δ-rings and Witt vectors [p-typical] (Joyal)

A - comm ring $\rightsquigarrow W_p(A)$ (or $W(A)$)

p prime

$$W_p(A) \xrightarrow{\text{comm rg}} A$$

$F : W_p(A) \circlearrowleft$ endomorphism

$Fa \equiv a^p \pmod{p}$

$W(A)$ is UNIVERSAL SUCH

1) Say, no p-torsion. Given $A, F : A \circlearrowleft$

$$\delta(a) = \frac{1}{p}\left[Fa - a^p\right] \qquad Fa - a^p \in pA$$

Properties

$$\delta(a+b) - \delta(a) - \delta(b) =$$

$$= \frac{1}{p}\left[\cancel{F(a+b) - Fa - Fb} - (a+b)^p + a^p + b^p\right]$$

$$= \underbrace{F_1(a,b)}_{} = \sum_{j=1}^{p-1} \frac{1}{p}\binom{p}{j} a^j b^{p-j}$$

$$\delta(ab) = \frac{1}{p}\left[F(ab) - a^p b^p\right] = \frac{1}{p}\Big[(a^p + p\delta(a))(b^p + p\delta(b)) - a^p b^p\Big] =$$

$$F(a) = a^p + p\delta(a)$$

$$= \underline{\underline{a^p}}\,\delta(b) + \delta(a)\,\underline{\underline{b^p}} + \underbrace{p\,\delta(a)\delta(b)}$$

$$\begin{cases} \delta(a+b) - \delta(a) - \delta(b) = F_1(a,b) \\ \delta(ab) = a^p\delta(b) + \delta(a)b^p + p\,\delta(a)\delta(b) \end{cases}$$

Called a δ-derivation; A a δ-ring.

$$\delta(1) = 0$$

$$\cancel{\delta(2)} = \cancel{\delta(1)} + \cancel{\delta(1)} + \frac{1}{p}\left[1^p + 1^p - \cancel{2^p}\right] \quad \left(= F_1(1,1)\right)$$

$$+$$

$$\cancel{\delta(3)} = \cancel{\delta(2)} + \cancel{\delta(1)} + \frac{1}{p}\left[\cancel{2^p} + 1^p - \cancel{3^p}\right] \quad \left(= F_1(2,1)\right)$$

$$+$$

$$\vdots$$

$$\cancel{\delta(p-1)} =$$

$$\delta(p) = \cancel{\delta(p-1)} + \cancel{\delta(1)} + \frac{1}{p}\left[\cancel{(p-1)^p} + 1^p - p^p\right]$$

$$\underline{\delta(p)} = \frac{1}{p}\left[p \cdot 1^p - p^p\right] = \underline{1 - p^{p-1}}$$

- $\delta(p) = 1 + p^{p-1}$.
- $\delta(p^2) = 2p^p \cdot \delta(p) + p\,\delta(p)^2 \equiv p \pmod{p^2}$

If $p = 0$: $\delta(p^2) = 0$

$\vdots$

$$\boxed{\delta(p^N) = (\text{unit})\, p^{N-1} + p^N \ldots}$$

N = minimal s.t. $p^N = 0$

$\Downarrow$

$\delta(0) = p^{N-1} = 0$ contrad.

A over $\mathbb{Z}/p^N\mathbb{Z} \implies$ no δ exists.

W(A) is a <u>universal</u> δ-ring

1)
$$\begin{tikzcd} \text{Rings} \arrow[rr, bend left, "L"] \arrow[rr, bend right, "R"'] & & \delta\text{-rings} \arrow[ll, "\text{forget}"'] \end{tikzcd}$$

$$\boxed{A \mapsto W(A)}$$

To A, <u>add</u>: $\delta(a), \delta^2(a), \dots$

and factor out all identities of δ-rings.

LA = comm ring freely generated/A by $\delta^n(a)$, $a \in A$, with relations:

Rings with <u>usual</u> derivations:

Rgs A Rgs B, $D \in \mathrm{Der}(B)$

$$A \xleftarrow{f} B$$

[and D is a derivation of B.

Given f and D:

produce many more elements of A.

$b \in B \mapsto f(b)$ and also: $f(Db)$ $f(D^2 b)$ ...

$$A \xleftarrow{f} B \circlearrowleft D \in \mathrm{Der}(B)$$

$$a_j^{(1)} = f(D^j(b_1))$$

$$a_j^{(2)} = f(D^j(b_2))$$

$$\begin{array}{cccc} f(b) & f(Db) & f(D^2b) & \ldots \\ \| & \| & \| & \\ (a_0, & a_1, & a_2 & \ldots) \end{array} \longleftarrow\!\!\shortmid \; b$$

LINEAR in $b \in B$

$$\longleftarrow\!\!\shortmid \; b_1 b_2$$

$$f(b_1 b_2) = f(b_1) f(b_2) = a_0^{(1)} a_0^{(2)}$$

$$a_0 = a_0^{(1)} \cdot a_0^{(2)} \qquad a_1 = a_0^{(1)} a_1^{(2)} + a_1^{(1)} a_0^{(2)}$$

$$a_j^{(1)} = f(D^j b_1) \qquad a_j^{(2)} = f(D^j b_2)$$

$$a_j = f(D^j b)$$

$$a'_j = f(D^j (\underline{Db}))$$

$$a_j = f(D^j(b_1 b_2)) = \sum_{k=0}^{j} a_k^{(1)} \underline{a_{j-k}^{(2)}} \cdot \binom{j}{k}$$

a_{j+1}

multiplication on sequences

$(a_0, a_1, a_2, \ldots)$

+ Derivation on sequences.

$$F(a) = a^p + p\,\delta(a) = W_1(a, \delta(a))$$

$$F^2(a) = W_2\big(a, \delta(a), \underbrace{\delta_2(a)}_{\text{exists universal in terms of } \delta}\big)$$

$$\underbrace{[a^p + p\,\delta(a)]}_{F(a)}{}^p + p\,\delta\big(\underbrace{a^p + p\,\delta(a)}_{F(a)}\big)$$

$$\| ?$$

$$\boxed{a^{p^2} + p\,\delta(a)^p} + p^2 \boxed{\delta_2(a)}$$

exercise?

In other words:

$$A \qquad\qquad (B, D)$$

$$A \hookleftarrow B \quad \text{rings}$$

$$\updownarrow$$

$$\boxed{A\{\{T\}\}, \frac{\partial}{\partial T}}$$

$$\parallel$$

$$\left\{ \sum_{j=0}^{\infty} a_j \frac{T^j}{j!}, \frac{\partial}{\partial T} \right\} \hookleftarrow (B, D)$$

Comm

$$\text{Rgs} \xleftarrow{F} \text{Rgs}$$

$$\text{Der}$$

$$R$$

$$R(A) = A\{\{T\}\}, \frac{\partial}{\partial T}$$

Same for δ-rings?

Obviously, $(a, Fa, F^2a, \ldots)$ are in the image described by Dwork's lemma!

$F(F^n a) = F^{n+1}(a) + 0$ ✓

Universal expressions

$\delta_n(a)$ in terms of a, $\delta^j(a)$, $+$, $\times$ exist.

$\delta_0(a) = a$, $\delta_1(a) = \delta(a)$, $\delta_2(a) = \ldots\ldots$

As we did with derivations:

$$A \qquad\qquad (B, \delta)$$

$$A \xhookrightarrow[\text{as rings}]{f} B \ni b$$

$$(a_0, a_1, a_2, \ldots) := (f(b), f(\delta(b)), f(\delta_2(b)), f(\delta_3(b)), \ldots)$$

$$a_0 = \mathbb{f}(b)$$

What happens to $\mathbb{f}(b)$?

If we add/multiply/ ~~aplly~~ apply F to b

Conclusion

$$\begin{tikzcd} \text{Rings} \\ (\text{Comm}) \arrow[rr, bend left=50, "L"] \arrow[rr, bend right=50, "R"'] & & \delta\text{-rings} \\ (\text{Comm}) \arrow[ll, "F"'] \end{tikzcd}$$

$$\boxed{R(A) = W(A)}$$ (p-typical Witt vectors)

(L was used in constructing R
(Dwork's lemma applied there))

Recall: Dwork's lemma

(A, F) $\quad Fa \equiv a^p \pmod p$

gh: $x_n = w_n(a_0, \ldots, a_n)$

~~the~~ image $\underline{(x_0, x_1, \ldots, x_n, \ldots)}$ of gh consists of: $x_{n+1} \equiv F(x_n) \bmod p^{n+1}$ $\forall n.$

L(Free Com)

Apply to the universal (free) δ-ring ✓