Cyclic object of a category C:

functor $\Lambda^{op} \to C$

Λ^{op}_{∞}: objects: $[n]$, $n \geq 0$

$$\Lambda^{op}_{\infty}\left([n],[m]\right) = \left\{ f : \frac{1}{(n+1)}\mathbb{Z} \to \frac{1}{(m+1)}\mathbb{Z} \ \text{nondecreasing, } \mathbb{Z}\text{-equivariant, i.e. } f(x+1) = f(x) + 1 \right\}$$

$\sigma \in \Lambda^{op}_{\infty}([n],[n])$

$f(x) = x + 1$

CENTRAL

$$\begin{tikzcd} {[n]} \arrow[r, "\sigma"] \arrow[d, "d"'] & {[n]} \arrow[d, "d"] \\ {[m]} \arrow[r, "\sigma"'] & {[m]} \end{tikzcd}$$

$$\boxed{\Lambda_\ell^{op} = \Lambda_\infty^{op} / \langle \sigma^\ell \rangle}$$

$$\Delta \xhookrightarrow{j'} \Lambda_\infty^{op}$$

Δ : objects:

$\{0, 1, \dots, n\} = [n], n \geq 0$

Morphisms: nondecreasing maps.

$$\Delta^{op} \xhookrightarrow[j]{} \Lambda_\infty^{op}$$

Λ_∞^{op} = natural operations

$(a_0, \dots, a_n) \longmapsto \dots \in X^{m+1}$ $\underline{\alpha \in Aut}$

X - any monoid,

Given: X, 1, $\alpha : X \circlearrowleft$

$$(a_0, \dots, a_5) \longmapsto \left(\alpha^2(a_3)\cdot a^2(a_4),\ \alpha(a_5)\,\alpha(a_0),\ \alpha(a_1 a_2),\ 1\right)$$

in $\Lambda_\infty([5],[3])$

<u>Recall:</u> gen. by

$(a_0, \dots, a_n) \to (\alpha(a_n), a_0, \dots, a_{n-1})$

$(a_0, \dots, a_n) \to (\dots, a_j a_{j+1}, \dots)$

$(a_0, \dots, a_n) \to (\dots, 1, \dots)$

1) cyclic order of $a_0, \dots, a_5$ the same

2) α

$\alpha(\cdot)$

$$d_n(a_0, \dots, a_n) = \alpha(a_n)\, a_0, \dots, a_{n-1}$$

$$d_j,\ j \geq 0 \qquad a_0, \dots, a_j a_{j+1}, \dots, a_n$$

Given an associative algebra $A/k \rightsquigarrow A^{\natural}$ cyclic object in k-modules:

$$A^{\natural}_n = A^{\otimes n+1}$$

And: Given an alg with $\alpha \in \mathrm{Aut}(A)$, $\alpha^{\ell} = id$:

$$(A^{\natural}_{\alpha})_n = A^{\otimes n+1}$$

Λ_{ℓ}^{op} – mod, i.e. an ℓ-cyclic obj of k-mod

$C^{full}_\bullet(A), b$ — the Hochschild complex

$\| $

$A^{\otimes \bullet + 1}$

$\uparrow$

$C_\bullet(A) = A \otimes \bar{A}^{\otimes \bullet}$

$$CC_\bullet(A) = \frac{C_\bullet(A)((u))}{u \cdot C_\bullet(A)[[u]]}$$

$$\begin{tikzcd}
 & \arrow[d] & \vdots \arrow[d, "b"] \\
u^{-2}C_0 \arrow[r, "B"] & u^{-1}C_1 \arrow[r, "B"] \arrow[d] & C_2 \arrow[d, "b"] \\
 & u^{-1}C_0 \arrow[r] & C_1 \arrow[d] \\
 & & C_0
\end{tikzcd}$$

1

Connes: $CC_\cdot(A) \cong A^\natural \otimes^{\mathbb{L}}_{\Lambda} k^\natural = \operatorname{hocolim}_{\Lambda^{op}} (A^\natural)$

$k^\natural$: the constant cocyclic obj $\Lambda \to k\text{-mod}$

the $\natural$ obj of the coalg k

Proof: the resolution of $k^\natural$:

$$
\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] \\
\mathbb{Z}\Lambda^{op}([\cdot],[0]) \arrow[r] & \mathbb{Z}\Lambda^{op}([\cdot],[1]) \arrow[r, "B"] \arrow[dl] & \mathbb{Z}\Lambda^{op}([\cdot],[2]) \arrow[d, "d_0-d_1+d_2"] \\
\mathbb{Z}\Lambda^{op}([\cdot],[0]) \arrow[rr, "B"] & & \mathbb{Z}\Lambda^{op}([\cdot],[1]) \arrow[d, "d_0-d_1"] \\
 & & \mathbb{Z}\Lambda^{op}([\cdot],[0])
\end{tikzcd}
$$

$h([2])$

one column class

$(A^{q})\otimes_{\Lambda} ?$

$\mathbb{Z}$

resolution of $\mathbb{Z}_q$

if it were Δ, not Λ: that's it.

$B : [1] \to [2]$

$$a_0 \otimes a_1 \mapsto (1 \otimes a_0 \otimes a_1 - 1 \otimes a_1 \otimes a_0)$$

$$- (a_0 \otimes a_1 \otimes 1 - a_1 \otimes a_0 \otimes 1)$$

$$\sum (-1)^{nj}\left[1 \otimes (\underline{a_j \otimes \cdots \otimes a_{j-1}}) - (-1)^{n+1}(\longrightarrow) \otimes 1\right]$$

Versions:

$$\begin{tikzcd}
{} \arrow[r, "1-\tau"] \arrow[d, "b'"'] & {} \arrow[r, "N"] \arrow[d, "b"] & {} \arrow[d, "b'"] & {} \arrow[d, "b"] \\
{} \arrow[r, "1-\tau"] \arrow[d, "b'"'] & {} \arrow[r, "N"] \arrow[d, "b"] & A^{\otimes 3} \arrow[r, "1-\tau"] \arrow[d, "b'"] & A^{\otimes 3} \arrow[d, "b"] \\
{} \arrow[r, "1-\tau"] \arrow[d, "b'"'] & {} \arrow[r, "N"] \arrow[d, "b"] & A^{\otimes 2} \arrow[r, "1-\tau"] \arrow[d, "b'"] & A^{\otimes 2} \arrow[d, "b"] \\
{} \arrow[r, "1-\tau"] & {} \arrow[r, "N"] & A \arrow[r, "1-\tau"] & A
\end{tikzcd}$$

2 1 0

$$b = \sum_0^n (-1)^j d_j \quad \Big| \quad b' = \sum_0^{n-1} (-1)^j d_j$$

$$\tau(a_0 \otimes \dots \otimes a_n) = (-1)^n a_n \otimes a_0 \otimes \dots \otimes a_{n-1}$$

$$N = 1 + \tau + \tau^2 + \tau^3 + \dots + \tau^{n-1}$$

__Lemma__ (By the same argum.)

the above computes

$$\operatorname{hocolim}_{\Lambda^{op}} (A^{\natural})$$

Same formulas for __any__ cyclic module $M_\bullet$.

$$\operatorname{hocolim}_{\Lambda^{op}} (M_\bullet)$$

Cor. $M_\bullet / \mathrm{im}(1-\tau)$, $b = \sum_0^n \pm d_j$

is a complex

$b(1-\tau) = (1-\tau)b'$

And it is quasi-isom to the above if:

$\mathbb{Q} \subset k$ **OR** M_n is free over $k[C_{n+1}]$
$\|$
$\mathbb{Z}/\mathbb{Z}(n+1)$

b/c

$\xrightarrow{N} \xrightarrow{1-\tau} \xrightarrow{N} \xrightarrow{1-\tau}$

are acyclic in both cases.

For Λ_ℓ : $\ell < \infty$:

same format double complex

Start with $b, b', 1-\tau, N$:

same thing. Now

The rest is as above.

$M_n \circlearrowleft t$ – the cyclic ...perm

$\boxed{N = 1 + \tau + \tau^2 + \dots + \tau^{(n+1)\ell - 1}}$ $\tau = (-1)^n t$

correct!

$\cancel{n \times 1}$

$\ell > 1$:

$\alpha(a_n) \otimes a_0 \otimes \dots \otimes a_n$

$\sigma^\ell = id$

sign $= (-1)^n$

$t^{(n+1)\ell} = id$.

$\sigma := t^{n+1}(a_0 \otimes \dots \otimes a_n) = \alpha(a_0) \otimes \dots \otimes \alpha(a_n)$

$$
\begin{array}{ccc}
HC_\cdot(M_\cdot) & \xrightarrow{\ \sim\ } & HH_\cdot(M) \\
\| & & \| \\
\operatorname{hocolim}_{\Lambda_\infty^{op}}(M_\cdot) & & \operatorname{hocolim}_{\Delta^{op}}(M_\cdot)
\end{array}
$$

<u>actually</u>: $\Delta^{op} \overset{j}{\hookrightarrow} \Lambda_\infty^{op}$ is

COFINAL

(Nikolaus-Scholze, App. B)

<u>Next</u>: The subdivision.

$$
\begin{tikzcd}
\sigma^\ell \in \Lambda_\ell^{op} \arrow[r, "\sigma \mapsto 1", "\pi_\ell"'] \arrow[r, bend right=60, "i_\ell = sd_\ell"'] & \Lambda^{op}
\end{tikzcd}
$$

$\boxed{[b, B_\alpha] = \mathrm{id} - \alpha}$

The B version:

$$B_\alpha(a_0 \otimes \dots \otimes a_n) = \sum \pm 1 \otimes (\alpha a_{j+1} \otimes \dots \otimes a_0 \otimes \dots \otimes a_j)$$

$B = (1-\tau) s_{-1} N$ where $s_{-1} = 1 \otimes a_0 \otimes a_1 \otimes \dots \otimes a_n$

we have seen it before...

$l = \infty$: actually,

$$\begin{tikzcd}
\arrow[d, "b'"'] & \arrow[d, "b"] \\
M_2 \arrow[r, "1-\tau"] \arrow[d, "b'"'] & M_2 \arrow[d, "b"] \\
M_1 \arrow[r, "1-\tau"] \arrow[d, "b'"'] & M_1 \arrow[d, "b"] \\
M_0 \arrow[r, "1-\tau"] & M_0
\end{tikzcd}$$

3 2 1 0

THIS computes $M_\bullet \overset{\mathbb{L}}{\underset{\wedge}{\otimes}} k^q$

Also: b' is contractible. $1 \otimes a_0 \otimes a_1 \otimes \dots$ is the contract homotopy

Given an algebra A:
get an algebra $A^{\otimes \ell}$

$$\alpha : A^{\otimes \ell} \circlearrowleft_{2}$$

$$\alpha(a_1 \otimes \ldots \otimes a_\ell) = a_\ell \otimes a_1 \otimes \ldots \otimes a_{\ell-1}$$

$$\left(A^\natural\right)_n = A^{\otimes n+1}$$

$$\left(A^\natural_\alpha\right)_n = \left(A^{\otimes \ell}\right)^{\otimes n+1} =$$

morphisms:
same.

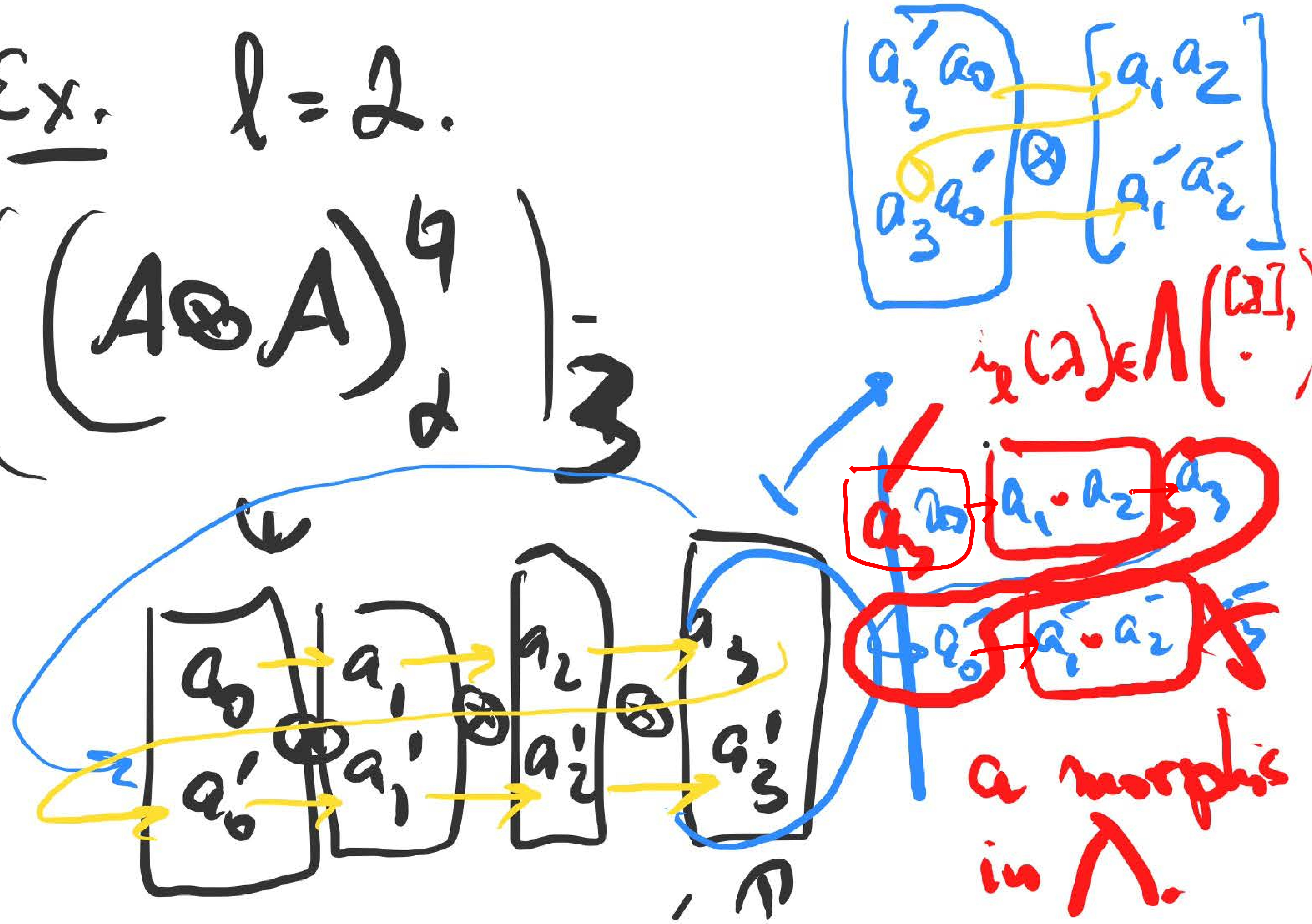
$$= A^{\otimes \ell(n+1)}$$

$$\| $$

$$\left(A_\natural\right)'_{(n+1)\ell + 1}$$

Ex. $\ell = 2$.

$$\left(\left(A\otimes A\right)^{q}_{d}\right|_{3}$$



$$A^{\otimes 2}\otimes A^{\otimes 2}$$

Morphism

$$b_0\otimes b_1\otimes b_2\otimes b_3 \mapsto \alpha(b_3)b_0\otimes b_1 b_2 \;?$$

$$\lambda \in^2 \Lambda([3],[1])$$

So:

$$i_\ell : \Lambda_\ell \longrightarrow \Lambda$$
$$[n] \longmapsto \ell(n+1)-1$$

<u>a functor</u>

$$\left(A^{\otimes \ell}\right)^{\natural}_{\alpha} \longleftarrow A^{\natural}$$
$$\cap$$
$$\Lambda^{op}\text{-mod}$$

$$\operatorname{hocolim}_{\Lambda_\ell^{op}} \left(i_\ell^* M_\bullet\right) \simeq$$
$$\simeq \operatorname{hocolim}_{\Lambda^{op}} (M_\bullet)$$

Same for Δ^{op}

<u>So</u>: $\overset{C}{HH}(A^G)$ has an interpretation for $\underline{\forall \ell}$

as

$$\operatorname*{hocolim}_{\Delta^{op}} \; \operatorname*{hocolim}_{\Lambda_\ell^{op}} i_\ell^* A^G$$

NC forms

~~Both~~ NC Frobenius, uses <u>that</u>.