Cartier isomorphism (comm and not), and Hodge-DeRham degeneration (commutative).

X - smooth projective / $\mathbb{F}_p$ (or a perfect field k, $\mathrm{char}(k) = p$).

① $p \geq \dim X$.

② X admits a smooth lifting over $\mathbb{Z}/p^2$ (or $W_2(k)$)

Thm: The Hodge-De Rham spectral sequence

$$E_1^{pq} = H^p(X, \Omega_X^q) \Rightarrow H_{DR}^{p+q}(X)$$

degenerates at E_1.

In particular: !

$$\dim H_{DR}^n(X) = \sum_{p+q=n} h^{pq}$$

$$h^{pq} = \dim H^p(X, \Omega_X^q) \qquad \overset{gr}{\underset{\|}{}} \; \check{C}^\cdot(X, \Omega_X^\cdot)$$

$\check{C}^\cdot(X, \Omega_X^\cdot)$, $\check{\partial} + d$

filtration $\check{C}^\cdot(X, \Omega_X^{\cdot \geq q})$

$$\Omega^\bullet_{A/k} \longrightarrow H^\bullet_{DR}(A/k)$$

$$\| $$

cohom of $\Omega^\bullet_{A/k}, d$

$$\left.\begin{array}{l} a \longmapsto a^p \\ b \longmapsto b^{p-1}db \end{array}\right\} \text{(mod im } d)$$

At the level of complexes?

Assume $\tilde{X}/W_2(k)$ smooth

Then $F_X : \mathcal{O}_X \to \mathcal{O}_X$

$$a \longmapsto a^p$$

has a lifting $\tilde{F}$.

(from dfn of smooth)

$$p\,\Omega^1_{\tilde{X}/W_2(k)} \longrightarrow \Omega^1_{\tilde{X}/W_2(k)}.$$

$$\Big\downarrow \text{reduction mod } p$$

$$\Omega^1_{X/k}$$

$$p\,\Omega^1_{\tilde{X}/W_2(k)} \Big/ p^2 \quad \longleftarrow \text{automatic}$$

$$\wr \Big\uparrow p. \qquad \text{(smoothness)}$$

$$\Omega^1_{\tilde{X}/W_2(k)} \Big/ p$$

$$db \mapsto d(b^p) = p\,b^{p-1}\,db = 0 \bmod p$$

$$\tilde{F} : \Omega^1_{\tilde{X}/W_2} \Big/ \cancel{p} \longrightarrow p\,\Omega^1_{\tilde{X}/W_2} \cong \Omega^1_{\tilde{X}/W_2} \Big/ p$$

So: $\varphi : \Omega^1_{X/k} \to \Omega^1_{X/k}$

$\cong \qquad \cong$

$(\Omega^1_{\tilde{X}})/p \to (p\,\Omega^1_{\tilde{X}})/p^2$

$\tilde{F}$ is ~~the~~ a lifting of F_X

If $b \longleftrightarrow \tilde{b}$. $\quad \tilde{b}^p + pu$

$b \in \mathcal{O}_X \quad \tilde{b} \in \mathcal{O}_{\tilde{X}} \quad \tilde{b}^p + pu = \tilde{F}(b)$

$\tilde{F}(db) = d(\tilde{b}^p + pu)$

$= p(\tilde{b}^{p-1}d\tilde{b} + du)$

$\| \quad \varphi(b)$

$F_X(a) = a^p$

$D(ab) = F_X(a)\,D(b) + D(a)\,F_X(b)$

Different $\tilde{F}$'s ?

$\tilde{F}_2 = \tilde{F}_1 + p \cdot D_{12}$

$\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$: $D_{12} + D_{23} = D_{13}$

in $\mathrm{Der}(\mathcal{O}_X, \mathcal{O}_X)$
tw
by F_X

$\boxed{\oplus \Omega^j[j] :}$ $\mathcal{O}_X \xrightarrow{0} \Omega^1_X \xrightarrow{0} \cdots \xrightarrow{0} \Omega^k_X$

of $\check{C}^\bullet(X, -)$ $\Big\downarrow$ quasi-isom ?

$\boxed{\tau DR_{\leq k} :}$ $\mathcal{O}_X \to \Omega^1_X \to \cdots \to Z^k_X \to 0$

$\ker(\Omega^k \xrightarrow{d} \Omega^{k+1})$

$k=0$:

$$\begin{array}{c} \mathcal{O}_X \\ \big\downarrow F_X \\ \mathcal{O}_X \end{array}$$

$k=1$:

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "0"] \arrow[d, "F_X"] & \Omega^1_X \arrow[d, "\varphi"] \\ \mathcal{O}_X \arrow[r, "d"] & Z^1_X \end{tikzcd}$$

✓ locally yes

$F_{X*}(\ \mathcal{O}_X \xrightarrow{d} Z^1_X\ )$

In general: $X = \bigcup \mathcal{U}_j$

$\tilde{X} = \bigcup \tilde{\mathcal{U}}_j$ open affine covers

$\tilde{F}_j$ on $\mathcal{O}_{\tilde{\mathcal{U}}_j} \Rightarrow \varphi_j$ on $\mathcal{U}_j$

$h_{jk} + h_{kl} = h_{jl}$

$$\boxed{\varphi_j - \varphi_k = [d, h_{jk}]} = d \circ h_{jk}$$

$h_{jk} \circ$ diff'l on l.h.s. $= 0$

$$\varphi_j(db) - \varphi_k(db) = $$

$$= d\, D_{jk}(b).$$

$h_{jk} = {}^{2}D_{jk}$

h_{jk} : $\cancel{\ } a\,db$ on Ω^1_X

$a\,db \mapsto a \circ D(b)$

a.k.a. ${}^{2}D_{jk}$

contraction by

$(\varphi_j) \in \check{C}^0(X, \underline{\mathrm{hom}}^0(\bigoplus_{j \le 1} \Omega^j_X[j],$

φ''

$\mathcal{O}_X \to \mathcal{Z}^1_X))$

h''

$(h_{jk}) \in \check{C}^1(X, \underline{\mathrm{hom}}^{-1}(\text{same}))$

$\check{\partial}\varphi = d \circ h \quad (= [\text{differential}, h])$

$\check{\partial} h = 0$

Cup product on $\check{C}^\bullet(X, \ldots)$:

$(s \cup t)_{j_0 \ldots j_{p+q}} = s_{j_0 \ldots j_p} \cdot t_{j_p \ldots j_{p+q}}$

$(\varphi + h)\cup : \check{C}^\bullet(X, \oplus \Omega^j[j]) \to \check{C}^\bullet(X, -$

In general:

$$\begin{tikzcd}
\mathcal{O}_X \arrow[r, "0"] \arrow[d] & \Omega^1_X \arrow[r, "0"] \arrow[d] & \cdots \arrow[r, "0"] & \Omega^k_X \arrow[d, "C^{-1}"] \\
F_{X*}\big(\mathcal{O}_X \arrow[r, "d"'] & \Omega^1_X \arrow[r, "d"'] & \cdots \arrow[r, "d"'] & \Omega^k_X\big)
\end{tikzcd}$$

Extend by multiplicativity.

$$\Omega^j_X = \Lambda^j_{\mathcal{O}_X}(\Omega^1_X)$$

$$\omega_1 \wedge \dots \wedge \omega_j \longmapsto \frac{1}{j!} \sum \pm C^{-1}(\omega_{\sigma 1}) \wedge \dots \wedge C^{-1}(\omega_{\sigma j})$$

As above, can extend to $\mathcal{C}^\bullet(X, -)$

$p \geq \dim X$:

$$\begin{array}{ccc} \Omega^0_X \xrightarrow{0} & \text{———} & \xrightarrow{0} \Omega^K_X \\ \downarrow & & \\ F_{X*}\big(\Omega^0_X \xrightarrow{d} & \text{———} & \xrightarrow{d} \Omega^K_X\big) \end{array}$$

Filter: 1) $\Omega^0_X \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_X$

by $\tau_{\leq K} \Omega^\bullet_X$

2) $\Omega^0_X \xrightarrow{0} \cdots \xrightarrow{0} \Omega^n_X$

by $\Omega^0_X \to \cdots \to \Omega^K_X$.

get a filtered morphism

inducing quis on gr.

$\Downarrow$

quis of spectral sequences.

Now use projectivity;
all dimensions are **finite**.

$$\dim H^n_{DR} = \sum_{p+q=n} h^{pq}$$

$\Downarrow$

all differential in the Hodge-DeRham spectral sequence are ZERO.