0. Done, or almost done:

Geometric realization of a cyclic set is a space w/ $\mathbb{T}$ action.

<u>From this</u>:

$$\left| X \times_\Delta \Lambda \times_\Delta \cdots \times_\Delta \Lambda \right| \simeq |X| \times_{\mathbb{T}} E\mathbb{T}$$

in particular

$$\left| pt \times_\Delta \Lambda \times_\Delta \cdots \times_\Delta \Lambda \times_\Delta pt \right| \simeq B\mathbb{T}$$

$$\simeq E\mathbb{T}/\mathbb{T}$$

From this, and "contractibility of Δ": should be easy that $\mathrm{Nerve}(\Lambda) \simeq B\mathbb{T}$

<u>Rmk</u> Fun detail: $\Delta^{op} \to \mathrm{Fin}_*$ that is used in sym monoidal vs monoidal ∞ categories is the standard S^1.
(significance?..)

1. For more general (than sets) categories $\mathcal{C}$: in what sense does a cyclic object in $\mathcal{C}$ lead to a simplicial object in $\mathcal{C}$ with a $\mathbb{T}$ action?

a). Presheaves on $B\mathbb{T}$. Given a cyclic object $X_\cdot$ of $\mathcal{C}$:

i) Category $\widetilde{N\Lambda^{op}}$ (what is the standard notation?..)

Objects: $[n] \in \mathrm{Ob}\,\Delta^{op}$; $\lambda \in N_n\Lambda$

$$\lambda = j_0 \xrightarrow{\lambda_1} j_1 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_n} j_n$$

Morphisms: for $[n] \xrightarrow{\delta} [m]$ in Δ^{op}

a morphism $[n], \lambda \longmapsto [m], \delta\lambda$

composition ...

2. Functor $\tilde{X} : \widetilde{N\Lambda^{op}} \to \mathcal{C}$

on obj.: $[n], \lambda \longmapsto X_{\cdot j_0}$

on morphisms: $[n], \lambda \to [m], \delta\lambda$

if $\lambda = \dot{j}_0 \xrightarrow{\lambda_1} \dot{j}_1 \xrightarrow{\lambda_2} \cdots \xrightarrow{\lambda_n} \dot{j}_n$

and δ transforms it into

$$\delta\lambda = \dot{j}_k \to \cdots \to \dot{j}_\ell$$

then $[m, \delta\lambda] \longmapsto X_{\dot{j}_k}$ and the morphism

$X_{\dot{j}_0} \to X_{\dot{j}_k}$ is $\lambda_k \cdots \lambda_1$.

So a cyclic object $X_\cdot$ of $\mathcal{C}$ produces a functor

$$\tilde{X} : \widetilde{N\Lambda}^{op} \to \mathcal{C}$$

We also have

$$\widetilde{N\Lambda}^{op} \xrightarrow{p} \Delta^{op}$$

$$[n], \lambda \longmapsto [n]$$

$p_! \tilde{X}, p_* \tilde{X}$ are Δ^{op}-objs in $\mathcal{C}$

(assumption on $\mathcal{C}$: colims exist).

A (good enough?) dfn of a $\mathbb{T}$-obj in $\Delta^{op}\mathcal{C}$: a functor $\widetilde{B\mathbb{T}} \to \mathcal{C}$, which is "same" as $\widetilde{N\Lambda}^{op} \to \mathcal{C}$;

$$\begin{tikzcd} \widetilde{B\mathbb{T}} \arrow[d, "p"'] \\ \Delta^{op} \end{tikzcd} \qquad \begin{aligned} p_! &= (-)_{h\mathbb{T}} \\ p_* &= (-)^{h\mathbb{T}} \end{aligned}$$

Not sure... maybe just a remark about this, and some reference to / reconciliation with Hoyois '15...

Also: from those to mixed complexes (as we go below from simplicial ab groups with $B\mathbb{Z}$ action), using some explicit generator of $H_1(N\Lambda)$...

2. From a cyclic object of $\mathcal{C}$ to a Δ^{op}-obj of $\mathcal{C}$ with an action of $B_\bullet \mathbb{Z}$.

$B_n \mathbb{Z} = \mathbb{Z} \times \dots \times \mathbb{Z}$ (n times)

$E_n \mathbb{Z} = (\underbrace{\mathbb{Z} \times \dots \times \mathbb{Z}}_{n \text{ times}}) \times \mathbb{Z}$

$$(a_1, \dots, a_n) \times a \overset{d_0}{\longmapsto} (a_2, \dots, a_n) \times a$$
$$\longmapsto (a_1 + a_2, \dots, a_n) \times a$$
$$\vdots$$
$$\underset{d_n}{\longmapsto} (a_1, \dots, a_{n-1})(a_n + a)$$

etc.; simplicial Abelian groups.

<u>Drinfeld-like realisation of a cyclic object of $\mathcal{C}$.</u>

Start with the following cocyclic object in simplicial sets with an action of $B\mathbb{Z}$.

$$\|\Lambda(m)\| = \left\{ \begin{array}{c} \text{Maps } \mathbb{Z} \xrightarrow{f} E_\bullet \mathbb{Z} \\ f(j+m+1) = f(j)+1 \end{array} \right\} \Big/ (f \mapsto f+1)$$

Simplicial set with an action of $B_\bullet\mathbb{Z} = E_\bullet\mathbb{Z}/\mathbb{Z}$

Recall: $\Lambda^{op}([n_1],[n_2]) = \{$ nondecreasing maps $\mathbb{Z} \xrightarrow{f} \mathbb{Z}$: $f(j+n_1+1) = f(j)+n_2+1\} / (f \mapsto f+n_2+1)\}$

So: $$\Lambda^{op}([m_1,m_2]) \times \|\Lambda(m_2)\| \to \|\Lambda(m_1)\|$$

<u>So</u>: $\|\Lambda(*)\|$ a cocyclic simplicial set with a $B\mathbb{Z}$ action

Compare: Drinfeld identifies the geometric realisation of $\Lambda([?],[m])$ with the topological space

$$\left\{ \text{Nondecreasing maps } \mathbb{Z} \xrightarrow{f} \mathbb{R} \text{ s.t. } f(j+m+1) = f(j)+1 \right\} / f \mapsto f+1$$

All homotopy equivalent to $\mathbb{T}$.

Roughly, the way to compare:

1) Drop the nondecreasing condition (in both cases you get (something

contractible, free action of $\mathbb{T}/_{\mathbb{T}}$);

2) $|E\mathbb{Z}| \sim \mathbb{R}$ as spaces with an action of $\mathbb{Z}$.

Next: For a cyclic object X of $\mathcal{C}$, $X \rightsquigarrow \|\Lambda(-)\|_{\bullet} \underset{\Lambda}{\times} X_{\bullet}$

And for a Δ^{op}-object of $\mathcal{C}$ $\mathcal{X}$ w/a $B\mathbb{Z}$ action:

$$\mathcal{X} \rightsquigarrow \mathrm{Hom}_{B\mathbb{Z}\text{-}\Delta^{op}\text{-sets}}(\|\Lambda(-)\|_{\bullet}, \mathcal{X}_{\bullet})$$

Cor... ?!

Next:

Cyclic Abelian groups vs mixed complexes

A mixed complex: $(M_{\bullet}, b, B)$ $\bullet \geq 0$

$|b| = -1$, $|B| = 1$,

$(b+B)^2 = 0$

(i.e. dg mods over $\mathbb{Z}[\epsilon]$, $\epsilon^2 = 0$, $|\epsilon| = 1$)

Cyclic object $\rightsquigarrow$ mixed complex
$M_\bullet$ $\qquad\qquad$ $(M_\bullet, b, B)$

An attempt at Dold-Kan:

Recall DK:

$$m \geq 0$$
$$\wr$$
A cosimplicial complex: $C^{nor}_\bullet(\Delta^m)$

$C^{nor}_\bullet$ = chain complex $/ \langle$im of $s_j\rangle$

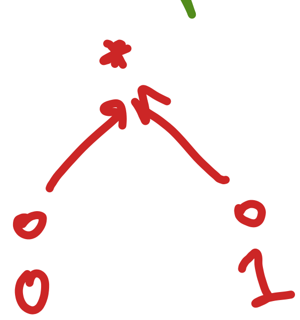
(recall $\Delta^m = \Delta([-],[m])$)

Complex $\quad A_0 \xleftarrow{\partial} A_1 \xleftarrow{\partial} A_2 \xleftarrow{\partial} \cdots$

$$\wr$$

$$m \longmapsto \mathrm{Hom}_{cplxs}(C^{nor}_\bullet(\Delta^m), A_\bullet)$$

simplicial Abelian group.



$$\longrightarrow \quad (A_0 \leftarrow A_1 \leftarrow \cdots)$$

$$a_0^{(1)} \xleftarrow{\partial} a^{(1)}$$
$$-a_1^{(1)}$$

$[0] \longmapsto A_0$

$[1] \longmapsto A_0 \oplus A_1$, namely: $a^{(1)}$ and $a_0^{(1)}$.

$[2] \longmapsto A_2 \oplus A_1^{\oplus 2} + A_0^{\oplus 2}$ etc.

In fact to define morphism of complexes $C^{nor}_{\bullet}(\Delta^m) \to A_{\bullet}$ is the same as define its values on all faces <u>not</u> on a flag e.g. $(0) \subset (01) \subset (02) \subset \dots \subset (01\dots m-1)$.

So it is natural to try:

A cyclic object in mixed complexes.

$$[m] \longmapsto C^{nor}\left(\Lambda\left([-],[m]\right)\right), b, B$$

$$\parallel$$

$$C^{nor}(\Lambda^m)$$

Might be a problem.

$$\mathrm{Hom}_{\text{mixed cplxs}}\left(C^{nor}(\Lambda^0), \underset{\substack{| \\ \text{mixed complex:}}}{A_{\bullet}}\right)$$

$$C^{nor}(\Lambda^0): \quad \underset{0}{(a_0)} \xrightarrow{B} \underset{1}{(1 \otimes a_0)}$$

(notation: $\Lambda([m],[n]) = \Lambda^{op}([n],[m])$

viewed as natural operations

$A^{\otimes m+1} \to A^{\otimes n+1}$; e.g.

$\Lambda([m],[0])$ consists of operations

$a_0 \longmapsto 1 \otimes \ldots \otimes a_0 \otimes \ldots \otimes 1$;

degenerate when we have 1 in position $\# i > 0$.

So: mod degeneracies, have

$$\begin{tikzcd} a_0 \arrow[r, mapsto, "B"'] & 1 \otimes a_0 \arrow[dl, bend left, "b"] \\ 0 & \end{tikzcd}$$

$$\mathrm{Hom}_{m.c.}\left(C^{nor}_\bullet(\Lambda^0), A_\bullet\right) = A_0$$

But next:

$C^{nor}_\bullet(\Lambda^1)$:

$$\begin{tikzcd} a_0a_1 \arrow[r, mapsto, "B"] & 1 \otimes a_0 a_1 & 1 \otimes a_0 \otimes a_1 \\ a_1a_0 \arrow[r, mapsto, "B"] & 1 \otimes a_1 a_0 & 1 \otimes a_1 \otimes a_0 \\ & a_0 \otimes a_1 \arrow[uur] \arrow[ur, "\underline{B}"] & \\ & a_1 \otimes a_0 \arrow[uuur, bend right] \arrow[uur] & \end{tikzcd}$$

$\underbrace{\qquad}_{0}$ $\underbrace{\qquad}_{1}$ $\underbrace{\qquad}_{2}$

The structure of a (b, B)-module on this:

Generators: $x_0 = \boxed{a_0 a_1}$

$$\begin{tikzcd} \boxed{a_0 a_1} \arrow[d] \\ 1 \otimes a_0 a_1 \end{tikzcd}$$

$$\begin{tikzcd} a_0 \otimes a_1 + a_1 \otimes a_0 - 1 \otimes a_0 a_1 \\ \boxed{1 \otimes a_0 \otimes a_1} \arrow[u] \end{tikzcd} = x_2$$

$x_1 = \boxed{a_0 \otimes a_1}$;

$$\begin{tikzcd} a_0 a_1 - a_1 a_0 \arrow[dr, bend left, "B"] & \\ \boxed{a_0 \otimes a_1} \arrow[u, "b"] \arrow[d] & 1 \otimes a_0 a_1 - 1 \otimes a_1 a_0 \\ 1 \otimes a_0 \otimes a_1 - 1 \otimes a_1 \otimes a_0 \arrow[ur] & \end{tikzcd}$$

So:

$$C_\bullet^{nor}(\Lambda^1) \simeq \mathbb{Z}[b, B]/(b) \cdot x_0$$
$$\oplus\ \mathbb{Z}[b, B] \cdot x_1$$
$$\oplus\ \mathbb{Z}[b, B]/(B) \cdot x_2$$

Therefore

$$\mathrm{Hom}_{m.c.}(C_\bullet^{nor}(\Lambda^1), A_\bullet) \simeq$$
$$\simeq A_0 \oplus A_1 \oplus (\ker B)_2$$

But this does not seem to behave well under colims, etc.

* Is there a better way? *

Another way from cyclic abelian groups to mixed complexes:

Cyclic ab group X.

$\big\downarrow$ 1

Simplicial group with a $B\mathbb{Z}$ action $\mathcal{X}$

$\big\downarrow$ 2

Mixed complex.

② is as follows:

$$B_1\mathbb{Z} \times \mathcal{X}_n \xrightarrow[EZ]{} \mathcal{X}_{n+1}$$ (Eil. Zilber)

The generator of $H_1(B\mathbb{Z})$:

$(1) \in B_1\mathbb{Z}$

Claim: $(1) \times_{EZ} (1) = 0$ in $B_2\mathbb{Z}$

Indeed:
general rule is:

$$(g) \times_{EZ} (h) = \Big((g|0) + (0|h)\Big) - \Big((0|g) + (h|0)\Big) =$$

$$= (g|h) - (h|g)$$

$$\dot{=}$$

<u>Claim</u> The two mixed

complexes obtained from a cyclic abelian group X. are equivalent.

Proof. ?

Next level: Filtered circle.

1. Cyclic object

$\downarrow$

Simplicial object with $B\mathbb{Z}$-acting

$\downarrow$ = (if k-modules)

Simplicial module over simplicial algebra $k[B\mathbb{Z}]$ $= C_\bullet(\mathbb{Z}, k)$

$\downarrow$ =

Simplicial comodule over cosimplicial coalgebra $\mathcal{O}(B_\bullet\mathbb{Z}) = \mathrm{Fun}(B_\bullet\mathbb{Z}, k) = C^\bullet(\mathbb{Z}, k)$

Now: replace $\mathcal{O}(B.\mathbb{Z})$ by something quasi-isomorphic; Simplicial comodules should be equivalent (how?)

① $\mathrm{char}(k) = 0$:

$$\begin{array}{ccc} \mathcal{O}(B.\mathbb{Z}) & \leftarrow & \mathcal{O}(B.\mathbb{G}_a) \\ & & \| \\ & & \mathcal{O}(\mathbb{G}_a^{\times \bullet}) \\ & & \| \\ \mathrm{Fun}(\mathbb{Z}^{\times \bullet}; k) & \leftarrow & k[x_1, \dots, x_\bullet] \end{array}$$

(just evaluate a polynomial in n variables on n integers $k_1, \dots, k_n \in \mathbb{Z}$)

Claim This induces quasi-iso (of differentials $d_0 - d_1 + \dots \pm d_n$)

(again, $\mathbb{Q} \subset k$).

Proof The right hand side is the Cobar construction of the coalgebra $k[x]$, $x \mapsto x \otimes 1 + 1 \otimes x$. Cohomologies on left and right both generated by a class in H^1: x, on the right, its image on the left.

(2) Moulinos-Robalo-Toën [MRT]

Would be still quiso if we could replace $k[x]$ by $k\langle x\rangle$ (divided powers).

But: evaluating at 1 or at integers makes no sense unless $\mathbb{Q} \subset k$. $\left(\frac{x^n}{n!} = \frac{1}{n!} \text{ for } x=1\right)$?

Instead: replace $k[x]$ with

$$\mathbb{Z}\{x\} = \{ p(x) \in \mathbb{Q}[x] \mid p(\mathbb{Z}) \subset \mathbb{Z} \}$$

The basis over $\mathbb{Z}$ is

$$\frac{x(x-1)\dots(x-n+1)}{n!}$$

This is a filtered (BI) algebra whose gr is $\mathbb{Z}\langle x \rangle$.

Because of that:

$$\mathrm{Fun}(\mathbb{Z}^{\times \bullet}) \xleftarrow{\ \mathrm{ev}\ } \mathbb{Z}\{x\}^{\otimes \bullet}$$

morphism of cosimplicial bialgebras and quis for $d = d_0 - d_1 + \dots \pm d_\bullet$

So, in some (simple?) sense:

Cyclic ab grps $\longrightarrow$ Simplicial $\mathbb{Z}\{x\}^{\otimes}$-comod

Over $\mathbb{Z}_{(p)}$ another (earlier?) version of [MRT]:

Divided powers, Frobenius, and the filtered circle. $\boxed{k = \mathbb{Z}_{(p)}}$

a) What are divided powers?

Over $\mathbb{Z}_{(p)}$: ker(Frob).

In fact:

p-typical Witt vectors

$W(A) = \{(a_0, a_1, a_2, \ldots) \mid a_j \in A\}$

Ghost variables:

$$x_n = w_n(a_0, a_1, \ldots, a_n) =$$
$$= a_0^{p^n} + p a_1^{p^{n-1}} + p^2 a_2^{p^{n-2}} + \ldots + p^n a_n$$

Universal $+, \times$ laws such

that

$$(a_0, a_1, \ldots) \longmapsto (x_0, x_1, \ldots)$$

is a ring morphism

$$W(A) \to \prod A$$

$F : W(A) \circlearrowleft$

On ghost variables:

$$(x_0, x_1, \ldots) \longmapsto (x_1, x_2, \ldots)$$

Formal scheme $\ker(F)$:

A-valued points are

$\ker(F : W(A) \circlearrowleft)$, OR:

$$w_n(a_0, \ldots, a_n) = 0, \quad n \geq 1$$

$$a_0^p + p a_1 = 0$$

$$a_0^{p^2} + p a_1^p + p^2 a_2 = 0$$

$$\ldots$$

By induction, we re-write

the above as

$$c_n \cdot a_n = a_0^{p^n} \qquad (c_n \in \mathbb{Z})$$

and check that $v_p(c_n) = v_p(p^n!)$)

Therefore:

$$\mathbb{Z}_{(p)}[\ker F] \simeq \mathbb{Z}_{(p)}\langle x \rangle$$

$(x = a_0)$.

Now, evaluation at int. does not make sense. Instead, consider

$$\mathrm{Fix}(A) = \{ a \in W(A) \mid Fa = a \}$$

$$= \{ (a_0, a_1, \ldots) \mid a_0^p + pa_1 = a_0 ;\ a_0^{p^2} + pa_1^p + p^2 a_2 = a_0; \ldots \}$$

$$\mathbb{Z}_{(p)}[\mathrm{Fix}] = \mathbb{Z}_{(p)}[a_0, a_1, a_2, \ldots] \Big/ \langle w_n(a_0, \ldots, a_n) - a_0 \mid n > 0 \rangle$$

<u>Now</u> we have

$\mathrm{ev}: \mathrm{Fun}(\mathbb{Z}, \mathbb{Z}_{(p)}) \leftarrow \mathbb{Z}_{(p)}[\mathrm{Fix}]$

(evaluation on $n \in \mathbb{Z} \subset W(\mathbb{Z})$).

As above, we get:

$$\begin{gathered}\text{Cyclic } \mathbb{Z}_{(p)}\text{-mods} \\ \downarrow \\ \text{Simplicial comods over the} \\ \text{cosimplicial } \underset{\text{(bi)}}{\text{coalgebra}}\ \mathbb{Z}_{(p)}[\mathrm{Fix}]^{\otimes \bullet} \\ \downarrow \\ \\ \downarrow\end{gathered}$$

What next?

[MRT] relate this to HKR-how exactly, in the above language?

Clearly, it is about the monoidal (vs symmetric monoidal?) structure on cyclic abelian groups.

Somehow, mixed complexes (as (sym?) monoidal category) are equiv. to Δ^{T}-comods / $k[\ker F]^{\times \bullet}$ rather than $k[\mathrm{Fix}]^{\times \bullet}$ ...

More precisely:

In [MRT], it seems:

Can replace $k[\ker F]^{\otimes \bullet}$ by the cosimplicial Hopf algebra which is the free cosimplicial coalgebra generated by an element of degree 1.

From this, they make a conclusion that $\Delta^{op}\ k[\ker F]^{\otimes \cdot}$- comodules $\sim$

$\sim$ Mixed complexes as sym. mon. cats.

And, apparently, this is not true for $\Delta^{op}\ k[Fix]^{\otimes \cdot}$- comodules.

I would think, the cyclic EZ/AW in the book (Getzler-Johnes-Petrack) tells that there is a $(A_\infty \ldots)$ functor from cyclic Ab groups to mixed complexes preserving monoidal (not symmetric monoidal) structure?

Also: we should clarify what [MRT] say about HKR, and note: HKR happens also for Hochschild cochains, while there is no circle there. Does the filtered circle enter?

Can this be looked at in terms of Adams operations and the related filtration?