Monday Lecture: just sent to you.
11/16 independent on the rest.

Today: NC Witt vectors, continued.

Last time: M a free $\mathbb{Z}$-module

$n > 0$

$$W_n(M) = \left(M^{\otimes p^n}\right)^{\sigma} \Big/ \{\text{norms}\}$$

$$\text{''} \quad \hat{H}^0(C_{p^n}, M^{\otimes p^n})$$

σ acts on $M^{\otimes p^n}$.
by permutation
$(\ )^{\sigma} = \sigma$-invariants

For now : FACT: W_n is a TRACE FUNCTOR

on free $\mathbb{Z}$-mods,

A trace functor: F : Monoidal category $\to$ (k-mods)

$\mathcal{C} \longrightarrow \mathcal{K}$

$\tau_{M,N} : F(M \otimes N) \xrightarrow{\sim} F(N \otimes M)$

$$F(\underbrace{M_1 \otimes M_2 \otimes M_3}_{\tau_{M_1, M_2 \otimes M_3}}) \overset{\tau}{\xrightarrow{\sim}} F(\underbrace{M_2 \otimes M_3}_{\tau_{M_2, M_3 \otimes M_1}} \otimes M_1) \overset{\tau}{\xrightarrow{\sim}} F(M_3 \otimes M_1 \otimes M_2) \overset{\tau}{\xrightarrow{\sim}} F(M_1 \otimes M_2 \otimes M_3)$$

$\tau_{M_3, M_1 \otimes M_2}$

id

<u>Example</u> For A-A-bimodules,

$$T_A(M) = M/_{[A,M]} = M \otimes_{A \otimes A^{op}} A = HH_0(A,M)$$

Actually for any ${}_A M_B$ ${}_B N_A$:

$$T_A(M \otimes_B N) \xrightarrow{\sim} T_B(N \otimes_A M)$$

$$m \otimes n \longmapsto n \otimes m$$

both are $A \otimes_{A \otimes A^{op}} M \otimes_B N$

$\cong$

$$\begin{tikzcd} M \arrow[r, bend left, leftarrow, "A"] \arrow[r, "B"'] & N \end{tikzcd}$$

$A=B$: $\mathcal{C} = A\text{-bimod}$ with $\otimes_A$ $\mathbb{1} = \mathbb{Z}\text{-mod}$

Next: given an **algebra** A in $\mathcal{C}$

(e.g. an associative algebra)

and a trace functor $F : \mathcal{C} \to \mathcal{K}$

construct a cyclic object in $\mathcal{K}$:

$$F^\natural_{(A)} : \Lambda^{op} \to \mathcal{K} \qquad F^\natural(A)_n = F\left(A^{\otimes n+1}\right)$$

at the level of generators $\underbrace{d_j, s_j, t}_{\text{of } \Lambda^{op}}$:

$d_j,\ 0 \leq j < n$:

$$\underbrace{A \otimes \ldots \otimes A}_{n+1} \longrightarrow \underbrace{A \otimes \ldots \otimes A}_{n} \qquad a_0 \otimes \ldots \otimes \boxed{a_j a_{j+1}} \otimes \ldots$$

$$d_j : F(A^{\otimes n+1}) \xrightarrow{F(d_j)} F(A^{\otimes n})$$

$0 \leq j < n$

s_j acts as $F(s_j)$

$a_0 \otimes \cdots \otimes 1 \otimes \cdots \otimes a_n$

$$t : F(A^{\otimes n} \otimes \underset{a_n}{A}) \longrightarrow F(\overset{a_n}{A} \otimes \overset{a_0 \otimes \cdots \otimes a_{n-1}}{A^{\otimes n}})$$

$\Downarrow$

$d_n = d_0 t$

$a_n \otimes \underbrace{a_0 \otimes \cdots \otimes a_{n-1}}$

t acts via $\tau_{A^{\otimes n}, A}$

So: for every n, any algebra A, any prime p:

$W_n(A)^{q}$

$\uparrow$

cyclic abelian group;

$HH_\cdot(W_n(A)^q)$, $HC_\cdot(W_n(A)^q)$, etc.

One important exact sequence:

A bit more on $W_n(M)$.

$$\sigma^{p^n} = \mathrm{id}$$

$$\begin{tikzcd}
0 \arrow[r] & W_n(M^{\otimes p}) \arrow[r, "V", "\mu"'] \arrow[d, equal] & W_{n+1}(M) \arrow[r, "R"] \arrow[dr, leftrightarrow, "\sim"] & W_n(M) = (M^{\otimes p^n})^{\sigma} / \text{norms} \\
& (\underbrace{M^{\otimes p^{n+1}}}_{\tau^{p^{n+1}} = \mathrm{id}})^{\sigma = \tau^p} & & W'_n(M) = (M^{\otimes p^n})^{\sigma} / p\text{-norms} \arrow[u, two heads]
\end{tikzcd}$$

$$\underline{1 + \tau + \dots + \tau^{p-1}}$$

$\mu^p = \mathrm{id}$ - the cyclic permutation of order p on $M^{\otimes p}$ $\rightsquigarrow$ induces $W_n(M^{\otimes p})$ ↺

Example

$p=2$ $\quad n=2$

invts / norms $= \mathbb{Z}/4 \cdot e_j^4 + \mathbb{Z}/2\left((e_j e_k)^2 + (e_k e_j)^2\right)$

$(e_j e_k)^2 - (e_k e_j)^2 \in \ker(V)$

$$\begin{tikzcd}
W_1(\underset{\mathbb{Z}^{\curvearrowright}_{\mu}}{M^{\otimes 2}}) \arrow[r] & W_2(M) \arrow[r] \arrow[dr, leftrightarrow] & W_1(M) \\
 & & W_1'(M) \arrow[u]
\end{tikzcd}$$

$= (1-\mu)(e_j e_k)^2$

sorting out monomials $e = e_{j_1} \cdots e_{j_4}$ by length of σ-orbit: 1, 2, 4

	$(M^{\otimes 2})^{\otimes 2}$	$M^{\otimes 4}$	$M^{\otimes 2}$
invts	$(e_j^2)^2$ $\quad (e_j e_k)^2$ $\quad$ rest	e_j^4 $\quad e_j e_k e_j e_k$ $\quad (e_j e_k)^2 + (e_k e_j)^2$ ~~and~~ norms$_4$	e_j^2 $\quad e_j e_k + e_k e_j$
norms	$\ldots (e_j e_k)^2 \bullet$ $\bullet (e_k e_j)^2$		$2e_j^2$ $\quad e_j e_k + e_k e_j$
invts / norms	$\in \ker(V)$	$4 \cdot e_j^4$ $\quad 2\left[(e_j e_k)^2 + (e_k e_j)^2\right]$ norms$_4$	$(\mathbb{Z}/2) \cdot e_j^2$

$$0 \to W_n(M^{\otimes p})_{C_p} \xrightarrow{V} W_{n+1}(M) \longrightarrow W_n(M) \longrightarrow 0$$

short exact

$$C_p \circlearrowright M^{\otimes p}$$

What does this give when $M = A^{\natural}_m = A^{\otimes m+1}$?

$$0 \to W_n\left((A^{\natural}_m)^{\otimes p}\right)_{C_p} \longrightarrow W_{n+1}(A)^{\natural}_m \longrightarrow W_n(A)^{\natural}_m \to 0$$

$$W_n\left(\left(A_m^\natural\right)^{\otimes p}\right) \cong i_p^*\left(W_n(A)^\natural\right)_m$$

$$\begin{tikzcd}
 & \Lambda_p^{op} \arrow[r, "i_p"] \arrow[d, "\pi_p"'] & \Lambda^{op} \\
 & \Lambda^{op} &
\end{tikzcd}$$

σ (w/ α) $\mapsto$ id

$\boxed{\sigma = \tau^{n+1},\quad \sigma^p = id}$

under $i_p : \Lambda_p^{op} \longrightarrow \Lambda^{op}$ (p^{th} subdivision)

$$\Big[\underbrace{i_p^* W_n(A)^\natural}_{\Lambda_p^{op}\text{-mod}}\Big]_\sigma \longrightarrow \overset{\Lambda^{op}\text{-mod}}{W_{n+1}(A)^\natural} \longrightarrow \overset{\Lambda^{op}\text{-mod}}{W_n(A)^\natural} \to 0$$

(yes, commute with action of morphisms in Λ^{op})

$$\begin{tikzcd} \Lambda_p^{op} \arrow[r, "i_p"'] \arrow[d] & \Lambda^{op} \\ \Lambda^{op} & \end{tikzcd}$$

a short exact sequence of cyclic abelian grps

$$0 \to \pi_{p!}\, i_p^* W_n(A)^q \longrightarrow W_{n+1}(A)^q \longrightarrow W_n(A)^q \to 0$$

taking coinvariants of $C_p = \langle \sigma \rangle$ (or $\langle \alpha \rangle$)