Big Witt vectors

A a comm ring

$$W_{big}(A) = \{(a_1, a_2, a_3, \ldots) \mid a_j \in A\}$$

$$W_n(a_1, \ldots, a_n) = \sum_{d \mid n} d\, a_d^{n/d}$$

Universal $+, \times$ on $W_{big}(A)$ such that

$$gh : (a_1, a_2, \ldots) \mapsto (w_1, w_2, w_3, \ldots)$$

is a morphism of rings

$$W_{big}(A) \longrightarrow A^\infty$$

more generally:

$S \subset \mathbb{N}$ $\quad n \in S, d | n \Rightarrow d \in S$

(in particular: <u>finite</u> subset S)

$W_S(A) = \{(a_n)_{n \in S}\}$

gh: $W_S(A) \longrightarrow A^S$

only w_n, $n \in S$

p-typical: $S = \{1, p, p^2, \ldots\}$

old $a_0, a_1, a_2, \ldots =$

$=$ new $a_1, a_p, a_{p^2}, \ldots$

Exp

W_{b}

$$W_{big}(A) = \left(1 + tA[[t]]\right)^{\times}$$

the addition will be $\times$ of power series.

Given $f(t) = 1 + c_1 t + c_2 t^2 + \cdots$

<u>Lemma</u> Uniquely:

$$\left.f(t) = \prod_{d=1}^{\infty} (1 - a_d t^d)^{-1}\right] \text{Uniquely}$$

$$f(t)^{-1} = 1 + b_1 t + b_2 t^2 + \cdots$$

$$\| $$

$$\underbrace{(1 + a_1 t)}_{\text{invertible}} (1 + \tilde{b}_2 t^2 + \cdots)$$

$$\underbrace{1 + \tilde{b}_d t^d}_{} + \tilde{b}_{d+1} t^{d+1} + \ldots = \underbrace{(1 + \tilde{b}_d t^d)}_{}(1 + \ldots t^{d+1} + \ldots)$$

ghost map:

$$(1+tA[[t]])^{\times} \longrightarrow A^{\infty} \simeq tA[[t]]^{+}$$

$$-t\frac{\partial}{\partial t}\log$$

$$(1-a_d t^d)^{-1} \longmapsto t\frac{\partial}{\partial t}\log(1-a_d t^d)$$

$$= t\frac{\partial}{\partial t}\sum_{n=1}^{\infty}(a_d t^d)^n/n =$$

$$= \underbrace{t\frac{\partial}{\partial t}}\sum_{n=1}^{\infty} a_d^n \frac{t^{dn}}{n} = \sum_{n=1}^{\infty} a_d^n \cdot n \cdot t^{\overset{N}{dn}}$$

$$= \sum_{N=1}^{\infty}\sum_{d|N} a_d^{N/d}(N/d)t^N =$$

$$= d \leftrightarrow N/d = \sum\left(a_{N/d}^{d}\, d\right)t^N$$

$$= \sum \widehat{w_N} \cdot t^N$$

Multiplication?

$a_1 = a$
$a_d = 0, d > 1$

Hint: $[a] = (1-at)^{-1}$

Ghost components: $a^n | n \in \mathbb{N}$

$$w_n(a, 0, 0, \ldots) = a^n$$

$$[ab] = [a][b]$$

$$(1 - a_d t^d) \times (1 - b_e t^e) = ?$$

(Extend the ring by all $\zeta_d^d = 1$ prim power series)

$$\prod_{k=0}^{d-1} \left(1 - a_d^{1/d} t \cdot \zeta_d^k\right) \prod_{l=0}^{e-1} \left(1 - b_e^{1/e} t \cdot \zeta_e^l\right)$$

$$\prod_{k,l} \left(1 - a_d^{1/d} b_e^{1/e} \cdot \boxed{\zeta_d^k \zeta_e^l}\right)$$

$$\prod_{k,l}\left(1-a_d^{1/d}\, b_e^{1/e}\, \zeta_d^{k}\zeta_e^{l}\cdot t\right)$$

$$=\prod_{0\leq m\leq l.c.m.(d,e)}\left(1-a_d^{1/d}\, b_e^{1/e}\, \zeta_{l.c.m.}^{m}\, t\right)^{(d,e)}$$

$$=\left(1-a_d^{\frac{l.c.m.}{d}}\, b_e^{\frac{l.c.m.}{e}}\, t^{l.c.m}\right)^{g.c.d(d,e)}$$

$$\|$$

$$(1-a_d t^d)\times(1-b_e t^e)$$

$$\|$$

$(d,e)=1:$

$$\left(1-a_d^{e}\, b_e^{d}\, t^{de}\right)$$

$d=e:$

$$\left(1-a_d\, b_d\, t^{d}\right)^{d}$$

$$\underbrace{\zeta_d^k \zeta_e^l}$$ $0 \le k < d$, $0 \le l < e$?

$(d,e)=1$: the collection

$\zeta_{de}^m \quad 0 \le m < de$

each enters once.

if $d=e$: $\zeta_d^m \quad 0 \le m < d$

each enters d times.

<u>In general</u>:

all roots $\zeta_{\text{l.c.m.}(d,e)}^m$

each enters $\underset{\text{g.c.d}}{\overset{}{(d,e)}}$ times

V, F on power series. $\forall n$:

$$(V_n f)(t) = f(t^n)$$ Res

$$(F_n f)(t) = \prod_{k=0}^{n-1} f(\zeta_n^k t^{1/n})$$ Norm

$$V_n(1-a_d t^d) = 1-a_d t^{dn}$$

$$F_n(1-a_d t^d) = $$ (depends on g.c.d and l.c.m)

$$\boxed{F_n V_n = n}$$

$$F_n V_n f(t) = \prod_{k=0}^{n-1} (V_n f)(\zeta_n^k t^{1/n})$$

$$= \prod_{k=0}^{n-1} f\left(\left(\zeta_n^k t^{1/n}\right)^n\right) = f(t)^n$$

$$F_n(1-a_d t^d) = \prod_{k=0}^{n-1}\left(1-a_d\left(\zeta_n^k t^{1/n}\right)^d\right)$$

$$= \prod_{k=0}^{n-1}\left(1-a_d \zeta_n^{kd} t^{d/n}\right)$$

Let $d = d'D \quad n = n'D \quad (n', d') = 1$

$$F_n(1-a_d t^d) = \prod_{k=0}^{n'D-1}\left(1-a_d \zeta_{n'\cancel{D}}^{k\cdot d'\cancel{D}} t^{d'/n'}\right)$$

$$= \prod_{k=0}^{n'-1}\left(1-a_d \zeta_{n'}^{k} t^{d'/n'}\right)^D$$

$$= \left(1-a_d^{n'} t^{d'}\right)^D = \left(1-a_d^{\frac{n}{gcd}} t^{\frac{d}{gcd}}\right)^{gcd}$$

Ex. $n = d$, $gcd = n$:

$$F_n(1-a_n t^n) = (1-a_d t)^n$$

$$\| $$

$$F_n V_n(1-a_n t)$$

recover $F_n V_n = n$