NC HKR and NC Cartier (iso)morphism

Cartier: A/k commutative

$$\Omega^{\bullet}_{A/k} \longrightarrow H^{\bullet}_{dR}(A/k)$$

$$C^{-1}$$

$$a \longmapsto a^p$$

$$db \longmapsto b^{p-1}db$$

Isomorphism when A smooth

HKR: $C_{\bullet}(A) = A \otimes \bar{A}^{\otimes \bullet}, b, B$

obs.

1) $(C_\cdot(A), b)$ is a diff graded algebra with shuffle product:

b is a derivation

$$(a_0 \otimes \ldots \otimes a_m) \cup (b_0 \otimes \ldots \otimes b_n) =$$

$$= \sum \pm a_0 b_0 \otimes \ldots \otimes a_j \otimes \ldots \otimes b_k \otimes \ldots$$

the order of each group $a_1, \ldots, a_m$; $b_1, \ldots, b_n$ is preserved (i.e. **shuffle perms**)

Defines $C_\cdot(A) \otimes C_\cdot(B) \to C_\cdot(A \otimes B)$

$a_i = a_i \otimes 1$; $b_j = 1 \otimes b_j$

Now, $B: C_\cdot \longrightarrow C_{\cdot+1}$ a derivation, but only at the level of homology:

$$B(a_0) \smile b_0 + a_0 \smile B(b_0) - B(a_0 \smile b_0) =$$

$$= (1 \otimes a_0) \smile b_0 + a_0 \smile (1 \otimes b_0) - 1 \otimes a_0 b_0$$

$$= b_0 \otimes a_0 + a_0 \otimes b_0 - 1 \otimes a_0 b_0 = b(1 \otimes a_0 \otimes b_0)$$

$$\begin{array}{c} Ba \\ \shortparallel \\ 1 \otimes a \end{array}$$

$$\underbrace{Ba \cup b +}_{(-1)^{|a|}} a \cup Bb - B(a \cup b) = \text{homotopic to zero}$$

homotopy: $(a_0 \otimes \ldots \otimes a_m) \otimes (b_0 \otimes \ldots \otimes b_n)$

$$\mathrm{I}$$

$u \cdot \sum \pm 1 \otimes \ldots \otimes \underset{=}{a_0} \otimes \ldots \otimes \underset{=}{b_0} \otimes \ldots$

all perms of $a_0, \ldots, a_m, b_0, \ldots, b_n$ such that:

1) The cyclic order of both groups a_0, — AND b_0 — preserved.

2) a_0 left of b_0

Recall $CC^-_\cdot(A) = C_\cdot(A)[[u]]$, $b+uB$

How about associativity?

$$m_n(a_0 \otimes \dots, b_0 \otimes \dots, \dots, x_0 \otimes \dots)$$

$$\|$$

$$u \cdot \sum \pm (1 \otimes \dots \otimes a_0 \otimes \dots \otimes b_0 \otimes \dots \otimes x_0 \otimes \dots)$$

(cyclic order of each group $a_0 \dots$, $b_0 \dots$, etc., $x_0 \dots$) preserved

$a_0 \to b_0 \to \dots \to x_0$

$n=2$

$$+ (a_0 \otimes \dots) \cup (b_0 \otimes \dots)$$

$CC^-_\cdot(A)$ $b+uB$ becomes an A_∞ algebra. (A comm.)

Exterior product:

$$CC_\cdot^-(A_1)\otimes\ldots\otimes CC_\cdot^-(A_n)\to$$

$$\to CC_\cdot^-(A_1\otimes\ldots\otimes A_n)$$

formulas as above; satisfy A_∞ relations. Similarly: $E_1,\ldots,E_n$ are cyclic objects: $E_1\otimes\ldots\otimes E_n$ the (exterior) $\otimes$: $[m]\mapsto \underbrace{E_1[m]\otimes\ldots\otimes E_n[m]}_{\text{diag action of } \Lambda^{op}}$

Cyclic objects $\longrightarrow$ Mixed complexes

$\boxtimes$ $\otimes$

"functor with higher homotopies"

? Morphism of MONOIDAL, not SYMMETRIC MONOIDAL, categories.

(highly asymmetric).

To take symm mono strre into account:

$$\begin{tikzcd}
\Lambda^{op}\text{-}k\text{-mods} \arrow[r, "\boxed{S^1_{fil}}"] & \text{Simplicial mods over cosimplicial coalg } \mathcal{O}(\mathrm{Fix}(F)) \arrow[d, "gr"]
\end{tikzcd}$$

MRT
Raksit

$\hookrightarrow$ $F: W(\cdot) \to W(\cdot)$

S.M.C. vs

Mixed cplexes $\otimes$ $\longleftrightarrow$ Simplicial mods over cosimplicial coalg $\mathcal{O}(\ker(F))$ $\otimes$

Anyway: $B : HH_\bullet(A) \to HH_{\bullet+1}(A)$

$HH_\bullet(A), \cup, B$ IS

a dga; $A \xrightarrow{\sim} HH^0 \subset HH(A)$

Enough to check for $A = k[[x_1, \dots, x_n]]$ by computing $HH_\bullet$ by Koszul

$$\Omega^\bullet_{A/k} \xrightarrow{\exists!} HH_\bullet(A)$$

$$d_{DR} \qquad\qquad B$$

$$a \longmapsto a$$

$$da \longmapsto Ba$$

For smooth A/k: ISOM.

NC Cartier?

$HH_\bullet \xrightarrow{\sim}$ cohomology of B

But in what sense?

Ω $\qquad$ d

Q cyclic k-mod $E_\bullet$
(say, $\mathbb{Z}$-mod)

$\mathbb{L}$ $\qquad$ the const cocyclic mod

Cyclic hom. $HC_\bullet(E_\bullet)$: $\quad E_\bullet \otimes_\Lambda \mathbb{Z}_q$

Found a resolution of $\mathbb{Z}_q$ and tensored by $E_\bullet$; got the standard $b, b', N, 1-\tau$ double complex:

$$\begin{tikzcd}
\cdots \arrow[r, "N"] & E_\bullet \arrow[r, "1-\tau"] \arrow[d, "b'"] & E_\bullet \arrow[r, "N"] \arrow[d, "b"] & E_\bullet \arrow[r, "1-\tau"] \arrow[d, "b'"] & E_\bullet \arrow[r] \arrow[d, "b"] & \cdots \\
 & {} & {} & {} & {} &
\end{tikzcd}$$

$\tau = (-1)^n t$ on E_{n+1}

$N = 1+\tau+\ldots+\tau^n$

Will obtain this by applying the functor to a res. of $E_\bullet$

Find a resolution of $E_\cdot$ s.t.

$[n] \longrightarrow$ free $\mathbb{Z}[C_{n+1}]$-module;

apply $\underbrace{\left(?/\mathrm{im}(1-\tau), b \right)}$ $\longrightarrow$ get the standard $\underbrace{b, b', N, 1-\tau \text{ cplx.}}$

$E_\cdot = \mathbb{Z}^{?} \quad [n] \mapsto \mathbb{Z}$

Recall: $\Lambda^{op}([n],[m]) = \begin{cases} \left(\text{nondecreasing } \mathbb{Z}\text{-equivar } \frac{1}{n+1}\mathbb{Z} \to \frac{1}{m+1}\mathbb{Z} \right) \\ \text{quotient by } \mathbb{Z} \end{cases}$

$E_\bullet / \mathrm{im}(1-\tau),\ b$

$$\begin{tikzcd} & & & {} \arrow[d, "b'"] & {} \arrow[d, "b"] \\ {} \arrow[r, "N"'] & {} \arrow[r, "1-\tau"'] & {} \arrow[r, "N"'] & {} \arrow[r, "1-\tau"'] & {} \end{tikzcd}$$

Case 1,2: quasi-isom

In the following two cases, $\underbrace{\longrightarrow\longrightarrow\longrightarrow}_{\text{horizontal}}$ is acyclic:

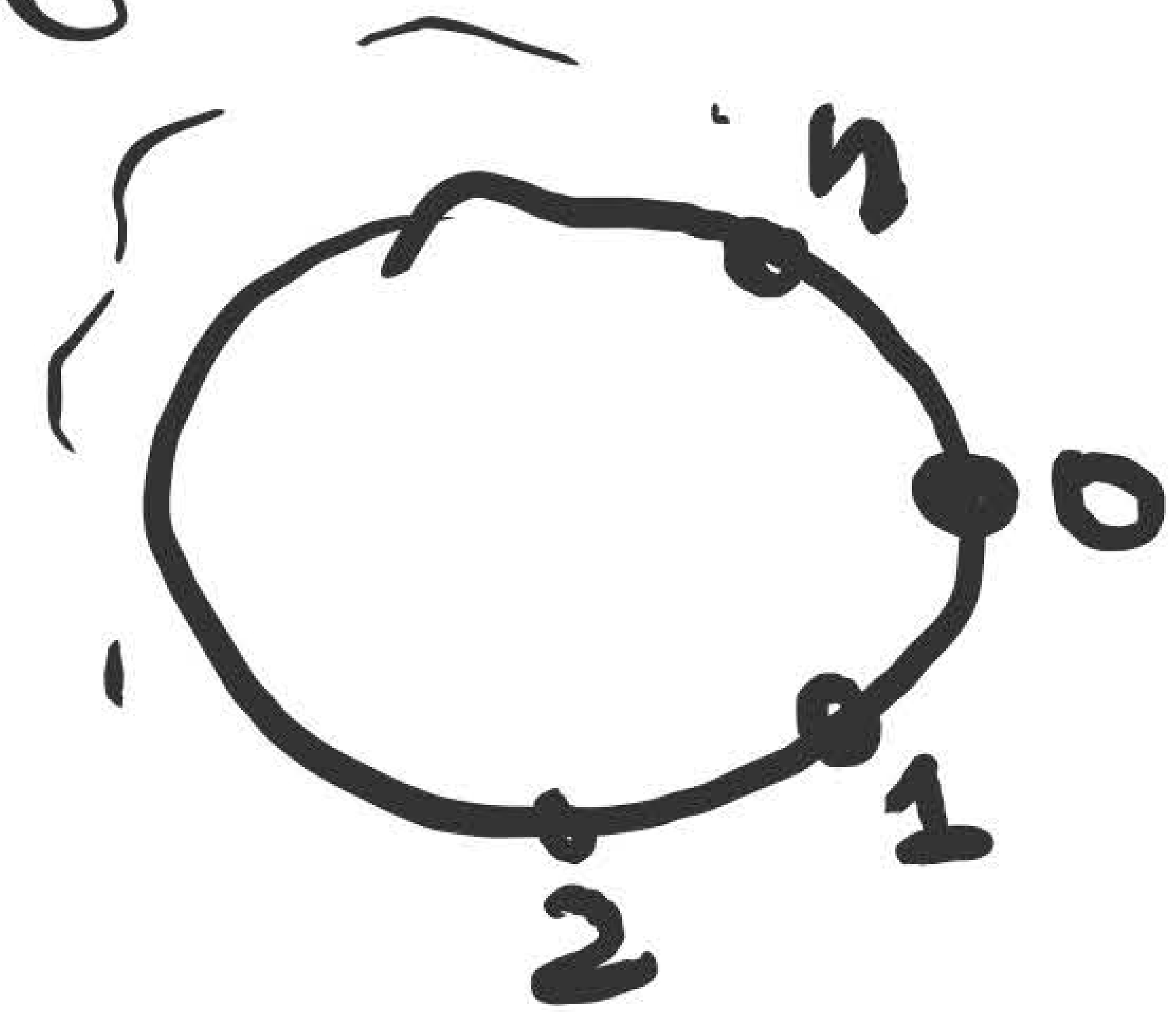
1) Over $\mathbb{Q}$

$\mathrm{Aut}_{\Delta^{op}}([n]) = C_{n+1} = \mathbb{Z}/n+1$.

$\boxed{\text{2) } E_n \text{ is a free mod over } \mathbb{Z}[C_{n+1}]}$

Triangulations of S^1:

T_n:



$C_\bullet(S^1, T_n)$ $\quad \bullet = 0, 1$

chain complex of triangulation T_n.

$\Lambda^{op}([n],[m]) = \{$clockwise nondecreasing cont. maps $(S^1, T^n) \to (S^1, T^m)$ of $\underline{\text{degree one}}$.

$C_1(S^1, T_n) \xrightarrow{\partial} C_0(S^1, T_n)$

$$
\begin{tikzcd}
C_1(S^1, T_n) \arrow[r, "\partial"] \arrow[d, "\cong"', no head] & C_0(S^1, T_n) \arrow[r, "\varepsilon"] \arrow[d, equal] & \mathbb{Z} \arrow[d, "{[S^1]}"] \\
\bigoplus_{j=0}^{n} \mathbb{Z}\cdot(j, j+1) \arrow[r] \arrow[d, no head] & \bigoplus_{j=0}^{n} \mathbb{Z}\cdot(j) \arrow[d, no head] & C_1(S^1, T_n) \\
\mathbb{Z}[C_{n+1}] \arrow[r, "1-t"] & \mathbb{Z}[C_{n+1}] &
\end{tikzcd}
$$

$[j] \mapsto 1$

$1 \mapsto \sum_j (j, j+1) \in C_1(S^1, T_n)$

$j+1 \mapsfrom j$

$j \mapsto j$

on the nose: Λ^{op}-invariant

$$\begin{array}{ccccccc}
C_1(S^1, T_n) & \xrightarrow{\varepsilon_{S^1}} & C_0(S^1, T_n) & \xrightarrow{\partial} & \boxed{C_1(S^1, T_n) \xrightarrow[\mathbb{Z}]{\varepsilon_{[S^1]}} C_0(S^1, T_n)} \\
\Big|\wr & & \Big|\wr & & \Big|\wr \quad \mathbb{K}_1(E.) \to \mathbb{K}_0(E.) \\
\mathbb{Z}[C_{n+1}] & \xrightarrow[N]{} & \mathbb{Z}[C_{n+1}] & \xrightarrow[1-t]{} & \mathbb{Z}[C_{n+1}]
\end{array}$$

$\mathbb{K}_\bullet(E)$

Get a complex of cyclic objects

in k-mod

$$\xrightarrow{B} \mathbb{K}_\bullet(E) \xrightarrow{B} \mathbb{K}_\bullet(E) \xrightarrow{B} \mathbb{K}_\bullet(E)$$

when $E = \mathbb{Z}^4$! by definition

length 2

Complex of cyclic modules

$$\cdots \xrightarrow{B} \mathbb{K}_\cdot(E) \xrightarrow{B} \mathbb{K}_\cdot(E) \xrightarrow{B} \cdots$$

$$\mathbb{K}_\cdot(E) \cong E_\cdot \boxtimes \mathbb{K}_\cdot(\mathbb{Z}^{\natural})$$

① This is a resolution of E.

② $HC_\cdot(\mathbb{K}_\cdot(E)) \simeq HH_\cdot(E_\cdot)$

$\mathbb{K}_\cdot(E)$: complex of Λ^{op}-k-mods

$$\begin{array}{lcc}
[2] \mapsto & \mathbb{Z}[C_3] \otimes E_2 & \xrightarrow{\partial} \mathbb{Z}[C_3] \otimes E_2 \\
[1] \mapsto & \mathbb{Z}[C_2] \otimes E_1 & \xrightarrow{\partial} \mathbb{Z}[C_2] \otimes E_1 \\
[0] \mapsto & \mathbb{Z}[C_1] \otimes E_0 & \xrightarrow{\partial} \mathbb{Z}[C_1] \otimes E_0
\end{array}$$

Λ^{op}-action...

Applying $?/\mathrm{im}(1-\tau)$, b to each

$\mathbb{K}_\cdot(E)$:

$$
\begin{tikzcd}
\vdots \arrow[r, "1-\tau"'] \arrow[d, "-b'"'] & \vdots \arrow[d, "b"] \\
E_1 \arrow[r, "1-\tau"'] \arrow[d, "-b'"'] & E_1 \arrow[d, "b"] \\
E_0 \arrow[r, "1-\tau"'] & E_0
\end{tikzcd}
$$

Computes HC, since $[n] \mapsto$ free $\mathbb{Z}[C_n]$-mod

b' acyclic

$\Rightarrow$ quasi-isom to $C_\cdot(E)$.

Next: recall $[n] \longmapsto [(n+1)\ell - 1]$

$$
\begin{tikzcd}
\Lambda_\ell^{op} \arrow[r, "i_\ell"] \arrow[d, "\pi_\ell"'] & \Lambda^{op} \\
\Lambda^{op} &
\end{tikzcd}
$$

$\sigma = t^{n+1} : [n] \circlearrowleft$

$$
\begin{array}{c}
\sigma^\ell = \mathrm{id} \\
\big\downarrow \\
\mathrm{id}
\end{array}
$$

$i_\ell^* A^\natural \longleftarrow\!\vert\; A^\natural$

$[n] \rightsquigarrow A^{\otimes n+1}$

$$i_\ell^* A^\natural = \left(A^{\otimes \ell}\right)^\natural_\alpha$$

$\alpha : A^{\otimes \ell} \circlearrowleft$ cyclic permutation

$\pi_{\ell !} = \mathbb{Z} \otimes$?

$\mathbb{Z}[C_\ell] = \mathbb{Z}[\sigma]$

<u>Claim</u>

$$HH_\cdot(i_\ell^* E) \simeq HH_\cdot(E)$$

as ℓ-cyclic mod

as cyclic mod

$$HC_\cdot(i_\ell^* E) \simeq HC_\cdot(E)$$

<u>Proof</u> (see e.g. [NT]). also [Nik.-Sch. app. 2]?

$E_\cdot = A^\natural$: for $HH_\cdot$:

$P_\cdot \xrightarrow{\sim} A$ free bimod resolution of A.

$$\mathrm{Proj}_{p!} = \bigotimes_{\Lambda_p^{op}} \mathbb{Z}_q$$

$$\begin{tikzcd}
\Lambda_p^{op}\text{-mod} \arrow[d, "\pi_{p!}"] \arrow[dd, bend right] & & \Lambda^{op}\text{-mod} \arrow[ll, "i_p^*"'] \arrow[dll, "\mathrm{Proj}_!"] \\
\Lambda^{op}\text{-mod} \arrow[d, "\mathrm{Proj}_! = \bigotimes_{\Lambda^{op}} \mathbb{Z}_q"] & & \\
k\text{-mod} & &
\end{tikzcd}$$

E

Same as $\leftarrow$ $\downarrow$ by above.

$$\underbrace{\mathbb{L}\pi_{p!}\left(i_p^* E\right)}_{\text{\underline{another} cplx of cyclic objects}}$$

Same HH, HC.

$$\begin{tikzcd}
\Lambda^{op} \arrow[d, "\pi_p"] \arrow[dd, bend right, "\mathrm{proj}"'] \\
\Lambda \arrow[d, "\mathrm{proj}"] \\
\mathrm{pt}
\end{tikzcd}$$

Therefore:

$$\underbrace{\pi_{p!}\,\mathbb{K}_{\bullet}(i_p^* A^\natural)}_{\text{Another complex of cyclic objects; its } HH_\cdot,\ HC_\cdot \text{ same as for } A^\natural.} \qquad E \quad \text{e.g. } A^\natural$$

$$\cdots \xrightarrow[\tilde{B}]{} \pi_{p!}\mathbb{K}_\bullet(i_p^* A^q) \xrightarrow[\tilde{B}]{} \pi_{p!}\mathbb{K}_\bullet(i_p^* A^q) \xrightarrow[\tilde{B}]{}$$

i.e. induced by B,

or $\pi_{p!}B\ldots$

$(\ell = 3)$

<u>forgot</u>: $\mathbb{K}_\bullet$ for p-cyclic modules?

for $\mathbb{Z}^\natural_\bullet$: it is

$$C_0(\,) \xrightarrow[\varepsilon]{} \xrightarrow[[S^1]]{} C_1(S^1, T_{\ell(n+1)-1}) \longrightarrow C_0(\,)$$

$\mathbb{K}_\bullet(\mathbb{Z}^\natural)\otimes E_\bullet$

Recall: M free k-module, $\mathrm{char}(k)=p$.

$$M \xrightarrow{\sim} (M^{\otimes p})^{\sigma} / \text{norms}$$

$$m \longmapsto m^{\otimes p}$$

σ acting on $M^{\otimes p}$

$\text{norms} = \mathrm{im}(1+\sigma+\ldots+\sigma^{p-1})$

$$(M^{\otimes p})^{\sigma}/\text{norms} = (M^{\otimes p})^{tC_p}$$

Tate cohom

Pf: basis of $M^{\otimes p}$:

monomials $e_{j_1} \otimes \ldots \otimes e_{j_p}$

C_p-orbits: length one or p.

$?^{tC_p} = 0$.

$$\xrightarrow[\widetilde{B}]{} \pi_{p!}\, \mathbb{K}.(i_p^* A^q) \xrightarrow[\widetilde{B}]{} \pi_{p!}\, \mathbb{K}.(i_p^* A^q) \xrightarrow[\widetilde{B}]{} \cdots$$

$$\xrightarrow[N_t]{} \mathbb{Z}[t] \otimes_{\mathbb{Z}[\sigma]} A^{\otimes (n+1)p} \xrightarrow[1-t]{} \mathbb{Z}[t] \otimes A^{\otimes (n+1)p} \xrightarrow[N_t]{}$$

$t^{(n+1)p} = 1;\ \sigma = t^{n+1}$

The cohomology of $\widetilde{B}$:

(the complex with induced different

$$\operatorname{coker}(N_t) \xrightarrow{1-t} \ker(N_t)$$

Compare to $\mathbb{K}.\left((i_p^* A^q)^{tC_p}\right)$:

$$\check{H}^0\left(C_p, A^{\otimes (n+1)p}\right) \xrightarrow{1-\bar{t}} \check{H}^0\left(C_p, A^{\otimes (n+1)p}\right)$$

$C_p = \langle \sigma \rangle$; $\bar{t}^{\,n+1} = \mathrm{id}$; induced by t

The basis of $A^{\otimes (n+1)p}$: monomials

$e_{j_1} \otimes \cdots \otimes e_{j_{(n+1)p}}$. Orbits under

the action of $\langle t \rangle$:

a) σ acts trivially.

b) σ acts freely.

If $n+1 = d \cdot e$, then these orbits are:

a) $C_{(n+1)p} / C_{dp}$ $\quad (\approx C_e)$

b) $C_{(n+1)p} / C_d$ $\quad (\approx C_{pe})$

In case b): both

$$\mathbb{Z}[C_{(n+1)p}] \underset{\mathbb{Z}[C_p]}{\otimes} E, \; N_t, \; 1-t \quad \text{and}$$

$$\mathbb{Z}[C_{n+1}] \otimes E^{tC_p}, \; N_{\bar{t}}, \; 1-\bar{t}$$

are acyclic. In case a):

we have

$$N_t = 0 \longrightarrow \mathbb{Z}[C_{dep}] \otimes_{\mathbb{Z}[C_{ep}]} \mathbb{Z} \xrightarrow{1-t} \cdots$$

identical to:

$$\boxed{\mathbb{Z}[C_{de}] \otimes_{\mathbb{Z}[C_e]} \mathbb{Z} \xrightarrow{1-\bar{t}} \cdots}$$

Get Kaledin's NC Cartier:

$$\mathbb{K}_\bullet(A^\natural) \xrightarrow{\sim} H^0\left(\xrightarrow{\tilde{B}} \pi_{p!}\mathbb{K}(i_p^* A^\natural) \xrightarrow{B}\right)$$

} more there...

$$A^{h} \xrightarrow{\sim} \left(i_p^* A^{h}\right)^{tC_p}$$

yes, commutes with the action of Λ^{op} on both sides.

Λ_p^{op} acts; induces the action of $\Lambda^{op} = \Lambda_p^{op}/(\sigma)$

Compare that to the cohomology of

$$\xrightarrow{\widetilde{B}} \pi_{p!}\left(i_p^* A^{h}\right) \xrightarrow{\widetilde{B}} \cdots \; ?$$

$$\underbrace{\qquad}_{\text{same}} \xrightarrow{N_{big}} \underbrace{\mathbb{Z}[C_{(n+1)p}] \otimes_{\mathbb{Z}[C_p]} A^{\otimes (n+1)p}} \xrightarrow{1-t} \underbrace{\qquad}_{\text{same}}$$

$$N_{big} = 1 + t + t^2 + \dots + t^{(n+1)p-1}$$

Apply $\pi_{p!} = \qquad \mathbb{K}.(i_p^* A^4)$

$= \mathbb{Z} \otimes_{\mathbb{Z}[C_p]} -$

By sort of Schapiro's Lemma: (?)

$$\xrightarrow{N_\sigma} A^{\otimes p(n+1)} \xrightarrow{1-\sigma} \dots$$

Note:

$$M^{\otimes p} \xrightarrow{N_\sigma} M^{\otimes p} \xrightarrow{1-\sigma} M^{\otimes p} \xrightarrow{N_\sigma} \cdots$$

cohomology is same in even and odd degree (i.e. $M^{\otimes p}$ is a tight $\mathbb{Z}[C_p]$-module)

Claim (need a crisper proof): The cohomology of the complex of cyclic k-mods

$$\cdots \xrightarrow{\tilde{B}} \pi_{p!}\left(i_p^* A^\natural\right) \xrightarrow{\tilde{B}} \pi_{p!}\left(i_p^* A^\natural\right) \xrightarrow{\tilde{B}} \cdots$$

is $\mathbb{K}_\cdot\left(\left(i_p^* A^\natural\right)^{tC_p}\right)$.

$$\uparrow ? \text{ frob}$$

$$\mathbb{K}_\cdot\left(A^\natural\right)$$

This is Kaledin's NC Cartier.

Finally: What kind of structure is there on Hochschild $\underbrace{}$ complexes?

(chain/cochain

Known: $\forall$ alg A, B

$C^\bullet(A,B)$ is actually a dg category; objects are morphisms $A \xrightarrow{f} B$ (or nice bimodules).

Morally: 2-category

$C^\cdot(A,B) \otimes C^\cdot(B,C)$

$\downarrow$ "associative"

$C^\cdot(A,C)$ actually A_∞...

$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet$

$\bullet \xrightarrow{gf} \bullet$

$$A \otimes C^\cdot(A,B) \to B$$

$$C^\cdot(A,B) \otimes C^\cdot(B,C) \to C^\cdot(A,C)$$

"all assoc. up to higher homotopies"

A_∞ functors

$C^\cdot(A,B) = \{$Hochschild cochains$\}$

= "Category in categories"

object $A \longmapsto$ algebra ($\Rightarrow$ category) A

"Module in categories"

Add Hochschild chains:

$C_\bullet(A,A)$ is a module over dg category $C^\bullet(A,A)$

$C_\bullet(A,A) = TR_A$

or: a functor $C^\bullet(A,A) \to k\text{-mod}$

$$f \longmapsto C_\bullet(A, {}_fA)$$

$$\begin{tikzcd}
C^\bullet(A,B)\otimes C^\bullet(B,A) \arrow[r, "\simeq"] \arrow[d, "m_{ABA}"'] & C^\bullet(B,A)\otimes C^\bullet(A,B) \arrow[d, "m_{BAB}"] \\
C^\bullet(A,A) \arrow[dr, "TR(A)"'] & C^\bullet(B,B) \arrow[d, "TR(B)"] \\
& k\text{-mod}
\end{tikzcd}$$

$$\tau_{AB} : m^*_{ABA} TR_A \xleftarrow{\sim} m^*_{BAB} TR_B$$

isomorphism of functors

such that for A, B, C ....

Similar to what appears in Alexander-Čech, ...

(up to all higher homotopies)

[NT] ← neater: $C^\circ(A,B)$ - "category in cocategories, etc....

So: $C^{\cdot}(A,B)$ "categ in categs"

$C_{\cdot}(A,A)$ "trace functor"

so: richer structure on chains

What about: $C_{\cdot}(A^{\otimes \ell},_{\alpha} A^{\otimes \ell}) \simeq C_{\cdot}(A,A)$

$C^{\cdot}(A^{\otimes \ell},_{\alpha} A^{\otimes \ell})$? — NOT

Another part of the structure:

Kontsevich–Vlassopoulos bracket...

Extends to K-V bracket on $C^{\bullet+1}(A, \mathcal{B})$ or $\otimes_{\mathcal{B}}$

$C^{\bullet}(A,B)$ Hochschild cochains

= "categ in categ" $\Longrightarrow$ brace struct.

Gerstenhaber bracket on

$C^{\bullet+1}(A, A)$ ($B = A$, $f = g = \mathrm{id}$...)

$C_{\bullet}(A, A)$ "trace functor"

$\Downarrow$

Cyclic differential B, Lie derivative L_{π}, etc.

Common structure including:

- Cat in cats
- Trace functor
- Gerstenhaber, calculus, etc
- Kontsevich-Vlassopoulos
- Frobenius, Cartier...

?

replace HH. by HHW.

And also: what serves as $C^\bullet(A,A)$ if you