On cyclic objects:

[Cyclic object in $\mathcal{C}$ $= \Lambda^{op} \longrightarrow \mathcal{C}$ functor] $\rightsquigarrow$ [a simplicial object in $\mathcal{C}$ with action of the simplicial group $B\mathbb{Z}$]

enough in $\mathcal{C}$ to do that: (co)limits,

[Nikolaus-Scholze app. B]

[Cyclic object X in $\mathcal{C}$ = Sets or Spaces] $\longrightarrow$ $|X|$ $\circlearrowleft$ $\mathbb{T}$ $= S^1$

2. Λ_ℓ^{op}; $\forall \ell = 1, 2, 3, \ldots \quad \infty$

$\Lambda^{op} := \Lambda_1^{op}$

ℓ-cyclic object: $\Lambda_\ell^{op} \to \mathcal{C}$

$\ell < \infty$:

ℓ-cyclic obj of $\mathcal{C} \longrightarrow$ Simplicial set w/ $B\mathbb{Z}$ action

ℓ-cyclic space set or, $X \longrightarrow \mathbb{T} = S^1$ acts on $|X|$

$$\Lambda^{op} \longleftarrow \Lambda_\ell^{op} \xrightarrow{i_\ell = s d_\ell} \Lambda^{op}$$

gen. by:

$\Delta^{op} = \langle s_j, d_j \rangle$

gen. by:

Δ^{op}

$t : [n] \circlearrowleft \quad t^{n+1} = id \Rightarrow t^{\ell(n+1)} = id.$

$id \longleftarrow\!\shortmid\; t^{n+1} =: \sigma$

$[n] \longmapsto [(n+1)\ell - 1]$

ℓ-cyclic obj of $A^{\otimes \ell}$, α = cyclic perm $\xleftarrow{i_\ell^*}\!\shortmid\; A^{\natural}$ / cyclic obj of an algebra A

$$\begin{array}{ccc} \Lambda_\ell^{op} & \longleftarrow & \Lambda^{op} \\ \cup & & \cup \\ \Delta^{op} & \longleftarrow & \Delta^{op} \end{array}$$

finite tot ord set

$S \times \{1, 2, \dots, \ell\} \longleftarrow S$

lexicographical

Prop. 2. The $B\mathbb{Z}$-equivariant simplicial objects are SAME (equivalent...) for M and $i_l^* M$

(to clarify / ☆ prove)

- For cyclic sets / spaces: geometric realizations are just homeom. as S^1-spaces.

One route from here:

$\Lambda^{op} \xrightarrow{X} \mathcal{C}$ $\rightsquigarrow$ Δ^{op}-object in $\mathcal{C}$ with the action of the simplicial group $B\mathbb{Z}$

cyclic object in $\mathcal{C}$

Say, $\mathcal{C} = Ab$

Simplicial CO module over the cosimplicial COALGEBRA $\mathbb{Z}[B.\mathbb{Z}]^{dual}$ $\longleftarrow$ Simplicial module over simplicial algebra $\mathbb{Z}[B.\mathbb{Z}]$

Simplicial comodules over cosimplicial coalgebra

$\left\{\begin{array}{l}\text{algebraic} \\ \text{functions}\end{array}\right. \mathcal{O}(B_\bullet\mathbb{Z}) = \mathcal{O}(\mathbb{Z}^{\times\bullet}) = \mathcal{O}(\mathbb{Z})^{\otimes\bullet}$

$\mathbb{Z}$-valued!

Functions$(B_\bullet\mathbb{Z})$ multiplication of functions

cosimplicial in that

$\| $

$\mathbb{Z}[B_\bullet\mathbb{Z}]^{\text{dual}}$: $n \mapsto$ $\left\{\begin{array}{l}\text{functions on} \\ B_n\mathbb{Z} = \mathbb{Z}^{\times n+1} \\ \text{comultiplication}\end{array}\right.$

comes from

$\mathcal{O}(\mathbb{Z}) = \mathbb{Q}[x]^{\mathbb{Z}\text{-valued}}_{\text{on } \mathbb{Z}}$

$B\mathbb{Z} \times B\mathbb{Z} \to B\mathbb{Z}$

This is over $\mathbb{Z}$. Over $\mathbb{Z}_{(p)}$ (i.e. when all $q \neq p$ are inverted):

Another cosimplicial coalgebra, coming from a "different $\mathcal{O}(\mathbb{Z})$":

$$\mathcal{O}(\mathbb{Z})' = \mathcal{O}(\mathcal{F}ix(F))$$

$\mathcal{F}ix(F)$ = group scheme $\mathcal{F}ix\,(F : W \to W)$

A-valued points: $a \in W(A) : Fa = a$

$a = (a_0, a_1, a_2, \dots) \in W(A)$ —

resp. Ker F:

$a_0^p + pa_1 = 0$

$\dots = 0$

$\dots$

Fix $\underline{Fa = a:}$

$a_0^p + pa_1 = a_0$

$a_0^{p^2} + pa_1^p + p^2 a_2 = a_1^p + pa_0$

$\vdots$

ker $Fa = 0$

Functions on $\mathbb{Z}$

$\mathcal{O}(\mathrm{Fix}\, F) = $ "$\mathcal{O}(\mathbb{Z})$" $= \mathbb{Z}_{(p)}[a_0, a_1, a_2, \dots] \Big/ \begin{matrix} a_0^p + pa_1 - a_0 \\ a_0^{p^2} + pa_1^p + p^2 a_2 - a_0 \\ \dots \end{matrix}$

$\mathbb{Z}_{(p)}$-valued

because $\mathbb{Z} \to \mathrm{Fix} \to W(A)$, $\mathbb{Z} \to W(A)$

$F1 = 1 \quad F2 = 2 \dots$

From cyclic ...
Abelian grps
to ~~cosimp~~ simplicial
comodules over
cosimplicial algebra $\mathcal{O}(\mathbb{Z})^{\otimes \bullet}$

$$\frac{x(x-\cancel{1})(x-\cancel{2})}{3!} \sim \frac{x^3}{3!}$$

$\mathcal{O}(\mathbb{Z}) = $ ~~ex~~ $\begin{cases} \mathbb{Z}\text{-val } P(x) & /\mathbb{Z} \\ \mathcal{O}(\text{Fix } F) & /\mathbb{Z}_{(p)} \end{cases}$

Both FILTERED.

$$\mathrm{gr}: \begin{cases} \mathbb{Z}\langle x \rangle \\ \mathcal{O}(\ker F) \end{cases}$$

divided power polynomials

2. Cyclic object $A^\natural$

$\wr$

$HH(A) \circlearrowleft S^1$ or $B\mathbb{Z}$ $\quad i_\ell^*(A^{\otimes \ell})^\natural$

Frobenius: $HH(A^{\otimes \ell}, \alpha)^{tC_p} \xleftarrow{T_\ell} HH(A)$

$\forall$ cyclic obj in place of $A^\natural$ $\quad$ $\mathfrak{A}$ $\quad HH(A^{\otimes \ell}, \alpha) \circlearrowleft S^1$ or $B\mathbb{Z}$,

α

3. Noncommutative with vectors / cyclic objects

BASED ON:

Given : a $\mathbb{Z}$-module M

(say, free).

a prime p.

cyclic group $\sigma \in C_p = \mathbb{Z}/p$ acts

$$\text{(not additive)}\ M \longrightarrow M^{\otimes p}$$

$$x \longmapsto x^{\otimes p} \quad (=: x^p)$$

$$M \longrightarrow (M^{\otimes p})^{\sigma} = \left\{ \sigma\text{-invariants} \right.$$

$$\text{norm} = 1 + \sigma + \dots + \sigma^{p-1}$$

And if M free mod over $\mathbb{F}_p$:
$M \underset{\varphi}{\xrightarrow{\sim}} \check{H}^0(C_p, M^{\otimes p})$

$$(M^{\otimes p})_\sigma \xrightarrow{\text{norm}} (M^{\otimes p})^\sigma$$

basis: $e_j^{\otimes p}$

now additive

$$M \longrightarrow \check{H}^0(C_p, M^{\otimes p}) = (M^{\otimes p})^\sigma / \{\text{norms}\}$$

$$(x+y)^p = x^p + y^p + (\text{norms})$$

$p=2$

$$(x+y)^2 = xx + yy + (xy + yx)$$

norm

$2 \cdot x^2 y$

Apply this to :

1) $M = HH(A)$

2) $M = \ldots$ get NC with vectors organised in a cyclic complex of Kaledin.

Given $M_\cdot, N_\cdot$ two cyclic objs (mods)

SMC

$\otimes$
$\shortparallel$
$M_\cdot \boxtimes N_\cdot$

$(M \boxtimes N)_n = M_n \underset{\mathbb{Z}}{\otimes} N_n$

diag action of Λ^{op}

to $\mathcal{O}(\mathbb{T})^{\otimes \cdot}$ – comodules

still SMC $\otimes$

Cyclic objects / mods $\longrightarrow$ Mixed cplxs

eq of (∞) M.C.

$$\begin{tikzcd} \cdots \arrow[r] & M_2 \arrow[r, "b"'] \arrow[l, bend right] & M_1 \arrow[r, "b"'] \arrow[l, bend right, "B"'] & M_0 \arrow[l, bend right, "B"'] \end{tikzcd}$$

$$(b+B)^2=0$$

a.k.a. dg mod / $\mathbb{Z}[\varepsilon]$, $\varepsilon^2=0$

SHC

$\otimes$

$B\otimes 1+1\otimes B$