Noncommutative Witt vectors
(Kaledin)

N free k-module (mainly $k = \mathbb{Z}$).

Basis $\{e_j\}$

$$\boxed{W_n(N) = \left(N^{\otimes p^n}\right)^{\sigma} \Big/ \{\text{norms}\}}$$

σ = generator of $C_{p^n} = \mathbb{Z}/p^n\mathbb{Z}$
acting on $N^{\otimes p^n}$ by permut.

$$\text{norm} = \sum_{0}^{p^n-1} \sigma^j$$

$$?^{\sigma} = \{C_{p^n}\text{-invariants}\}$$

1) $n = 1$ $\quad N^{\otimes p} = N_1 \oplus N_p$

C_p-orbit of length 1 $\quad N_1 \cong \bigoplus \mathbb{Z} e_j^{\otimes p}$ $\quad N_p \cong \bigoplus \mathbb{Z} \cdot e$

of length p $\quad$ $e \neq e_j^{\otimes p}$

monomials

$$N_1^{\sigma}/\text{norms} = \bigoplus (\mathbb{Z}/p)\cdot e_j^{\otimes p}$$

$$N_p^{\sigma}/\text{norms} = 0$$

$$\begin{tikzcd}
N \arrow[d] & & W_1(N) \\
N/pN \arrow[urr] & &
\end{tikzcd}$$

$n=2$:

$$N^{\otimes p^2} = \underset{\parallel}{N_1} \oplus \underset{\parallel}{N_p} \oplus N_{p^2}$$

$$\bigoplus_j \mathbb{Z}\, e_j^{\otimes p^2} \oplus \bigoplus \mathbb{Z}\, e^{\otimes p} \oplus \bigoplus \mathbb{Z}\, e$$

$e \neq e_j^{\otimes p}$

the rest of monomials

$$\text{norm}_k := 1 + \sigma + \dots + \sigma^{p^k - 1}$$

$$(N^{\otimes p^2})^{\sigma} = \bigoplus \text{norm}_0(e_j^{\otimes p^2})\cdot(\mathbb{Z}/p^2) \oplus$$

$\oplus \oplus \mathrm{norm}_1(e^{\otimes p}) \cdot \mathbb{Z}/p \oplus$

$\oplus \mathrm{norm}_2(e)$

Proposition There are

$e_j(x,y) \in (\mathbb{Z}x \oplus \mathbb{Z}y)^{\otimes p^j}$,

$j = 1, 2, \dots$ such that

$$(x+y)^{p^n} - x^{p^n} - y^{p^n} = \sum_{j=1}^{n} \mathrm{norm}_j \, c_j(x,y)^{p^{n-j}}$$

for all n.

Examples

$$(x+y)^p - x^p - y^p = \sum \sum_{j=0}^{p-1} \sigma^j (\dots)$$

e.g.

$$(x+y)^2 - x^2 - y^2 = \mathrm{norm}_1(xy)$$

$$(x+y)^4 - x^4 - y^4 =$$

$\boxed{c_1(x,y) = = xy}$

$= \mathrm{norm}_2(x^3y + x^2y^2 + xy^3) +$

$+ \mathrm{norm}_1((xy)^2)$

$$\boxed{c_2(x,y) = x^3y + x^2y^2 + xy^3}$$

...

Why plausible? $(x+y)^{p^n} - x^{p^n} - y^{p^n}$ is σ-invariant; have seen that it is sum of $\mathrm{norm}_k(e_{p^{n-k}})$.

Comparison to Witt vectors:

$(x, 0, 0, \dots) + (y, 0, 0, \dots) =$

$= (x+y, -\tilde{c}_1, -\tilde{c}_2, -\tilde{c}_3, \dots)$

where $\tilde{c}_n$ = image of c_n in $Sp^n(\mathbb{Z}x \oplus \mathbb{Z}y) \leftarrow$ $\mathrm{norm}_k \mapsto p^k$

Recall:

$w_n = a_0^{p^n} + p a_1^{p^{n-1}} + p^2 a_2^{p^{n-2}} + \dots + p^n a_n)$

Plausible, b/c of the decomposition of $N^{\otimes p^n}$ into $\mathrm{norm}_k(e^{\otimes n-k})$

<u>A closer look:</u> (at W_n).

0) Teichmüller representatives:

$$\begin{tikzcd} N \arrow[r, "T_n"] \arrow[d] & W_n(N) \\ N/p \arrow[ur, dashed, "?"] & \end{tikzcd} \qquad x \mapsto x^{\otimes p^n}$$

$(x+y)^{p^n} =$

$= x^{p^n} + p\text{-norms}$

$$\boxed{(x+py)^n - x^n = p \cdot \text{norm}}$$

$p=2$

ex. $(x+2y)^4 = x^4 + \text{norm}_2\left(x^3 \cdot py + \ldots (py)^2 + \ldots (py)^3\right)$

$+ 2\,\text{norm}_1\left(x \cdot (2y) \cdot x \cdot \boxed{2}y\right)$

$= 2 \cdot \text{norm}$

$$N \xrightarrow{T_n} W_n(N)$$

$$T_n' : N \dashrightarrow W_n'(N) \to W_n(N)$$

$W_n'(N) \cong (N^{\otimes p^n})^\sigma / p \cdot \text{norms}$

$\downarrow 2$

$W_{n+1}(N)$

$(N^{\otimes 2})^\sigma \bmod 2\text{ norms}$ $(N^{\otimes 4})^\sigma / \text{norms}$

$\underbrace{x^2 \quad y^2}_{\bmod 4}$ $\qquad \underbrace{xy + yx}_{\bmod 2}$ $\qquad x^4 \bmod 4$ $\qquad xyxy + yxyx$ $((\bmod 2$

$\cancel{xy}$

not σ-inv

$\text{norm}_1(xyxy)$

$$\cancel{(N^{\otimes p^n})^\sigma} = W_n'(N) \xrightarrow{\sim} W_{n+1}(N)$$

$$p\text{ norms} \to (N^{\otimes p^n})^\sigma / \text{norms} = \qquad W_n'(N) \to W_n(N) \xleftarrow{R} W_{n+1}(N)$$

Verschiebung: $W_n(N^{\otimes p}) \xrightarrow[V]{1+\sigma+\dots+\sigma^{p-1}} W_{n+1}(N)$

$$\begin{tikzcd} W_n(N^{\otimes p}) \arrow[r, "1+\sigma+\dots+\sigma^{p-1}", "V"'] \arrow[d, equal] & W_{n+1}(N) \arrow[d, equal] \\ (N^{\otimes p^{n+1}})^{\sigma^p} / 1+\sigma^p+\sigma^{p^2}+\dots & (N^{\otimes p^{n+1}})^{\sigma} / 1+\sigma+\sigma^2+\dots \end{tikzcd}$$

$$\begin{tikzcd} W_n(N^{\otimes p}) \arrow[r, "V"] & W_{n+1}(N) \arrow[dr, "R"] & \\ W_n'(N) \arrow[ur, "S"] \arrow[rr] & & W_n(N) \end{tikzcd}$$

Proof of Proposition: $V^j \cdot T'_{n-j}(c_j)$

$$(x+y)^{p^n} - x^{p^n} - y^{p^n} = \sum_{j=1}^{n} \text{norm.}\, c_j(x,y)^{p^{n-j}}$$

$T'_n(x+y) - T'_n(x) - T'_n(y)$

$W_n' = T\left(N^{\otimes p^n}\right)^\sigma \Big/ p\cdot\text{norms}$

Assume true for n.

l.h.s. is σ-invariant. in $\left(N^{\otimes p^n}\right)^\sigma$

$$W_{n+1} = \left(N^{\otimes p^{n+1}}\right)^\sigma \Big/ \text{norms}$$

$$T_n'(x+y) - T_n'(x) - T_n'(y) = \boxed{\text{l.h.s.} - \text{r.h.s.} = \ldots}$$

$= 0$ mod norms

$$= \sum V^j T_{n-j}' c_j(x,y)$$

$\boxed{= \text{norm } c_n(x,y)}$ in

$W_n'(N)$

$\shortparallel$

$W_{n+1}(N)$

$\boxed{\text{in } W_{n+1}}$

$$T_{n+1}(x+y) \ldots V^j \cdot T_{n-j+1} = 0$$

Next: for an associative algebra $A/\mathbb{F}_p$:

$\forall n$

$$\boxed{\begin{array}{c} W_n(A)^{q}_{m} = \tilde{W}_n\left(A^{\otimes m+1}\right) \\ {\scriptstyle m \geq 0} \end{array}}$$

form a cyclic ~~object~~ abelian group;

HC, HH are nc DRWitt.