Bhatt-Lurie-Mathew:

$W\Omega^\bullet_{A/k}$ will be the <u>saturation</u> of $\underbrace{\Omega^\bullet_{W(A)/W(k)}}$

1) Dieudonné complexes

$M^\bullet, d$ complex $\quad F : M^\bullet \to M^\bullet$

(of deg 0)

(for now: all over $\mathbb{Z}$) $\to \mathbb{Z}_p \ldots$

$$dF = pFd$$

(or over $W(k)$; semi-linear

$$F(\alpha m) = F_{W(k)}\alpha \cdot m$$

<u>Example</u>: $\quad M^\bullet = \Omega^\bullet_{W(A)/W(k)}$

1) More on DRW complex

2) Crystalline cohomology (directly, without Witt vectors)

$A/k \quad \mathrm{char}(k) = p > 0$

$W(A)$

$\Omega^{\bullet}_{W(A)/W(k)}$ not <u>quite</u> good:

$A = k[T] \quad VT$

$[T] =: \underline{T} \in W(A)$

VT

$$\boxed{(VT)^p = p^p T}$$

$\cancel{p}(VT)^{p-1} d(VT)$

$\cancel{p}^{p-1} dT$

NO?

$$\eta(M)^n = \{ m \in p^n M^n \mid dm \in p^{n+1} M^{n+1} \}$$

1) $d\eta(M)^n \subset p^{n+1} M^{n+1} \cap \ker(d)$

$\Rightarrow$ subcomplex. $dF = pFd$

2). If $(M^\cdot, d, F)$ is a Dieudonné complex:

$\alpha_F : M^\cdot \to \eta_p M^\cdot$

morphism of complxs.

$$M^n \ni x \longmapsto p^n F x \in \eta(M)^n$$

$$\underbrace{p^n F x}_{} \xrightarrow{d} p^{n+1} F dx$$

Why morphism of complexes?

$$\begin{array}{ccc} x \in M^n & \longmapsto & p^n F \\ \downarrow & & \downarrow \\ & & p^n dFx \\ dx \in M^{n+1} & \xrightarrow[d_F]{} & p^{n+1} Fdx \end{array}$$

$$d_F : M^\bullet \to \eta_p M^\bullet$$

$$a_0\, da_1\, da_2 \cdots \mapsto Fa_0\, dFa_1 \cdots$$

When $M^\bullet = \Omega^\bullet_{R/W(k)}$

this is just the natural extension of F from R to $\Omega^\bullet_{R/W(k)}$.

$\Omega^{\cdot}_{R/W(k)}$

$F(a) = a^p + p\delta(a)$

Recall F on $\Omega^{\cdot}_{R/W(k)}$:

$F(a) = \checkmark$

$F(da) = d\delta(a) + a^{p-1}da$

$$pF(da) = d[p\delta(a)] + pa^{p-1}da$$

$$\| \qquad\qquad\qquad \|$$

$$\alpha_F(da) \qquad d[p\delta(a) + a^p]$$

$$\|$$

$$dF(a)$$

$\alpha(a_0\, da_1 \dots) = \alpha(Fa_0\, dFa_1 \dots)$

$$\alpha_F : M^\bullet \to \eta_p M^\bullet$$ morphism of cplxs

$$\begin{array}{ccc}
m^n & \overset{\alpha_F}{\longmapsto} & p^n F m^n \\
\cap & & \Big\downarrow d \\
M^n & & p^{n+1} F d m^n
\end{array}$$

$$\boxed{d(Fm) \in pM^{\bullet+1}} \qquad (dF = pFd)$$

Dfn A Dieudonné complex is <u>saturated</u> if: 1) it is torsion-free 2) $F : M^\bullet \xrightarrow{\sim} \{m \mid dm \in pM^\bullet\}$

Given a saturated Dieudonné module.

$$V(x) = \text{the unique } y \text{ s.t.}$$

$$Fy = \underbrace{px}$$

$$FV = p$$

$$\exists! \xrightarrow[F]{} d(px) \in pM^{\bullet+1}$$

$$FV(x) = px \qquad \exists!\, V(x)$$

$$F : M^{\bullet} \xrightarrow{\sim} \{ m \mid dm \in pM^{\bullet+1} \}$$

<u>Lemma</u> $FV = VF = p$

$$FdV = d \qquad Vd = pdV$$

$$dF = pFd$$

$FV = p$ ✓ $\quad FVF = pF \quad F \hookrightarrow$, so ok.

$$pd = dFV = pFdV$$

no p-torsion $\Downarrow$

$$d = FdV$$

$$Vd = VFdV = p \quad ✓$$

L. If $M^\cdot$ is saturated then

$$F^r : M^\cdot \xrightarrow{\sim} \{x \in M^\cdot \mid dx \in p^r M^{\cdot+1}\}$$

0) $dF^r = p^r F^r d$ ✓ $\quad$ 2) onto?

1) $F^r : \hookrightarrow$ $(F \hookrightarrow)$

<u>Assume</u>: $dx \in p^r M^{\bullet+1}$

why is it $F^r(\dots)$?

induction in r.

r=1: ✓

$$dx = p^r y \qquad dy = 0$$

(no p-torsion; $ddx = 0$) $\Downarrow$

$$y = Fy' \qquad \Leftarrow dy = p \cdot 0$$

$$dx = p^r y \Rightarrow dx = p \cdot (p^{r-1} y)$$

$\Downarrow$ $M^\bullet$ saturated

$$x = Fx'$$

$$d(Fx') = p^r (Fy')$$

$$\|$$

(no torsion)

$F : \hookrightarrow$

$$\cancel{p}F(\underbrace{dx'}) = \underbrace{p^{r-1}\cancel{F}(y')}$$

$$dx' = p^{r-1} y'$$

Next: M° is saturated

if no torsion

$$\Updownarrow$$

$$\alpha_F : M^\bullet \xrightarrow{\sim} \eta_p(M^\bullet) \quad \star$$

From that:

Saturation of $M^\bullet$.

1). Kill (p-)torsion.

2). $\varinjlim_n \eta_p^n(M^\bullet) \quad = \quad \mathrm{sat}(M^\bullet)$

$$M^\circ \xrightarrow[\alpha_F]{} \eta_p M^\bullet \xrightarrow[\alpha_F]{} \eta_p^2 M^\bullet \to \cdots$$

is saturated.

Properties:

$$\boxed{\widehat{\mathrm{sat}(M^\bullet)} = \varprojlim \mathrm{sat}(M^\bullet)_n}$$

$$M^\bullet \longrightarrow \mathrm{sat}(M^\bullet)$$

sat is left adjoint to forgetful functor: left adj

$$\begin{tikzcd}
\text{Dieudonné cplxs} \arrow[r, bend left, "\mathrm{sat}"] & \text{Saturated Dieudonné cplxs} \arrow[l, "\text{Forget}"]
\end{tikzcd}$$

Completion:

$$\mathrm{sat}(M^\bullet)_n := \mathrm{sat}(M^\bullet) \Big/ \left(\mathrm{im}(V^n) + \mathrm{im}(dV^n)\right)$$

power!

$$(\mathbb{Z}_p)_n := \mathbb{Z}/p^n$$

$$\mathrm{sat}(M^\bullet)_n \underset{V}{\overset{F}{\rightleftarrows}} \mathrm{Sat}_{n-1}(M)$$

or $\Omega^{i}_{R/\mathbb{Z}_p}$ R $\mathbb{Z}_p$-alg, tors free with a Frob lifting

(last time) $\underbrace{}_{\text{or a } \delta\text{-ring}}$

The functors η_p:
For any complex $M^\cdot$

$$\eta_p(M^\cdot) = \text{subcomplex}$$

$$\eta_p(M)^n = \{m \in p^n M^n \mid dm \in p^{n+1} M^{n+1}\}$$