\in \Delta^{op}$ $\in \Delta^1$.

$$(a_2 a_3, a_4 a_0, 1, a_1)$$

$$\mapsto (a_3 a_4 a_0, 1, a_1 a_2).$$

in Δ^{op}

$\Lambda \simeq \Lambda^{op}$

$a = (a_0, \dots, a_n)$ - $b = (b_0, \dots, b_m)$

$\lambda : [n] \to [m]$ in Λ^{op}

$\langle a, b \rangle = tr(a_0 b_0 a_1 b_1 \cdots a_k b_k)$

$\langle \lambda a, b \rangle = \langle a, \lambda^\circ b \rangle$

$\therefore$ λ° exists, unique.

not symm

whenever $tr : X \to ?$

$tr(ab) = tr(ba)$

$$\begin{array}{ccccc} & & \mu & \longmapsto & \mu^\circ \\ \Delta^{op}_{\ell} & \xrightarrow{\sim} & \Delta_{/1} & \xrightarrow{\sim} & \Delta^{op}_{r} \\ \lambda & \longmapsto & & & \lambda^\circ \end{array}$$

(I think):

Δ_l^{op} = {those that a_0 lands on leftmost place}

Δ_r^{op} = {those that a_{last} lands on rightmost place}

$$a_0, a_1, a_2, a_3 \to a_3\, a_0\, a_1\, a_2, 1 \quad \Delta_l^{op}$$

$$a_0, a_1, a_2, a_3 \to a_1, a_2, a_3\, a_0 \quad \text{in } \Delta_r^{op}$$

$\lambda \in \Delta_l^{op}$

$\lambda = a_3 a_0, a_1, 1, a_2$

$\langle \lambda a, b \rangle$

$\lambda^\circ = b_0, b_1, b_2, b_3, 1$

Δ^-

$$t\lambda(a_3 a_0 \cdot \underline{b_0} \cdot a_1 \cdot \underbrace{b_1 \cdot 1 \cdot b_2} \cdot a_2 \cdot b_3)$$

Variation: Given: $X, 1$ unital Λ_ℓ^{op} monoid

Note: α could be supposed to be an ENDOmorphism ($\ell = \infty$)

$\alpha : X \circlearrowleft$ automorphism

$\alpha^\ell = id_X$ (could be $\ell = \infty$)

Good Dold-Kan, but a bit confusing

All operations

$$X^{n+1} \longrightarrow X^{n+1}$$

generated, wrt composition:

$(\alpha a_n, a_0, a_1, \ldots, a_{n-1})$

$\uparrow$

$(a_0, \ldots, a_n) \mapsto (a_0 a_1, a_2, \ldots)$

$\ldots a_j a_{j+1}, \ldots$

$\ldots 1 \ldots$

$\Delta' = \{$ nondecreasing maps

$\{0, 1, \ldots, n\} \to \{0, \ldots, m\}$

$\| \qquad\qquad \|$

$[n] \qquad\qquad [m]$

$\rightsquigarrow$ in

[NS]: this may be

[Kaledin] $[n+1]_\Lambda$

$\Lambda^{op} = \{$ maps $[n] \xrightarrow{f} [m]$ with

linear order on

each $f^{-1}(j)$ $j \in [m]$

such that the correct cyclic order

$f^{-1}(0) = \{y < 0\}$

$f^{-1}(0) \quad f^{-1}(1) \quad \ldots \quad f^{-1}(m)$

$a_0\, a_0 \qquad a_1\, a_2 \qquad 1 \qquad a_3$

$\underline{\text{OR}}$: $\Lambda^{op}_{\infty}([n],[m]) =$

$= \{$ nondecreasing maps

$$\frac{1}{n+1}\mathbb{Z} \xrightarrow{f} \mathbb{Z}\cdot\frac{1}{m+1}$$

$$f(j+1) = f(j)+1 \}$$

$$\begin{array}{ccc} f(j+1) & = & f(j)+1 \\ \| & & \| \\ & (\sigma f)(j) & \end{array}$$

a_0, a_1, a_2, a_3, a_4

$\alpha(a_2), \alpha(a_3 a_4),$

$$\boxed{\Lambda^{op}_{\ell} = \Lambda^{op}_{\infty}/\sigma^{\ell}}$$

0 x x x 1 ?? 0 x x 1 x

$\frac{1}{4}$ $\frac{1}{2}$ $\frac{3}{4}$ $\frac{1}{3}$ $\frac{2}{3}$ $1\frac{2}{3}$

$$a_0 \otimes \cdots \otimes a_4$$

$$\downarrow$$

$$\alpha(a_2 a_3) \otimes \alpha(a_4) \otimes a_0 a_1 \otimes 1$$

f

$$0 \quad \tfrac{1}{5} \quad \tfrac{2}{5} \quad \tfrac{3}{5} \quad \tfrac{4}{5} \quad 1 \quad \tfrac{6}{5} \quad \tfrac{7}{5}$$

$$0 \quad \tfrac{1}{4} \quad \tfrac{2}{4} \quad \tfrac{3}{4} \quad 1 \quad \tfrac{5}{4} \quad \tfrac{6}{4} \quad \tfrac{7}{4} \quad 2$$

$$f(j+1) = f(j) + 1$$

$$f : \tfrac{1}{5}\mathbb{Z} \longrightarrow \tfrac{1}{4}\mathbb{Z}$$