Witt vectors

A - comm. ring

$W(A)$ - a comm. ring

$\shortparallel$

$\{(a_0, a_1, \dots), \; a_j \in A\}$ $\qquad W(A) \longrightarrow A^\infty$

prod of rings A

$$gh(a_0, a_1, \dots) = (w_0(a_0), w_1(a_0, a_1), \dots)$$

$$w_n(a_0, \dots, a_n) = a_0^{p^n} + p a_1^{p^{n-1}} + \dots + p^n a_n$$

a morphism of rings

$V : W(A) \to W(A)$ - of ab grps $\qquad (a_0, a_1, \dots) \mapsto (0, a_0, a_1, a_2, \dots)$

$F : W(A) \to W(A)$ - of rings $\qquad \mapsto \dots$

$V, F : W(A) \to W(A)$ $\quad (x_0, x_1, x_2, \ldots) \mapsto (0, px_0, px_1, px_2, \ldots)$

$V : (a_0, a_1, \ldots) \mapsto (0, a_0, a_1, \ldots)$ $\longrightarrow$ of ab grps

$F : (w_0, w_1, w_2, \ldots) \mapsto (w_1, w_2, w_3, \ldots)$

1) morphism of $A^\infty \to A^\infty$
2) and well-defined

$\Rightarrow$ morphism of rings $W(A) \to W(A)$

The same argument as for $\times, +$. (later)

V at ghost level: $w_0 = 0$ $\quad w_1 = 0^p + pa_0 = pa_0$ $\quad w_2 = 0^{p^2} + pa_0^p + p^2 a_1$
$= p(a_0^p + pa_1)$
$= p\,w_1(a_0, a_1)$

Constructing Frobenius:

$$F(a_0, a_1, a_2, \ldots) = (c_0, c_1, c_2, \ldots)$$

$$c_0 = a_0^p + pa_1$$

$$c_0^p + pc_1 = a_0^{p^2} + pa_1^p + p^2 a_2 \;;$$

$$(a_0^p + pa_1)^p + pc_1 = a_0^{p^2} + pa_1^p + p^2 a_2 \;;$$

$$\cancel{a_0^{p^2}} + p^2 \_ + \_ \; + \underbrace{pc_1} = \cancel{a_0^{p^2}} + \underbrace{pa_1^p} + p^2 a_2 \;;$$

$$pc_1 = pa_1^p + p^2 \ldots$$

Existence automatic from Dwork's lemma.

$W(A)$; $(a_0, a_1, a_2, a_3, \ldots) \mapsto p \cdot (a_0, a_1, a_2, \ldots) = (c_0, c_1, c_2, \ldots)$

$$(x_0, x_1, x_2, \ldots) \mapsto (px_0, px_1, px_2, \ldots)$$

$\boxed{p = 0 \text{ in } A}$ $\Rightarrow$ $(0, a_0^p, a_1^p, a_2^p, \ldots)$

$c_0 = 0 + p\ldots$ ✓

$$c_0 = pa_0$$
$$c_0^p + pc_1 = p(a_0^p + pa_1)$$
$$c_0^{p^2} + pc_1^p + p^2c_2 = p(a_0^{p^2} + pa_1 + p^2 a_2)$$
$$\ldots\ldots$$

$pc_1 = p(a_0^p + pa_1) - (pa_0)^p$

$= pa_0^p + p^2 \ldots$

$c_1 = a_0^p + p\ldots$

$\boxed{c_n = a_{n-1}^p + p\ldots}$ $\leftarrow$ easy by induction

$VF = p = FV$ if $A/\mathbb{F}_p$

Teichmüller representatives.

$W(A)$ is complete in top.: $(0,0,\ldots,0,a_{n+1},a_{n+2},\ldots)$

$a \in A \qquad [a] := (a, 0, 0, \ldots, 0, \ldots)$

$\downarrow$ ghost

$(a, a^p, a^{p^2}, a^{p^3}, \ldots)$

$W(\mathbb{Q}) = \mathbb{Q}^\infty$

$\boxed{[a][b] = [ab]}$ in $W(A)$ $\Longleftarrow$

$$a = \sum_{n=0}^{\infty} V^n([a_n])$$

$a = (a_0, a_1, a_2, \ldots)$

$\downarrow$

$[a_0] + V([a_1]) + V^2([a_2]) + \ldots$

$\quad = (a_0, a_1, a_2, \ldots, a_n, \ldots)$

check:

$$(a_0, a_1, a_2, \ldots) = [a_0] + V[a_1] + V^2[a_2] + \cdots$$

$$\Big\downarrow gh$$

$$(a_0, a_0^p, a_0^{p^2}, a_0^{p^3}, \ldots) = gh([a_0])$$

$$+ \quad (0, pa_1, pa_1^p, pa_1^{p^2}, \ldots) = gh(V[a_1])$$

$$(0, 0, p^2 a_2, \ldots) = gh(V([a_2])$$

$$+ \quad \vdots \qquad + \quad \vdots$$

$$gh(a) = (a_0, a_0^p + pa_1, a_0^{p^2} + pa_1^p + p^2 a_2, \ldots)$$

$$[a][b] = [ab]$$

$$a = \sum V^n([a_n])$$

$VF = p$ if $A/\mathbb{F}_p$

$FV = p$ always

$$W(\mathbb{Q}) = \prod_{n=0}^{\infty} \mathbb{Q}$$

(the ghost map is an ~~isom~~ onto for $\mathbb{Q}$-algebras)

(resolve at every step

$$x_n = \underline{\text{know}} + p^n a_n$$

Big Witt vectors. $W_{big}(A)$

$$\sum_{k=0}^{n} p^k a_{p^k}^{p^{n-k}}$$

$\cancel{k}\ p^n$

$$w_n((a_j \mid j \in S)) = \sum_{d \mid n} d\, a_d^{n/d}$$

Or rather: start with $S \subset \mathbb{N}$

$n \in S,\ d \mid n \Rightarrow d \in S$

$S = \{1, p, p^2, p^3, \dots\}$ $\qquad$ $S = \mathbb{N}$

$(a_1, a_p, a_{p^2}, a_{p^3}, \dots) =$ formerly $(a_0, a_1, a_2, \dots)$ $\qquad$ $(a_1, a_2, a_3, a_4, a_5, \dots)$

Same as before: $\exists!$ $+, \times$ law

$$\mathbb{Z}[c_j \mid j \in S] \to \mathbb{Z}[a_j \mid j \in S] \otimes \mathbb{Z}[b_j \mid j \in S]$$

s.t.

$$(a_j) \longmapsto (w_n(a_1, \text{---}) \mid n \in S) \in A^S$$

is a ring morphism.

Same proof as above; Dwork lemma.

Case $S = \mathbb{N}$: explicit construction of $\times, +$.