Plan: live lectures: today + Wednesday

31 | Prisms; examples; Čech/Alex.
32 | complexes

30: will record: noncommutative Cartier isomorphism (Kaledin)

(sequel to: Deligne-Illusie degen.

as well as: to NC Witt vectors

maybe some more.

From last time: Have defined the cyclic objects $\mathbb{Z}_p$-modules $W_n(A)^\natural$ for any algebra A, prime p, and $n \geq 0$;

Thm (Kaledin) $HH\,W_n(A)^\natural$ is isom to DRW forms of A if A is smooth commutative / $\mathbb{F}_p$.

Relation of cyclic to DRW cohom

Defining a Cartier divisor; "locally principal" (p prime)

Prisms

a prism is: a δ-ring A; an ideal I; A is I-adically complete, and:

$$\boxed{p \in I + \phi_A(I) \cdot A}$$

Recall: a δ-ring is a ring with a δ-derivation $\delta : A \to A$ axioms designed so that:

$\delta(a+b) = \ldots$ $\delta(ab) = \ldots$

$$\phi_A(a) = a^p + p\delta(a) \in \mathrm{End}(A)$$

$\phi_A(I)$ not necessarily ideal

assume: $I = (d)$ for an element $d \in I$.

Then

$*$ $\quad I + \phi_A(I)\cdot A \ni p \iff \delta(d)$ is invertible in A.

d is a distinguished element of a δ-ring A.

Pf. $\quad a\cdot d + b\cdot\phi_A(d) = p \qquad ad + b(d^p + p\,\delta(d)) = p$

$\overset{?}{\Longrightarrow}$: obvious

$\underbrace{b\,\delta(d)}\,p + (ad + bd^p) = p \qquad (b = \delta(d)^{-1}$

$p(b\cdot\delta(d) - 1) + d\,(a + bd^{p-1})$ mod $d)$;

$$p(b\cdot\delta(d)-1)\pm(ad+bd^p)=0$$

If $\delta(d)$ is invertible mod d:

$\Downarrow$

$$b_0\cdot\delta(d)=1+dx$$

$$p(b_0\cdot\delta(d)-1)=pdx \overset{?}{=} ad+b_0d^p$$

$$pdx-b_0d^p=ad$$

$$a=px-b_0d^{p-1}$$

$\delta(d)$ invertible

$\Downarrow$ $\not\Uparrow$?

prism

Example ① A p-adically complete, no p-torsion,

$\phi_A : A \to A$ lifts Frobenius;

$I = (p)$

($\mathbb{Z}_p$-equivar ϕ_A)

Btw: $\delta(p)$ <u>is</u> invertible:

$$\phi_A(p) = (p^p - p)/p = p^{p-1} - 1$$

$\phi_A(q) = q^p$, $\mathbb{Z}_p$-lin.

② $A = \mathbb{Z}_p[[q-1]]$ q-formal parameter.

$$I = ([p]_q) = 1 + q + \dots + q^{p-1} = \frac{q^p - 1}{q - 1}$$

$[p]_q$ is distinguished. Indeed:

$$\delta([p]_q) = \frac{1}{p}\left([p]_q^p - \varphi_A([p]_q)\right) =$$

$$= \frac{1}{p}\left((1+q+\dots+q^{p-1})^p - (1+q^p+\dots+q^{p(p-1)})\right)$$

= polynomial in $\mathbb{Z}[q]$; at 1, equal to

$$\frac{1}{p}(p^p - p) = p^{p-1} - 1.$$

so: $p^{p-1} - 1 + (q-1)\dots$ invertible in $(p, q-1)$-adic completion.

③. $(A_{inf}, \ker(\theta))$

Generality: $E \supset \mathbb{Q}_p$ finite extension

$$\begin{array}{c} E \\ \cup \\ \pi \in \mathcal{O}_E \end{array} = \{\text{algebraic over } \mathbb{Z}_p\}$$

$$\boxed{\mathcal{O}_E/(\pi) \simeq \mathbb{F}_q}$$

Mostly: $E = \mathbb{Q}_p$; $\mathcal{O}_E = \mathbb{Z}_p$;

$\pi = p = q$

In <u>this</u> situation:

modification of Witt vectors

$$W_{n,\pi}(x_0,\dots,x_n) = X_0^{q^n} + \pi X_1^{q^{n-1}} + \dots + \pi^n X_n$$

based on <u>those</u> Witt vectors, all theory goes through; get:

$$W_{E,\pi}(A) \text{ for any } A$$

in part. $\mathbb{F}_q$-algebra A

no real dependence on π.

$\pi = p$, $E = \mathbb{Z}_p$:
usual Witt vectors.

A - an algebra over $\mathcal{O}_E$; complete π-adically

$A/\pi A$ an $\mathbb{F}_q$-algebra.

$$\varphi := \varphi_q = F_q : A/\pi A \to A/A\pi$$
$$a \longmapsto a^q$$

$\mathbb{F}_q$ - linear

Usually, very much not so.

$A/\pi A$ PERFECT if φ_q is an isom.

Solve $x^{q^n} = a$, $\forall a \ldots \forall n \ldots$

$$A^\flat = \varprojlim_{u} \left(A/_{\pi} A \right)$$

Usually $\pi = p$

$$\cdots \xrightarrow{\varphi_q} A/_{\pi} \xrightarrow{\varphi_q} A/_{\pi} \xrightarrow{\varphi_q} A/_{\pi}$$

$$a \longmapsto a^q$$

$A/_{\pi}A = \mathbb{F}_q[x]$:

$$A^\flat = \mathbb{F}_q$$

motivation:

$$E \subset \bar{E} \subset \hat{\bar{E}} =: \mathbb{C}$$

$$\mathcal{O}_{\mathbb{C}} \subset \mathbb{C}$$

integers

$$W\left(\left(\mathcal{O}/_{\pi}\right)^\flat\right) = A$$

$$A \supset \pi$$

rest of today: just look at it, and define I.

Observation A $\quad E$-algebra, π-adically complete.

$$A^\flat = \varprojlim_{n} \left(A/\pi A \right) \qquad \cdots \qquad A^\flat$$

$$a \longmapsto a^q \qquad\qquad\qquad \|$$

Lemma:
$$\varprojlim_{n} A \xrightarrow[\text{mod } \pi \ \sim]{\text{reduction}} \varprojlim_{n} \left(A/\pi A \right)$$

$$a \longmapsto a^q$$

(a monoid)

isomorphism

Proof inverse:

$$\begin{tikzcd}
 & \vdots \arrow[d, "I"] \\
\tilde{x}_2 \arrow[r, mapsto] & x_2 \arrow[d, "I"] \\
\tilde{x}_1 \arrow[r, mapsto] & x_1 \arrow[d, "I"] \\
\tilde{x}_0 \arrow[r, mapsto] & x_0 = x_1^q = x_2^{q^2}
\end{tikzcd}$$

A^b

here: $\pi \in I$!

$\tilde{x}_j$ lifting of each x_j

$x_2^q = x_1 \implies \tilde{x}_2^q \equiv \tilde{x}_1 \pmod{I}$

$\Downarrow$ LEMMA

$\tilde{x}_2^{q^2} \equiv \tilde{x}_0 \mod I^2$

$\boxed{\begin{array}{l} \underline{\text{Lemma}}:\ a \equiv b\ (I^k) \\ \qquad \Downarrow \\ a^q \equiv b^q\ (I^{k+1}) \end{array}}$

$y_k = \lim \tilde{x}_{n+k}^{q^n}$

$\downarrow$ ⋮

$y_1 = \lim\limits_{n\to\infty} \tilde{x}_{n+1}^{q^n}$

$\downarrow$

$y_0 = \lim\limits_{n\to\infty} \tilde{x}_n^{q^n}$

$\uparrow$

$x_n = x_0^{q^{-n}}$;

$\tilde{x}_n$ – a root "$\tilde{x}_0^{q^{-n}}$" mod I^n...

$$\begin{array}{ccc}
\vdots & & \vdots \\
 & & \downarrow \\
\tilde{x}_2 & \longmapsto & x_2 \\
 & & \downarrow \\
\tilde{x}_1 & \longmapsto & x_1 \\
 & & \downarrow \\
\tilde{x}_0 & \longmapsto & x_0
\end{array} \Bigg\} \in \varprojlim A/\pi$$

<u>some</u> lifting to A

$(y_j) \in \varprojlim A$

<u>and</u> the two are inverse.

The morphism θ.

$$\begin{tikzcd}
W(A) \arrow[r, "w_n"'] \arrow[d] & A \arrow[r] & A/\pi^{n+1} \\
W(A/\pi) \arrow[urr, bend right, "\exists!"'] & &
\end{tikzcd}$$

w_n = the n^{th} ghost component

$(a_0, a_1, a_2, \dots) \longmapsto a_0^p + \pi a_1$

~~out~~ mod π^2: only depends on a_j mod π

$$\begin{tikzcd}
W(A/\pi) \arrow[r, "w_n"'] \arrow[d, "F"'] & A/\pi^{n+1} \arrow[d, "\text{can proj}"] \\
W(A/\pi) \arrow[r, "w_{n-1}"'] & A/\pi^{n}
\end{tikzcd}$$

$w_n = \theta_n$

$w_{n-1} = \theta_{n-1}$

$$\boxed{\varprojlim_{F} W(A/\pi) \xrightarrow{\theta} A}$$

b/c Frob acts on ghost comps by shift.

$$
\begin{tikzcd}
\varprojlim\limits_{F} W_E(A/\pi) \arrow[r, "\theta"] & A \\
W_E\left(\varprojlim\limits_{\varphi} A/\pi\right) \arrow[u] \arrow[ur, "\theta"'] &
\end{tikzcd}
$$

apply that to:

$A = \mathcal{O}_{\mathbb{C}} = \{|x| \le 1\}$

$\cap$

$\mathbb{C} = \widehat{\overline{E}}$

$\mathbb{A} = W(\mathcal{O}_{\mathbb{C}}^{\flat})$

$I = \ker \theta$

(did for $\mathbb{F}_p$)

Using: For algebras $R/\mathbb{F}_q$ Frobenius on $W(R)$ is induced by $\varphi : R \to R$, $a \mapsto a^q$

$$\mathcal{A} = W(\mathcal{O}_{\mathbb{C}}^{b})$$

$\delta(d)$ invertible; in fact $\delta([\pi^b]) = 0$

$$I = \ker(\theta : W(\mathcal{O}_{\mathbb{C}}^{b}) \to \mathcal{O}_{\mathbb{C}})$$

$\delta(a+b) = \ldots$

$$d = [\pi^b] - \pi$$

$\boxed{\text{Teichmüller rep.}}$

Since $\mathbb{C}$ alg. closed: choose

$\lim \mathcal{O}_{\mathbb{C}}/\pi = \mathcal{O}_{\mathbb{C}}^{b} \sim$ in $\lim \mathcal{O}_{\mathbb{C}}$

$\pi^{1/p^2} \to \pi^{1/p} \to \pi$