Operations on Hochschild / cyclic complexes

$C_\bullet(A)$, b, B for an assoc. alg. A

$$C_n(A) = A \otimes \bar{A}^{\otimes n}, \quad n \geq 0$$

$$b^2 = B^2 = (b+B)^2 \qquad |b| = -1 \qquad |B| = +1$$

MIXED COMPLEX

$$C^n(A,A) = \mathrm{Hom}_k(\bar{A}^{\otimes n}, A)$$

Much more general alg. structure esp. when include Hoch. cochains:

$$\bar{A} = A/k \cdot 1$$

1. What is (the ?) algebra of operations on $C_\cdot(A), b, B$?

$\mathfrak{g}^\cdot_A, [\,,\,]_{Gerst}, \delta : \mathfrak{g}^\cdot_A \to \mathfrak{g}^{\cdot+1}_A$

$$\mathfrak{g}^\cdot_A = C^{\cdot+1}(A,A) = \qquad (\cdot \geq -1)$$

$$= \mathrm{Hom}_k(\bar{A}^{\otimes \cdot+1}, A)$$

$m(a_1, a_2) = a_1 a_2$

$[m, \cdot]_{Gerst} = \delta$

Gersterhaber: $\mathfrak{g}^\cdot_A$ is a dg Lie algebra.

e.g. $\cdot = 0 \quad C^1(A,A) = \mathfrak{g}^0_A = \{\bar{A} \to A\}$

and $[\,,\,]_{Gerst}$ is commutator.

$U(\mathfrak{g}^\bullet_A)$ = associative ;
actually, a dg Hopf algebra.

Look at the $\underset{\text{dg}}{\wedge}$ coalgebra $U(\mathfrak{g}^\bullet_A)$

Do the dual construction : Hochschild/cyclic cohomology.

$$C^\bullet(\bar{U}): \; U^\bullet \xrightarrow{b'} U \otimes \bar{U}^\bullet \xrightarrow{b'} U \otimes \bar{U} \otimes \bar{U}^\bullet \xrightarrow{b'} .$$

$$\boxed{CC^{\bullet}_{II}(U):}$$

Π or $\cancel{\oplus}$?

$$C^{\bullet}_{II}(U) \xrightarrow{B^\vee} C^{\bullet}_{II}(U) \xrightarrow{B^\vee} C^{\bullet}_{II}(U) \xrightarrow{B^\vee}$$

Cyclic complex of a coalgebra

<u>But</u>: $U \otimes U \longrightarrow U$ morphism of coalgebras, assoc.

$$\begin{tikzcd}
 & CC^{\bullet}_{II}(U\otimes U) \arrow[r] \arrow[d, shift right] & \boxed{CC^{\bullet}_{II}(U)} \\
\text{A-W vs E-Z} \arrow[r] & CC^{\bullet}_{II}(U)\otimes CC^{\bullet}_{II}(U) \arrow[u, shift right] \arrow[ur] &
\end{tikzcd}$$

A_∞ ALGEBRA

Thm Naturally, $CC^-_\bullet(A)$ is an A_∞ module over the A_∞ algebra

$$CC^\bullet_{II}(\underbrace{U(\mathfrak{g}^\bullet_A)}_{\text{as coalgebra (dg)}})$$

the (A_∞) product comes from the algebra structure on U.

$\forall \mathfrak{g}$: $CC^{\bullet}_{\mathrm{II}}(\underbrace{U(\mathfrak{g})})$ is an A_∞ algebra.

or any bialgebra U

What is it?

I know : when U is cocommutative Hopf algebra

U

(Hopf) algebra

$\boxed{d(u) = \sum (u^{(1)})(u^{(2)})}$

$\boxed{\text{coBar}(\overline{U})}$

coalgebra

$\overline{U} = \ker(\epsilon)$

dg algebra

$(u_1|\dots|u_n)$ a.k.a. $(u_1)(u_2)\dots(u_n)$

Freely generated by (u), $u \in \overline{U}$

$\mathcal{U}$ cocomm. Hopf algebra

U acts upon $\mathrm{cobar}(\overline{U})$

Hopf alg dg algebra (free as a gr alg)

$u \cdot (u_1 | \dots | u_n)$, mult., differential.

$$u(v) = \sum \left(u^{(1)} \cdot v \cdot S u^{(2)} \right)$$

$$U \ltimes \mathrm{cobar}(\overline{U}) \simeq C^{\bullet}_{\mathbb{I}}(U)$$

How about $\underline{B}$?

Thm

$$\boxed{\left(U \ltimes \mathrm{cobar}(\overline{U})\right)[\![u]\!] \quad \underbrace{b}_{A} + uB} \cong CC^{\bullet}_{\mathrm{II}}(U)$$

$\underbrace{b}_{A}$: diff in $U \ltimes \mathrm{cobar}(\overline{U})$

A_∞ isom

B : on generators

$u \longmapsto 0$

$\uparrow$

(u)

free gens of $\mathrm{cobar}(\overline{U})$

well-defined when U is cocommutative.

$$M = \sum c_{k,l}\, m^k \underbrace{(m^l)}_{\text{free generators of } \mathrm{cobar}(\overline{U})}$$

$$\sum c_{k,l}\, x^k y^l$$

$$\|$$

$$F(x,y)$$

$$\|$$

$$\tilde{F}(m,(m))$$

$$F(x,y) = \sum_{n=1}^{\infty} \frac{x(x-y)(x-2y)\cdots(x-(n-1)y)}{n!\,u^n}$$

Corollary: even, say, if we have a product on a $\mathbb{Z}$-module A which is associative mod p:

$$CC^{per}_{\cdot}(A)^{\wedge} = A \otimes \bar{A}^{\otimes \cdot}((u))$$

p-adic

$$M + uB$$

defines a differential on it.

(nc crystalline cplx of A/pA).

Cyclic objects and action of S^1.

Λ - cyclic category of Connes

$\Lambda \cong \Lambda^{op}$

$$\underbrace{\Delta^{op}}_{\text{simplicial}} \hookrightarrow \Lambda^{op}$$

$$\underline{\Lambda^{op}} \to \mathcal{C} \quad \text{is a cyclic object of } \mathcal{C}$$

$$\left(\begin{array}{c} \uparrow \\ \Delta^{op} \end{array} \right)$$

Cyclic object of $\mathcal{C}$ $\rightsquigarrow$ A simplicial object of $\mathcal{C}$ (or something) with an action of S^1.

Cyclic <u>set</u> $X_\cdot$ $\rightsquigarrow$ $\underbrace{|X_\cdot|}_{\text{geom. realisation}}$ is a S^1-space

Cyclic object $X_\cdot$ of $\mathcal{C}$ ← some conditions $\rightsquigarrow$ <u>Simplicial</u> obj of $\mathcal{C}$ + $B_\cdot\mathbb{Z}$ action

$$B_n\mathbb{Z} = \mathbb{Z}\times \dots \times \mathbb{Z} \quad n \text{ times}$$

$$\ni (m_0 | \dots | m_{n-1})$$

$E_n\mathbb{Z} = B_n\mathbb{Z}\times\mathbb{Z}$

a variation on the Drinfeld's constr. of $|X_\cdot|$

$$\begin{tikzcd} B_2\mathbb{Z} \arrow[d, shift left=2] \arrow[d] \arrow[d, shift right=2] \\ B_1\mathbb{Z} \end{tikzcd}$$

$$\begin{tikzcd} & (m_0|m_1) \arrow[dl, "d_0"'] \arrow[d] \arrow[dr, "d_2"] & \\ (m_1) & (m_0+m_1) & (m_0) \end{tikzcd}$$

Simplicial (Abelian) group.

$$\underset{\text{cyclic obj}}{X_\cdot} \rightsquigarrow \|X\|_\cdot \quad \text{simplicial object}$$

$$B_\cdot\mathbb{Z}\times\|X_\cdot\| \to |X_\cdot|$$

$$X_\bullet \rightsquigarrow \|X_\bullet\| \circlearrowleft B_\bullet\mathbb{Z}$$

a mixed complex (when $\mathcal{C} = \mathcal{A}b$)

Classically:

$$(X_\bullet, b_\bullet, B)$$

$$X_\bullet^{cplx} = (\|X_\bullet\|, b = \underbrace{d_0 - d_1 + \dots \pm d_n})$$

$$X_\bullet^{mix\,cplx} = (\|X\|_\bullet, b, B)$$

$$B = ?$$

$$\begin{array}{c} (\mathbb{Z}G_\bullet)^{cplx} \otimes X_\bullet^{cplx} \\ \big\downarrow \\ (\mathbb{Z}G \otimes X_\bullet)^{cplx} \\ \big\downarrow \text{action of } G_\bullet \\ X_\bullet^{cplx} \end{array}$$

$\|X_\bullet\|$, $\Sigma \pm d_j$

$E\mathbb{Z} \neq E\text{-}Z$

eil. zilber

$\uparrow$ of Δ^{op}-~~sets~~s

$\mathbb{Z}$-

$\boxed{\epsilon_{E\text{-}Z}\,\epsilon = 0}$

$G_\bullet = B_\bullet\mathbb{Z}$

$(\mathbb{Z}G_\bullet)^{cplx}$

$\|$

$\mathbb{Z}G_\bullet$ with

$d_0 - d_1 + d_2 - \cdots$

$\boxed{\epsilon} \in (\mathbb{Z}G_1)^{cplx}$

$\|$

$(1) \qquad 1 \in \mathbb{Z}$

$$X_\cdot^{\text{mix cplx}} = \left(\| X_\cdot \|,\ b,\ B = \underset{E \cdot Z}{\in X?} \right)$$

<u>**Claim**</u> equiv. to the usual one

(M. Hoyois)

How does this relate to $\otimes$?

(Interesting) $(X_\cdot \boxtimes Y_\cdot)^{m.c.}$ + two cyclic Ab grps

vs

$b \otimes 1 + 1 \otimes b,\ B \otimes 1 + 1 \otimes B$ $X_\cdot^{m.c.} \otimes Y_\cdot^{m.c.}$

MRT

Moulinos - Toën - Robalo

A. Raksit

1. Δ^{op}- mods / $B.\mathbb{Z}$

$\updownarrow$

Comodules over the cosimplicial Hopf algebra

$k = \mathbb{Z}_{(p)}$: FILTERED $\underbrace{\mathcal{O}(B\mathrm{Fix})}$

in terms of Witt vectors

$\mathrm{gr}\,\mathcal{O}(B\mathrm{Fix})$
$\|$
$\mathcal{O}(B\ker)$

Comodules over that are

$\downarrow$

$\mathbb{Z}$ ⊗ Mix. cplxs

⊗"

First step / $\mathbb{Z}$:

basis: $\frac{x(x-1)\dots(x-n+1)}{n!}$

Simplicial mods over $\mathbb{Z}[B_\bullet\mathbb{Z}]$

$\updownarrow$ dual

Simplicial comods over $\mathrm{Fun}(\mathbb{Z}^{X_\bullet})$

$= \mathrm{Fun}_{\mathbb{Z}}(\mathbb{Z})^{\otimes \bullet}$

Polynomial functions $\mathbb{Z} \to \mathbb{Z}$ $\to$ $\mathrm{Fun}(\mathbb{Z})$

Cyclic $\mathbb{Z}$-mod $\rightsquigarrow$ simplicial module over $\mathbb{Z}[B.\mathbb{Z}] \longrightarrow$

mixed complex

$$X_\bullet \rightsquigarrow \|X\|_\bullet \rightsquigarrow \|X\|_\bullet,$$

$$b = d_0 - \ldots \pm d_\bullet,$$

$$\left(\varepsilon = (1) \in \mathbb{Z}[B_1\mathbb{Z}]\right) \quad B = \varepsilon \underset{E-Z}{\times} {}^{-}$$

Does this respect the $\otimes$ on a) cyclic $\mathbb{Z}$-mods ($\boxtimes$) and b) on mixed complexes?

$$(b \otimes 1 + 1 \otimes b,\ B \otimes 1 + 1 \otimes B)$$

As far as I understand, the difficulty is: