De Rham Witt complex

Crystalline cohomology (next week; from scratch)

Recall: $X = \mathbb{A}^1$ over $\mathbb{F}_p$

$$A = \mathbb{F}_p[T]$$

$W(\mathbb{F}_p[T])$: basis over $\mathbb{Z}_p$: $p^a T^\lambda$

$k \in \mathbb{Z}$ — $\lambda = p^k m$ $\quad (m,p)=1$

$$a = \begin{cases} 0, & k \geq 0 \\ -k, & k<0 \end{cases}$$

For $x, y \in W\Omega^{\bullet}_{A}$: $xy = (-1)^{|x||y|} yx$; (g comm alg) dga

$d(xy) = dx \cdot y + (-1)^{|x|} x\,dy$

$V, F : W\Omega^{\bullet}_{\mathbb{Q}}$ "$\frac{dT}{T}$ is invariant for both"

On Ω^0:

$p^a T^\lambda \xmapsto{V} p^{a+1} T^{\lambda/p}$

$p^a T^\lambda \xmapsto{F} p^a T^{p\lambda}$

$V(a \cdot \frac{dT}{T}) = V(a)\frac{dT}{T}$

$F(a\frac{dT}{T}) = Fa \cdot \frac{dT}{T}$

$FV = VF = p$

$$V(T^\lambda \cdot dT^\mu) = V\left(\mu \cdot T^\lambda T^\mu \frac{dT}{T}\right) = \mu p T^{\frac{\lambda+\mu}{p}} \cdot \frac{dT}{T}$$

should be p^a in front

makes sense if we tensor by $\mathbb{Q}_p$ and compute there.

$$pT^{\frac{\lambda}{p}} \cdot d\left(pT^{\frac{\mu}{p}}\right)$$

$$\shortparallel$$

$$p^{2} \cdot \frac{\mu}{\cancel{p}} \cdot T^{\frac{\lambda+\mu}{p}}$$

$$V(T^{p-1}dT \cdot T^\lambda)$$

$$\shortparallel$$

$$V\left(T^{p+\lambda} \cdot \frac{dT}{T}\right) = p \cdot T^{\cancel{1}+\frac{\lambda}{p}} \frac{dT}{T}$$

$$dT \cdot V(T^\lambda)$$

✓

$$F(dT) = F\left(T \cdot \frac{dT}{T}\right) = T^{p} \cdot \frac{dT}{T} = \cancel{F(T)}$$

$$T^{p-1} dT$$

$$V(x\, dy) = V(x)\, dV(y) \quad ✓ \quad ★$$

$$V\left([x]^{p-1} d[x] \cdot y\right) = d[x] \cdot V(y) \quad ★$$

checked when $x = T$

So is with $F(d[x]) = [x]^{p-1} d[x]$

Dfn The De Rham Witt complex of p-typical $\underline{A}$ over k is the universal $\boxed{\boxed{\text{cdga}}}$ $W\Omega^{\bullet}_{A/k}, d$ such that:

over $W(k)$

1) $W\Omega^{0}_{A/k} \cong \underline{W(A)}$

2) $V : W\Omega^{\bullet} \circlearrowleft$ $\quad V =$ the usual on $W\Omega^{0}$

* $V(x\,dy) = Vx \cdot dVy \qquad \forall x, y \in W\Omega^{\bullet}$
* $V(\underline{[x]^{p-1} d[x]}) = d[x] \cdot V(y)$

Frob of

$$\begin{array}{c} \Omega^\bullet_{W(A)/W(k)} \\ \downarrow \\ W\Omega^\bullet_{A/k} \end{array}$$

b/c $\Omega^0 = W(A)$

a cdga

$p(VT)^{p-1} d(VT)$
$\shortparallel$
$p^p dT$

$(VT)^{p-1} d(VT)$
$\shortparallel$
$p^{p-1} dT$

$(A = \mathbb{F}_p[T])$

So: WHAT do we have to do to $\Omega^\bullet_{W(A)/W(k)}$ to get to $W\Omega^\bullet_{A/k}$?

1) add V, etc.

2): SATURATE

Bhatt, Lurie, Mathew : <u>saturation</u>

<u>Obs. 1</u>: If A is a δ-ring (e.g. $W(A)$) then

F can be extended to $\Omega^{\bullet}_{A/K}$

A, $\delta : A \to A$ δ-derivation

e.g. $F(a) = a^p + p\delta(a) \in \mathcal{E}nd(A)$

Now define $F : \Omega^{\bullet}_{A/k} \circlearrowleft$

1) F a morphism of algebras;
so need to know it only on
a, da $a \in \mathcal{A}$.

$\boxed{F(da) = a^{p-1}da + d\delta a}$ ✓

as in $\mathcal{A}$

expect: $\boxed{dF = pFd}$

$$dFa = pFda$$

$$dFa = \underbrace{da^p}_{pa^{p-1}da} + pd\delta(a)$$

Recall:

$$dF(T) = pT^{p-1}dT$$

$F(dT)$

So, in addition to T^n, $n \in \mathbb{Z}_{\geq 0}$:

basis over $\mathbb{Z}_p$ of $W\Omega^0 = W(\mathbb{F}_p[T])$

$$p^k T^{m/p^k} \qquad k>0 \text{ and } (m,p)=1.$$

$$p^a T^\lambda \longmapsto p^a \lambda T^\lambda \frac{dT}{T}$$

$$\Omega^0: \ p^a T^\lambda \longrightarrow \Omega^1: \ T^\lambda \frac{dT}{T}$$

$$0 < \lambda = p^k m \qquad (m,p)=1 \longleftarrow \boxed{\lambda > 0}$$

$$a = \begin{cases} 0, & k \geq 0 \\ -k, & k<0 \end{cases}$$

$\boxed{\text{OR}}$ $\lambda = 0$

$$\frac{dT}{T} \notin \Omega^1 !$$