Witt vectors; De Rham Witt complex

Plan

Idea: $X/\mathbb{F}_p$ $\mathcal{O}(X)$ $\mathcal{O}_X$

affine

De Rham complex over $\mathbb{Z}_p$ (or $\mathbb{Q}_p$)

completely natural;

$W(\mathbb{F}_p) = \mathbb{Z}_p$

$X = \mathbb{A}^1$: $\mathcal{O}(X) = \mathbb{F}_p[T]$

How natural?

$\mathbb{Z}_p[T] \xrightarrow{d} \mathbb{Z}[T]\,dT$?

$\mathbb{Q}_p$ $\mathbb{Q}_p$

Well, in fact, natural ENOUGH

LATER

Ok, but $W(\mathbb{F}_p[T])$ grows naturally out of $\mathbb{F}_p[T]$?

Our 0-forms: $W(\mathbb{F}_p[T])$ or $W(\mathcal{O}_X)$?

$W(\mathbb{F}_p[T])$ vs $\mathbb{Z}_p[T]$?

$[T] =: T$

$F : T \longmapsto T^p$

$[a] \longmapsto [a^p] \quad \forall a$

$W(\mathbb{F}_p[T])$

$W(\mathbb{F}_p) = \mathbb{Z}_p$

$\boxed{V(T)?}$

$[T] =: T$

The Teichmüller

VT say, formally: $VT = a\,T^{\lambda}$?

$$FVT = p\,T \qquad a\left(T^{\lambda}\right)^{p} = p\,T$$

$$a = p \qquad \lambda = \frac{1}{p}$$

Seems that:

new element $p \cdot T^{1/p}$ $(= VT)$

More precisely: $V([T])^{p} = p^{p}[T]$?

Recall: $pV(ab) = V(a)V(b)$

(look at ghost components:

$$V(x_0, x_1, x_2, \dots) = (0, px_0, px_1, \dots))$$

So: if $T = [T] \in W(\mathbb{F}_p[T])$:

$$V(T^m)^p = p^{p-1} V(T^{mp}) = p^{p-1} V(FT^m)$$

$$= p^p T^m$$

$$V(T^m) = p\, T^{m/p}$$

If $p \mid m$: this happens in $\mathbb{Z}_p[T]$.

If $(m,p)=1$: we get new elements

$$V^n(T^m) = p^n T^{m/p^n}$$

Claim The basis of $W(\mathbb{F}_p[T])$ over $\mathbb{Z}_p$ (as a completed free mod):

$$T^n ; \; p^n T^{m/p^n}, \quad (m,p)=1, \; n \geq 0.$$

Multiplication:

We are in $W(\mathbb{F}_p[T])$ $\quad \mathbf{T} := [T]$

Teich.

$$(V\mathbf{T})^p = p\mathbf{T}$$

$$\boxed{V\mathbf{T} = p\,\mathbf{T}^{1/p}}$$

formally,

$\mathbf{T}^{1/p} \notin W$!!

of $W(\mathbb{F}_p[T])$ as (p)-complete free $\mathbb{Z}_p$-mod

$$\vdots$$

$$V^n\mathbf{T} = \boxed{p^n\,\mathbf{T}^{1/p^n}}$$

Next guess: BASIS over $\mathbb{Z}_p$? **YES**

$\mathbf{T}^n,\ n \geq 0$ **and** $V^n\mathbf{T},\ n > 0$.

Very natural DeRham complex:

$$
\begin{array}{ccc}
\vdots & & \vdots \\
T^3 & & \\
T^2 & \longmapsto & 2T\,dT \\
T & \longmapsto & dT \\
1 & \longmapsto & 0 \\
\hline
pT^{1/p} & \xrightarrow{\ \times\ } & T^{1/p-1}\cdot dT \\
p^2T^{1/p^2} & \xrightarrow{\ \times\ } & T^{1/p^2-1}\cdot dT \\
p^3T^{1/p^3} & & \vdots
\end{array}
$$

$T^n dT$

$n \geq 0$

proposed basis for Ω^1:

Cohom vanishes HERE

$\boxed{W\Omega^{\bullet}_{\mathbb{A}^1_{\mathbb{F}_p}}}$

$$T^n \qquad\qquad T^n\, dT$$

$$p^k T^{1/p^k} \xmapsto{d} T^{1/p^k - 1}\, dT$$

$$\shortparallel$$

$$T^{1/p^k} \cdot \frac{dT}{T}$$

$$\boxed{\text{Can we extend } V \text{ and } F\text{?}}$$

$1/p^k > 0$

as small as we want, but $\neq 0$.

$\cancel{\Omega}$

$W\Omega$

Recall Cartier isomorphism:

$$db \mapsto b^{p-1}db \ ?$$

$$T \mapsto T^p; \quad dT \mapsto T^{p-1}dT \ ?$$

makes sense:

$$\frac{dT}{T} \mapsto \frac{T^{p-1}dT}{T^p} = \frac{dT}{T}$$

$$T^\lambda \frac{dT}{T} \xrightarrow{V} V(T^\lambda)\frac{dT}{T} \qquad T^\lambda \cdot \frac{dT}{T} \mapsto T^{p\lambda} \cdot \frac{dT}{T}$$

Properties of V:

actually $x, y \in W\Omega$:

$$V(x \cdot dy) = Vx \cdot dVy$$

$[V(x_1 x_2) \neq$
$\neq Vx_1 \cdot Vx_2$!

How V "commutes" with $d[x] \cdot$?

$$(d[x]) \cdot Vy =$$

$$= V(\underbrace{[x]^{p-1} d[x]}_{\text{"}Fd[x]\text{"}} \cdot Y)$$

So: we get $W\Omega^{\bullet}_X$ $(X = \mathbb{A}^1)$ with F

1) $\boxed{W\Omega^0 = \overbrace{W\mathcal{O}_X}^{\text{universal}}}$ $\Rightarrow$ Frob on Ω^0 second dfn: build on THAT

2) V acts, and properties a, b satisfied.

F never appears; it FOLLOWS

$W\Omega_X$ is universal such.

Makes sense for all X.

$$p^{n+n'} T^{m/p^n} T^{m'/p^{n'}} = p^{n+n'} T^{\frac{m}{p^n}+\frac{m'}{p^{n'}}}$$

$$= p^{n+n'} T^{\frac{mp^{n'}+m'p^n}{p^{n+n'}}}$$ makes sense

DRW complex: basis of $W\Omega^1$ is

$$T^n \frac{dT}{T}, \ n>0$$

$$T^{m/p^k} \frac{dT}{T}, \ (m,p)=1, \ k>0$$

Action of V, F:

$$V\left(T^\lambda \frac{dT}{T}\right) = p T^\lambda \frac{dT}{T} \qquad V(T^\lambda) = p T^{\lambda/p}$$

$$F\left(T^\lambda \frac{dT}{T}\right) = T^{\lambda p} \frac{dT}{T} \qquad F(T^\lambda) = T^{\lambda p}$$

Properties $\quad FV = VF = p$

$$d[x]\cdot V(y) = V\left([x]^{p-1} d[x]\cdot y\right)$$

$$V\left(T^{p-1} dT \cdot T^\lambda\right) =$$

$$= V\left(T^{p+\lambda} \frac{dT}{T}\right) = p T^{1+\lambda/p} \frac{dT}{T}$$

$$= dT \cdot p T^{\lambda/p} = dT \cdot V(T^\lambda)$$

$$V\left(T^{\lambda}\cdot d\left(T^{\mu}\right)\right) = V\left(T^{\lambda+\mu-1}\,dT\right)$$

$$= V\left(T^{\lambda+\mu}\,\frac{dT}{T}\right) = p\,T^{\frac{\lambda}{p}+\frac{\mu}{p}}\,\frac{dT}{T}$$

$$= p\,T^{\lambda/p}\cdot T^{\frac{\mu}{p}}\,\frac{dT}{T} = p\,T^{\lambda/p}\,d\left(p\,T^{\frac{\mu}{p}}\right)$$

gives

$$V(x\cdot dy) = Vx\cdot dVy$$