Prismatic cohomology

<u>Prism</u>: $I \subset A$ $\quad A$ (I,p)-adically completed

I defines a Cartier divisor: locally principal...

<u>for us</u>: $I = (d)$

A is a δ-ring $(\Rightarrow$

$\Rightarrow \phi_A : A \to A \quad \phi_A(a) \equiv a^p \ (p)$

$$p \in I + \phi_A(I) \cdot A$$

$$\Updownarrow$$

$\delta(d)$ is invertible

(by def.: d is DISTINGUISHED)

Example: · $I=(p)$ δ-ring A p-adically complete

· $\mathbb{Z}_p[[q-1]]$, $I=([p]_q)$

· $A_{inf}(R)$, $I=\ker(\theta)$

...

a few remarks.

1) Rigidity of the ideal:

$$\begin{array}{ccc} A & \longrightarrow & C \\ \cup & & \cup \\ I & \longrightarrow & J \end{array} \qquad \boxed{J = I\cdot C}$$

Proof when I, J are principal:

$$\begin{array}{ccc} A & \longrightarrow & C \\ \cup & & \cup \\ (d) & \longrightarrow & (e) \end{array}$$

(say, inclusion)

f invertible $\Rightarrow (d) = (e)$

$$d = e \cdot f \qquad f \in C$$

invertible

$$\delta(d) = \delta(e \cdot f) = e^p \delta(f) + \delta(e) f^p + p\,\delta(e)\delta(f)$$

inv.

small

definition of δ-der; designed so that $\phi(a) = a^p + p\,\delta(a)$ is an endom.

f invertible

C is (p, e)-adically complete.

$\delta(e) \cdot f^p =$ inv + small = inv ✓

BTW: If d is distinguished and u is invertible then $u \cdot d$ is distinguished.

(Any generator of $I = (d)$ is distinguished).

$$\underbrace{\delta(ud)}_{\Downarrow} = \underbrace{u^p}_{\text{inv}} \cdot \underbrace{\delta(d)}_{\uparrow\ \text{inv}} + \delta(u) \cdot d^p + p\,\delta(\,)\delta(\,)$$

small

invertible.

Localization: A δ-derivation extends uniquely to localization A_S.

$$\delta(s^{-1}) = ? \qquad \delta(s\cdot s^{-1}) = \delta(1) = 0$$

$$\| $$

$$\underbrace{s^p\cdot\delta(s^{-1})}_{?} + \delta(s)\cdot s^{-p} + p\,\delta(s)\,\delta(s^{-1})$$

$$0 = \delta(s)\cdot s^{-p} + \underbrace{\delta(s^{-1})}_{?}\,\underbrace{(s^p + p\,\delta(s))}_{\|}$$

$$\delta(s^{-1})\cdot\phi_A(s)$$

$$\delta(s^{-1}) = -\,\delta(s)\cdot s^{-p}\cdot\underbrace{\phi_A(s)^{-1}}_{\text{invertible in localiz.}}$$

$$\boxed{\begin{array}{l} D(s^{-1}) = -s^{-2}D(s) \\ \text{for usual} \\ \text{derivation}\end{array}}$$

b/c $= s^p + p\ldots$

δ-der and pd structures.

1). In a δ-ring: (say, no torsion)

$z \in A$ $\qquad \dfrac{z^p}{p!} \in A \Rightarrow \dfrac{z^n}{n!} \in A, \forall n.$

<u>Proof.</u> 1) Observe: enough to check for $n = p^k$, $k = 2, 3, \ldots$

2) check here for $k=2$:

$\dfrac{z^{p^2}}{p^2!} \in A?$ $\qquad v_p(p^2!) = p+1$

i.e. $p^2! = p^{p+1} \cdot \underset{\substack{\uparrow \\ \text{unit in } \mathbb{Z}_p}}{\epsilon}$

btw

$\dfrac{z^p}{p!} \in A \Leftrightarrow \dfrac{z^p}{p} \in A$

$v_p(p!) = p$

Recall: $\nu_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \dots$

enough to check: $\nu_p(n!) \leq n$, $\forall n$

for $n = p^k$

$\nu_p(p!) = 1$, $\nu_p(p^2!) = p+1, \dots$

$$\frac{z^{p^2}}{p^{p+1}} \in A \overset{?}{\Longleftarrow} \frac{z^p}{p} \in A$$

work inside $A[p^{-1}]$

$$\delta\left(\frac{z^p}{p}\right) = \frac{1}{p}\left(\left(\frac{z^p}{p}\right)^p - \phi_A\left(\frac{z^p}{p}\right)\right)$$

$$= \frac{z^{p^2}}{p^{p+1}} - \frac{\phi_A(z)}{p^2}$$

$$\delta\left(\frac{z^p}{p}\right) = \frac{z^{p^2}}{p^{p+1}} - \frac{1}{p^2}\phi(z)^p$$

$$= \underset{\in A}{\frac{z^{p^2}}{p^{p+1}}} - \underbrace{\frac{1}{p^2}\left(z^p + p\,\delta(z)\right)^p}$$

$$\frac{1}{p^2} z^{p^2} + \frac{p^2}{p^2} \cdots$$

$$\frac{1}{p^2} z^{p^2} = p^{p-2}\Big(\underset{\in A}{\frac{z^p}{p}}\Big)^p$$

$\gamma_p(z) \in A \implies \gamma_n(x) \in A$, $\forall n$ (in a $\underline{\delta\text{-ring}}$) ✓

Also: Claim
(as with localization):
If you add $\gamma_n(x)$ to a δ-ring (say, no torsion), the δ-ring structure is inherited.

Sketch of proof. Trying to extend δ to $\gamma_n(x) = \frac{x^n}{n!}$:

Since no torsion : just extend the Frobenius ϕ_A.

$$\phi_A\left(\frac{x^n}{n!}\right) = \frac{\phi_A(x)^n}{n!} = \sum_{i=0}^{n} a_{i,n}\,\gamma_{pi}(x)\,\delta(x)^{n-i}$$

$$= \frac{(x^p + p\,\delta(x))^n}{n!} =$$

$$= \sum_{i=0}^{n} \frac{1}{n!}\binom{n}{i} x^{pi}\, p^{n-i}\,\delta(x)^{n-i}$$

$$= \sum_{i=0}^{n} \frac{1}{\cancel{n!}}\,\frac{\cancel{n!}\,p^{n-i}}{i!\,(n-i)!}\,(pi)! \cdot \frac{x^{pi}}{(pi)!}\cdot\delta(x)^{n-i}$$

$$= \sum_{i=0}^{n} \frac{(pi)!}{i!}\cdot\frac{p^{n-i}}{(n-i)!}\cdot\gamma_{pi}(x)\,\delta(x)^{n-i}$$

$\leftarrow$ in $\mathbb{Z}_p$

Corollary Assume that

(A, I) ← "ground ring", e.g. $\mathbb{Z}_p, (p)$

(B, J)

is a torsion-free δ-ring and $J = (f_1, \dots, f_r)$ an ideal in A.

Let $B\left\{\frac{\phi(f_1)}{p}, \dots, \frac{\phi(f_r)}{p}\right\}$ be the smallest δ-subring of $\cancel{A}B\left[\frac{1}{p}\right]$ containing $\frac{\phi(f_1)}{p}, \dots, \frac{\phi(f_r)}{p}$. Then $\cancel{A}B\left\{\frac{\phi(f_1)}{p}, \dots, \frac{\phi(f_r)}{p}\right\}$ coincides with rel pd envelope of I,

PROVIDED:

$(f_1, \dots, f_r)$ is a REGULAR SEQUENCE mod p.

(cannot freely operate inside $A[\frac{1}{p}]$, etc., if in general).

$B\left\{\frac{\phi(f_1)}{p}, \dots, \frac{\phi(f_r)}{p}\right\}$ is the smallest δ-ring /sub containing

$$\frac{\phi(f_j)}{p} = \frac{f_j^p}{p} + \delta(f_j)$$

and the pd envelope is the smallest subring cont.

BUT: δ extends uniquely when add pd...

Alexander-Čech complex for prismatic cohomology.

Given: a prism A, I;
an A/I-algebra R.

e.g. $A, I = \mathbb{Z}_p, p : \ldots$

$$
\begin{tikzcd}
I \arrow[r] \arrow[d, hook] & J \arrow[d, hook] \\
A \arrow[r] \arrow[ddr] & B \arrow[d, phantom, "\perp"] \\
& {} \\
& R
\end{tikzcd}
$$

formally: free / A, comm

note: free δ-ring : $A\{x_1, \ldots, x_n\} = A[x_j ; \delta x_j ; \delta^2 x_j ; \ldots]$

NOW: free δ-ring over $A \Rightarrow$ free com rg!

Now:

$$\begin{tikzcd}[column sep=large]
 & & R & & \\
A \arrow[r] & B \arrow[r, shift left] \arrow[r, shift right] \arrow[urr, bend left] & B^{\otimes_A 2} \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] \arrow[u] & B^{\otimes_A 3} \arrow[ul, bend right] & \\
 & J \arrow[u, phantom, "\subset" rotate=90] & J^{(2)} \arrow[u, phantom, "\subset" rotate=90] \arrow[r] & J^{(3)} \arrow[u, phantom, "\subset" rotate=90] &
\end{tikzcd}$$

The Alexander-Čech as usual.

$$\begin{tikzcd}[column sep=huge]
B^{\otimes n} \arrow[r] & C^{(n)} \\
J^{(n)} \arrow[u, phantom, "\subset" rotate=90] \arrow[r] & IC^{(n)} \arrow[u, phantom, "\subset" rotate=90] \\
 & \arrow[u]
\end{tikzcd}$$

$$\boxed{H^{\bullet}_{\Delta}\left(\mathrm{Spf}(R)/A\right)}$$

the universal prism

$$C^{(1)} \rightrightarrows C^{(2)} \, \substack{\rightarrow \\ \rightarrow \\ \rightarrow} \, C^{(3)} \, \substack{\rightarrow \\ \rightarrow \\ \rightarrow \\ \rightarrow} \cdots$$

get the complex

Comparison to crystalline:

$I \subset A$ (e.g. $A = \mathbb{Z}_p$)

$\shortparallel$

(p)

$\lambda = 1, \dots, n$

$B^{(n)} = A\{x_j^{(\lambda)}, \delta x_j^{(\lambda)}, \dots\}$

$\{\ \} =$ smallest δ-subring containing..

$\cup$

$J^{(n)} = (f_1, f_2, f_3, \dots)$

$\begin{array}{ccc} & & \\ \text{(crys)} & & \text{(prismatic)} \end{array}$

pd envelope of $J^{(n)}$:

$B^{(n)}\left\{\dfrac{\phi(f_j)}{p}\right\}$

universal prism

$B^{(n)}\left\{\dfrac{f_j}{p}\right\}$

This leads to:

$$\check{C}^{\bullet}_{crys}(\mathrm{Spf}(R)) \simeq \check{C}^{\bullet}_{prism}(\mathrm{Spf}(R)) \otimes^{L} A$$

$$\phi_A$$

$$\shortparallel$$

(roughly)

$$\phi_A^{*}\, \check{C}^{\bullet}_{prism}(\mathrm{Spf}(R))$$

will record² also:

lect. 30 (NC Cartier/Kaledin)