$\mathcal{D}_X$- modules

X complex analytic smooth

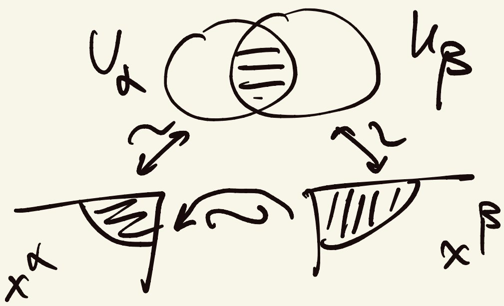
$\mathcal{D}_X^{\leq N} = \{$operators on $\mathcal{O}_X$ locally of the form $\sum_{|I| \leq N} \underbrace{P_I(x)}_{\text{holomorphic}} \partial_x^I$

$\partial_x^I = \partial_{x_1}^{i_1} \cdots \partial_{x_n}^{i_n}$

Fact: this is independent of local coords (will see more).

$$\mathcal{D}_X = \bigcup_{N=0}^{\infty} \mathcal{D}_X^{\leq N} \qquad \mathcal{D}_X^{\leq N} \mathcal{D}_X^{\leq M} \subseteq \mathcal{D}_X^{N+M}$$

$X = \bigcup_\alpha U_\alpha \qquad x_\alpha = x_1^\alpha, \ldots, x_n^\alpha$ holo coords



$x^\alpha = g_{\alpha\beta}(x_\beta)$

Identify

$\mathcal{D}_{U_\alpha} \cong \mathcal{O}_{U_\alpha}[\xi_1, \ldots, \xi_n]$ $\ast_\alpha$

$f \circ \partial_{x_1}^{j_1} \circ \cdots \circ \partial_{x_n}^{j_n} \overset{\alpha}{\longleftrightarrow} f \cdot \xi_1^{j_1} \cdots \xi_n^{j_n}$

$\mathcal{D}_X^\hbar = \operatorname{Rees} \mathcal{D}_X = \sum_{k=0}^{\infty} \hbar^k \mathcal{D}_X^{\leq k}$ $\mathbb{C}[\hbar]$-algebra

$\mathcal{D}_{U_\alpha}^\hbar \cong \mathcal{O}_{U_\alpha}[\xi_1, \ldots, \xi_n, \hbar]$ $\quad f \circ (\hbar \partial_{x_1})^{j_1} \circ \cdots \circ (\hbar \partial_{x_n})^{j_n} \longleftrightarrow f \xi_1^{j_1} \cdots \xi_n^{j_n}$

Fact: ① $$f \underset{\alpha}{*} g = \sum_{|J| \geq 0} \frac{\hbar^{|J|}}{J!} \partial_\xi^J f \cdot \partial_x^J g$$

for the product on $\mathcal{O}_{U_\alpha}[\xi, \hbar]$

② Identify further: $$\mathcal{O}_{U_\alpha}[\xi,\hbar] = \mathcal{O}_{T^*U_\alpha}^{\text{polynomial in } \xi}[\hbar]$$

Then $$\phi_{\alpha\beta} : \mathcal{O}_{T^*U_\alpha}^{poly}\Big|_{U_{\alpha\beta}} \xleftarrow{\sim} \mathcal{O}_{T^*U_\beta}^{poly}\Big|_{U_{\alpha\beta}}$$

are of the form:

$$\phi_{\alpha\beta} = id + \hbar T_{\alpha\beta}^{(1)} + \hbar^2 T_{\alpha\beta}^{(2)} + \dots$$

where $T_{\alpha\beta}^{(j)}$ are differential operators

Proof: $$f_\alpha(x_\alpha) \longmapsto f_\alpha(g_{\alpha\beta}(x_\beta))$$

$$\frac{\partial}{\partial x_\beta}\left[f_\alpha(g_{\alpha\beta}(x_\beta))\right]$$

$$\partial_{x_\beta} g_{\alpha\beta}(g_{\alpha\beta}^{-1}(x_\alpha)) \cdot \partial_{x_\alpha} f_\alpha(x_\alpha) \longleftarrow \frac{\partial}{\partial x_\beta}\left[f_\alpha(g_{\alpha\beta}(x_\beta))\right]$$

$$= \partial_{x_\beta} g_{\alpha\beta}(x_\beta) \cdot (\partial_{x_\alpha} f_\alpha)(g_{\alpha\beta}(x_\beta))$$

Therefore $\quad \Phi_{\alpha\beta}:$

$$\partial_\beta g_{\alpha\beta}\left(g_{\alpha\beta}^{-1}(x_\alpha)\right)\cdot \xi_\alpha \longleftarrow \xi_\beta$$

$$g_{\alpha\beta}(x_\alpha) \longleftarrow x_\beta$$

these are precisely transition isoms of T^*X.

Denote this coordinate change

$$\mathrm{Fun}(x_\alpha, \xi_\alpha) \longleftarrow \mathrm{Fun}(\xi_\beta, x_\beta)$$

by $G_{\alpha\beta}^{T^*}$. Then

$$\Phi_{\alpha\beta} = \left(1 + \hbar T_{\alpha\beta}^{(1)} + \hbar^2 T_{\alpha\beta}^{(2)} + \dots\right)\circ G_{\alpha\beta}^{T^*}$$

$T_{\alpha\beta}^{(k)}$ dif. ops.

We get a sheaf of algebras

$\mathcal{O}^{\hbar}_{T^*X}$ defined by $*_\alpha$, $\phi_{\alpha\beta}$

$$\begin{array}{c} T^*X \\ \downarrow{\scriptstyle pr} \\ X \end{array}$$

$pr^{-1}\operatorname{Rees}\mathcal{D}_X \to \mathcal{O}^{\hbar}_{T^*X}$ inclusion of sheaves of algs

Now: given $(\mathcal{M}, \mathcal{F}_\ell)$ $\mathcal{D}_X$-module with filtration $\mathcal{F}$ compatible w/ order filtration: $\mathcal{D}_X^{\le k} \cdot \mathcal{F}_\ell \mathcal{M} \subset \mathcal{F}_{k+\ell}\mathcal{M}$

$$\operatorname{Rees}(\mathcal{M}) = \sum_{k=0}^{\infty} \hbar^k \mathcal{F}_k \mathcal{M}$$

$$\mathcal{M}^{\hbar} = \mathcal{O}^{\hbar}_{T^*X} \otimes_{pr^{-1}\operatorname{Rees}\mathcal{D}_X} pr^{-1}\operatorname{Rees}\mathcal{M}$$

(microlocalization of $\mathcal{M}$). $SS(\mathcal{M}) := \operatorname{Supp}\mathcal{M}^{\hbar}$

Ex. $\mathcal{M} = \mathcal{O}_X$ $\qquad F_\ell$?

$$\mathcal{D}^{\le n}\cdot F_\ell \subset F_{\ell+n} \qquad F_0 = F_1 = \dots = \mathcal{O}_X$$

$\mathrm{Rees}(\mathcal{M}) = \mathcal{M}^\hbar$:

$$\mathcal{O}_X[\hbar]\cdot 1 \qquad i\hbar\partial_\xi \cdot 1 = 0$$

$X = \mathbb{A}^1$:
$$f(x,\xi)\cdot g(x)\cdot 1 = \sum_{n=0} \frac{\hbar^n}{n!}\partial_\xi^n f(x,0)\cdot \partial_x^n g(x)$$

$$\mathcal{O}^\hbar_{\mathbb{A}^2} \otimes_{pr^{-1}\mathcal{D}^\hbar_{\mathbb{A}^1}} pr^{-1}\mathcal{M}^\hbar \simeq \mathcal{O}_{\mathbb{A}^1}[[\hbar]]$$

with this action

supported on $\{\xi = 0\}$

e.g. $\xi\cdot g(x)\cdot 1 = \cancel{g(x)\xi}\cdot 1 + \hbar g'(x)\cdot 1$

Ex. $\mathcal{M} = \mathcal{O}_X$; ∂_x acts via $\frac{d}{dx} + \varphi(x)$

x ———— x

$$\mathcal{M} \sim \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1}\cdot(\partial_x - \varphi(x))\cdot 1$$

$$\wr\wr$$

$$\mathcal{O}(x)$$

$$\mathcal{M}^\hbar = \mathcal{O}^\hbar_{\mathbb{A}^2} \otimes_{pr^{-1}\mathcal{D}^\hbar} pr^{-1}\mathrm{Rees}(\mathcal{M}) \simeq \mathcal{O}^\hbar_{\mathbb{A}^2} / \mathcal{O}^\hbar_{\mathbb{A}^2}\cdot(\xi - \hbar\varphi(x))$$

still supported at $\xi = 0$.

Ex. $\mathcal{M} = \delta_0 = \mathcal{D}_{\mathbb{A}^1} / \mathcal{D}_{\mathbb{A}^1} \cdot x \simeq \bigoplus_{n \geq 0}^{\infty} \mathbb{C} \cdot \delta_0^{(n)}$

$F_n \mathcal{M} = \mathbb{C} \cdot \delta^{(\leq n)}$ $\qquad$ $\mathrm{Rees}(\mathcal{M}) =$

$$= \bigoplus \mathbb{C}[\hbar] \cdot \hbar^n \delta^{(n)}$$

$$\xi \cdot \hbar^n \delta_0^{(n)} = \hbar^{n+1} \delta_0^{(n+1)}$$

$$\mathcal{M}^{\hbar} = \mathcal{O}^{\hbar}_{\mathbb{A}^2} \underset{pr^{-1}\mathcal{O}_{\mathbb{A}}[\xi,\hbar]}{\otimes} pr^{-1}\mathrm{Rees}(\mathcal{M}) \simeq \mathcal{O}^{\hbar}_{\mathbb{A}^2} / \mathcal{O}^{\hbar}_{\mathbb{A}^2} \cdot x$$

$$\cong \mathcal{O}(\xi)[[\hbar]]$$

$\mathrm{Supp}\, \mathcal{M}^{\hbar} = \{x = 0\}$

$$f(x,\xi) \cdot \varphi(\xi)\delta_0 = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\partial_x^n f \cdot \partial_\xi^n \varphi\right)(0,\xi) \cdot \delta_0$$

e.g.

$$x \cdot \varphi(\xi)\delta_0 - \cancel{\varphi(\xi) \cdot x\delta_0} = -i\hbar\, \varphi'_\xi(\xi) \cdot \delta_0$$

Ex. $X = \mathbb{A}^1$ $\quad$ $\mathcal{M} = \mathcal{O}[x, x^{-1}]$ $\quad$ ∂_x acts via $\frac{d}{dx} + \frac{c}{x}$

$X \longrightarrow X$

Filtration:

$1 \quad x \quad x^2 \quad x^3 \quad \ldots \quad F_0$

$x^{-1} \quad F_1$

$x^{-2} \quad F_2$

$\vdots$

$$\hbar\left(\frac{d}{dx} + \frac{c}{x}\right)(\hbar x^{-1}) = -\hbar^2 x^{-2} + c\hbar^2 x^{-2}$$

Basis of $\mathrm{Rees}(\mathcal{M})$:

$\hbar + \hbar c \ldots$

$1, x, x^2, x^3, \ldots$

$c \quad \hbar x^{-1} \quad \hbar$

$c-1 \quad \hbar^2 x^{-2} \quad \hbar$

$\vdots$

$\xi \quad x$

$$\boxed{x\xi\mathbb{1} = c\hbar\mathbb{1}}$$

$$\mathcal{M}^\hbar = \mathcal{O}^\hbar_{\mathbb{A}^2} \otimes_{pr^{-1}\mathcal{D}^\hbar_{\mathbb{A}^1}} pr^{-1}\mathrm{Rees}\,\mathcal{M} \simeq \mathcal{O}^\hbar_{\mathbb{A}^2} / \mathcal{O}^\hbar_{\mathbb{A}^2}(x\xi - \hbar)$$

invertible away from $x\xi = 0$.

$$\mathrm{supp}(\mathcal{M}^\hbar) = \{x\xi = 0\}$$

Ex) An $\mathcal{O}^\hbar_{T^*X}$ - module not coming from a $\mathcal{D}_X$ - module $\mathcal{M}$:

$$V_f = \mathcal{O}^\hbar_{\mathbb{A}^2} / \mathcal{O}^\hbar_{\mathbb{A}^2}(\xi - f'(x)) \simeq \mathcal{O}(x)[[\hbar]];$$

x acts by x

ξ — by $\hbar\partial_x + f'(x)$

$$\mathrm{supp}(V_f) = \{\xi = f'(x)\}$$

Formally, V_f comes from $\mathcal{M} = e^{f/\hbar} \cdot \mathcal{O}_{\mathbb{A}^1}$

but $e^{f/\hbar} \notin$ anything we know.

Lecture 2-3

More on $\mathcal{O}^{\hbar}_{T^*X}$:

can define also when X is <u>algebraic</u> smooth over $\mathbb{C}$.

In fact: T_X is locally free of rank n;

easy to see $\underset{\substack{\uparrow \\ \text{filtered}}}{\mathcal{D}_X} \underset{loc}{\simeq} \underset{\substack{\uparrow \\ \text{gr}}}{\mathrm{Sym}_{\mathcal{O}_X}} (T_X)$

<u>Also</u>: (bi) dif. ops extend to localizations; if $f \in \mathcal{O}(U)$, $U \subset T^*X$, is invertible on U then $f + \hbar \ldots$ is invertible in $\mathcal{O}^{\hbar}(U)$

$$\left(\underset{\substack{\| \\ \mathrm{Spec}\left(\mathcal{O}\left(pr^{-1}(U_\alpha)\right)_f\right)}}{U} \subset T^*X \right) \longmapsto \left(\mathcal{O}\left(pr^{-1}(U_\alpha)\right)_f [[\hbar]], * \right)$$

principal open subset

Another ex.: $\mathcal{M} = \mathbb{C}[x, x^{-1}]$

x via x

∂_x via $\frac{d}{dx} + \frac{c}{x^2}$

$$\begin{array}{llllll} 0 & 1 & x & x^2 & x^3 & \cdots \\ 1 & x^{-1} & x^{-2} & & & \\ 2 & x^{-3} & x^{-4} & & & \\ & \vdots & & & & \end{array}$$

$$\boxed{x^2\xi \cdot 1 = c\hbar \cdot 1}$$

Rees($\mathcal{M}$):

$$\begin{array}{llll} 1 & x & x^2 & x^3 \ \cdots \\ \hbar x^{-1}, \hbar x^{-2} & & & \\ \hbar^2 x^{-3}, \hbar^2 x^{-4} & & & \\ \vdots & & & \end{array} \otimes \mathbb{C}[\hbar]$$

$$\mathcal{M}^\hbar \simeq \mathcal{O}^\hbar_{\mathbb{A}^2} / \mathcal{O}^\hbar_{\mathbb{A}^2}(\underbrace{x^2\xi - c\hbar}_{\text{invertible outside } x\xi \neq 0})$$

Another way to allow infinite sums:

$U \subset T^*X$

$$p = \sum_{k=-N}^{\infty} p_k(x,\xi) \qquad p_k \text{ homogeneous of degree } -k$$

Fact: $*$ and transition functions G_{uv}:

$$p * q = \sum (p*q)_\ell$$

each $(p*q)_\ell$ is a finite sum; similarly for G_{uv}.

(b/c

$$f * g = \sum_{k=(k_1,\dots,k_n)} \frac{\partial_\xi^k f \cdot \partial_x^k g}{k!}$$

(no $\hbar$ involved).

But $\partial_\xi^k / k!$ decreases degree by 1

∂_x^k keeps it the same

$$\hat{\mathcal{E}}(U) = \left\{ \sum_{k=-N}^{\infty} p_k \,\middle|\, p_k \in \mathcal{O}(U) \text{ of deg } -k \right\}$$

Growth condition: $|p_k| \leq k!\, C_{U_0}^k$ on $\forall U_0 \Subset U$

Fact: preserved by $*$, G_{uv}.

[No chance for C^∞ case; but $\frac{\partial_\xi^k}{k!} p$ has an estimate using integral Cauchy formula].

$$\underset{\text{w/growth condition}}{\mathcal{E}} \subset \hat{\mathcal{E}}$$ sheaves of rings on T^*X

(note: $\mathcal{E}(T^*U) = \mathcal{D}(U)$ $U \subset X$

But away from zero sections: lots of elements of $\mathcal{E}$, $\hat{\mathcal{E}}$

$\hbar$ appears:

$$\mathcal{E}_{T^*(X \times \mathbb{C})}$$

x t

ξ τ

$$\boxed{\tau = \frac{1}{\hbar}}$$

Polesello-Schapira:

$$\{P \subset \mathcal{E} \mid [P, \tfrac{\partial}{\partial t}] = 0\}$$

Away from $\tau = 0$:

$$\left\{ \sum_{k=-N}^{\infty} p_k\left(x, \frac{\xi}{\tau}\right) \cdot \tau^{-k} \right\}$$

Recover $\mathcal{O}^{\hbar}_{T^*X}$ on open subsets $U \times \{\tau \neq 0\}$

Polesello-Sch., D'Agnolo ...

Kashiwara ...

Classification/construction of deformations $\mathcal{O}^{\hbar}_{M}$ (M, ω) holomorphic

__Locally:__ Darboux coords; $\mathcal{O}^{\hbar}_{U}$ using them. __Fact__: any two such are isomorphic.

$$\mathcal{O}^{\hbar}_{U} \underset{G_{UV}}{\overset{\sim}{\longleftarrow}} \mathcal{O}^{\hbar}_{V} \quad \text{on} \quad U \cap V$$

__Fact:__ Any automorphism of $\mathcal{O}^{\hbar}_{U}$ is __inner__

$$\mathcal{O}^{\hbar}_{U} \overset{\sim}{\leftarrow} \mathcal{O}^{\hbar}_{V} \overset{\sim}{\leftarrow} \mathcal{O}^{\hbar}_{W} \quad \text{on} \quad U \cap V \cap W$$

$$G_{UV} G_{VW} = \mathrm{Ad}(c_{UVW}) \cdot G_{UW} \qquad c_{UVW} \in \mathcal{O}^{\hbar \times}_{U \cap V \cap W}$$

Now: $$G_{uv}\, G_{vw}\, G_{wz}$$

$$Ad(c_{uvw})\, Ad(c_{uwz}) \cdot G_{uz} = Ad\, G_{uv}(c_{vwz}) \cdot Ad(c_{uvz})\, G_{uz}$$

Extra work: $c_{uvw} = 1 + \hbar \ldots$

$$c_{uvw}\, c_{uwz} = G_{uv}(c_{vwz})\, c_{uvz}$$

We get an algebroid stack.

From that: sheaf of categories

$U \rightsquigarrow \mathcal{C}(U)$ category

$U \supset V \rightsquigarrow$ functor $\mathcal{C}(U) \xleftarrow{G_{uv}} \mathcal{C}(V)$

$U \subset V \subset W \rightsquigarrow$ natural transf.

$$\begin{tikzcd} \mathcal{C}(u) & \mathcal{C}(v) \arrow[l, "G_{uv}"'] \arrow[u, phantom, "\Uparrow"] & \mathcal{C}(w) \arrow[l, "G_{vw}"'] \arrow[ll, bend right, "G_{uw}"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathcal{C}(u) & \mathcal{C}(v) \arrow[l] & \mathcal{C}(w) \arrow[l] & \mathcal{C}(z) \arrow[l] \arrow[lll, bend right] \arrow[ll, phantom, "\Downarrow \quad = \quad \Downarrow"] \end{tikzcd}$$

Construction: objects of $\mathcal{C}(U)$:
collections

$$U = \bigcup_\alpha U_\alpha \qquad \mathcal{M}_\alpha \in \mathcal{O}^{\hbar}_{U_\alpha}\text{-mod}$$

$$+ \quad g_{\alpha\beta} : \mathcal{M}_\alpha|_{U_{\alpha\beta}} \xrightarrow{\sim} \mathcal{M}_\beta|_{U_{\alpha\beta}}$$

s.t. $\quad g_{\alpha\beta} \circ g_{\beta\gamma} = c_{\alpha\beta\gamma} \cdot g_{\alpha\gamma}$

morphisms: ...

Rmk This is a more natural object than a deformation of the sheaf $\mathcal{O}_M$. How do the latter sit inside the former?

Non-trivial question. Nest-Ts., Bezrukavnikov-Kaledin: the first Rozansky-Witten invariant $RW_\Theta \in H^{0,2}(M)$ appears; G. Papayanov's thesis, to appear.

Rmk Part of the reasoning similar to Maslov class, metaplectic rep, etc. will become closer for enhanced $\mathcal{O}_M^{\hbar}$-modules.

Back to RH

$$\mathcal{D}\text{-mods} \xrightarrow{DR} \text{Sheaves (complexes of)}$$

(holonomic)

(later)

dim Supp $(\mathcal{M}^{\hbar})$ minimal

ex. $\mathcal{M} = \mathbb{C}[x, x^{-1}]$

$(X = \mathbb{A}^1$

∂_x via d/dx or $d/dx + \frac{c}{x^2}$ $\rightsquigarrow$ Same sheaf (triv local system on $x \neq 0$)

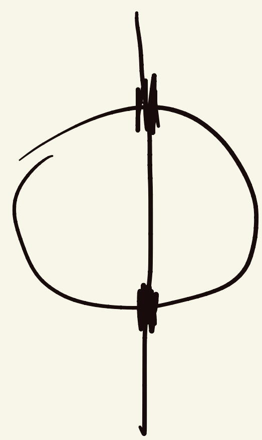
Solution 1: ignore the latter (it is irregular)

Solution 2: use enhanced sheaves on the right.

Trivializing the loc syst of $\frac{d}{dx} + \frac{c}{x^2}$: $s(x) = e^{-c/x}$

Say, $c = -1$

$e^{1/x}$ SMALL



$e^{1/x}$ BIG

(Stokes zones)

So: (irreg) $\mathcal{D}_X$-mod $\rightsquigarrow$ Sheaf + growth conditions

Idea: this is described by a completely algebraic theory of Ind-sheaves (later)

And, to distinguish btw $\frac{d}{dx} - \frac{1}{x^2}$ and $\frac{d}{dx} - \frac{2}{x^2}$: Ind-sheaves on $X \times \mathbb{C}$

Historically:

Polesello-Sch.-D'Agn.:

$X \times \mathbb{C}_t$

Dima T.:

enhanced sheaves = Sheaves on $X \times \mathbb{R}$

(+ microsupport condition)

D'Agnolo-Kashiwara:

use $X \times \mathbb{C}$ in irregular RH

$DR(\mathcal{M})$ as Tor

$$\mathcal{O}_X \leftarrow \mathcal{D}_X \otimes \Lambda^1 T_X \leftarrow \mathcal{D}_X \otimes \Lambda^2 T_X \leftarrow \cdots$$

$$\sum (-1)^{j-1} P f_j \otimes (\ldots \wedge \hat{f_j} \wedge \ldots) \longleftarrow P \otimes (l_1 \wedge \ldots \wedge l_k)$$

$$+ \sum_{i<j} (-1)^{i+j} P \otimes [l_i, l_j] \wedge \ldots \wedge \hat{l_i} \wedge \ldots \wedge \hat{l_j} \ldots$$

$\partial^2 = 0$; well-def., i.e. $Pf \otimes (l_1 \wedge \ldots \wedge l_k) - P \otimes (f l_1 \wedge \ldots) \mapsto 0$

get a locally-free resolution of $\mathcal{O}_X$

For a right $\mathcal{D}_X$-mod $\mathcal{N}$:

$$\mathcal{N} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X = \mathcal{N} \underset{\mathcal{O}_X}{\otimes} \Lambda^\bullet T_X, \text{ same differential}$$

$\mathcal{M}$ left $\mathcal{D}_X$-mod $\rightsquigarrow$ $\underset{\substack{\| \\ \omega_X}}{\Omega_X^{top}} \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$ - right $\mathcal{D}_X$-mod

$l \in T_X$: $\quad l^+(\mathrm{vol} \otimes m) = -(\mathrm{vol} \otimes l m) - \mathrm{vol} \otimes \frac{L_l \mathrm{vol}}{\mathrm{vol}} \cdot m$

for a local generator

$\mathrm{vol} \in \omega_X$

- well-def. under $\mathrm{vol} \rightsquigarrow g \cdot \mathrm{vol}$, $g \neq 0$

This, together with multiplication by functions, defines a right action of $\mathcal{D}_X$.

Indeed:

$$\mathrm{div}_{\mathrm{vol}}(l) := \frac{L_l \mathrm{vol}}{\mathrm{vol}};$$

$$\mathrm{div}(a l) = a\,\mathrm{div}(l) + l(a)$$

$$-(a \cdot l)^+ = a l + \mathrm{div}(a l) = a(l + \mathrm{div}(l)) + l(a)$$

$$\| \qquad\qquad$$

$$-l^+ \cdot a$$

also: transforms well when $\mathrm{vol} \rightsquigarrow g \cdot \mathrm{vol}$, $g \neq$

Define $\qquad \mathcal{D}\mathcal{M} = \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{M}$

$\qquad\qquad\qquad$ left $\qquad$ right

$$DR(\mathcal{M}) = \mathcal{D}\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{D}_X}{\otimes}} \mathcal{O}_X[-n]$$

as well as

$$DR(\mathcal{M}) = R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{M})$$

Inverse image of $\mathcal{D}_X$-mods

$X \xrightarrow{f} Y \qquad\qquad f^*\mathcal{M} = \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{M}$

$\qquad\qquad \mathcal{M}$

affine case: $\quad \mathcal{O}_X \leftarrow \mathcal{O}_Y$

$$f^*\mathcal{M} = \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{M}$$

The left action of $\mathcal{D}_X$: $\mathcal{O}_X$ as usual

$$\mathcal{M} \xrightarrow{\nabla} \Omega^1_Y \underset{\mathcal{O}_Y}{\otimes} \mathcal{M}$$

[Same as for (flat) connect.

$$\mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\mathcal{M} \xrightarrow{f^{-1}\nabla} \mathcal{O}_X \underset{f^{-1}\mathcal{O}_Y}{\otimes} f^{-1}\Omega^1_Y \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M} \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} (\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{M})$$

Ex. $\begin{array}{ccc} \mathbb{A}^n & \xrightarrow{f} & \mathbb{A}^m \\ \| & & \| \\ X & & Y \end{array}$

$x_1, \dots, x_n \qquad y_1, \dots, y_m$

$$D_{X\to Y} := \mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} D_Y = \mathbb{C}[x_1, \dots, x_n ; \partial_{y_1}, \dots, \partial_{y_m}]$$

$$\| \wr$$

$$\mathbb{C}[x, \partial_y] \simeq \mathbb{C}[x, y, \partial_y] / (y - f(x))\cdot$$

x_j acts via x_j

$$\frac{\partial}{\partial x_j} \longrightarrow \frac{\partial}{\partial x_j} + \sum \frac{\partial f_i}{\partial x_j} \cdot \frac{\partial}{\partial y_i}$$

and D_Y acts on the right.

Identify $\frac{\partial}{\partial y_i}$ with η_i (we are in D_Y, not $\mathcal{O}_{T^*Y}$)

$$D_{X\to Y} = \mathbb{C}[x, \eta];$$

∂_x via $\frac{\partial}{\partial x} + f'(x)\eta$ $\qquad$ ∂_y via η

x via x $\qquad$ y via $-\frac{\partial}{\partial \eta} + f(x)$

$$D_{X\to Y} = \left(D_X \otimes D_Y^{op}\right) / \left(D_X \otimes D_Y^{op}\right)\left(y - f(x);\ \partial_x - f'(x)\partial_y\right)$$

Also

$$D^{\hbar}_{X\to Y} \cong \mathcal{O}^{\hbar}_{T^*X\times T^*Y^-} / \mathcal{O}^{\hbar}\cdot\left(y-f(x);\ \xi-f'(x)\eta\right)$$

more precisely:

$$y_i - f_i(x_1,\dots,x_n) \qquad 1\le i\le m$$

$$\xi_j - \sum_{i=1}^{m} \frac{\partial f_i}{\partial x_j}\cdot\eta_i \qquad 1\le j\le n$$

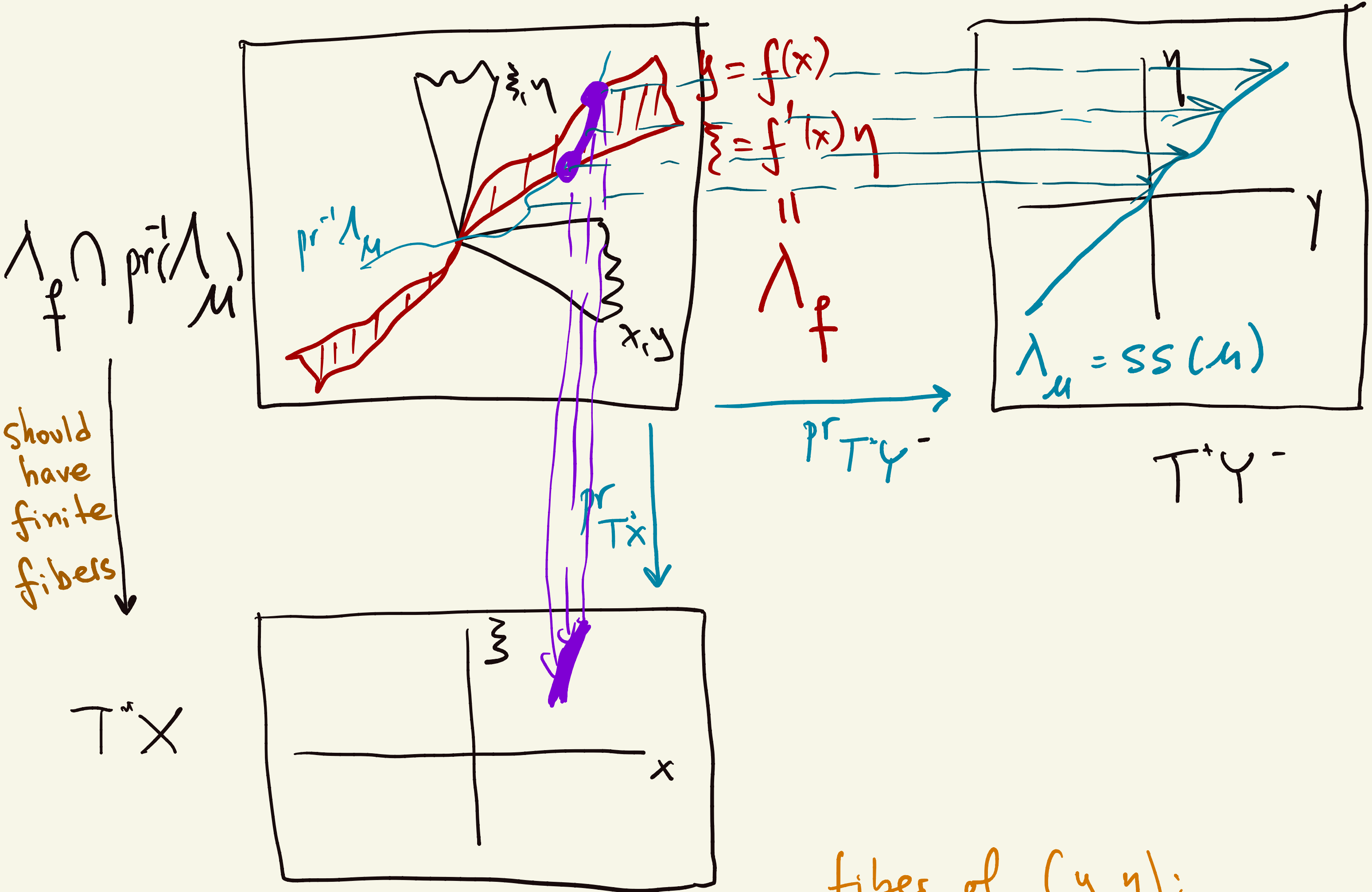
(Note: the $n+m$ functions commute in $\mathcal{O}^{\hbar}_{T^*X}\otimes\mathcal{O}^{\hbar\,op}_{T^*Y}$, and $\overrightarrow{\text{also}}$ Poisson commute wrt $d\xi\cdot dx + \underbrace{dy\cdot d\eta}_{=\,\omega_{T^*Y^-}}$

Support = conormal to graph(df)

By dfn: $f^*\mathcal{M} = D_{X\to Y}\otimes_{\mathcal{O}_Y}\mathcal{M}$

as above.

When should we expect this to be finitely generated$/D_X$?



Reduces to:

fiber of (y, η):

$$(x, \eta) \xmapsto{f^{-1}(\{y\})} (f(x), \eta)$$

$$\begin{array}{ccc} (x,\eta) \quad X \times_Y T^*Y & \longrightarrow & T^*Y^- \hookleftarrow \Lambda_\mu \\ (x,\eta) \mid \eta \in T^*_{f(x)}Y & & (y, \eta) \\ \downarrow & & \\ (x, f'(x)\eta) \quad T^*X & & \\ (x, \xi) & & \end{array}$$

fiber of (x, ξ):

$f'(x)\eta = \xi$

affine space in $T^*_{f(x)}$

Thm Given $X \xrightarrow{f} Y$

$\mathcal{N}$ $\mathcal{D}_X$-mod

f noncharacteristic for $\mathcal{N}$

$\mathbb{D}f_* :=$ left derived of f_*

(i.e. using $\otimes^L$).

Then: $H^k(\mathbb{D}f^*\mathcal{N}) = 0$ for $k \neq 0$

$H^0(\mathbb{D}f^*\mathcal{N}) = f^*\mathcal{N}$ coherent / $\mathcal{D}_X$

$SS(f^*\mathcal{N}) = pr_{T^*X}\left(pr^{-1}_{T^*Y}(SS(\mathcal{N})) \cap \lambda_f\right)$

Direct image:

$$X \longrightarrow Y$$

$$\underset{\shortparallel}{\mathbb{R}f_*}\left(\mathcal{M} \overset{\mathbb{L}}{\underset{\mathcal{O}_X}{\otimes}} \mathcal{D}_{X\to Y}\right)$$

$\mathbb{R}f_+$ coherent if f is a projective morphism

Looking at fibers, expect:

Microlocal version of f_*, f^*:

$$\begin{tikzcd} T^*X \times T^*Y^- \arrow[r] \arrow[d] & T^*Y^- \\ T^*X & \end{tikzcd}$$

$$\mathbb{R}(pr_{T^*Y})_*\left[pr^{-1}_{T^*X}(\mathcal{M}^\hbar) \overset{L}{\underset{pr^{-1}\mathcal{O}^\hbar_{T^*X}}{\otimes}} \mathcal{V}_{N^*(\mathrm{graph}\, f)}\right]$$

preserves coherence if f is projective

$$\mathcal{V}_{N^*(\mathrm{graph}\, f)} = \mathcal{D}^\hbar_{X\to Y}$$

no need, since fibers are affine spaces

$$\mathbb{R}(pr_{T^*X})_*\left[\mathcal{V}_{N^*(\mathrm{graph} f)} \overset{L}{\underset{pr^{-1}\mathcal{O}^\hbar_{T^*Y^-}}{\otimes}} pr^{-1}_{T^*Y^-}(\mathcal{M}^\hbar)\right]$$

Nice symmetry, esp. when we allow $\Lambda \subset T^*X \times T^*Y^-$ to be arbitrary.

Holonomic modules:

Thm $SS(\mathcal{M})$ is conical involutive
(Gabber theorem)

in smooth pts: $T^{\perp}_{SS(\mathcal{M})} \subset T_{SS(\mathcal{M})}$

in some Darboux coordinates: locally,

$$\{\xi_1 = \ldots = \xi_\ell = 0\} \qquad \ell \leq n$$

Cor $\dim SS(\mathcal{M}) \geq n$ (Bernstein ineq)

$\mathcal{M}$ **holonomic** if $SS(\mathcal{M})$ Lagrangian

Lemma Any Lagrangian subspace in a symplectic vector space is in some Darboux coordinates of the form

$$\xi_1 = \ldots = \xi_n = 0 \qquad (\text{or } \xi = 0)$$

And in given Darboux coordinates (x,ξ)

Any (smooth) Lagrangian subvariety is locally of the form

$$\xi_1 = F_{x_1}(x_1, \xi_2)$$
$$x_2 = -F_{\xi_2}(x_1, \xi_2)$$

for some subdivision $x = (x_1, x_2)$
$\xi = (\xi_1, \xi_2)$

linear version:

$$\xi_1 = A_{11} x_1 + A_{12}\xi_2$$
$$-x_2 = A_{12}^t x_1 + A_{22}\xi_2$$

$A_{11}^t = A_{11}$, $A_{22}^t = A_{22}$

Next:

$$\begin{tikzcd}
& \Lambda \arrow[d, phantom, "\cap" description] & \\
& V \times W \arrow[dl, "pr_V"'] \arrow[dr, "pr_W"] & \\
V & & W \\
L & & L \circ \Lambda
\end{tikzcd}$$

$L \circ \Lambda$
$\|$
$pr_W(pr_V^{-1} L \cap \Lambda)$

(by Linear Algebra Done not Right)

$\xi = (\xi_1, \xi_2)$
$x = (x_1, x_2)$
$y = (y_1, y_2)$
$\eta = (\eta_1, \eta_2)$

$$\begin{matrix} \xi_1 & \xi_2 & \eta_1 & \eta_2 \\ x_1 & x_2 & y_1 & y_2 \end{matrix}$$

— coordinates on $V \times W$

$$\xi_1 = A_{11} x_1 + A_{12} \xi_2 + A_{13} y_1 + A_{14} \eta_2$$
$$-x_2 = A_{12}^t x_1 + A_{22} \xi_2 + A_{23} y_1 + A_{24} \eta_2$$
$$\eta_1 = A_{13}^t x_1 + A_{23}^t \xi_2 + \boxed{A_{33} y_1 + A_{34} \eta_2}$$
$$-y_2 = A_{14}^t x_1 + A_{24}^t \xi_2 + \boxed{A_{34}^t y_1 + A_{44} \eta_2}$$

$\boxed{L: \xi_1 = \xi_2 = 0}$

$\xi_2 = 0$

Any Lagrangian Λ in $V \times W$ for some subdivision $(\star)$

make those $= 0$ (changing $A_{13}, A_{14}, A_{23}, A_{24}$ (same sympl. coord change as when reducing any $L' \subset \{(y, \eta)\}$ to standard form)

$$0 = A_{11} x_1 + A_{13} y_1 + A_{14} \eta_2$$

$L \circ \Lambda$:
$$\eta_1 = A_{13}^t x_1$$
$$-y_2 = A_{14}^t x_1$$
(for some x_1)

$$B = \begin{bmatrix} A_{13} & A_{14} \end{bmatrix} : \left\{ \begin{bmatrix} y \\ \eta \end{bmatrix} \right\} \to \{ x_1 \}$$

$$B^t = \begin{bmatrix} A_{13}^t \\ A_{14}^t \end{bmatrix} : \left\{ \begin{bmatrix} y \\ \eta \end{bmatrix} \right\} \leftarrow \{ x \}$$

$$\zeta = \begin{bmatrix} \eta_1 \\ -y_2 \end{bmatrix} \qquad z = \begin{bmatrix} y_1 \\ \eta_2 \end{bmatrix} \qquad \begin{aligned} x &:= x_1 \\ A &:= A_{11} \end{aligned}$$

Then $L \circ \lambda$:

$$(z, \zeta) : \underline{\text{for some}}\ x : \begin{cases} \zeta = B^t x \\ Bz + Ax = 0 \end{cases}$$

$(A^t = A)$

① It is isotropic:

$$\langle z, \zeta' \rangle - \langle \zeta, z' \rangle = \langle z, B^t x' \rangle - \langle z', B^t x \rangle$$

$$= \langle Bz, x' \rangle - \langle Bz', x \rangle = - \langle Ax, x' \rangle + \langle Ax', x \rangle$$

$$= 0 \quad \checkmark$$

$\dim(\Lambda \circ L)$: can assume $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

$$\begin{bmatrix} B_1 z + x_1 \\ B_2 z \end{bmatrix} = 0$$

$$\zeta = B_1^t x_1 + B_2^t x_2$$

or:

$$\zeta = -B_1^t B_1 z + B_2^t x_2$$

for some x_2

$$B_2 z = 0$$

New Darboux coords $\zeta + B_1^t B_1 z$; z

$\zeta = B_2^t x_2$, $\exists x_2$;

$B_2 z = 0$

✓

But

$\dim \mathrm{Im}(B_2^t) + \dim \ker B_2 = \frac{1}{2} \dim W$

Expect:

Thm f_*, f^* send holonomic to holonomic
(true)

A $\qquad\qquad$ ${}_A\mathcal{M}_B$ $\qquad\qquad$ B

$x_2, \dots, x_n$

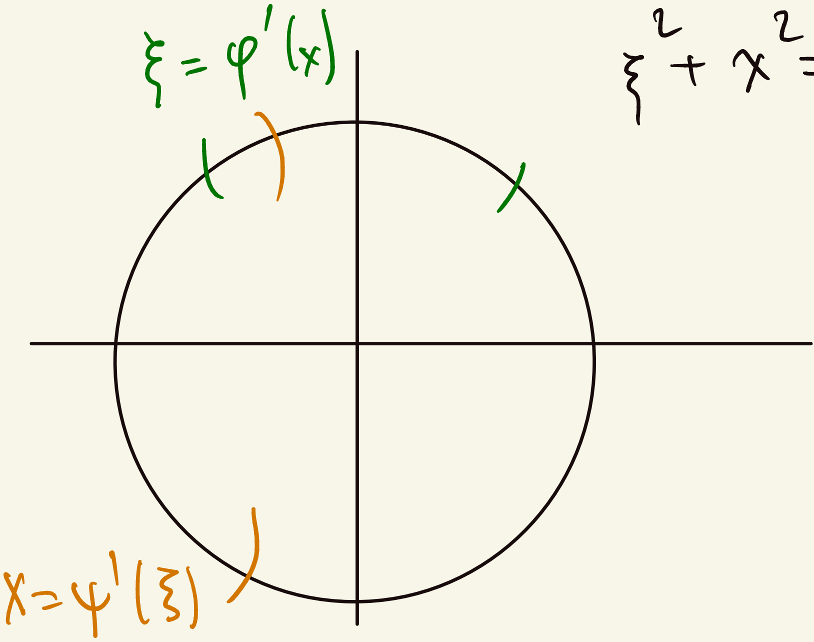
$\xi_2, \dots, \xi_n$

$$A \otimes B^{op} \Big/ \begin{matrix} y_1;\ y_2 - x_2; \dots; y_n - x_n \\ \xi_2 - \eta_2; \dots; \xi_n - \eta_n \end{matrix} \qquad \Bigg| \begin{matrix} y_1, \dots, y_n \\ \eta_1, \dots, \eta_n \end{matrix}$$

$$\mathrm{Hom}_{\substack{x_2 - x_n \\ \xi_2 - \xi_n}}\left(\underline{\qquad}\ ,\ \begin{matrix} y_1 - y_n \\ \eta_1 - \eta_n \end{matrix} \right)$$

As a $\begin{matrix} x_2 - x_n \\ \xi_2 - \xi_n \end{matrix}$ module: $\quad A[\eta_1]$

$\mathrm{RHom}($ $\qquad \xi \quad \eta^*$



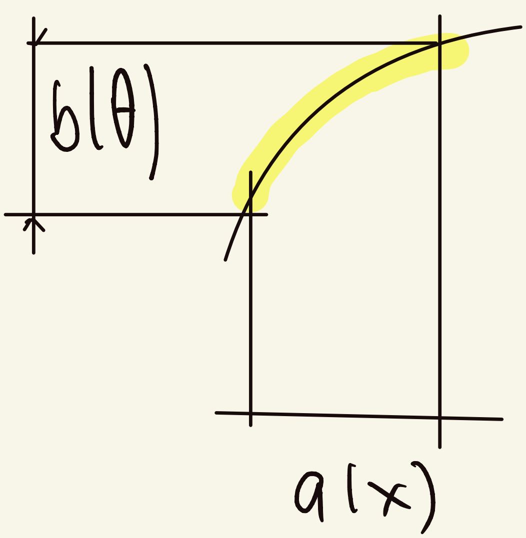
$$\xi^2 + x^2 = R$$

$$\Psi(x,\theta) = x\theta - \psi(\theta)$$

$$\Psi_\theta(x,\theta) = 0 :$$

$$\xi = \Psi_x(x,\theta) :$$

$$\boxed{\begin{aligned} x &= \psi'(\theta) \\ \xi &= \theta \end{aligned}}$$



$$u_\hbar(a)(x) = e^{\frac{1}{i\hbar}\varphi(x)} a(x)$$

$$v_\hbar(b)(x) = \frac{1}{(2\pi\hbar)^{1/2}} \int e^{\frac{1}{i\hbar}(x\theta - \psi(\theta))} b(\theta)\,d\theta$$

When do $a(x)$ and $b(\theta)$ produce same function $u_\hbar$ (up to $O(\hbar^\infty)$)?

1) apply stationary phase method:

Recall:

$$\frac{1}{(2\pi\hbar)^{1/2}}\int e^{\frac{\varphi(\theta)}{i\hbar}}\cdot b(\theta)\,d\theta =$$

$$= e^{\frac{\varphi(\theta_0)}{i\hbar}}\,\frac{e^{\frac{\pi i}{4}\operatorname{sgn}\varphi_{\theta\theta}(\theta_0)}}{\sqrt{|\varphi_{\theta\theta}(\theta_0)|}}\cdot\left(b_0+\sum_{n=1}^{\infty}\hbar^n (D_n b)(\theta_0)\right)$$

$$+\,O(\hbar^\infty)$$

θ_0 - isolated critical point

given ξ;

$(x,\xi)\in L$

$$\frac{1}{(2\pi\hbar)^{1/2}}\int e^{\frac{1}{i\hbar}(x\theta-\psi(\theta))}\,b(\theta)\,d\theta$$

$$\|$$

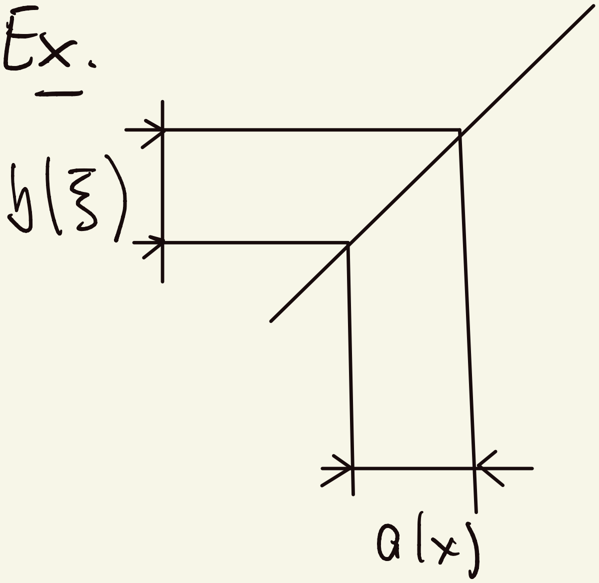
$$e^{\frac{1}{i\hbar}(x\xi-\psi(\xi))}\cdot\frac{e^{\frac{\pi i}{4}\operatorname{sgn}(-\psi_{\xi\xi}(\xi))}}{\sqrt{|\psi_{\xi\xi}(\xi)|}}\cdot\left[b(\xi)+\hbar\ldots\right]$$

$$\|$$

$$e^{\frac{1}{i\hbar}\varphi(x)}\cdot a(x)$$

$$+\,O(\hbar^\infty)$$

Ex.



$\xi = kx \qquad x = k^{-1}\xi$

$$\psi(\xi) = k^{-1}\frac{\xi^2}{2}$$

$$e^{\frac{1}{i\hbar}(x\xi - \psi(\xi))} = e^{\frac{1}{i\hbar}\left[x\cdot kx - k^{-1}(kx)^2/2\right]} =$$

$$= e^{\frac{1}{i\hbar}\cdot k\frac{x^2}{2}} = e^{\frac{1}{i\hbar}\varphi(x)}$$

$$-\psi_{\xi\xi}(\xi) = -k^{-1}$$

$$a(x) = \frac{e^{-\frac{\pi i}{4}\operatorname{sgn}(k)}}{\sqrt{|k^{-1}|}}\cdot\left[b(kx) + \hbar\dots\right]$$

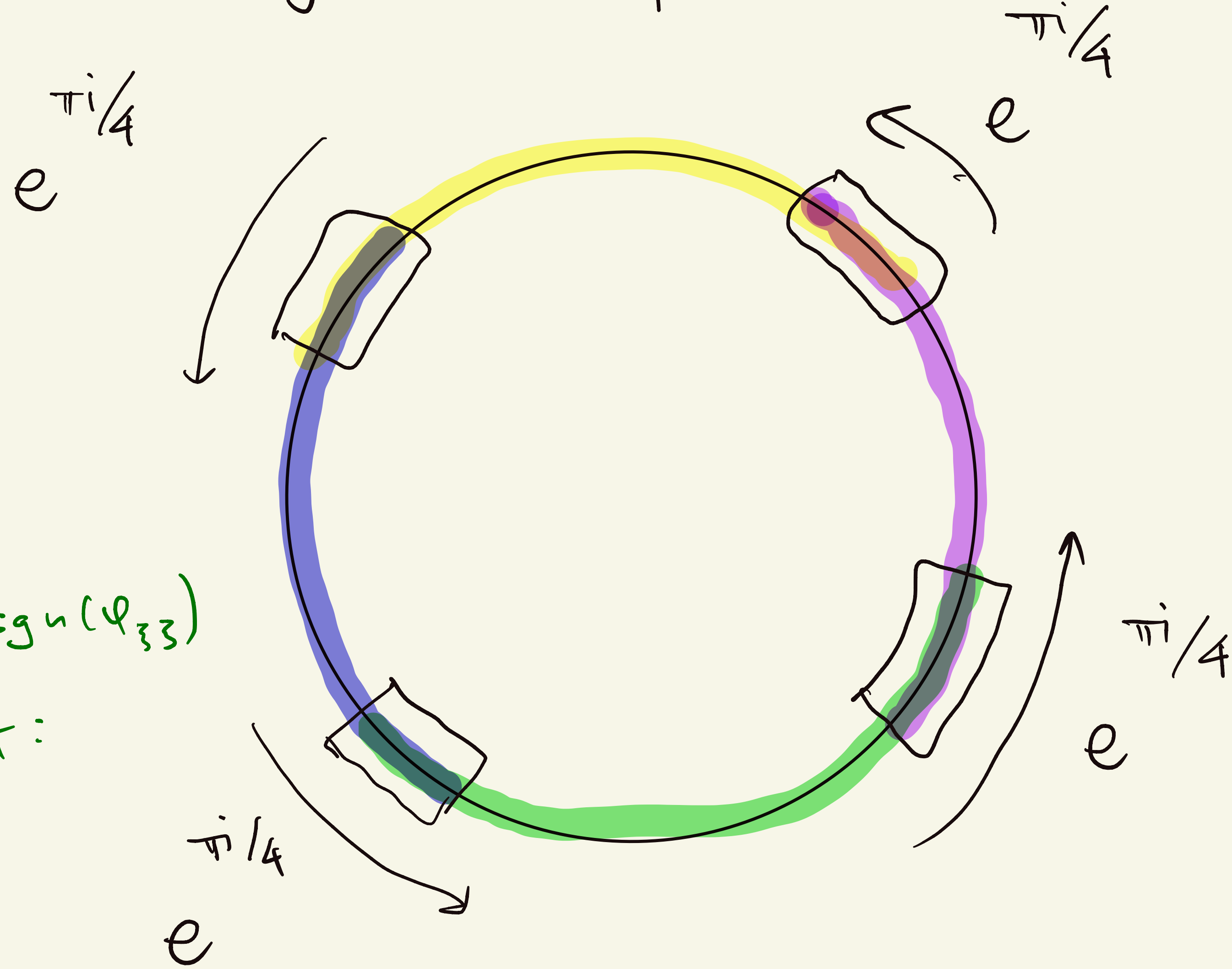
$$\parallel$$

$$e^{-\frac{\pi i}{4}\operatorname{sgn}(k)}\cdot\left[b(kx)\sqrt{|k|} + \hbar\dots\right]$$

in gen.: $\quad x\xi - \psi(\xi) - \varphi(x) = \underline{\text{const}}$ on $\underline{L}$

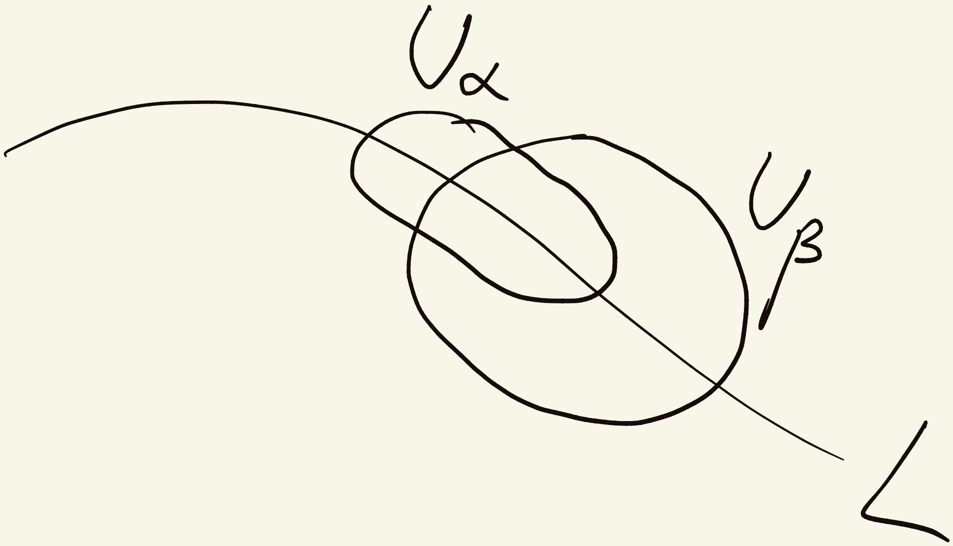
$$\begin{cases} \partial_x\big(x\theta - \psi(\theta) - \varphi(x)\big) = x - \psi'(\theta) = 0 \\ \partial_\theta\big(\qquad\qquad\qquad\qquad\big) = \theta - \varphi'(x) = 0 \end{cases}$$

[φ, ψ defined up to a constant]



The $e^{-\frac{\pi i}{4}\operatorname{sgn}(\varphi_{\xi\xi})}$ part:

The $e^{\frac{1}{i\hbar}(x\theta - \varphi(x) - \psi(\theta))}$ part: easier to do in general.



choose local coords

x_α, ξ_α $\qquad\qquad$ x_β, ξ_β

$\varphi_\alpha(x_\alpha, \theta_\alpha)$ $\qquad\qquad$ $\varphi_\beta(x_\beta, \theta_\beta)$

$$\dim\{\theta_\alpha\} = n_\alpha$$

$$L \cap U_\alpha \simeq \{(x_\alpha, \theta_\alpha) \mid \varphi_{\alpha,\theta}(x_\alpha, \theta_\alpha) = 0\}$$

$L \cap U_\beta$ $\quad \ldots$

$\varphi(x, \theta)$ now a local function on L.

$$\boxed{d\varphi(x, \theta) = \xi \, dx \big|_L}$$

Proof $\quad d\varphi(x,\theta) = \varphi_x(x,\theta)\, dx + \varphi_\theta \cdot d\theta$

$= \varphi_x \cdot dx = \xi\, dx \qquad$ on L

So: $\quad u_{\alpha\beta} = \varphi_\alpha(x_\alpha, \theta_\alpha) - \varphi_\beta(x_\beta, \theta_\beta)$

constant on $U_\alpha \cap U_\beta \cap L$

By def. defines the DeRham class of $\xi\, dx|_L$.

So we get:

$$M = \bigcup_\alpha U_\alpha$$

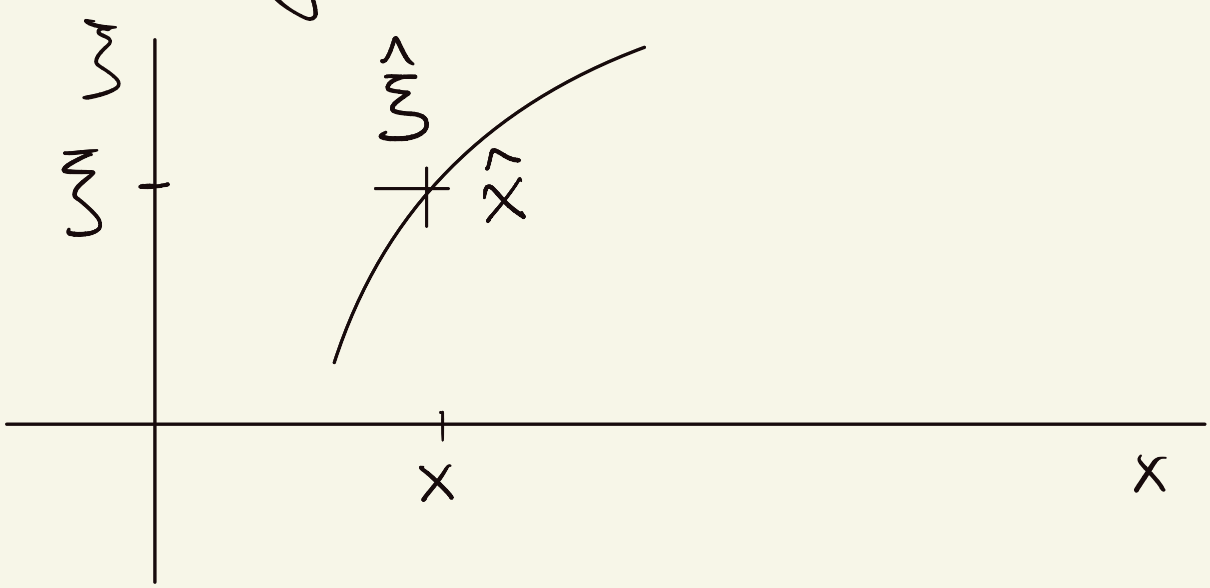
$$V_\alpha = |\Omega^{1/2}|_{L \cap U_\alpha} [[\hbar]] \langle e^{\frac{c}{i\hbar}} \rangle_{c \in \mathbb{R}}$$

$\mathcal{O}^\hbar_{U_\alpha}$ - modules transition isomorphisms:

$$e^{\frac{1}{i\hbar}\lambda_{\alpha\beta}} \cdot e^{\frac{\pi i}{4}\mu_{\alpha\beta}} \cdot \left[1+\sum_{n=1}^{\infty} \hbar^n D_n\right]$$

dif. ops

Remark: local reduction to the case of no linear terms



$$e^{\frac{1}{i\hbar}\varphi(x+\hat{x})} a(x;\hat{x})$$

$$\Phi(x;\hat{x}) = \varphi(x+\hat{x}) - \varphi(x) - \varphi'(x)\hat{x}$$

$$\Psi(\theta;\hat{\theta}) = \dots$$

$$\|$$

$$\frac{1}{(2\pi\hbar)^{1/2}} \int e^{\frac{1}{i\hbar}\left((x+\hat{x})(\theta+\hat{\theta}) - \psi(\theta+\hat{\theta})\right)} b(\theta;\hat{\theta})\, d\hat{\theta}$$

$$e^{\frac{1}{i\hbar}\varphi(x)} \cdot e^{\frac{1}{i\hbar}\cancel{\varphi'(x)}\hat{x}}\, e^{\frac{1}{i\hbar}\Phi(x;\hat{x})} a(x;\hat{x})$$

$$\|$$

$$\frac{1}{(2\pi\hbar)^{1/2}} \int e^{\frac{1}{i\hbar}(x\xi - \psi(\xi))} \cdot e^{\frac{1}{i\hbar}\left[(x\cancel{\hat{\theta}} + \cancel{\theta}\hat{x}) - \cancel{\psi'(\theta)\hat{\theta}}\right]} \cdot e^{\frac{1}{i\hbar}(\hat{x}\hat{\theta} - \Psi(\theta;\hat{\theta}))} b(\theta;\hat{\theta})\, d\hat{\theta}$$

$$e^{\frac{1}{i\hbar}(\varphi(x) + \psi(\xi) - x\xi)} \cdot a(x;\hat{x})$$

$$\|$$

$$\frac{1}{(2\pi\hbar)^{1/2}} \int e^{\frac{1}{i\hbar}[\hat{x}\hat{\theta} - \Psi(\theta;\hat{\theta})]} b(\theta;\hat{\theta})\, d\hat{\theta}$$

We reduce to the case:

$$(\text{const})\cdot e^{\frac{1}{i\hbar}\Phi(\hat{x})}\, a(\hat{x}) = \frac{1}{(2\pi\hbar)^{1/2}} \int e^{\frac{1}{i\hbar}(\hat{x}\hat{\theta} - \Psi(\hat{\theta}))}\, b(\hat{\theta})\, d\hat{\theta}$$

$$L: \quad \hat{\xi} = \Phi'(\hat{x}), \qquad \hat{x} = \Psi'(\hat{\xi})$$

Φ, Ψ : no const/linear terms; nondegenerate critical point 0.

This can be done formally, at the level of power series:

$$\Phi(\hat{x}) = \sum_{n=2}^{\infty} \frac{\varphi_n}{n!} \hat{x}^n \qquad 0 \neq \varphi_2 = \psi_2^{-1}$$

$$\Psi(\hat{\xi}) = \sum_{n=2}^{\infty} \frac{\psi_n}{n!} \hat{\xi}^n$$

$$\text{Fourier}_\hbar\left(b(\hat{\theta})\cdot e^{\frac{1}{i\hbar}\sum_{n=3}^{\infty} \frac{\psi_n}{n!} \hat{\xi}^n}\cdot e^{\frac{1}{i\hbar}\psi_2 \frac{\hat{\theta}^2}{2}} \right) =$$

$$= b(i\hbar\partial_{\hat{x}})\cdot e^{\frac{1}{i\hbar}\sum_{n=3}^{\infty} \frac{\psi_n}{n!}(i\hbar\partial_{\hat{x}})^n} \left[\frac{e^{?\pm\frac{\pi i}{4}\operatorname{sgn}\psi_2}}{\sqrt{|\psi_2|}} \cdot e^{-\frac{1}{i\hbar}\varphi_2 \frac{\hat{x}^2}{2}} \right]$$

But $$\frac{\psi_n}{n!}(i\hbar\partial_{\hat{x}}^n)\left(e^{-\frac{1}{i\hbar}\frac{\varphi_2 \hat{x}^2}{2}}\right) = \left(\frac{\psi_n}{n!}(\varphi_2\hat{x})^n + \hbar\ldots\right) e^{\frac{1}{i\hbar}\ldots}$$

Easy to see:

$$a(x;\hat{x}) = \underbrace{e^{\frac{1}{i\hbar}[x\xi-\varphi(x)-\psi(\xi)]}}\, e^{?\frac{\pi i}{4}\operatorname{sgn}\Phi_{\hat{x}\hat{x}}} \cdot \sqrt{|\Phi_{\hat{x}\hat{x}}|}\cdot \underbrace{\left(id+\sum \hbar^n D_n\right) b(\Phi'(\hat{x}))}_{\text{coordinate change in half-densities}}$$

So we refined the construction of the module V_{L_ω}:

Sections on U_α:

$$\mathrm{Jets}|\Omega|^{1/2}_L[[\hbar]]\langle e^{\frac{c}{i\hbar}}\rangle_{c\in\mathbb{R}}$$

(jets of half-densities)

Transition isomorphisms:

$$G_{\alpha\beta} = e^{\frac{1}{i\hbar}S_{\alpha\beta}}\, e^{\frac{\pi i}{4}\mu_{\alpha\beta}}\left(id+\sum_{n=1}^{\infty}\hbar^n D_n\right)$$

D_n: ((x,ξ)-dependent) dif. ops on jets

This is a bundle of modules over the bundle of algebras $\mathrm{Jets}\,\mathcal{O}^{\hbar}_{\mathbb{R}^2}$

(fiber: $\mathbb{C}[[\hat{x},\hat{\xi}]][[\hbar]]$)

Both carry compatible flat connections (later)

Remarks

1. Bohr-Sommerfeld condition:

$$\frac{1}{i\hbar}\left[\xi dx|_L\right] + \mathrm{Maslov}_L = 0 \quad \text{in } H^1(L)?$$

ex.

$$\frac{1}{i\hbar}\underbrace{\pi R^2} + \pi i = 2\pi i N$$

$$\frac{\pi}{i}\left[\frac{R^2}{\hbar} - 1\right] = 2\pi i N$$

$$\frac{R^2}{\hbar} - 1 = -2N \qquad R^2 = \hbar(1-2N), \quad N \in \mathbb{Z}$$

what is the status of this? Our calculation was formal in $\hbar$; now we treat $\hbar$ as actual number. Is it just a hunch/suggestion?

R small.

2. We can try:

solve

$$(\xi^2 + x^2)v = \lambda v \quad \text{in actuall}$$

$$V_{L_R}, \quad R > 0.$$

$$\lambda = R + \hbar \ldots ?$$

a) Probably no chance for a smooth solution; will have discontinuity

b) Does it have any connection to (asymp. of) actuall eigenvalues of $\xi^2 + x^2$ when $\hbar \to 0$, $n \to \infty$, $(n+\frac{1}{2})\hbar \to R$?

3. One thing flashed and disappeared: (got cancelled out)

expressions

$$e^{\frac{c}{i\hbar}} \cdot e^{\frac{1}{i\hbar}\left(\varphi_2 \frac{\hat{x}^2}{2} + \varphi_3 \frac{\hat{x}^3}{3!} + \dots\right)} \cdot a(\hat{x})$$

They will re-appear later when we study enhanced $\mathcal{O}_M^\hbar$ - modules.

4. More general questions of deformation quantization of Lagrangian (and coisotropic) are numerous and interesting:

DQ modules of Kashiwara, Schapira et al; Baranovsky - Ginzburg; ...

5. Half-densities: we described deformation quantization $\mathcal{O}^\hbar_{T^*X}$ in terms of $\mathrm{Rees}(\mathcal{D}_X)$. That is fine; but the better, "canonical" deformation quantization is defined same way in terms of $\mathrm{Rees}(\mathcal{D}_X(|\Omega|^{1/2}))$. (Its Fedosov characteristic class is $\frac{1}{i\hbar}\omega$)

$A = A^\hbar = \mathcal{O}^\hbar(U) = \mathcal{O}(U)[[\hbar]], *$

U a Darboux chart

$\mathrm{Aut}(A)$ (local automs, as of a sheaf of algs):

$$\mathrm{Ad}(\Phi) = \Phi * \_ * \Phi^{-1}$$

$\Phi \in A$

Compare to $\mathrm{Der}(A)$: $\frac{1}{i\hbar}[\Phi, -]$

$$0 \to \frac{1}{\hbar}\mathbb{C}[[\hbar]] \longrightarrow \frac{1}{\hbar}A \longrightarrow \mathrm{Der}(A) \to 0$$

$\left.\begin{array}{l} \mathrm{Der}(A) = \frac{1}{\hbar}A / \frac{1}{\hbar}\mathbb{C}[[\hbar]] \\ \mathrm{Aut}(A) = A^\times / \mathbb{C}[[\hbar]]^\times \end{array}\right\} \begin{array}{l} \text{– ders} \\ \text{– auto} \end{array}$ of A

Try to rectify:

$\left.\begin{array}{l} \frac{1}{\hbar}A \\ A^\times \end{array}\right\} \begin{array}{l} \text{– ders} \\ \text{– auto} \end{array}$ of what?

of A + extra structure

1. Polesello-Schapira:

(Quantized) contact transformations

later

$$(x,u) \xmapsto{\ \gamma\ } (y,v)$$

$\cup \qquad\qquad \cup$

symplectomorphism of Darboux charts

upgrade to:

$$(x,t;\xi,\tau) \longmapsto (y,s;\eta,\tau)$$

1) Homogenous under:

$(x,t;\xi,\tau) \mapsto (x,t;\lambda\xi,\lambda\tau)$ $\qquad (y,s;\eta,\tau) \mapsto (y,s;\lambda\eta,\lambda\tau)$

2) $\tau dt + \xi dx \longmapsto \tau ds + \eta dy$

3) If $u = \xi/\tau$ $\qquad v = \eta/\tau$

then

$$(x,u) \longmapsto (y,v) = \gamma(x,u)$$

(if forget t,s,τ

i) For any γ there is $\uparrow$.

Given γ: $\quad (x,u) \xmapsto{\ \gamma\ } (y,v)$

$$du\,dx = dv\,dy \qquad u\,dx = v\,dy - da(x,u)$$

Look for: $\quad (x,t;\xi,\tau) \longmapsto (y,s;\eta,\tau)$

extend γ to:

$$\tau dt + \xi dx \quad = \quad \tau ds + \eta dy$$
$$\| $$
$$\tau(ds + v\,dy)$$
$$\| $$
$$\tau(dt + u\,dx - da(x,u))$$

$$ds = d(t + a(x,u))$$

$$\boxed{s = t + a(x,u)}$$

Any two extensions of γ differ by

$$(x,t;\xi,\tau) \longmapsto (x, t+c; \xi,\tau)$$

$$=$$

Examples 1 γ: $(x,u) \longmapsto (x, u + f'(x))$

$$v\,dy - u\,dx = df(x) \qquad a(x,u) = f(x)$$

$$(x,t;\xi,\tau) \longmapsto (x, t - f(x); \xi + \tau f'(x), \tau)$$

$$(\xi + \tau f'(x))\,dx + \tau d(t - f(x))$$
$$\|$$
$$\xi dx + \tau dt \quad = \quad \xi dx + \cancel{\tau df} + \tau dt - \cancel{\tau f'(x) dx}$$

Infinitesimal generator: $\displaystyle -f(x)\frac{\partial}{\partial t} + \tau f'(x)\frac{\partial}{\partial \xi}$

$= \{\tau f(x), -\}$

(2) $(x,u) \longmapsto (x-\psi'(u), u)$

$$v\,dy = u\,dx - u\psi'(u)\,du$$

$$a(x,u) = -\int u\psi'(u)\,du = -\int u\,d\psi' = -u\psi' + \psi(u)$$

$$(x,t;\xi,\tau) \longmapsto \left(x-\psi'\left(\tfrac{\xi}{\tau}\right); t-\psi\left(\tfrac{\xi}{\tau}\right)+\left(\tfrac{\xi}{\tau}\right)\psi'\left(\tfrac{\xi}{\tau}\right), \tau\right)$$

Infinitesimal generator:

$$-\psi'\left(\tfrac{\xi}{\tau}\right)\frac{\partial}{\partial x} - \left(\psi\left(\tfrac{\xi}{\tau}\right) - \left(\tfrac{\xi}{\tau}\right)\psi'\left(\tfrac{\xi}{\tau}\right)\right)\frac{\partial}{\partial t}$$

$$\left\{\tau\psi\left(\tfrac{\xi}{\tau}\right), -\right\} = \left(\psi\left(\tfrac{\xi}{\tau}\right) - \tfrac{\xi}{\tau}\psi'\left(\tfrac{\xi}{\tau}\right)\right)\frac{\partial}{\partial t} + \psi'(\xi)\frac{\partial}{\partial x}$$

$$-\left\{\tau\psi\left(\tfrac{\xi}{\tau}\right), -\right\}$$

(3) $\gamma: \quad x \longmapsto u \longmapsto -x$

$$(y,v) = (-u,x)$$

$$v\,dy = -x\,du \qquad v\,dy - u\,dx = -d(xu)$$

$$a(x,u) = -xu$$

$$(x,t;\xi,\tau) \longmapsto \left(\underset{y}{-\tfrac{\xi}{\tau}}, \underset{s}{t+x\tfrac{\xi}{\tau}}; \underset{\eta}{\tau x}, \underset{\tau}{\tau}\right)$$

$$\left\{\frac{\xi^2-x^2}{2}, -\right\} = \xi\partial_x - x\partial_\xi$$

$$\tau\,ds + \eta\,dy =$$
$$= \tau\,dt + \tau\,d\left(\frac{x\xi}{\tau}\right) + \tau x\,d\left(-\frac{\xi}{\tau}\right)$$
$$= \tau\,dt + \xi\,dx + \cancel{x\,d\xi} - \cancel{\frac{1}{\tau}d\tau}\cdot x\xi - \cancel{x\,d\xi} + \cancel{\frac{d\tau}{\tau}}\cdot x\xi$$
$$= \tau\,dt + \xi\,dx$$

3 $(y,v) = (\cos\varepsilon . x + \sin\varepsilon . u,\ -\sin\varepsilon . x + \cos\varepsilon . u)$

$$v\,dy - u\,dx = (-\sin\varepsilon . x + \cos\varepsilon . u)(\cos\varepsilon . dx + \sin\varepsilon . du)$$

$$= -\sin\varepsilon . \cos\varepsilon . x\,dx + \cos\varepsilon . \sin\varepsilon . u\,du$$
$$- \sin^2\varepsilon . x\,du + \cos^2\varepsilon . u\,dx - u\,dx$$

$$= \sin\varepsilon\cos\varepsilon\,(u\,du - x\,dx) - \sin^2\varepsilon\,(x\,du + u\,dx) =$$

$$= \sin\varepsilon . \cos\varepsilon . d\left(\frac{u^2}{2} - \frac{x^2}{2}\right) - \sin^2\varepsilon . d(xu)$$

$$a(x,u) = \sin\varepsilon\cos\varepsilon\left(\frac{u^2}{2} - \frac{x^2}{2}\right) - \sin^2\varepsilon . xu$$

$$(y,s;\eta,\tau) = \left(\cos\varepsilon . x + \sin\varepsilon . \frac{\xi}{\tau},\ t - \sin\varepsilon\cos\varepsilon\left(\frac{\xi^2}{2\tau^2} - \frac{x^2}{2}\right) + \sin^2\varepsilon . \frac{x\xi}{\tau};\ -\tau\sin\varepsilon . x + \tau\cos\varepsilon . u,\ \tau\right)$$

Infinitesimal generator:

$$\frac{\xi}{\tau}\frac{\partial}{\partial x} - \left(\frac{\xi^2}{2\tau^2} - \frac{x^2}{2}\right)\frac{\partial}{\partial t} - x\tau\frac{\partial}{\partial \xi} =$$

$$= \left\{\frac{\xi^2}{2\tau} + \tau\frac{x^2}{2},\ -\right\}$$

Contact transformations and phase functions (compare to [GKS])

$$\gamma : U \xrightarrow{\sim} V$$

$\mathrm{graph}(\gamma) \subset U \times V^-$ Lagrangian

Another way to construct the function a (primitive to $v\,dy - u\,dx$):

$\varphi(x, y; \theta)$ phase function of graph(γ)

$$\text{graph}(\gamma) = \{(x, y, \quad) \mid u = \varphi_x(x, y, \theta);\ v = \varphi_y(x, y, \theta) \text{ for some } \theta \text{ s.t. } \varphi_\theta(x, y, \theta) = 0\}$$

Then: $\text{graph}(\gamma) \simeq \{(x, y, \theta) \mid \varphi_\theta(x, y, \theta) = 0\}$
$$\downarrow \varphi$$
$$\mathbb{C}$$

is our function $a(x, u)$.

Ex. 1 $\varphi(x, y, \theta) = (x-y)\theta + \phi(y)$ $\qquad \varphi|_{\text{graph}} = \phi(x) = \phi(y)$

$x = y;\ u = \theta;\ v = -\theta + \phi'(y)$

2 $\varphi(x, y, \theta) = (x-y)\theta + \psi(\theta)$

$$\left.\begin{array}{l} x - y + \psi'(\theta) = 0 \\ u = \theta \quad v = -\theta \end{array}\right\} \quad \begin{array}{l} y = x + \psi'(u) \\ v = -u \end{array}$$

$$\varphi|_{\text{graph}} = -\psi'(\theta)\theta + \psi(\theta)$$

3 ...

Quantized contact transformations

Recall: $\hat{\mathcal{E}}_{\mathcal{V}} = \left\{ \sum_{j=-\infty}^{N} P_j(x, \xi) \mid P_j \text{ homog of deg } j \right\}$

UCT^*X
Darboux chart

We make a change: They extend dif. ops on half-forms (densities) $\Omega_X^{top/2}$. (not on functions $\mathcal{O}_X$) $\hat{\mathcal{E}}^{\sqrt{v}}_{T^*X}$

This means: coordinate isoms are:

$$\xi \longmapsto$$

$$x \longmapsto g_{uv}(x)$$

$$f(g_{uv}(x))\, g'_{uv}(x)^{1/2} dx \longleftarrow f(x)\,|dx|^{1/2}$$

$$i\hbar\partial_x \downarrow$$

$$\left.\begin{array}{l} g'_{uv}(x).\, i\hbar\partial_x f(g_{uv}(x)) \cdot g_{uv}^{1/2}\, dx \\ + \tfrac{1}{2} \cdot i\hbar \cdot \dfrac{g'_{uv}(x)}{g_{uv}(x)} \cdot f(g_{uv}(x))\, dx \end{array}\right\} \longrightarrow \begin{array}{l} g'_{uv}(g_{uv}^{-1}(x))\, i\hbar f \\ + \tfrac{1}{2} \cdots \end{array}$$

which leads to:

$$\xi \longmapsto \tfrac{1}{2}\left(g'_{uv}(g_{uv}^{-1}(x)\right) \underset{w}{*} \xi$$

$$x \longmapsto g_{uv}(x)$$

here $a \underset{w}{*} b = \frac{1}{2}(a * b + b * a)$

The advantage of this:

$x \mapsto x$

$\xi \mapsto -\xi$

is an anti-autom $\hat{\mathcal{E}}^{\sqrt{v}}_{T^*X}$ ↺

$$\mathcal{W}^{\sqrt{v}}_{T^*X} \subset \mathcal{E}^{\sqrt{v}}_{\dot{T}^*(X\times\mathbb{C})}:$$

$$\dot{T}^*(X\times\mathbb{C}) = \{\tau\neq 0\} \times t$$

Those who commute with $\partial_t = \tau$

They are of the form

$$\sum_{j=N}^{\infty} P_j(x, \xi/\tau)\cdot\tau^{-j}$$

a QCT:

$$\hat{\mathcal{E}}^{\sqrt{v}}_{U\times\dot{T}^*\mathbb{C}} \xleftarrow{\sim} \hat{\mathcal{E}}^{\sqrt{v}}_{V\times\dot{T}^*\mathbb{C}}$$

lifting a contact transformation $U \xrightarrow{\Xi} V$ and preserving $*$ (the anti-involution).

Fact: if $\gamma = \mathrm{id}$, any Q.C.T is uniquely of the form:

$$T_c \circ \mathrm{Ad}(P) \qquad T_c(x,\xi,t,\tau) = (x,\xi,t+c,\tau)$$

P of ord 0, $\sigma(P) = 1$, $PP^* = 1$;

<u>Now</u>: choose G_γ - a QCT over any γ.

<u>Then</u>:

$$G_{\gamma_1} G_{\gamma_2} = Ad\left(c(\gamma_1,\gamma_2)\right) \cdot G_{\gamma_1\gamma_2}$$

where $c(\gamma_1,\gamma_2)$ is the $\mathcal{I}$ for the QCT over id, namely:

$$G_{\gamma_1} G_{\gamma_2} G^{-1}_{\gamma_1\gamma_2}$$

<u>and automatically</u>:

$$c(\gamma_1,\gamma_2)\cdot c(\gamma_1\gamma_2,\gamma_3) = G_{\gamma_1}\left(c(\gamma_2,\gamma_3)\right)\cdot c(\gamma_1,\gamma_2\gamma_3)$$

(Recall: (a good example should be included)

$$G_{\gamma_1} G_{\gamma_2} G_{\gamma_3} = Ad\left(c(\gamma_1,\gamma_2)\right)\cdot G_{\gamma_1\gamma_2} \cdot G_{\gamma_3}$$

$$\| \qquad \cdot = Ad\left(c(\gamma_1,\gamma_2)\cdot c(\gamma_1\gamma_2,\gamma_3)\right)\cdot G_{\gamma_1\gamma_2\gamma_3}$$

$$G_{\gamma_1} \cdot Ad\left(c(\gamma_2\gamma_3)\right)\cdot G_{\gamma_2\gamma_3} = Ad\left(G_{\gamma_1} c(\gamma_2,\gamma_3)\cdot c(\gamma_1,\gamma_2\gamma_3)\right))$$

So: get the <u>canonical</u> stack/sheaf of categories) on any (M,ω).

Can get other ones by twisting c_{uvw} by any 2-cocycle in $H^2(M, \mathbb{C}[[\hbar]]^\times)$

We will later see other ways of rectifying automorphisms of $\mathcal{O}^{h}$.

<u>But first</u>: back to D-modules.

$$f : \mathbb{A}^n \longrightarrow \mathbb{C} \qquad f^*$$
$x = (x_1, \dots, x_n)$ $\quad t$

$$\mathbb{C}[x_1, \dots, x_n] \longleftarrow \mathbb{C}[t]$$

$$\mathbb{C}[x_1, \dots, x_n] \underset{\mathbb{C}[t]}{\otimes} \mathbb{C}[t, \partial_t] \simeq \mathbb{C}[x_1, \dots, x_n, \tau]$$
$\partial_t = \tau$

$x_1, \dots, x_n$ act via x_i.

∂_{x_i} via $\underbrace{\dfrac{\partial}{\partial x_j} + \dfrac{\partial f}{\partial x_j} \cdot \tau}_{\text{the role of pulling back connection}}$

$t, \dfrac{\partial}{\partial t}$ act <u>on the right</u>:

τ by multiplication by τ;

$$t = \frac{\partial}{\partial \tau} + f(x_1, \dots, x_n)$$

Going from right to left $\mathbb{C}[t, \partial_t]$-mod:

$$x \quad \text{via} \quad x \cdot$$
$$\partial_x \quad \text{via} \quad \partial_x + \tau f'(x)$$
$$t \quad \text{via} \quad -\frac{\partial}{\partial \tau} - f(x)$$
$$\tau \quad \text{via} \quad \tau$$

Recognize the QCT. Or in other words:
Take the standard module

$$x \quad \text{via} \quad x \qquad \partial_x \text{ via } \partial_x$$
$$t \quad \text{via} \quad -\frac{\partial}{\partial \tau} - f(x)$$
$$\tau \quad \text{via} \quad \tau$$

and twist it by a formal expression

$$\exp(\tau f(x))$$

And, consistent with the above: put

$$\tau = 1/\hbar$$

$$s = \hbar \frac{\partial}{\partial \hbar}$$

Now: two formal exercises

1) $e^{\frac{f(x)}{\hbar}} \cdot \delta_0$ 2) $f^s \cdot \delta_1$ (will need to invert f)

Just start acting by x_j, ∂_{x_j}, $\hbar\partial_\hbar$, $\hbar$

$$
\begin{tikzcd}
f'(x)\cdot e^{f/\hbar}\cdot\delta_0 & e^{f(x)/\hbar}\,\delta_0 \arrow[l, "\hbar\partial_x"'] \arrow[r, "x"] & x\, e^{f(x)/\hbar}
\end{tikzcd}
$$

$\vdots$ x^2 $\vdots$

$$u(x,\hbar) \xrightarrow{\hbar\partial_x} \underline{(\hbar u' + f')}\, e^{f/\hbar}$$

$$\mathbb{C}[\hbar, x]\cdot e^{\frac{f(x)}{\hbar}}$$

closed under $\mathbb{C}[x, \hbar\partial_x]$

$$\hbar\partial_\hbar\left(u\cdot e^{f(x)/\hbar}\right) = \left(\hbar\partial_\hbar u - \frac{f(x)}{\hbar}u\right) e^{f/\hbar}$$

(as above; $\partial_\hbar$ acts via $\partial_\hbar - \frac{1}{\hbar^2}$, and $\tau = \frac{1}{\hbar}$)

Now:

$$v(x,s)\cdot f^s$$

$\mathbb{C}[x,s]$ by multiplication

$$\partial_x\left(v(x,s)\cdot f^s\right) = v'(x,s) + \frac{sf'}{f}\cdot v(x,s) f^s$$

$$\begin{tikzcd}
e^{f/\hbar} \arrow[d, "s"] & & f^s \arrow[d] \\
-\frac{f}{\hbar}\cdot e^{f/\hbar} \arrow[d] \arrow[rr, leftrightarrow] & & s f^s \arrow[d] \\
\left(+\frac{f}{\hbar}+\frac{f^2}{\hbar^2}\right)\cdot e^{f/\hbar} \arrow[rr, leftrightarrow] & & s^2 f^s \arrow[d] \\
= \left(-\frac{f^3}{\hbar^3} - 3\frac{f^2}{\hbar^2} - \frac{f}{\hbar}\right) e^{f/\hbar} \arrow[rr, leftrightarrow] & & s^3 f^s
\end{tikzcd}$$

$$-\left(\frac{f}{\hbar}+\frac{f^2}{\hbar^2}\right)\frac{f}{\hbar} - \frac{f}{\hbar} - 2\frac{f^2}{\hbar^2} =$$

$$+\frac{f}{\hbar} e^{f/\hbar} = -s f^s$$

$$\left(\frac{f}{\hbar}\right)^2 e^{f/\hbar} = (s^2+s) f^s$$

$$(f/\hbar)^3 e^{f/\hbar} = -(s^3+3s^2+2s) f^s$$

$$\cdots$$

$$(f/\hbar)^n = (-1)^n s(s-1)\dots(s-n+1)$$

Two tangents from

$$e^{f/\hbar} \quad \text{vs} \quad f^s$$

1). Deligne's way to rectify $\mathrm{Aut}(\mathcal{O}_U^\hbar)$

U Darboux chart $\mathcal{O}_U^\hbar = A_U = A = \dots$

$$= \mathcal{O}_U[[\hbar]], *$$

enlarge the <u>Lie algebra</u>:

$$\mathrm{Der}_{\mathbb{C}[[\hbar]]}(\mathcal{O}_U^\hbar) = \tfrac{1}{\hbar}\mathcal{O}_U^\hbar \Big/ \tfrac{1}{\hbar}\mathbb{C}[[\hbar]]$$

$$a*b - b*a$$

enlarge <u>that</u>:

$$\mathcal{L}(A)$$

$$\|$$

$\{$Ders of the form $\hbar\frac{\partial}{\partial \hbar} + \underbrace{F}_{\mathbb{C}[[\hbar]]\text{-linear}}$

e.g. $\hbar\frac{\partial}{\partial\hbar} + \xi\frac{\partial}{\partial\xi}$

$\hbar\frac{\partial}{\partial\hbar} + \frac{1}{2}\left(x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial\xi}\right)$

Two such differ by $\frac{1}{\hbar}[\Phi, -]$

And locally you <u>choose</u> a lifting:

$$\begin{tikzcd}
\frac{1}{\hbar}\mathbb{C}[\hbar] \arrow[r, hook] \arrow[d] & \frac{1}{\hbar}A \arrow[r] \arrow[d] & \tilde{\mathcal{L}}(A) \arrow[r, two heads] \arrow[d] & \mathbb{C}\cdot\hbar\partial_\hbar \arrow[d, equal] \\
0 \arrow[r] & \frac{1}{\hbar}A/\frac{1}{\hbar}\mathbb{C}[\hbar] \arrow[r] & \mathcal{L}(A) \arrow[r, two heads] & \mathbb{C}\cdot\hbar\frac{\partial}{\partial\hbar}
\end{tikzcd}$$

<u>All such $\tilde{\mathcal{L}}(A)$ are isomorphic.</u>

<u>Indeed</u>: <u>choose</u> $D_0 \longmapsto \hbar\partial_\hbar$

We have: $[D_0, \mathrm{ad}(\frac{1}{\hbar}\phi)] = \mathrm{ad}(\frac{1}{\hbar}D_0(\phi))$

So $\left[D_0, \frac{1}{\hbar}\phi\right] = \frac{1}{\hbar}D_0(\phi) + \frac{1}{\hbar}\underset{\substack{\cap \\ \mathbb{C}[\hbar]}}{\alpha(\phi)}$

$$\left[D_0, \left[\tfrac{1}{\hbar}\phi, \tfrac{1}{\hbar}\psi\right]\right] = \left[\tfrac{1}{\hbar}D_0(\phi) + \tfrac{1}{\hbar}\alpha(\phi), \tfrac{1}{\hbar}\psi\right]$$

$$+\left[\tfrac{1}{\hbar}\phi, \tfrac{1}{\hbar}D_0(\psi) + \tfrac{1}{\hbar}\alpha(\psi)\right] = D_0\left(\left[\tfrac{1}{\hbar}\phi, \tfrac{1}{\hbar}\psi\right]\right)$$

$$D_0\left(\left[\tfrac{1}{\hbar}\phi, \tfrac{1}{\hbar}\psi\right]\right) + \tfrac{1}{\hbar}\alpha\left(\tfrac{1}{\hbar}[\phi,\psi]\right)$$

But $\frac{1}{\hbar}[A, A] = A$, so $\alpha = 0$

Automorphisms of $\widetilde{\mathcal{L}}(A)$ (as a sheaf, i.e. over the symplectomorphism id_U) are all of the form $Ad(u)$, $u \in A^\times$:

namely, on $\frac{1}{\hbar}A$: $\frac{1}{\hbar}\phi \mapsto \frac{1}{\hbar}u\phi u^{-1}$

for chosen $D_0 \mapsto \hbar\partial_\hbar$:

$$D_0 \longmapsto D_0 - \hbar\partial_\hbar(u).u^{-1}$$

$\hbar \mapsto -\hbar$ defines an anti-involution on A and on $\widetilde{\mathcal{L}}(A)$; we consider automorphisms preserving the anti-invol.

They are of the form $Ad(\mathbb{P})$ where $\mathbb{P}\mathbb{P}^* = 1$.

Proceeding exactly as with the Polesello-Schapira WKB, we construct for $\gamma : U \xrightarrow{\sim} V$ $G(\gamma) : \widetilde{\mathcal{L}}(U) \xrightarrow{\sim} \gamma^{-1}\widetilde{\mathcal{L}}(V)$ and $c(\gamma_1, \gamma_2)$ s.t. $G(\gamma_1)G(\gamma_2) = Ad\, c(\gamma_1, \gamma_2) G(\gamma_1\gamma_2)$

In fact $\tilde{\mathcal{L}}(\mathcal{V})$ is (almost) identical to the Lie algebra of $\hat{\mathcal{G}}_{\mathcal{V}\times T^*\mathbb{C}}$:

$\hbar = \frac{1}{\tau}$; $t = -\partial_\tau = +\frac{1}{\tau}(\hbar\partial_\hbar)$

Second tangent to $\hbar\frac{\partial}{\partial\hbar} = s$

$(\mathfrak{g}^\cdot; \delta)$ - Lie algebra (dg graded)

u - formal parameter

$\mathfrak{g}^\cdot[\varepsilon, u], \delta + u\frac{\partial}{\partial\varepsilon}$ dga $\quad |\varepsilon| = -1$, $|u| = 2$

$\varepsilon^2 = 0$

$m \in \mathfrak{g}^1$; $R = \delta m + \frac{1}{2}[m, m] = \delta m + m^2 \in \mathfrak{g}^2$

(say, $\frac{1}{2} \in k$)

Can flatten m in $\mathfrak{g}^\cdot[\varepsilon, u][u^{-1}]$

$$\left(\delta + u\frac{\partial}{\partial\varepsilon}\right)\left(m - \frac{\varepsilon R}{u}\right) + \left(m - \frac{\varepsilon R}{u}\right)^2 = \delta m + m^2 \underset{\overset{\shortparallel}{0}}{- R} - \underbrace{\frac{\varepsilon}{u}(\delta R + [m, R])}_{\overset{\shortparallel}{0}}$$

Ex. $\mathfrak{g}^\cdot = \Lambda^{\cdot+1} T_X$, $[\,,\,]_{Sch}$, $\delta = 0$

$\mathfrak{g}^\cdot[\varepsilon, u]$ acts on $(\Omega^\cdot_X[[u]], ud)$

$a + \varepsilon b$ by $L_a + u\,\iota_a$

$\mathfrak{g}^\cdot = C^{\cdot+1}(A)$, $[\,,\,]_{Gerst}$, δ_{Hoch} :

$\mathfrak{g}^\cdot[\varepsilon, u]$ does act on $C_{-\cdot}(A)^{[[u]]}$, $b + uB$

up to homotopy ($\mathbb{Q} \subset k$)

Smthng quasi-isom to $U(\mathfrak{g}^\cdot_A[\varepsilon, u])$ acts.

How about smthg quasi-isom to $U(\varepsilon \cdot \mathfrak{g})[u]$?

First guess : $U(\varepsilon\mathfrak{g}) \simeq Sym(\mathfrak{g}[1])$

$$Sym(\mathfrak{g}[1]) \twoheadleftarrow Cobar\, C_\cdot(\mathfrak{g}[1])$$

$\mathfrak{a} = k \cdot R$
one even element:

$$\mathrm{Cobar}\,(C_\bullet(\varepsilon\mathfrak{a})) = \mathrm{Cobar}\,(U^+(\mathfrak{a}))$$

augm. ideal as <u>coalgebra</u>

<u>In other words:</u> what replaces $U(\mathfrak{a}[\varepsilon])$:

R; (R^n), $n>0$

R commutes with all R^n

<u>Differential:</u> $\partial(R^n) = \sum_{k=1}^{n-1} \binom{n}{k}(R^k)(R^{n-k})$

$$\left.\begin{array}{r} \cdots \\ (R)(R) \xrightarrow{\sim} (R^2) \end{array}\right\} \text{all contractible}$$

$R \qquad (R) \qquad\qquad (R)^2 \sim 0$

<u>Now:</u> on $k[R] \otimes k\langle (R), (R^2), \ldots \rangle [u]$:

what replaces $u\,\partial/\partial\varepsilon$?

<u>One</u> suggestion: $(R) \longmapsto R$

uB: $(R^n) \longmapsto 0$, $n>1$

<u>Another</u>: uB: $(R^n) \longmapsto R^n$, $\forall n \geq 1$

THE SECOND VERSION DOES ACT ON $C_{-\bullet}(A)[[u]]$

How to flatten m?

$$(\delta + \partial + uB)(m - (R) + \dots) + (m - (R) + \dots)^2 = 0$$

our options:

$$m - (R) + \sum_{n=2}^{\infty} \sum_{j=0}^{n-1} c_{n,j} R^j (R^{n-j})$$

Notation: given $F(x,y) = \sum_{j,k} a_{jk} x^j y^k$

$$m_F(R) = m + \sum_{j,k} a_{jk} R^j (R^k)$$

Lemma To make $m_F(R)^2 = 0$:

★ $$F(x,y) = \frac{1}{n!\,u^n} x(x-y)\dots(x-(n-1)y)$$

In fact:

look for $F(R,y) = \sum_{n=1}^{\infty} x_n(R) \cdot y^n$

i) $F(y_1+y_2) - F(y_1) - F(y_2) = F(y_1)F(y_2)$

ii) $u\,F(R) = R$

i) gives

$$F(y) = \sum_{n=1}^{\infty} x(R)^n \cdot \frac{y^n}{n!}$$

ii) gives ☆

(note: the "classical" choice of the differential uB would give:

ii') $\quad x_1(R) = 1$

so for example:

$$F(R,y) = e^{Ry} - 1 \quad \text{will do.}$$

Rmk On $g[\varepsilon]$, the $U(g)$-valued CE cochain

p: $\quad p(\varepsilon x) = x \quad p(x) = 0$

Then two versions of B:

$p: \varepsilon x \mapsto x \quad \varepsilon x_1 \ldots \mapsto 0 \quad e^p - 1 : (\varepsilon x)^n \mapsto x^n$

$n > 1$ RELATION TO TODD/index?..

$$i_+ \mathcal{M} \xrightarrow{\ \sim\ } f^s \cdot \mathcal{M}[s]$$

when f is invertible on $\mathcal{M}$.

Proof:

$i_+\mathcal{O}_X$: $i_+\mathcal{M} \simeq \mathcal{M}[\tau]$

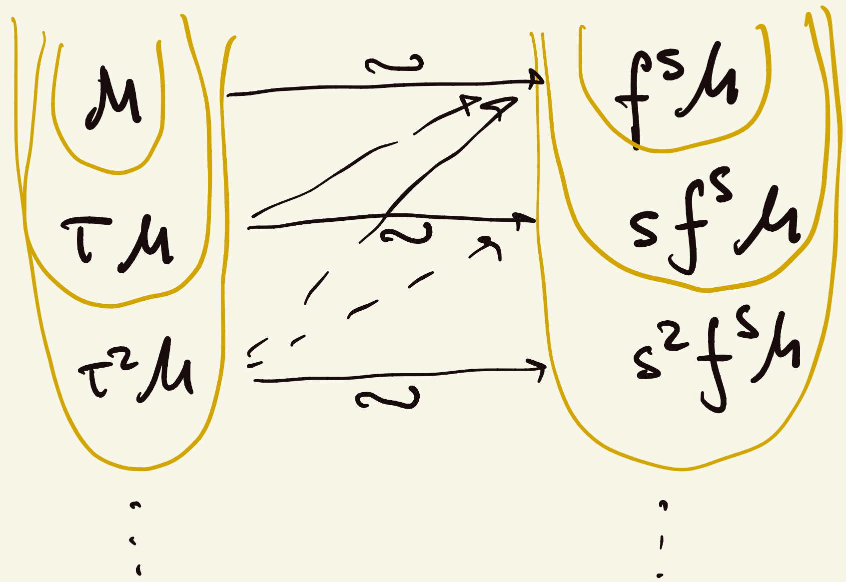
x by x. τ by τ.

∂_x by $\underset{\text{in } \mathcal{M}}{\partial_x} + \tau f'$ $\quad$ t by $-\partial_\tau - f(x)$

Formally:

$$-s = e^{-\tau f} \cdot \tau\partial_\tau \cdot e^{\tau f} = \tau\partial_\tau + \tau f$$

$$\mathcal{M}[\tau] \longleftrightarrow \mathcal{M}[s]\cdot f^s$$



Filtered isomorphism intertwining s with $(\tau\partial_\tau + \tau f)$:

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & f^s \arrow[d, "s"] \\
-\tau f \arrow[r, leftrightarrow] \arrow[d] & s f^s \arrow[d, "s"] \\
(\tau f)^2 + \tau f \arrow[r, leftarrow] & s^2 \\
\dots &
\end{tikzcd}$$

$(\tau f) \longleftrightarrow -s$

$(\tau f)^2 \longleftrightarrow s^2 + s$

$(-1)^n (\tau f)^n \longleftrightarrow s(s+1)\dots(s+n-1)$

This intertwines $-(\tau\partial_\tau + \tau f)$ with s (yes), and also x with itself (obviously). But also: $-(\tau\partial_\tau + \tau f) = -\tau t$ commutes with (the action of) ∂_x. Therefore the horizontal isom intertwines ∂_x with itself. (on M:

$$\begin{tikzcd}
m \arrow[r, leftrightarrow] \arrow[d, "\partial_x"'] & f^s m \arrow[d, "\partial_x"] \\
\partial_x m + \underline{\tau f' \cdot m} & f^s \cdot \partial_x m + \underline{s f' \cdot f^{s-1}}
\end{tikzcd}$$

correct the signs?

Formula for t on the right:

<u>On the left</u>: $-\partial_\tau - f$

$s \sim \tau t$; $[t, s] = -t$ $\qquad ts - st = -t$

$t(s+1) = st \qquad ts = (s-1)t$

$t \cdot s^n m = (s-1)^n f m$

Work over $\mathbb{C}(s)$

$$X_s = \operatorname{Spec} \mathbb{C}(s) \times_{\operatorname{Spec} \mathbb{C}} X$$

$$U_s = \{f \neq 0\}$$

$$\mathcal{N} = f^s \cdot \mathcal{D}_{X_s}$$

<u>Claim</u>: $\mathcal{N}$ is holonomic

1) $\mathcal{N}|_{U_s}$ is holonomic. In fact: as we saw, $\mathcal{N}|_{U_s} \simeq \mathcal{D}_{U_s}(s) / (\partial_x + s\frac{f'}{f})$

2) $i_* \mathcal{M} \longrightarrow f^s \cdot \mathcal{M}[s]$ is <u>always</u> defined, not just when f is invertible.

As we saw: $\partial^n m \mapsto s(s-1)\dots(s-n+1) f^{s-n} m$

But by definition

$$f^s \mathcal{M}[s] := \bigoplus_{n \in \mathbb{Z}} f^{s-n} \mathcal{M}[s] / \sim$$

$$f \cdot f^{s-n} \sim f^{s-n+1}$$

Next: yes, $i_* \mathcal{M}$ is holonomic as a $\mathcal{D}_{X \times \mathbb{A}^1}$- module (its Char is $\{ \xi = \tau f'(x) ; t = f(x) \}$). But: is it holonomic as a $\mathcal{D}_{X_s} = \mathcal{D}_X(s)$- module? far from it.

Ex. $f = 0$:

$$s = -\tau \partial_\tau ; \qquad s(\tau^n m) = n \tau^n m ;$$

$$i_* \mathcal{M} \otimes_{\mathbb{C}[s]} \mathbb{C}(s) = 0$$

So, $N|_{U_s}$ does not have an obvious (holonomic) extension to X_s, for us to start with. Instead:

$N' =$ Maximal Holonomic Submodule

$= \{ n \in N \mid \mathcal{D}_{X_s} \cdot n \text{ is holonomic} \}$

It is a submodule b/c subquotients of holonomic are holonomic.

<u>Rmk</u> N' is the lowest (nth) term of the Gabber filtration:

$$G_k N = \{u \mid \dim \mathrm{Char}(\mathcal{D}_X u) \leq k\}$$

(later; together with the Sato–Kashiwara filtration based on $\mathcal{E}xt^j_{\mathcal{D}}(-)$)

<u>Fact</u>: $N' = N$. In fact:

$$N'|_{U_s} = N|_{U_s}$$

not $\mathbb{C}(s)$-linearly, $\mathbb{C}(s)$-semilin., but ok

N/N' supported on $\{f=0\}$

$\exists k: f^k(f^s \mathcal{M}) \in N'$ $\quad f^{s+k}\mathcal{M}$ holonomic

But $f^{s+k}\mathcal{M} \simeq f^s \mathcal{M}$

$P(s) f^s$
$\updownarrow$
$P(s+k) f^{s+k}$

$\mathcal{N}$ holonomic $\Rightarrow$ finite length. From this:

$$f^{k+s}\mathcal{D}_{X_s} = f^{k+1+s}\mathcal{D}_{X_s}$$

s instead of $k+s$

$$f^s = P(x, \partial_x, s) \cdot f^{s+1}$$

with coeffs in $\mathbb{C}(s)$

Common denom: $b(s)$

$$b(s) f^s = P(x, \partial_x, s) \cdot f^{s+1}$$

with coeffs in $\mathbb{C}[s]$

e.g. $(s+1)x^s = \partial_x x^{s+1}$

Holonomic $\Rightarrow$ finite length:

1) $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ if $k < \operatorname{codim} Ch(\mathcal{M})$

2) $\operatorname{Codim}(\mathcal{E}xt^k(\mathcal{M}, \mathcal{D}_X)) \geq k$

Pf True in comm alg:

$\mathcal{M}$ coh $\mathcal{O}_Z$-mod, Z smooth:

- $\mathcal{E}xt^k_{\mathcal{O}_Z}(\mathcal{M}, \mathcal{O}_Z)$ $\mathcal{O}_Z$-module; codim supp $\geq k$
- $\mathcal{E}xt^k_{\mathcal{O}_Z}(\mathcal{M}, \mathcal{O}_Z) = 0$ for $k <$ codim supp$(\mathcal{M})$

<u>Corollary</u> $\mathcal{E}xt^k_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ <u>unless</u> $k = n$

if $\mathcal{M}$ <u>holonomic</u>

<u>And</u>: $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ is <u>ALSO</u> holonomic (right) $\mathcal{D}_X$-module

<u>So</u>: get <u>duality</u> on <u>holonomic</u> $\mathcal{D}_X$-modules. [Reflexive, since $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \simeq R\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$

Cor. $\mathcal{M}$ Noetherian $\Rightarrow$ increasing chains stabilize. But: on its DUAL, same. So: on $\mathcal{M}$, decreasing chains stabilize.

One hole remaining in the proof:

Why is N/N' coherent?

N' = the biggest holonomic submodule of N

ok, $N'|_{U_s} = N|_{U_s}$; but what if, when we touch $\{f=0\}$, we somehow lose sections u s.t. $\mathcal{D}_{X_s} u$ are holonomic?

Ex. What if, by some accident,

$N' = N|_{U_s}$ extended to X_s by zero?

Then $N/N' = i_* \{\text{germs of sections of } N\}$

$$i : \{f=0\} \hookrightarrow X_s$$

not true that f^k kills them for some k.

Answer: No, that does not happen. N' has cohomological interpretation, coherent by constr.

Later

We compare two computations:

one arising in $\mathcal{D}_X$- modules and comparing $i_{f+}\mathcal{O}_X$ to $f^s \cdot \mathcal{O}_X$ where $f: X \to \mathbb{C}$ and

$$i_f : X \longrightarrow X \times \mathbb{C} \quad x \mapsto (x, f(x));$$

another arising in noncommutative geometry and the study of nc GM connection, nc crystalline cohomology, etc.

COMPUTATION 1

$\partial_x ; x ; \tau ; -\partial_\tau (=t)$ $\qquad$ $\partial_x ; x ; s ; T_s (=t)$

acting on: $\qquad (T_s P)(s) = P(s+1)$

$u(x,\tau)$ $\qquad$ $v(x,s)$

$e^{\tau f}$ $\quad$ Twisted by: $\quad$ f^s

become:

$\partial_x + \tau f' ; x ; \tau ; -\partial_\tau - f$ $\qquad$ $\partial_x + s\dfrac{f'}{f} ; x ; s ; T_s f$

$s = -\tau t = \tau \partial_\tau + \tau f$ $\qquad$ $\tau = -st^{-1} = -s T_{-s} f^{-1}$

$$\begin{tikzcd}
1 & 1 \arrow[d, "s"] \arrow[r, leftrightarrow] & 1 \arrow[d, "s"] & 1 \\
\tau f & \tau f \arrow[d, "s"] \arrow[r, leftrightarrow] & s \arrow[d, "s"] & s \\
(\tau f)^2 & \tau f + (\tau f)^2 \arrow[d, "s"] \arrow[r, leftrightarrow] & s^2 \arrow[d, "s"] & s^2 - s \\
(\tau f)^3 & \tau f + 3(\tau f)^2 + (\tau f)^3 \arrow[r, leftrightarrow] & s^3 & s^3 - 3s + 2
\end{tikzcd}$$

$$\frac{1}{n!}(\tau f)^n \longleftrightarrow \frac{1}{n!}s(s-1)\dots(s-n+1)$$

<u>Formally</u>:

(and using notation $\tau = \frac{1}{\hbar}$)

$$e^{z\frac{f}{\hbar}} \longleftrightarrow (1+z)^s$$

Or, remembering twists:

$$e^{(z+1)\frac{f}{\hbar}} \longleftrightarrow (1+z)^s f^s$$

COMPUTATION 2

Question: find $F(x,y)$

1) $F(x, y_1+y_2) - F(x,y_1) - F(x,y_2)$

$$+F(x,y_1)F(x,y_2) = 0$$

2)$_{cl}$: $u \cdot \partial_y F(x,0) = -x$

or

2)$_q$: $uF(x,x) = -x$

1): $$F(x,y) = 1 - e^{a(x)y}$$

2)$_{cl}$: $a(x) = \frac{1}{u}$ $\quad 1 - F(x,y) = e^{y/u}$

2)$_q$: $$1 - e^{a(x)x} = -\frac{x}{u}$$

$$e^{a(x)x} = 1 + \frac{x}{u}$$

$$a(x)x = \log\left(1 + \frac{x}{u}\right)$$

$$e^{a(x)y} = \left(1+\frac{x}{u}\right)^{y/x}$$

$$F(x,y) = 1 - \left(1+\frac{x}{u}\right)^{y/x}$$

$$-F(x,y) = \sum_{n=1}^{\infty} \frac{y(y-x)\cdots(y-(n-1)x)}{u^n n!}$$

(for 1)$_{cl}$: $\displaystyle\sum_{n=1}^{\infty} \frac{y^n}{u^n n!}$

Compare formally:

$$e^{\frac{z}{\hbar}f} \rightsquigarrow (1+z)^s$$

$$e^{\frac{y}{u}} \rightsquigarrow \left(1+\frac{x}{u}\right)^{\frac{y}{x}}$$

?

Motivation for Computation 2.

Start with a Lie algebra $\mathfrak{g}$
(or dg LA $(\mathfrak{g}; \delta)$)

$\mathfrak{g}[\varepsilon, u]$, $u \frac{\partial}{\partial \varepsilon}$

$|\varepsilon| = 1$; $|u| = 2$; $\varepsilon^2 = 0$

Resolve it over $k[u]$ in two different ways:

1) (cl):

$U(\mathfrak{g}) \ltimes \mathrm{Cobar}\, \mathrm{Sym}^+(\mathfrak{g})$

2) (q):

$U(\mathfrak{g}) \ltimes \mathrm{Cobar}\, U^+(\mathfrak{g})$

Generators:

(i) subalgebra $U(\mathfrak{g})$

(ii) (x) of degree 1

$\ker(U(\mathfrak{g}) \xrightarrow{\varepsilon} k)$

$\|$

$x \in \mathrm{Sym}^+(\mathfrak{g})$ (cl) or $x \in U^+(\mathfrak{g})$ (nc)

Relations:

$$[x,(y)] = (\mathrm{ad}_x(y))$$

$x \in \mathfrak{g}$ $\quad y \in \mathrm{Sym}^+(\mathfrak{g})$

or $U^+(\mathfrak{g})$

Differential: $\partial x = 0,\ x \in \mathfrak{g};$

1) $\partial(x) = \sum (x')(x'')$

where $\Delta x = \sum x' \otimes x''$

2) $uB : x \mapsto 0;$

$$\begin{tikzcd} ux, \ x \in \mathrm{Sym}^1(\mathfrak{g});\ 0,\ x \in \mathrm{Sym}^{>1}(\mathfrak{g}) & (x) \arrow[l, mapsto, "(cl)"'] \arrow[r, mapsto, "(q)"] & ux,\ x \in U^+(\mathfrak{g}) \end{tikzcd}$$

Get two DGAs:

$\mathcal{U}_{cl}(\mathfrak{g})$ and $\mathcal{U}_{nc}(\mathfrak{g})$

Flattening an odd element of $\mathfrak{g}$

Now let $\mathfrak{g}$ be graded; $m \in \mathfrak{g}^1$; $m^2 = \frac{1}{2}\underbrace{[m,m]}_{\in\, \mathfrak{g}^2} =: R$

Looking for: $\mu = m + \dots$

$$d\mu + \mu^2 = 0$$ (whatever the differential d is)

In $g[\varepsilon, u^{\pm 1}]$: $\mu = m - \frac{\varepsilon R}{u}$

$$u\frac{\partial}{\partial\varepsilon}\left(m - \frac{\varepsilon R}{u}\right) + \left(m - \frac{\varepsilon R}{u}\right)^2 =$$

$$= -R + m^2 + \frac{\varepsilon}{u}\cancel{[m,R]} = 0$$

In resolved algebras:

in $\mathcal{U}_{cl}(g)[u^{-1}]$:

$$\mu = m - \sum_{n=1}^{\infty} \frac{(R^n)}{u^n n!}$$

in $\mathcal{U}_{nc}(g)[u^{-1}]$:

$$\mu = m - \sum_{k,l} c_{k,l} \frac{R^k (R)^l}{u^{k+l}(k+l)!}$$

where $$1 + \sum_{k,l} c_{k,l} \cdot \frac{x^k y^l}{(k+l)!} = \sum \frac{y(y-x)\dots(y-(n-1)\dots}{n!}$$

Because: If we look for $\mu = m + \sum ?_{k,l} R^k (R)^l \dots$

then $$(\partial + uB)\mu + \mu^2 = 0$$

give equations 1), $2)_{cl}$ or $2)_q$.

Motivation for $\mathfrak{g}[\varepsilon, u]$, $\mathcal{U}_{cl}$, $\mathcal{U}_{nc}$:

$\mathfrak{g} = \mathrm{Vect}(X)$: $\mathfrak{g}[\varepsilon, u]$ acts on $\Omega_X^{-\bullet}[[u]], ud$

$X + \varepsilon Y$ act by $L_X + \iota_Y$

$\mathfrak{g} = \mathrm{Der}(A)$: $\mathfrak{g}[\varepsilon, u]$ acts on $C_{-\bullet}(A)[[u]]$, $b + uB$

(negative cyclic complex)

up to homotopy in characteristic zero.

What is actually acting is $\mathcal{U}_{nc}(\mathfrak{g})$.

Explicitly:

$$C_n(A) = A \otimes \bar{A}^{\otimes n} \qquad \bar{A} = A/k\cdot 1$$

$$C_\bullet(A) \underset{b}{\overset{B}{\rightleftarrows}} C_{\bullet+1}(A)$$

$$b(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \dots \otimes a_i a_{i+1} \otimes \dots \otimes a_n$$
$$+ (-1)^n a_n a_0 \otimes a_1 \otimes \dots \otimes a_{n-1}$$

$$B(a_0 \otimes \dots \otimes a_n) = \sum_{i=0}^{n} (-1)^{ni} 1 \otimes a_i \otimes \dots \otimes a_n \otimes a_0 \otimes \dots \otimes a_{i-1}$$

now, for $x \in U^+(\mathfrak{g})$, $x : A \to A$ is the action of $U(\mathfrak{g})$.

$$a_0 \otimes \cdots \otimes a_n \overset{(x)}{\longmapsto} a_0 x(a_1) \otimes a_2 \otimes \cdots \otimes a_n +$$

$$+ \sum_j \pm 1 \otimes \underbrace{x(a_j \otimes \cdots \otimes a_n)}_{\text{action of } U(\mathfrak{g}) \text{ on } A^{\otimes n-j+1}} \otimes a_0 \otimes \cdots \otimes a_{j-1}$$

(essentially Rinehart's formula).

$$a_0 \otimes \cdots \otimes a_n \longmapsto \sum_{0 \le k \le n} a_0 \otimes \cdots \otimes x(a_k) \otimes \cdots \otimes a_n$$

$x \in \mathfrak{g}$

This defines the action of $\mathcal{U}_{cl}(\mathfrak{g})$ on $C_{-\bullet}(A)[[u]], b + uB$

which is a nc analog of $\Omega^{-\bullet}_X [[u]], ud$

This extends to the full dgLa $\mathfrak{g}_A$ of Hochschild cochains with Gerstenhaber bracket ($\mathrm{Der}(A)$ is the Lie subalgebra of zero-cocycles in it). The action is a) A_∞ b) rather inexplicit

Why flatten elements?

$\mathcal{U}_{cl}(\mathfrak{g}_A^\bullet)$ (A_∞) acts on $C_{-\bullet}(A)[[u]]$
$b+uB$

$m \in \mathfrak{g}_A^1$ can be flattened into $\mu \in \mathcal{U}_{cl}(\mathfrak{g}_A^\bullet)$

What m to take?

Getzler's GM connection:

S scheme; $\mathcal{O}_S$-algebra $\mathcal{A}$;

$$CC^{per}_{\mathcal{O}_S,\bullet}(\mathcal{A}) = \mathcal{A}^{\otimes_{\mathcal{O}_S}(\bullet+1)}((u)),\ b+uB$$

complex of $\mathcal{O}_S$-modules

$$\mathcal{L}_{\mathcal{A}}^\bullet = \Omega_S^\bullet \otimes_{\mathcal{O}_S} \mathfrak{g}_{\mathcal{A}}^\bullet$$

<u>Assume:</u> ∇ a connection in the $\mathcal{O}_S$-<u>module</u> $\mathcal{A}_S$ (<u>not</u> preserving product).

(Say, $\mathcal{A}$ is a <u>free</u> $\mathcal{O}_S$-module).

$$a_1, a_2 \longmapsto m_2(a_1, a_2) \in \Omega_S^0 \otimes \mathfrak{g}_A^1$$

$$a_1 \longmapsto \nabla(a_1) \in \Omega_S^1 \otimes \mathfrak{g}_A^0$$

$$m = \nabla + m_2 \in \mathcal{L}_{\mathcal{A}}^1$$

a flattening μ of m, via the action of $\mathcal{U}_{nc}(\mathcal{L}_A)$, defines a superconnection

$$\nabla_{GM} : \Omega^\bullet_S \otimes_{\mathcal{O}_S} CC^{per}_{\mathcal{O}_S;\bullet}(\mathcal{A}_S) \circlearrowleft +1$$

$$\nabla^2_{GM} = 0$$

(itself a (higher) $\mathcal{D}_S$ - module...)
(derived)

2

a related application:

nc crystalline cohomology

$$\underbrace{A/\mathbb{F}_p}_{\text{algebra}} \qquad \underbrace{\tilde{A}/\mathbb{Z}_p}_{\substack{\text{just a} \\ \text{module} \\ \text{lifting}}} \qquad \tilde{A}/p \xrightarrow{\sim} A$$

Product on A lifts to one on $\tilde{A}$; associative modulo p. $m(\tilde{a}_1, \tilde{a}_2) = \widetilde{a_1 a_2}$

$m \in \mathfrak{g}^1_{\tilde{A}}$ (viewed, say, as an alg w/ zero multiplication)

$\mu = m + \dots$

defines a differential on the p-adic completion

$$CC_\bullet^{per}(\tilde{A})^\wedge$$

Everything well-defined up to homotopy (and all higher homotopies).

When m is associative on $\hat{A}$:
just get $CC_\bullet^{per}(\tilde{A})$ algebra.

Questions 1) Duflo/Todd/... flavor:

p vs $e^p - 1$

CE (Cartan–Eilenberg) cochains of a Lie algebre $\mathfrak{a}$: $\mathrm{Hom}(\Lambda^p \mathfrak{a}, K)$
where K is a ring on which $\mathfrak{a}$ acts. Cup product:

$$(\varphi \cup \psi)(x_1, \dots, x_{p+q}) =$$

$$= \sum_{\text{shuffles}} \pm \varphi(x_{i_1}, \dots, x_{i_p}) \psi(x_{i_{p+1}}, \dots, x_{i_{p+q}})$$

$\underline{\text{Ex.}}$ $\quad \mathfrak{a} = \underbrace{\varepsilon \cdot \mathfrak{g}}_{\uparrow}$ abelian

concentrated in degree 1

$$C^p(\mathfrak{a}, K) = \mathrm{Hom}(\underbrace{\mathrm{Sym}^k(\mathfrak{g})}_{\text{in degree } (-1)+1=0}, K)$$

$$\mathrm{Sym}^1(\mathfrak{g}) \xrightarrow{\rho} U(\mathfrak{g})$$
$$\shortparallel$$
$$\Lambda^1(\mathfrak{g}\varepsilon)$$

$$x \longmapsto x$$

$$\rho^{\cup n}(x, \dots, x) = n!\,\rho(x) \cdots \rho(x)$$

$$\frac{\rho^n}{n!}(x, \dots, x) = \mathrm{PBW}(x^n) \in U(\mathfrak{g})$$

This is why

$$uB_{cl}(x^n) = \rho(x, \dots, x); \quad uB_{nc}(\mathrm{PBW}(x^n)) = (e^{\rho} - 1)(x, \dots, x)$$

2). JLO vs Connes-Moscovici

A graded $\quad D \in A^1$; $[D,\cdot] \in (\mathfrak{g}^0_A)^{odd}$

$(\mathbb{Z}/2)$-?

$D^2 \in (\mathfrak{g}^{-1}_A)^{ev}$

$$m = [D,\cdot] + D^2$$

an odd element of $\mathfrak{g}^{\bullet}_A$

Flattening it: $\mu = m + \dots$; Using U_{cl}:

Acting by μ on $CC^{per}_{\bullet}(A)$ and applying a trace $tr: A \to k$, get the JLO cocycle of D. Using U_{nc}: get something else, perhaps the CM cochains used in their local index formula, or Perrot's cocycle, or...

The Sato-Kashiwara filtration on a $\mathcal{D}$-module

$\mathcal{M}$ $\mathcal{D}_X$-mod

$\mathbb{D}\mathcal{M} = R\mathcal{H}om(\mathcal{M}, \mathcal{D}_X)$ $\quad \mathbb{D}\mathbb{D}\mathcal{M} \simeq \mathcal{M}$

spectral sequence of the composition:

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{E}xt^{-j}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X), \mathcal{D}_X) \Rightarrow \mathcal{M}$$

<u>More precisely</u>:

<u>locally</u> $A = D(U)$ $\quad M$ A-mod (<u>perfect</u>)

$$0 \to P_d \to \dots \to P_0 \to M \to 0$$

$$\mathbb{D} P_j = P_j^\vee = \mathrm{Hom}_A(P_j, A) \leftarrow \text{right } A\text{-mod}$$

$$C^\bullet(P_*^\vee, \underline{A})$$

standard complex for computing $\mathrm{Ext}^\bullet_A(-, \underline{A})$

e.g.: $C^\bullet(P_*^\vee, A) = \mathrm{Hom}_A(P_*^\vee, I^\bullet)$

$$0 \to A \to I^0 \to I^1 \to \dots$$

or: $C^\bullet(N, A) = \mathrm{Hom}_k(A^{\otimes \bullet}, \mathrm{Hom}_k(N, A))$

$= \mathrm{Hom}^\bullet_k(A^{\otimes \bullet} \otimes N, A)$ $\quad C^\bullet$ is an A-bimodule

$$
\begin{tikzcd}
C^d(P_d^\vee, A) \arrow[r] & \cdots \arrow[r] & C^d(P_0^\vee, A) \\
\vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] \\
 & C^1(P_1^\vee, A) \arrow[r] \arrow[u] & C^1(P_0^\vee, A) \arrow[u] \\
C^0(P_d^\vee, A) \arrow[r] \arrow[u] & \cdots \arrow[r] & C^0(P_2^\vee, A) & C^0(P_1^\vee, A) \arrow[r] \arrow[uul] & C^0(P_0^\vee, A) \arrow[uul]
\end{tikzcd}
$$

Sato-Kashiwara filtration:

$$S^\ell = C^{\geq \ell}(P_*^\vee, A)$$

$$
\begin{tikzcd}
\vdots & & \vdots \\
C^d(\mathrm{Ext}_A^d(M,A), A) \arrow[r] \arrow[u] & \cdots & C^d(\mathrm{Ext}_A^0(M,A); A) \arrow[u] \\
 & & \vdots \arrow[u] \\
 & C^1(\mathrm{Ext}_A^1(\,), A) \arrow[r] & C^1(\mathrm{Ext}_A^0(M,A); A) \arrow[u] \\
C^0(\mathrm{Ext}_A^d(M,A), A) \arrow[r] & \cdots\ C^0(\mathrm{Ext}_A^1(\,), A) \arrow[r] \arrow[u] & C^0(\mathrm{Ext}_A^0(M,A); A) \arrow[u]
\end{tikzcd}
$$

$\mathrm{Ext}^d(\mathrm{Ext}^d, A)$ $\qquad\qquad\qquad\qquad$ $\mathrm{Ext}^d(\mathrm{Ext}^0, A)$

$\vdots$

$\mathrm{Ext}^1(\mathrm{Ext}^0, A)$

$\mathrm{Ext}^0(\mathrm{Ext}^d, A)$ $\qquad\qquad\qquad\qquad$ $\mathrm{Ext}^0(\mathrm{Ext}^0, A)$

Recall: if $\operatorname{codim} Ch(M) \overset{=}{=} \ell$:

$$\mathrm{Ext}^0_A(M, A) = \ldots = \mathrm{Ext}^{\ell-1}_A(M, A) = 0$$

In general: $Ch(\mathrm{Ext}^j_A(M, A))$ is of codim $\geq j$

$$\begin{array}{c} Ch(\mathrm{Ext}^j_A(M,A)) \\ \cap\!\mid \\ Ch(M) \end{array}$$

Ex. M holonomic: $\mathrm{Ext}^n_A(M, A)$ is the only $\neq 0$

b/c $\operatorname{codim} \leq n$

$\mathrm{Ext}^n_A(\mathrm{Ext}^n_A, A)$ is the only $\neq 0$ term.

Ex. $\operatorname{codim} Ch(M) = n-1$:

$\mathrm{Ext}^{n-1}_A(M, A)$ $\quad$ codim $\geq n-1$ (i.e. $\begin{smallmatrix} n-1 \\ n \end{smallmatrix}$)

$\mathrm{Ext}^n_A(M, A)$ $\quad$ codim $\geq n$ (i.e. $= n$)

$\mathrm{Ext}^n(\mathrm{Ext}^n, A)$ $\quad$ $\mathrm{Ext}^n(\mathrm{Ext}^{n-1}, A)$

$\mathrm{Ext}^{n-1}(\mathrm{Ext}^{n-1}, A)$

The picture is <u>triangular</u>:

$\mathrm{Ext}^{\ell}_A(\mathrm{Ext}^{\ell}_A, A)$

$\mathrm{Ext}^{\ell-1}(\mathrm{Ext}^{\ell}, A)$

$0 =$

$\vdots$

$\mathrm{Ext}^{1}(\mathrm{Ext}^{\ell}, A)$

$0 =$

$\mathrm{Ext}^{0}(\mathrm{Ext}^{\ell}, A)$

$0 =$

codim $\geq \ell$

$\mathrm{Ext}^{0}(\mathrm{Ext}^{0}_A, A)$

all this vanishes if codim $Ch(M) \geq \ell$

$\mathrm{Ext}^{0}(\mathrm{Ext}^{\ell}, A)$ $\qquad$ $\mathrm{Ext}^{0}(\mathrm{Ext}^{0}, A)$

So: codim $Ch(M) \geq \ell$:

$\mathrm{Ext}^{\ell}(\mathrm{Ext}^{\ell}, A)$

nothing outside

Recall: $G^\ell \mathcal{M} = \{u \mid Ch(\mathcal{D}_X u) \text{ is of codim} \geq \ell\}$

① $S^\ell \mathcal{M} \subseteq G^\ell \mathcal{M}$

$\uparrow$

ind. by $C^{\geq \ell}(\ldots, -)$; codim $Ch(\mathcal{E}xt^{\geq \ell}) \leq \ell$.

② $$\begin{tikzcd} S^\ell(G^\ell \mathcal{M}) \arrow[d] \arrow[r, equal] & G^\ell \mathcal{M} \arrow[dl] \\ S^\ell(\mathcal{M}) & \end{tikzcd}$$

mutually inverse. Conclusion:

$$G^\ell = S^\ell$$

<u>Theorem</u> Let j be an open embed. $U \hookrightarrow X$ and $\mathcal{M}$ a holonomic $\mathcal{D}_U$-mod.

Then $j_* \mathcal{M}$ is holonomic.

<u>Idea</u>: relate to f^s. $\mathcal{M}$ holonomic $\mathcal{D}_U$-mod;

m local section of $\mathcal{M}$; (also, $\mathcal{M}_0 = \mathcal{O}_X \cdot m$ coherent $\mathcal{O}_X$-module)

can assume $\mathcal{M} = \mathcal{D}_U \cdot m$

$$\mathcal{D}_X[s]\cdot(mf^s) \subset \mathcal{D}_X[s]\cdot(mf^{s+\mathbb{Z}})$$

$=$ after $\otimes_{\mathbb{C}[s]} \mathbb{C}(s)$ by b-function Lemma.

$f^{-1}\partial = \partial f^{-1} - f'f^{-2}$, etc.

$\cap$ $=$ b/c $f^{-k}\cdot\mathcal{D}_X \subset \mathcal{D}_X\cdot f^{\mathbb{Z}}$

$$f^{\mathbb{Z}}\cdot\mathcal{D}_X[s]\cdot(mf^{s+\mathbb{Z}})$$

$\cup$ $=$ b/c:

$$P(x,\partial_x,s,f^{-1})\cdot(f^s m) = f^s\cdot\big(\underbrace{P\big(x,\partial_x+s\tfrac{f'}{f},s,f^{-1}\big)}_{\text{automorphism of } \mathcal{D}_U[s]}\cdot m\big)$$

$$f^{\mathbb{Z}}(\mathcal{D}_X[s]m)\cdot f^s$$

$$= \boxed{j_*(\mathcal{D}_U[s]m)\cdot f^s}$$

$\downarrow$

Therefore: $j_*(\mathcal{M}(s))\cdot f^s$ is holonomic over $\mathbb{C}(s)$ (where $\mathcal{M} = \mathcal{D}_X\cdot m$) and generated over $\mathcal{D}_X(s)$ by one section $m\cdot f^s$.

So: over $\mathbb{C}(s)$, $j_* \mathcal{U}(s) \cdot f^s$ generated by f^s over $\mathcal{D}_X(s)$.

$$j_* \mathcal{U}(s) \cdot f^s = \mathcal{D}_X(s) / I$$

And is holonomic $/ \mathcal{D}_X(s)$.

I - ideal inside $\mathcal{D}_X(s)$

fin. gen. by $P_1(s, x, \partial_x)/f_1(s); \dots; P_m(\ )/f_m(s)$

greatest common denom.: $f(s)$.

Outside its zeroes, we can specialize to a number s.

In particular: Can specialize to $n \ll 0$.

$I(n)$ kills $m \cdot f^{-n}$ [could the actual $\operatorname{ann}(mf^{-n})$ be even larger?]

Next: localize $\mathbb{C}[s]$ by finitely many $x - s_1, \dots, x - s_N$

Can specialize to $s \neq s_j$ denote by $\mathbb{K}_0$

$$\dim_{\mathbb{C}(s)} \left(\operatorname{gr} \mathcal{D}_X(s) / \operatorname{gr} I(s) \right) = n = \tfrac{1}{2} \dim (T^*X)$$

<u>Claim</u>: if specialize to $s \neq s_j$:

same true over $\mathbb{C}$ for almost any s.

<u>After that</u>:

$$\mathcal{D}_X \cdot (m f^n) \quad n \ll 0$$

are all <u>holonomic</u>. $n \to -\infty$:

increasing chain; <u>must</u> stabilize.

<u>So</u>: $j_* \mathcal{M}$ $\underline{\text{is}}$ holonomic.

<u>What about its characteristic variety?</u>

In $\underset{x,\xi}{T^*X} \times \underset{s}{\mathbb{C}}$:

$$\left(x, \xi + s \frac{df}{f}(x) \Big|_{f(x) \neq 0} \right) \subset T^*U \times \mathbb{C}$$

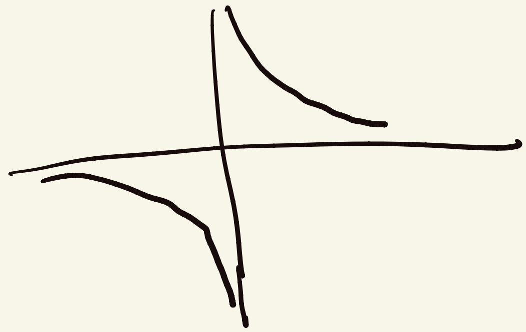
<u>Its closure</u>: $\subset T^*X \times \mathbb{C}$

Its intersection with $\{s = 0\}$

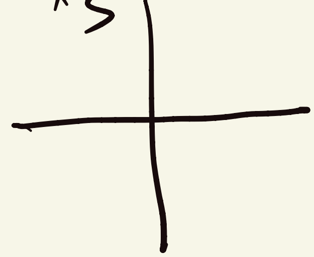
$\|$

$SS(j^* \mathcal{M})$ IF $\mathcal{M}$ is holonomic REGULAR on U.

Ex. $f = x$ $\qquad \left\{ \xi = \dfrac{s}{x} \right\}$ close, $\cap \{s=0\}$:



$x\xi = 0$



Ex. $f = x_1^2 + \dots + x_n^2$

$$\xi_j = s \cdot \frac{2x_j}{x_1^2 + \dots + x_n^2}$$

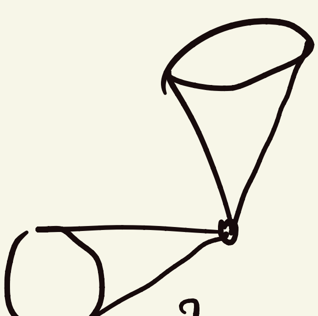
$$\Downarrow$$

$$\begin{cases} (\xi_1^2 + \dots + \xi_n^2)(x_1^2 + \dots + x_n^2) = 4s^2 \\ \dfrac{\xi_1}{x_1} = \dots = \dfrac{\xi_n}{x_n} \end{cases}$$

$\cap \{s=0\}$:

$$\begin{cases} (\xi_1^2 + \dots + \xi_n^2)(x_1^2 + \dots + x_n^2) = 0 \\ \xi_i x_j = \xi_j x_i, \ \forall\, i,j \end{cases}$$

$\sum \xi_j^2 = 0$



$\sum x_j^2 = 0$

The comp. $\sum \xi_j^2 = 0$.

yes, Lagrangian $\varphi(x_n; \xi_1, \dots, \xi_{n-1})$

$= \pm x_n \sqrt{-\xi_1^2 - \dots - \xi_n^2}$

$\xi_n = \varphi_{x_n}$; $x_j = -\varphi_{\xi_j}$, $j < n$

$$\left(\partial_{x_1}^2 + \dots + \partial_{x_n}^2\right)(x_1^2 + \dots + x_n^2)^{s+1} =$$

BTW:

$$= \sum_{j=1}^{n} \partial_{x_j}\left[(s+1)\,2x_j \cdot (x_1^2 + \dots + x_n^2)^s\right] =$$

$$= \sum_j \left[2(s+1)(x_1^2 + \dots + x_n^2)^s + 4s(s+1)\cdot x_j^2 (\quad)^{s-1}\right]$$

$$= \left[2(s+1)n + 4s(s+1)\right](x_1^2 + \dots + x_n^2)^s$$

The b-function for $\sum_1 x_j^2$:

$$(s+1)\left(s+\frac{n}{2}\right)$$

<u>Theorem</u> (Kashiwara): roots of $b(s)$ are rational and negative (They always include $s=-1$).

<u>Ex.</u> $f(x) = x_1^{a_1} \dots x_n^{a_n}$

1) $\partial_x^a \left(x^{a(s+1)}\right) = a(s+1)\left(a(s+1)-1\right)\dots\left(a(s+1)-a+1\right) \cdot x^{as}$

$$a^a \cdot \prod_{\nu=1}^{a}\left(s+\frac{\nu}{a}\right) = (as+a)(as+a-1)\dots(as+1)$$

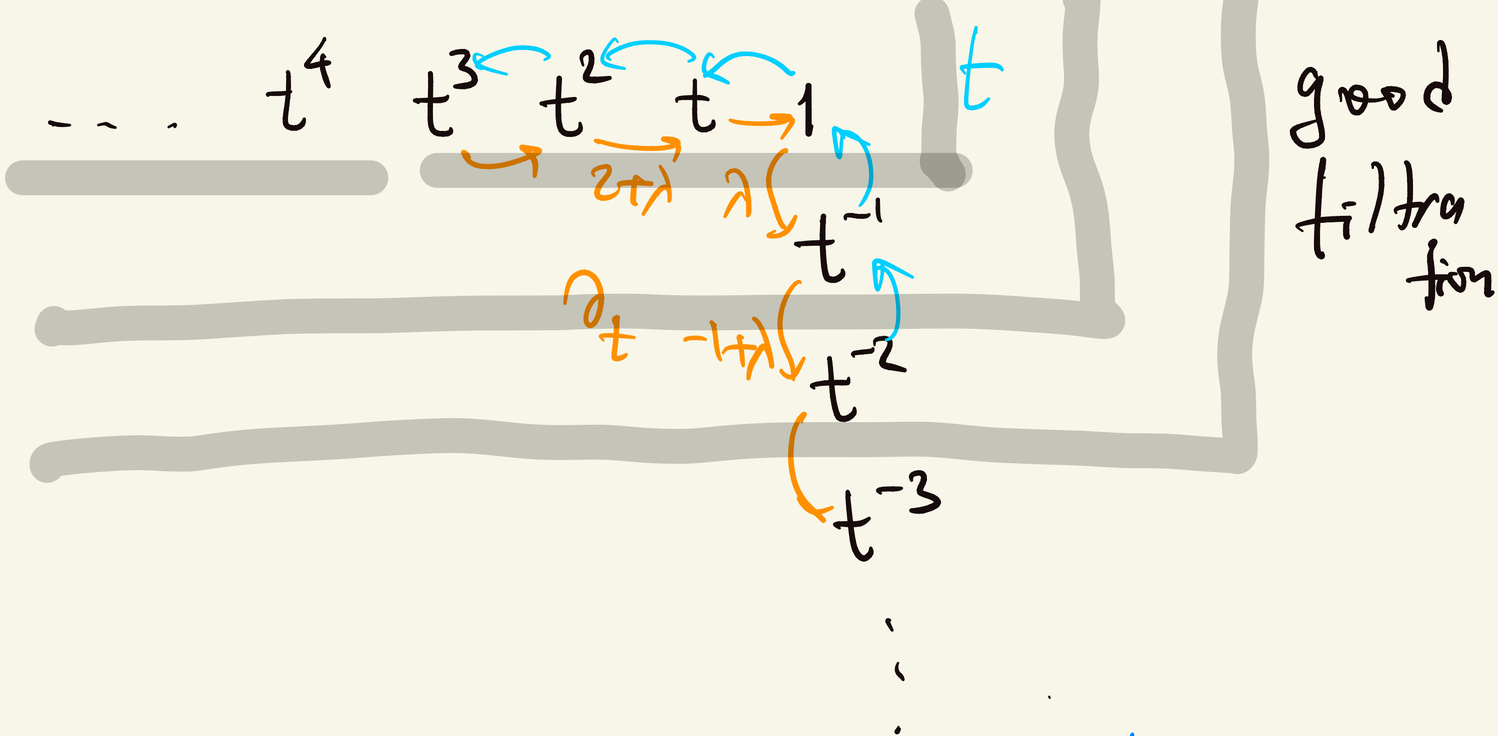
$$b(s) = \prod_{j=1}^{n} \prod_{\nu=1}^{a_j}\left(s+\frac{\nu}{a_j}\right)$$

General case uses resolution of singul.

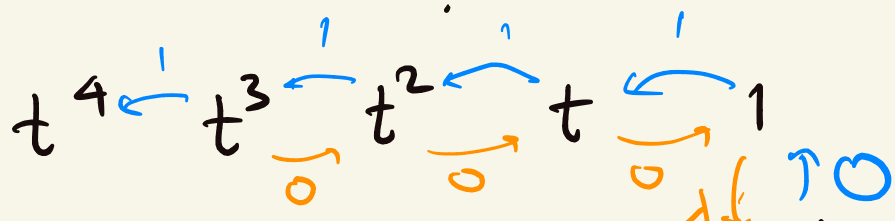
Regular singularities

Ex. ∂_t acts by $\frac{\partial}{\partial t} + \lambda/t$:

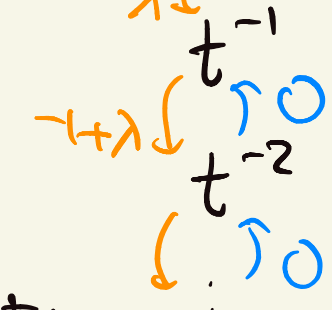
$M = \mathbb{C}[t, t^{-1}]$



On gr:



$e_n = \hbar^n \cdot (-1)^n \cdot (n-1)!\, t^{-n}$, $n > 0$



$$\mathcal{O}^\hbar / \mathcal{O}^\hbar \cdot (x\cdot\xi - \lambda) \subset \underbrace{\mathcal{O}^\hbar(x) \oplus \mathcal{O}^\hbar(\xi)}_{\text{same value at } (0,0)}$$

(switch to x from t)

may be worthwile to look at the nature of the formulas explicitly:

$$\xi \cdot f(x) = \hbar f'(x) + f(x) * \xi =$$

$$= \hbar f'(x) + \frac{f(x)-f(0)}{x} \cdot x\xi + f(0)\xi$$

$$= \hbar f'(x) + \hbar \frac{f(x)-f(0)}{x} \cdot \lambda + f(0) \cdot \xi$$

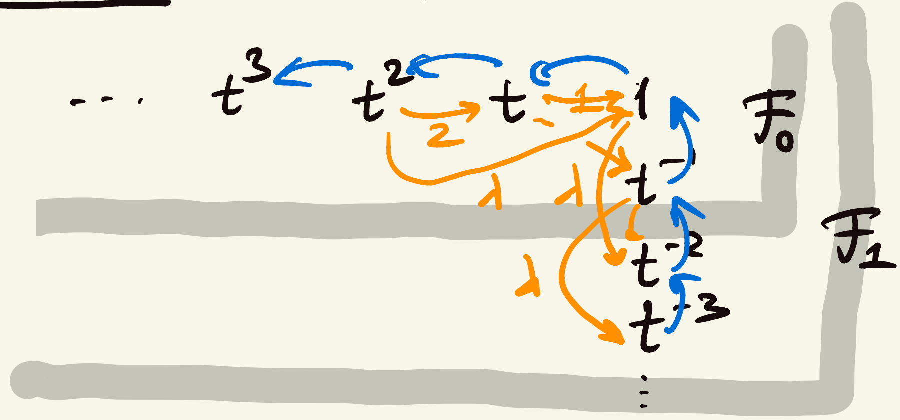
$$x \cdot f(x) = xf$$

and same for $g(\xi)$.

In particular: if $\varphi(x,\xi) \in (x\xi)$ then the action of $\varphi(x,\xi) = 0 \mod \hbar$.

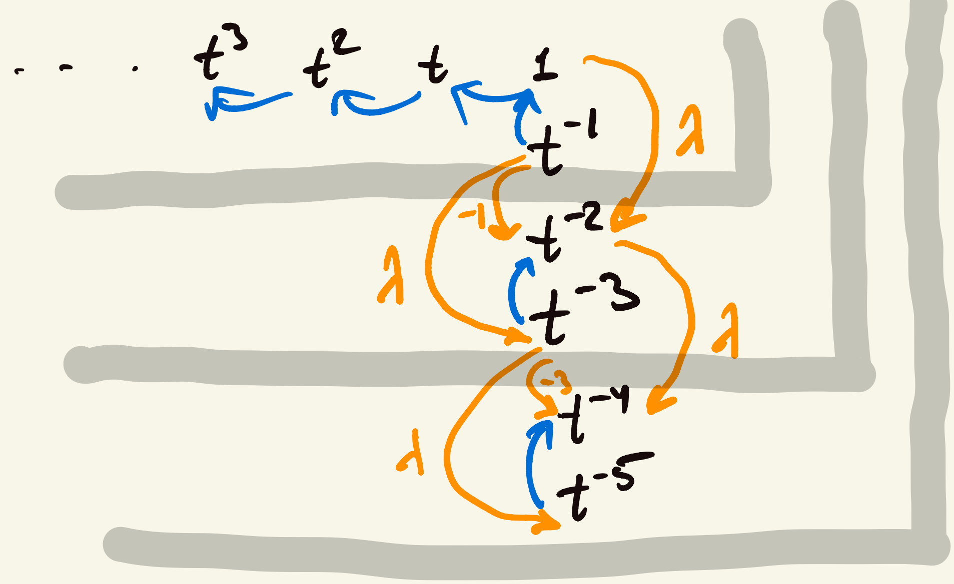
($\operatorname{gr} \mathcal{M}$ reduced over $\operatorname{gr} \mathcal{D}_X$, or $\mathcal{M}^\hbar/\hbar$ reduced over $\mathcal{O}_{T^*X}$).

Compare to: ∂_t via $\frac{\partial}{\partial t} + \frac{\lambda}{t^2}$



good filtration

After passing to gr:



On the $t=0$ component: t^2 acts by 0 but t does not.

Not reduced; irregular singularity.

$\mathcal{M}^{\hbar}$: $\mathrm{Rees}(\mathcal{M})$ basis over $\mathbb{C}[\hbar]$:

. . . x^3 x^2 x 1

x^{-1}

$\hbar x^{-2}$

$\hbar x^{-3}$

$\hbar^2 x^{-4}$

$\hbar^2 x^{-5}$

$\vdots$

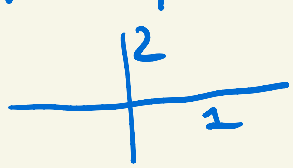
$\mathrm{Rees}(\mathcal{D}_X) \circlearrowright \mathrm{Rees}(\mathcal{M})$:

$$(x^2\xi - \hbar\lambda)1 = 0$$

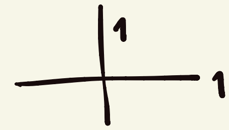
Microlocalize:

$$\mathcal{M}^{\hbar} = \mathcal{O}^{\hbar}_{\mathbb{A}^2} / \mathcal{O}^{\hbar}_{\mathbb{A}^2} \cdot (x^2\xi - \hbar\lambda)$$

Characteristic cycle:



(multiplicities)

<u>Note</u>: $\lim_{s\to 0} \left(SS(\mathcal{M}) + s\frac{df}{f} \right) =$ 

At least at the level of char. <u>cycles</u>: no way to distinguish $\partial_t + \frac{\lambda}{t}$ from

$$\underbrace{\partial_t + \frac{\lambda}{t^k}, \quad k > 1}_{\text{irreg sing}}$$

<u>General definition</u>: If X projective: $\mathcal{M}$ is <u>regular</u> if there is a good filtration s.t. $\mathcal{M}^{\hbar}/\hbar\mathcal{M}^{\hbar}$ is a <u>reduced</u> module over $\mathcal{O}^{\hbar}/\hbar\mathcal{O}^{\hbar} = \mathcal{O}_{T^*X}$.

In general: if there is a resol. of singularities $X \xrightarrow[\text{open}]{j} \overline{X}$ such that $j_*\mathcal{M}$ is regular on $\overline{X}$.

b-functions and V-filtration

$\mathcal{M}$ holonomic $\mathcal{D}_U$-module (e.g. $\mathcal{O}_U$)

$$U = \{f \neq 0\} \qquad f : X \to \mathbb{C}$$

$$i : X \hookrightarrow X \times \mathbb{C}$$

$$x \longmapsto (x, f(x))$$

$$i_+ \mathcal{M} = \mathcal{M}[\tau] \cdot e^{f\tau}$$

formal element

in this definition, $\mathcal{M}$ can be just an $\mathcal{D}_X$-mod

$$x(m \cdot e^{f\tau}) = xm \cdot e^{f\tau}$$

$$\partial_x (\quad) = (\partial_x m + \tau f'(x) m) \cdot e^{f\tau}$$

$\underbrace{\tau \text{ acts by mult.}; t \text{ by } \partial_\tau;}_{\text{right action of } \mathcal{D}_{\mathbb{A}^1}\text{, comm. with left action of } \mathcal{D}_X.}$ $t(m \cdot e^{\tau f}) = (\partial_\tau m + fm) e^{f\tau}$

In Polesello-Schapira approach: $\tau = \frac{1}{\hbar}$; $\mathcal{D}_X^\hbar = \mathrm{Rees}(\mathcal{D}_X)$ acts in the standard way, twisted by the autom $\xi \mapsto \xi + f'(x)$, $x \mapsto x$.

Also: $s = \tau \frac{\partial}{\partial \tau}$ $\qquad (s \cdot m e^{\tau f} = (\tau \partial_\tau + \tau f) m \cdot e^{\tau f}$

$$ts = (s+1)t$$

Rmk For (M,ω): $\mathcal{O}^\hbar_M$-modules are interesting but some version of enhanced $\mathcal{O}^\hbar_M$-modules are (prob.) more interesting. An enhanced $\mathcal{O}^\hbar_M$-module is an $\mathcal{O}^\hbar_M$-module on which certain symmetries of $\mathcal{O}^\hbar_M$ act compatibly.

Example: $Sp(2n) \circlearrowright \mathcal{D}^{alg}_{\mathbb{R}^n}$; $\widetilde{Sp}(2n) \circlearrowright \mathcal{D}^{alg}_{\mathbb{R}^n}$ act on $L_2(\mathbb{R}^n)$ (Weil, or metaplect.)

$\left(V_f = e^{f\tau}\mathbb{C}[x,\tau] \text{ is not like this.}\right]$

a parabolic subgroup $\tilde{P}_f \subset \widetilde{Sp}(2n)$ does act on it.

We can enhance V_f by: 1) induce from $\tilde{P}_f$ to $\widetilde{Sp}$ (and to a bigger groupoid); 2) introduce formally $e^{c/i\hbar}$, $c \in \mathbb{R}$; c) complete).

What we have here is rather: $\frac{1}{\hbar}\mathcal{O}^\hbar_{T^*X} \hookrightarrow \tilde{\mathcal{L}}_{T^*X} \twoheadrightarrow \frac{\hbar\partial\mathbb{C}}{\hbar}$

(Deligne's sheaf of Lie algebras that he used to construct deformation quantization);

$$\tilde{\mathcal{L}}_{T^*X} \circlearrowright \mathcal{O}^\hbar_{T^*X} \circlearrowright \mathcal{O}_U$$
compatibly

Rmk If $\hbar = \frac{1}{\tau}$:

$$t = -\hbar^2 \frac{\partial}{\partial \hbar} + f$$

In our approach to enhanced $\mathcal{O}^{\hbar}$-modules: Something very close to a filtration by gen. eigenvalues of $-\underline{\underline{t}}$ appears. Indeed:

Sections of the enhanced module $\mathcal{V}_f$ are expressions

$$\sum e^{\frac{\varphi_n}{\hbar}} \cdot a_n(x, \xi, \hbar)$$

(sorry, $\hbar \in i\mathbb{R}$)

$$e^{\frac{\varphi}{\hbar}} a(x,\xi,\hbar) \mapsto (\varphi - f) e^{\frac{\varphi}{\hbar}} a - e^{\frac{\varphi}{\hbar}} \cdot \underbrace{\hbar^2 \partial_\hbar}_{\text{pronilpotent wrt powers of } \hbar} \cdot a$$

The filtration on $\mathcal{V}_f$:

$\mathrm{Fil}^0 = \sum_{c \geq f} e^{\frac{c}{\hbar}} a_c$ plays key role.

Observe: $\mathcal{D}_X$, t, s (and τ) also act on
or $\mathcal{M}$

$\mathcal{O}_U[s]\cdot f^s$ another formal symbol

$$\partial_x(m\cdot f^s) = \left(\partial_x m + s\frac{f'}{f}m\right)\cdot f^s \qquad t(m(s)\cdot f^s) =$$
$$= m(s+1)\cdot f^{s+1}$$

x, s act via multiplication.

Prop. $\mathcal{O}_U[\tau]\cdot e^{\tau f} \sim \mathcal{O}_U[s]\cdot f^s$

as $\mathcal{D}_U \otimes \mathcal{D}^{op}_{\mathbb{A}^1_{t,\tau}}[s]$ – modules (also, t is invertible on both)

$\tau = st^{-1}$

Pf $\quad s^n e^{f\tau} = (\tau\partial_\tau)^n(e^{f\tau}) = (\tau\partial_\tau + \tau f)^n(e^{f\tau})$

$$[\tau(\partial_\tau + f)]^n = P_n(s)$$

polynomial of degree n

$$\tau\cdot P_n(s)(\partial_\tau + f) = P_n(s-1)\cdot s = P_{n+1}(s)$$

$\tau s = (s-1)\tau$

$$P_n(s) = s(s-1)\dots(s-n+1)$$

$$\tau^n f^n e^{\tau f} = s(s-1)\dots(s-n+1)\cdot e^{\tau f}$$

Also: $\partial e^{\tau f} = \tau f' \cdot e^{\tau f} = \tau f\cdot\frac{f'}{f}\cdot e^{\tau f} = s\cdot\frac{f'}{f}\cdot e^{\tau f}$

Compare: $\partial f^s = s f^{s-1} f' = s\cdot\frac{f'}{f}\cdot f^s$

Therefore

$$\tau^n u(x)\cdot e^{\tau f} \longleftrightarrow s(s-1)\dots(s-n+1)\, u(x) f^{s-n}$$

is the desired isomorphism.

Rmk This is a (formal) Laplace+Mellin transform:

$$M_f(u(x,\tau)) = \int e^{\tau f(x)} u(x,\tau)\, \tau^{-s} \cdot \frac{d\tau}{\tau} \Big/ \Gamma(-s)$$

(intertwines $\tau\partial_\tau + \tau f$ with s)

When f is invertible:

$$M_f(u) = \int e^{-\tilde{\tau}} u\left(x, \frac{-\tilde{\tau}}{f}\right) \tilde{\tau}^{s-1} d\tilde{\tau} \cdot f^s \Big/ \Gamma(-s)$$

(when $u=1$, numerator $= \Gamma(-s)$). OR, Remembering $\hbar = 1/\tau$:

$$u(x,\hbar) \mapsto -\int e^{-\frac{1}{\hbar}} u\left(x, -\frac{f}{\hbar}\right) \hbar^{s-1} d\hbar \Big/ \Gamma(-s) \approx$$

Interesting to see how/whether this sort of things appears in resurgent WKB, etc.

So, again: $\quad i_+\mathcal{O}_U \simeq j_*\mathcal{O}_U[s]\cdot f^s$

$$\cup$$

$$U(\tilde{\mathfrak{d}}_X)\cdot \mathcal{D}_X[s]\cdot e^{\tau f} \leftrightarrow \mathcal{D}_X[s]\cdot f^s =: \mathcal{N}_f$$

$e^{\tau f}$ (lattice inside)

$$s(t u) = t(s-1)u$$

$$\begin{array}{ccc} s: \mathcal{N}_f & \longrightarrow & \mathcal{N}_f \\ \cup & & \cup \\ t\mathcal{N}_f & \longrightarrow & t\mathcal{N}_f \end{array}$$

$b(s)$ = minimal polynomial of s on $\mathcal{N}/t\mathcal{N}$

$b(s)N \subset tN$

$t\,b(s)N \subset t^2N$

$\quad \shortparallel$

$b(s+1)tN$

$b(s+1)$ – min poly of s on tN/t^2N

. . .

Note $\mathcal{N}_f[t^{-1}] = j_*\mathcal{O}_U[s].f^s$

get a filtration on the right h.s.

V-filtration on N_f and $j_* \mathcal{O}_U[s] f^s$:

$$V^\alpha N = \bigoplus_{-\lambda \geq \alpha} (\text{generalized eigenspaces of } s)$$

(basically).

α rational; discrete (i.e. jumps at discrete set of α's); $tV^\alpha \subset V^{\alpha+1}$

$\tau V^\alpha \subset V^{\alpha-1}$

$s V^\alpha \subset V^\alpha$

Examples 1) $f = x$ $X = \mathbb{A}^1$

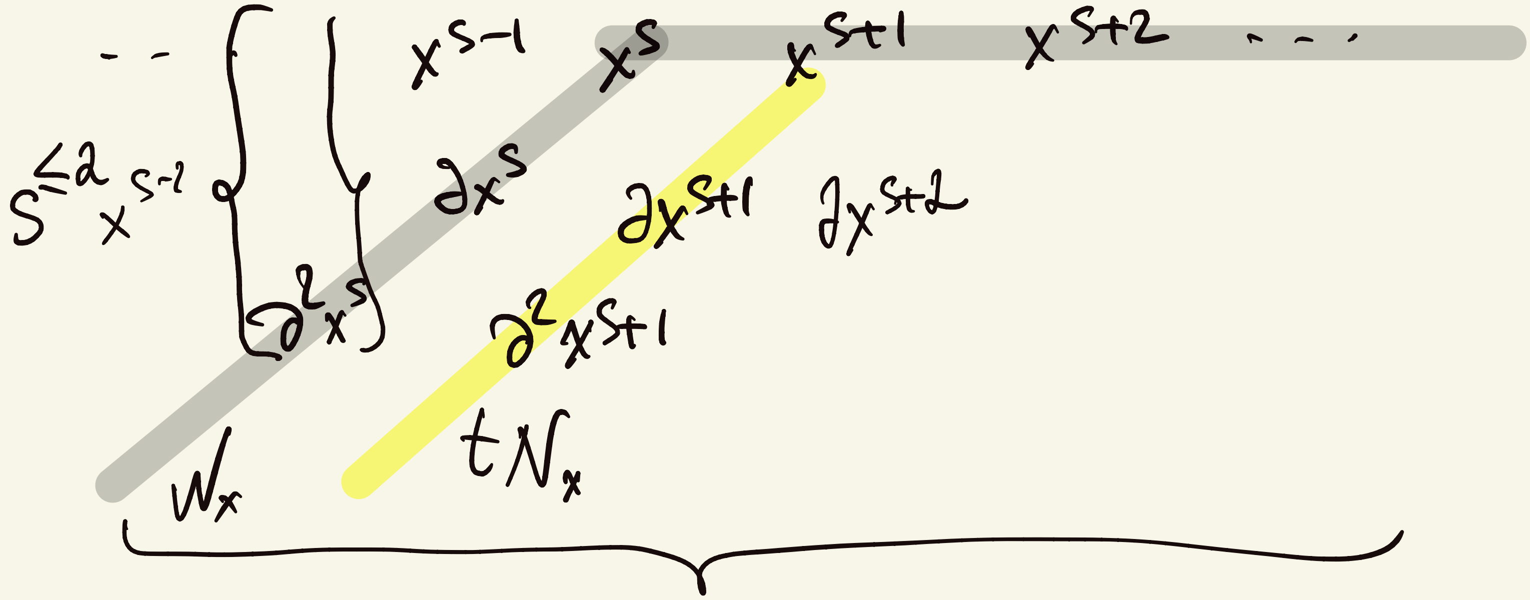
$e^{\tau x} \bullet$

$$\begin{array}{lll} 1 & x & x^2 \ \cdots \\ \tau & \tau x & \\ \tau^2 & \tau^2 x & \\ \tau^3 & & \\ \vdots & & \end{array}$$

$i_+ \mathcal{O}_X$:

$$\longleftrightarrow \quad \partial_x^n x^s = s(s-1)\cdots(s-n+1) x^{s-n}$$

$$\partial_x = \frac{\partial}{\partial x} + \tau \qquad \partial_x^n 1 = \tau^n$$



$$N_x \cong i_+ \mathcal{O}_X = \mathbb{C}[x,\tau]$$

In this case, $V^1 = N_x$

$V^2 = tN_x$

$\cdots$

$V^{n-1} = t^n N_x$

$\vdots$

$t^m \tau^n =$

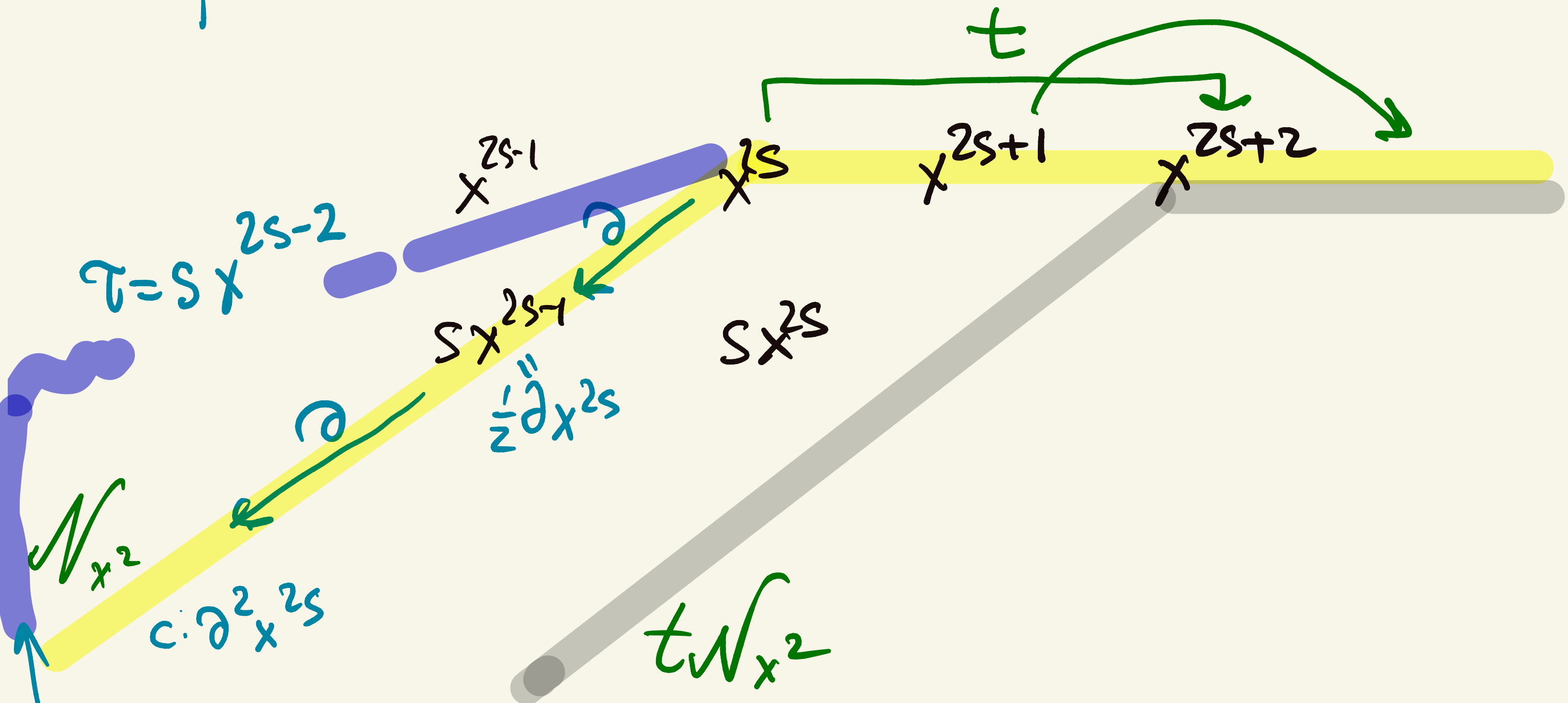
$= (\partial_\tau + x)^m \tau^n$

of weight $m+1$

Q. Can we somehow define $V^\bullet$ more invariantly, without fixing $s \in \widetilde{\mathbb{Z}}$?

2). $f = x^2$

$$\tau^n \longleftrightarrow (s-1)\dots(s-n+1)\, x^{2(s-n)}$$



$i_+ \mathcal{O}_X$ bigger than $\mathcal{N}_{x^2}$

On $\mathcal{N}_{x^2} / t\mathcal{N}_{x^2}$,

$i_+ \mathcal{O}_X$

two eigenvalues of s: -1 and $-\frac{1}{2}$

$$t \cdot 2s(2s-1)\, x^{2s} = t\, \partial^2 x^{2s}$$

$$4(s+1)\left(s+\tfrac{1}{2}\right) \cdot x^{2s} = t\, \partial^2 x^{2s}$$

V^{ϵ}: (will see more below)

Also: $V^0 = t^{-1} N_{x^2}$, etc.

Intersecting with $V^\bullet$, will get a filtration on $i_+ \mathcal{O}_X$.

$$\tau \in i_+ \mathcal{O}_X \subset (j_* \mathcal{O}_U)[s] \cdot x^{2s}$$

$$\downarrow$$

$$s \cdot x^{2s-2} = t^{-1}[\underbrace{(s+1) x^{2s}}_{\substack{\cap \\ \ker(s+\frac{1}{2}) \bmod t}}]$$

So $\tau \in V^1$, not in $V^{+1/2}$.

Vanishing and nearby cycles

$$\tau t = s \circlearrowright gr^\alpha_V \underset{\tau}{\overset{t}{\rightleftarrows}} gr^{\alpha+1}_V \circlearrowleft t\tau = s+1$$

$$s = -\alpha \qquad s+1 = -1-\alpha+1 = -\alpha$$

So: $gr^\alpha_V \underset{\sim}{\overset{\sim}{\rightleftarrows}} gr^{\alpha+1}_V$ unless $\alpha = 0$

And: they inherit the $\mathcal{D}_X$-mod struct.

Example $f = x$

$$\begin{array}{ccccccc}
\cdots \rightleftarrows & t^{-1} & \underset{=}{\overset{t}{\longrightarrow}} & 1 & \rightleftarrows & t & \cdots \\
& t^{-1}\tau & \underset{=}{\longrightarrow} & \tau & & t\tau & \\
& t^{-1}\tau^2 & \underset{=}{\longrightarrow} & \tau^2 & \rightleftarrows & t\tau^2 & \\
& t^{-1}\tau^3 & & \tau^3 & & t\tau^3 & \\
& \vdots & \vdots & \vdots & & \vdots &
\end{array}$$

$$\mathrm{gr}^0_V \underset{t=1}{\overset{\tau=0}{\rightleftarrows}} \mathrm{gr}^1_V$$

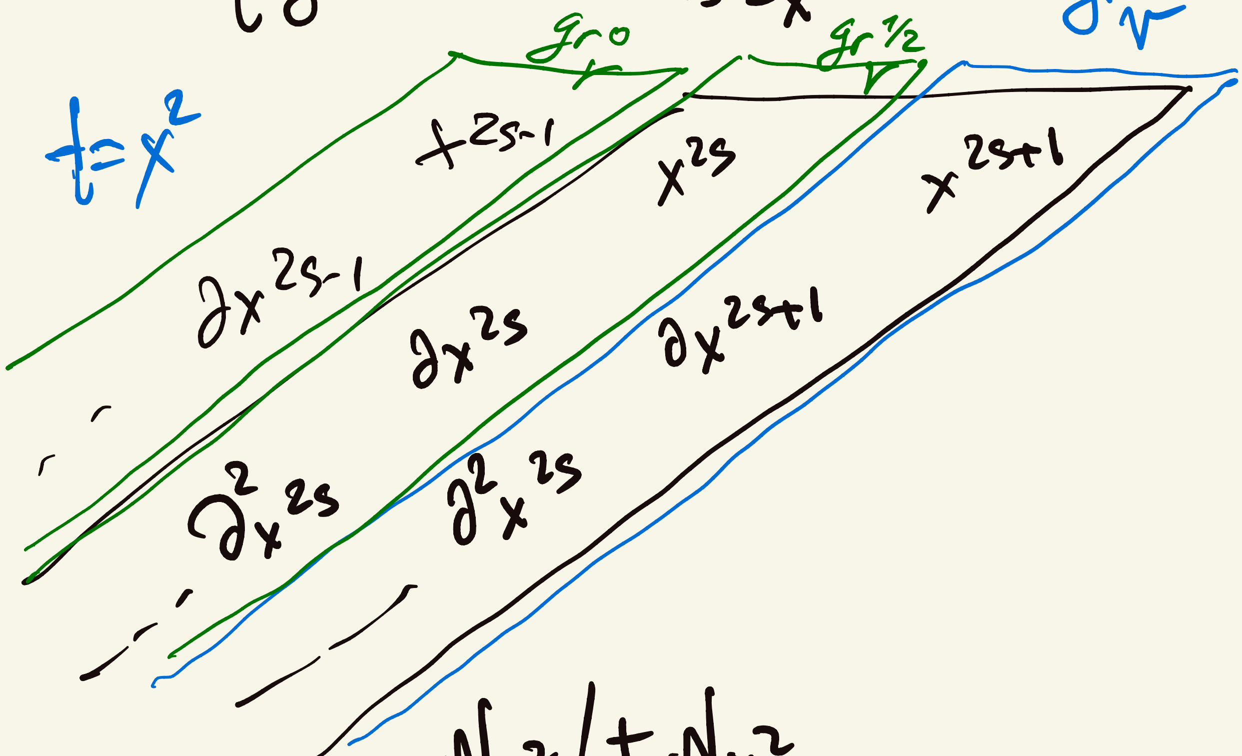
$\mathcal{D}_x$ acting on gr^1_V:

$$x \cdot 1 = x = t \cdot 1 \sim 0$$

$$\partial_x \cdot 1 = \tau$$

$$(s+1)x^{2s+1} = t\partial$$

$\mathrm{gr}^1_V \simeq$ as $\mathcal{D}_x$-mod $\delta_0 = i_{0+}(\mathbb{C})$



$$(s+1)\partial_x^n x^{2s+1} = \frac{1}{2} t\partial \partial_x^n x^{2s}$$

$$\mathcal{N}_{x^2} / t\mathcal{N}_{x^2}$$

<u>Idea</u>: instead of sheaves on X:

inductive systems of sheaves with compact supports.

(<u>every</u> sheaf is $\varinjlim$ of sheaves w/compact support; now: view it as the ind <u>system</u>, and allow other such systems).

To what extent can those Ind sheaves be treated as <u>sheaves</u>?

(restrict in various ways to open/locally closed subspaces; extend by 0; etc.).

Answer: to a large extent.

<u>Motivation</u>: given a real-valued function, can restrict (in certain ways) sheaves to $\{f < a\}$, and look at ind systs of those.

Ind-sheaves

$\mathcal{C}$ category

$\hat{\mathcal{C}} = \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Sets})$ ex.

$\mathcal{C} \to \hat{\mathcal{C}}$

$X \mapsto h_X : Y \mapsto \mathcal{C}(Y, X)$

An ind-object: an object of $\hat{\mathcal{C}}$ isomorphic (in $\hat{\mathcal{C}}$) to

$$\text{"}\operatorname*{colim}_{I}\text{"}(\alpha) : \quad Y \mapsto \operatorname*{colim}_{i \in I} \mathcal{C}(X, \alpha(i))$$

$\alpha : \underset{\text{small}}{I} \to \mathcal{C}$ functor

and: I is filtered (Recall: $\forall i_1, i_2\ \exists$ $\begin{matrix} i_1 \searrow \\ i_2 \nearrow \end{matrix}$

$\forall\, i_1 \rightrightarrows i_2 \xrightarrow{\exists} i$ equalizer)

Rmk Given such: can choose I, α for it. Namely: $A \in \hat{\mathcal{C}}$; $A : \mathcal{C}^{op} \to \mathrm{Sets}$

Objects of I_A: (X, x) $X \in \mathrm{Ob}(\mathcal{C})$
$x \in A(X)$

Morphisms $(X, x) \longrightarrow (Y, y)$:

$$X \xrightarrow{f} Y \quad \text{in } \mathcal{C}$$
$$A(X) \xleftarrow{A(f)} A(Y)$$
$$x \longleftarrow\!\shortmid\, y$$

$\alpha_A : I_A \to \mathcal{C}$ $\qquad (X, x) \longmapsto X$

Always: $A(X) = \operatorname{colim}_{I_A} \mathcal{C}(X, \alpha_A(Z, z))$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| \quad$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad A(Z)$

If $A(X) = \operatorname{colim}_{I} \mathcal{C}(X, \alpha(i))$ and I is filtered
then I_A is filtered.

Ex. k a field; $k_0^\infty = \operatorname{colim}(k^n \hookrightarrow k^{n+1} \hookrightarrow \cdots$

$$\text{"colim"}(k^n)(V) = \operatorname{colim}(\mathrm{Hom}(V, k^n))$$
$$\neq$$
$$\mathrm{Hom}(V, k_0^\infty)$$

unless $\dim V < \infty$.

$$\hat{\mathcal{C}}\Big(\underset{I}{\text{"colim"}}\,\alpha,\ \underset{J}{\text{"colim"}}\,\beta\Big) = \lim_{I}\ \operatorname*{colim}_{J}\ \mathcal{C}(\alpha(i), \beta(j))$$

__Thm__ Ind-objects of an abelian category form an abelian category.

Idea for constructing ker, coker:

Roughly, morphism $\underset{i}{\text{"lim"}}\, \mathcal{F}_i \to \underset{j}{\text{"lim"}}\, \mathcal{G}_j$:

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{F}_i \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{comm.}"] & \cdots \arrow[r] & \mathcal{F}_{i'} \arrow[r] \arrow[d] \arrow[dr, phantom, ""] & \cdots \arrow[r] & \mathcal{F}_{i''} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{G}_j \arrow[r] & \cdots \arrow[r] & \mathcal{G}_{j'} \arrow[r] & \cdots \arrow[r] & \mathcal{G}_{j''} \arrow[r] & {}
\end{tikzcd}$$

Ker: $(\ker \mathcal{F})_i = $ " $\{x \in \mathcal{F}_i \mid x \xrightarrow{\ \exists\ } 0$

$\qquad \|$

$\ker(\mathcal{F}_i \to \varinjlim_j \mathcal{G}_j)$

Clearly, $\mathcal{E} \to \mathcal{F} \to \mathcal{G}$ with composite 0 $\iff$ $\mathcal{E} \to \ker(\mathcal{F} \to \mathcal{G})$

coker?

$$\begin{tikzcd}
\cdots \arrow[r] & \mathcal{F}_i \arrow[r] \arrow[dr] & \cdots \arrow[r] & \mathcal{F}_{j'} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{F}_{j''} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & & \mathcal{G}_j \arrow[r] \arrow[dl] & \mathcal{G}_{j'} \arrow[r] \arrow[d] & \cdots \arrow[r] & \mathcal{G}_{j''} \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \mathcal{H}_k \arrow[r] & \cdots \arrow[r] & \mathcal{H}_{k'} \arrow[r] & \cdots \arrow[r] & \mathcal{H}_{k''} \arrow[r] & \cdots
\end{tikzcd}$$

composition $= 0$:

for what $x_{j'} \in \mathcal{G}_{j'}$:

$$\begin{tikzcd}
x_{j'} \arrow[d, bend right] & & ? \\
{} \arrow[rr, "\exists"] & & 0
\end{tikzcd}$$

$$\exists\, y_{i''} \mapsto x_{j''}$$

$$x_{j'} \xmapsto{\exists} x_{j''}$$

Factor those $x_{j'}$ out; this is our $(\mathrm{Coker})_{j'}$.

Make definition rigorous; check that

$$\left\{ f : \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(\mathcal{G}, \mathcal{H}) \;\; \begin{tikzcd} \mathcal{F} \arrow[r] \arrow[rr, bend right, "0"'] & \mathcal{G} \arrow[r] & \mathcal{H} \end{tikzcd} \right\} = \mathrm{Hom}_{\mathrm{Ind}\,\mathcal{C}}(\mathrm{Coker}, \mathcal{H})$$

Stacks

Stack on X: $\boxed{\text{PRESTACK on } X:}$

0) $X \supset U$ (open) $\rightsquigarrow$ category $\mathcal{C}(U)$

$?|_U = \rho_{UV}$ $\quad U \subset V$: $\mathcal{C}(U) \leftarrow \mathcal{C}(V)$ functor

θ_{UVW} $\quad U \subset V \subset W$:

$$\begin{tikzcd} \mathcal{C}(U) & \mathcal{C}(V) \arrow[l] & \mathcal{C}(W) \arrow[l] \arrow[ll, bend left, phantom, "\Downarrow"] \arrow[ll, bend left] \end{tikzcd}$$

isom of functors

1) $U \subset V \subset W \subset T$:

$$\begin{tikzcd} \mathcal{C}(U) & \mathcal{C}(W) \arrow[l, bend right] \arrow[l, bend left] \arrow[l, phantom, "\Downarrow = \Downarrow"] \end{tikzcd}$$

Morphisms form a sheaf:

$\mathcal{F}, \mathcal{G} \in \mathrm{ob}\, \mathcal{C}(U) \quad \mathrm{Hom}_{\mathcal{C}(U)}(\mathcal{F}, \mathcal{G})$

get a presheaf on U:

$V \subset V \rightsquigarrow \mathrm{Hom}_{\mathcal{C}(V)}(\mathcal{F}|_V, \mathcal{G}|_V)$

restriction: $\mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ in $\mathcal{C}(U)$

$$\begin{tikzcd} \mathcal{C}(U) \arrow[d, "\text{functor } |_V"] \\ \mathcal{C}(V) \end{tikzcd}$$

$\boxed{\text{ST1}}$

$\mathcal{F}|_V \xrightarrow{\varphi|_V} \mathcal{G}|_V$

This presheaf is required to be a sheaf.

2) Descent data come from objects:

Descent datum: $\mathcal{F}(U)$ in $\mathrm{Ob}\, \mathcal{C}(U)$;

$\mathcal{F}(U)|_{U\cap V} \xleftarrow{\simeq} \mathcal{F}(V)|_{U\cap V}$; $\quad \mathcal{F}(U)|_{U\cap V\cap W} \overset{\mathrm{II}}{\leftleftarrows} \mathcal{F}(W)|_{U\cap V\cap W}$

$(\forall$ descent data on any $T)\ \exists\, \mathcal{F} \in \mathrm{ob}\, \mathcal{C}(T): \mathcal{F}(U) \cong \mathcal{F}|_U$ $\quad \boxed{\text{ST2}}$

Proper stack of Abelian categories:

a priori not a stack (nor proper in any obvious sense).

X Hausdorff locally compact

- Prestack of Abelian categories; ρ_{uv} are exact functors.

- Satisfies ST1.
- $\mathcal{C}(U)$ admits small filtered lims, colims that commute with ρ_{uv}.

- Denote by $i_U^{-1} : \mathcal{C}(X) \to \mathcal{C}(U)$ $(i_U^{-1} = \rho_{UX} = ?|_U)$

it admits left adjoint $\boxed{i_{U!}}$

$\mathrm{id}_{\mathcal{C}(U)} \xrightarrow{\sim} i_U^{-1} i_{U!}$

(recall: for sheaves, $i_{U!}$ = extension by zero)

This neatly axiomatizes the properties of <u>sheaves</u>. More intuitive properties that are satisfied by <u>sheaves</u> follow in this generality.

<u>E.g.</u> 

$i_{V!}\mathcal{F}|_V \to i_{V'!}\mathcal{F}|_{V'}$

<u>Lemma</u> $i_{U!}\mathcal{F} \simeq \operatorname{colim}_{V \Subset U} i_{V!}(\mathcal{F}|_V)$

sheaf-theoretical colim involves <u>sheafification</u> of the presheaf $U \mapsto \operatorname{colim}_j \mathcal{F}(U)$

Also: $\mathrm{supp}(\mathcal{F})$ makes sense for $\mathcal{F} \in \mathrm{Ob}\,\mathcal{C}(U)$ (b/c we have object 0 now).

<u>Lemma</u> $\mathrm{supp}(i_{U!}\mathcal{F}) \subseteq \overline{\mathcal{F}}$

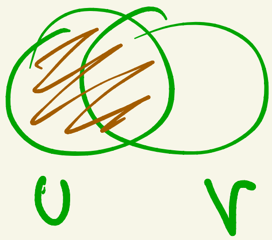
If $\mathrm{supp}(\mathcal{F}) \subset U$: $i_{U!}\, i_U^{-1}\mathcal{F} \xrightarrow{\sim} \mathcal{F}$

<u>In general:</u> $i_{U!}\, i_U^{-1}\mathcal{F} \hookrightarrow \mathcal{F}$

$\downarrow\wr$ always

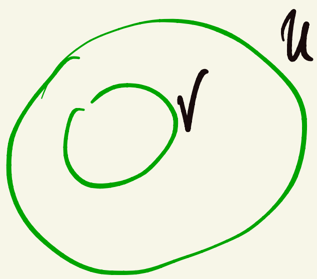
$i_U^{-1}\, i_{U!}\mathcal{F}$

e.g. $i_{U!}$ is exact

$\mathcal{F} \in \mathcal{C}(U)$ 

$$i_{V!}\, i_V^{-1}\, i_{U!}\mathcal{F} \simeq i_{U\cap V!}\mathcal{F}$$

e.g.



$i_U^{-1} i_{V!}$ left adjoint to ρ_{VU}

$\mathcal{C}$ is a proper stack on $X \Rightarrow \mathcal{C}|_U$ is a proper stack on U.

Now:

$$\underbrace{\mathcal{H}om_{\mathcal{C}}(i_{U!}\mathcal{F}, \mathcal{G})}_{\text{sheaf on } X \text{ by ST1}} \simeq i_{U*}\underbrace{\mathcal{H}om(\mathcal{F}, \mathcal{G}|_U)}_{\text{sheaf on } U \text{ by ST1 for } U}$$

just sheaf-theoretical i_{U*}. i_{U*} has not been defined for $\mathcal{F} \in \mathcal{C}(U)$

Now define: $\mathcal{F} \in \mathcal{C}(X)$;

$\mathcal{F}_U := i_{U!}\, i_U^{-1}\mathcal{F}$ ← a) subobject of $\mathcal{F}$ b) supported on $\bar{U}$

have seen: (FOR NOW)

$$\underbrace{\Gamma}_{\text{sections on } U}(U, \underbrace{\mathcal{H}om_{\mathcal{C}}(\mathcal{F}, \mathcal{G})}_{\text{sheaf on } X}) \simeq \mathrm{Hom}_{\mathcal{C}(X)}(\underbrace{\mathcal{F}_U, \mathcal{G}}_{\text{two objects of } \mathcal{C}(X)})$$

★

Also: $\mathcal{F}_V \to \mathcal{F}_U$ for $V \subset U$

(agrees with/ follows from ★) $\quad X = \bigcup_i U_i$

We are doing "sheaves", not "ab grps", i.e. two morphisms in the abelian category $\mathcal{C}(X)$.

$$\bigoplus_{i,j\in I} F_{U_{ij}} \overset{\alpha}{\underset{\beta}{\rightrightarrows}} \bigoplus_{i\in I} F_{U_i}$$

$$\left(\begin{array}{ccc} F_{U_{ij}} & \to & F_i \\ \downarrow & & \\ F_j & & \end{array} \right.$$

Lemma $\operatorname{coker}(\alpha-\beta) \cong F$ $\quad (*)$

Theorem A proper stack is a stack.

We have to prove: any descent data comes from an object. How to construct the object?

Descent data: $\mathcal{F}_k \in \operatorname{Ob} \mathcal{C}(U_k)$, $\forall k$

$\mathcal{F}_j|_{U_{jk}} \overset{\sim}{\leftarrow} \mathcal{F}_k|_{U_{jk}}$ s.t. ...

That is enough to repeat the coequalizer construction $(*)$.

The coequalizer is our object. ◀

We have seen: axiomatics of proper stacks is : like sheaves, but: i_U^{-1} $i_{U!}$ $\cancel{i_{U*}}$ ← SO FAR

What about closed embeddings?

$S \subset X$ closed. $\quad 0 \to \mathcal{F}_{X \setminus S} \to \mathcal{F} \to \boxed{\mathcal{F}_S} \to 0$

know: mono $\qquad$ By DFN

Z locally closed: $Z = S \cap U$

$$\mathcal{F}_Z = (\mathcal{F}_U)_S \quad \text{(depends on } Z \text{ only)}.$$

Lemma

$$\operatorname*{colim}_{U \in X} \mathcal{F}_U \xrightarrow{\sim} \mathcal{F} \xleftarrow{\sim} \lim_{K \subset X} \mathcal{F}_K$$

K compact

And now: $i_{U*} \mathcal{G} = \lim\limits_{\substack{\longleftarrow \\ K \Subset U}} i_{U!}(G_K)$

Right adjoint to i_U^{-1}

$\Downarrow$

$$i_{U!} \longrightarrow i_{U*}$$

(morphism of functors $\mathcal{C}(U) \to \mathcal{C}(X)$)

And then for $Z \subset X$ locally closed:

$$\Gamma_Z(-) : \mathcal{C}(X) \to \mathcal{C}(X)$$

right adjoint to $\mathcal{F}_Z \leftarrow\!\shortmid \mathcal{F}$

Z open: $\Gamma_U = i_{U*} i_U^{-1}$ $\quad ?_U = i_U^{-1} i_{U!}$

Z closed: $0 \to \Gamma_Z(\mathcal{F}) \to \mathcal{F} \to \Gamma_{X \setminus Z}(\mathcal{F}) \to 0$

So: from axioms of a proper stack recovered:

$$\mathcal{C}(U) \underset{i_{U*}}{\overset{i_{U!}}{\rightrightarrows}} \mathcal{C}(X), \quad \mathcal{C}(X) \xrightarrow{i_U^{-1}} \mathcal{C}(U) \qquad U \subset X \text{ open}$$

$$\mathcal{F} \longmapsto \mathcal{F}_Z$$

$$\mathcal{C}(X) \underset{\Gamma_Z}{\overset{}{\rightleftarrows}} \mathcal{C}(X) \qquad Z \subset X \text{ locally closed}$$

Next: <u>Indization</u>

Goal: 0) Ind-objects of a category $\mathcal{C}$ ✓

1) Ind-objects of $\mathcal{C}(U)$, $\forall U$: those do not glue

2). How to modify that?

Why do not they glue?

$\mathcal{C}(U) = \{\text{Sheaves on } U\}$

$U \mapsto \operatorname{Ind}\mathcal{C}(U)$? $\quad X = \mathbb{R}$ $\quad \mathcal{F} = \text{"colim"}\, k_{[n,\infty)}$

$\mathcal{F} \in \operatorname{Ind}\mathcal{C}(X)$: $\quad k_{[0,\infty)} \to k_{[1,\infty)} \to k_{[2,\infty)} \to$

$\mathcal{F}|_{(-n,n)} = 0, \ \forall n$ but $\mathcal{F} \neq 0$ in $\hat{\mathcal{C}}$

in fact $\widehat{\mathcal{C}(\mathbb{R})}(k_{\mathbb{R}}, \mathcal{F}) = \varinjlim_n \mathrm{Hom}(k_{\mathbb{R}}, k_{[n,\infty)})$

$= k$

not a sheaf on morphisms.

Change the definition: for every proper stack $\mathcal{C}$:

$I\mathcal{C}(U) = \mathrm{Ind}\, \mathcal{C}_c(U)$; $\mathcal{C}_c(U) = \{$ objects with compact support in $\mathcal{C}_U \}$

$\mathcal{C}(U) \overset{i_U}{\hookrightarrow} I\mathcal{C}(U)$

since $\mathcal{F} = \operatorname{colim}_{V \Subset U} i_{V!} i_V^{-1} \mathcal{F}$

$\boxed{\begin{array}{l} \text{or: } i_X : \mathcal{C}(X) \to I\mathcal{C}(X) \\ \mathcal{F} \mapsto \mathrm{Hom}_{\mathcal{C}(X)}(?, \mathcal{F}) \,\big|\, \hat{\mathcal{C}}_c(X) \end{array}}$

Need: ① restriction

$U \hookrightarrow V$

$I\mathcal{C}(U) \xleftarrow{\rho_{UV}} I\mathcal{C}(V)$

② [≈automatic; just $i_{U!}$

$I\mathcal{C}(U) \xrightarrow{i_{U!!}} I\mathcal{C}(X)$

$U \hookrightarrow X$

$i_{U!!}$: "colim" $\mathcal{F}_i \longmapsto$ "colim" $i_{U!} \mathcal{F}_i$

Restriction

$\rho_{UX} : \mathcal{IC}(U) \leftarrow \mathcal{IC}(X)$

of course the usual $\rho_{UX} : \mathcal{C}_c(U) \cancel{\leftarrow} \mathcal{C}_c(X)$

$$\mathcal{C}_c(X) \to C(U)$$

intuitively:

$$\mathrm{Ind}\,\mathcal{C}_c(X) \to C(U) \to \mathrm{Ind}\,\mathrm{Ind}\,\mathcal{C}_c(U)^{\omega}$$

Actually:

$$\rho_{UX}\mathcal{F} : \underset{\mathcal{C}_c(U)}{\overset{\cap}{g}} \longmapsto \mathrm{Hom}_{\mathcal{IC}(X)}(i_{U!}g, \mathcal{F})$$

$$\mathcal{IC}(X) = \mathrm{Hom}_{\hat{\mathcal{C}}_c(X)}$$

by this reason:

if $\mathcal{F} \in \mathrm{Ind}\,\mathcal{C}_c(X)$

then is the above presheaf on $\mathcal{C}_c(U)$ is in $\mathrm{Ind}\,\mathcal{C}_c(U)$.

Last but not least, $\varprojlim$ (small) exist in $\mathcal{IC}(X)$.

(slightly counterintuitive...)

Enough to check for products. Given:

$\mathcal{F}_i \in \mathcal{C}_c(X)$ $\quad \prod \mathcal{F}_i$ in $\mathcal{C}_c(X)^{\wedge}$. Why is this a filtered "colim"? over what? over $U \Subset X$.

$$\mathcal{C}_c(X) \ni g \mapsto \mathrm{Hom}_{C(X)}(g, \prod \mathcal{F}_i) = \prod \mathrm{Hom}_{C(X)}(g, \mathcal{F}_i) \in \varinjlim_{U \Subset X} \mathrm{Hom}(g, \prod \mathcal{F}_{i_U})$$

since $\mathrm{supp}\, g \subset U$

A sample of a computation: why is $i_{U!!}$ left adjoint to i_U^{-1}?

$$\mathrm{Hom}_{IC}\left(\text{"}\varinjlim_i\text{"} \mathcal{F}_i, G\right) = \varprojlim_i \underbrace{\mathrm{Hom}_{IC}(\mathcal{F}_i, G)}_{\substack{\| \\ \varinjlim_j (\mathcal{F}_i, G_j)}}$$

So:

$$\mathrm{Hom}_{IC(X)}\left(i_{U!!}\mathcal{F}, G\right) = \varprojlim_i \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}\mathcal{F}_i, G)$$

$$= \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}(X)}(i_{U!}\mathcal{F}_i, G_j)$$

$$= \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathcal{C}(X)}(\mathcal{F}_i, i_U^{-1} G_j) \qquad \text{"}\varinjlim_{j;V\in U}\text{"} i_V i_U^{-1} G_j$$

$\|$ b/c $\mathcal{F}_i \in \mathcal{C}_c(X)$

compare to:

$$\varprojlim_i \mathrm{Hom}_{IC(X)}\left(\mathcal{F}_i, i_U^{-1}\left(\text{"}\varinjlim_j\text{"} G\right)\right)$$

Now on locally closed subset $Z \subset X$:

Recall for any proper stack:

$\mathcal{F}_Z$ and $\Gamma_Z(\mathcal{F})$

Applying that to $\mathcal{IC}$: if $Z = \overset{op}{U} \cap \overset{cl}{S}$

$$ {}_Z\mathcal{F} = \text{“}\lim_{\substack{i \\ V \Subset U;\, W \supset S}}\text{”} (\mathcal{F}_i)_{V \cap \overline{W}} $$

whenever $\mathcal{F} = \text{“}\lim_i\text{”} \mathcal{F}_i$

Why ${}_Z\mathcal{F}$ and not $\mathcal{F}_Z$? b/c

$$ \iota_X({}_Z\mathcal{F}) \not\cong \iota_X(\mathcal{F})_Z \quad \text{for } \mathcal{F} \in \mathcal{C}(X) $$

whereas

$$ \iota_X(\Gamma_Z(\mathcal{F})) \simeq \Gamma_Z(\iota_X \mathcal{F}) $$

L. $\underbrace{\mathcal{H}om_{\mathcal{IC}}(G, \mathcal{F})}_{\text{sheaf on } X}{}_Z \simeq \mathcal{H}om_{\mathcal{IC}}(G, {}_Z\mathcal{F})$

If $\mathcal{C} = \{\text{sheaves of } \mathcal{A}\text{-modules}\}$, $\mathcal{A}$ – a sheaf of rings on X: ok.

Further relation between $\mathcal{C}(X)$ and $I\mathcal{C}(X)$

$$\mathcal{C}(X) \underset{\alpha_X}{\overset{\iota_X}{\rightleftarrows}} I\mathcal{C}(X)$$

$$\alpha_X : \text{"}\varinjlim\text{"} \mathcal{F}_i \longmapsto \varinjlim \mathcal{F}_i$$

$$\text{"}\varinjlim_U\text{"} \iota_U \mathcal{G}$$

$$\operatorname{Hom}_{\mathcal{C}(X)}(\alpha_X \mathcal{F}, \mathcal{G}) = \operatorname{Hom}_{I\mathcal{C}(X)}(\mathcal{F}, \iota_X \mathcal{G})$$

$$\varprojlim_i \mathcal{C}(X)(\mathcal{F}_i, \mathcal{G}) \qquad \varprojlim_i \varinjlim_{U \in X} \mathcal{C}(X)(\mathcal{F}_i, \iota_U \mathcal{G})$$

$=$

again: b/c all $\mathcal{F}_i$ are in $\mathcal{C}_c$

So: α_X left adjoint to ι_X.

How about a left adjoint to α_X?

$$\operatorname{Hom}_{\mathcal{C}(X)}(\mathcal{F}, \alpha_X(\mathcal{G})) \doteq \operatorname{Hom}_{\mathcal{C}(X)}(\mathcal{F}, \varinjlim_j \mathcal{G}_j)$$

$$\mathcal{G} = \text{"}\varinjlim_j\text{"} \mathcal{G}_j$$

? compactness condition for $\mathcal{F}$

$$\operatorname{Hom}_{I\mathcal{C}(X)}(\text{"}\varinjlim_?\text{"} \mathcal{F}_?, \mathcal{G})$$

$$\varprojlim_? \varinjlim_j \operatorname{Hom}_{\mathcal{C}(X)}(\mathcal{F}_?, \mathcal{G}_j) \overset{?}{=} \varinjlim_j \operatorname{Hom}_{\mathcal{C}(X)}(\mathcal{F}, \mathcal{G}_j)$$

Say, $\mathcal{F} \in \mathcal{C}_c(X)$: $\mathcal{F}_? = \mathcal{F}$

Can say about the desired left adjoint

$$\beta_X : \mathcal{C}(X) \longrightarrow I\mathcal{C}(X):$$

If $U \in X$ and $\mathcal{F} \in \mathcal{C}(X)$ and $\mathcal{F}_U$ compact object of $\mathcal{C}(U)$:

$$\beta_X(\mathcal{F}_U) = (\text{constant ind syst})\ \mathcal{F}_U.$$

If we can represent $\mathcal{F}$ as

☆ $$\operatorname{coker}\left(\bigoplus_a (\mathcal{F}_a)_{U_a} \to \bigoplus_b (\mathcal{G}_b)_{U_b}\right)$$

where each $(\mathcal{F}_a)_{U_a}$, $(\mathcal{G}_{U_b})_{U_b}$ are compact,

then:

$\beta_X(\mathcal{F})$ = same coker, viewed as of morphism of constant ind systems

☆ In Kashiwara and Schapira's language: $\mathcal{C}$ has enough light objects.

In which case:

$$\begin{tikzcd} \mathcal{C}(X) \arrow[r, bend left, "\iota_X"] \arrow[r, bend right, "\beta_X"'] & I\mathcal{C}(X) \arrow[l, "\alpha_X"'] \end{tikzcd}$$

$\mathcal{C} = \mathcal{A}$-mod, $\mathcal{A}$=sheaf of rgs: OK

Grothendieck topologies

$\mathcal{C}$ category; objects: $U, V, \dots$

$\mathcal{C}_U = \{ V \to U \}$ objs $/U$

$S \subset \mathrm{Ob}(\mathcal{C}_U)$ (i.e. family of morphisms to U):

S_1, S_2: $S_1 \preceq S_2$ if $\forall\ V \to U$ in S_1 $\exists\ V' \to U$ in S_2:

(S_1 refinement of S_2)

$$\begin{tikzcd} V \arrow[d] \arrow[dr] & \\ V' \arrow[r] & U \end{tikzcd}$$

Grothendieck topology on $\mathcal{C}$:

$U \in \mathcal{C} \rightsquigarrow$ family $\mathrm{Cov}(U)$ of subsets of $\mathcal{C}_U$

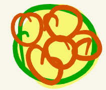
GT1 $\quad U \xrightarrow{=} U \in \mathrm{Cov}(U)$

GT2 $\quad S_1 \preceq S_2 \in \mathrm{Cov}(U) \Rightarrow S_1 \in \mathrm{Cov}(U)$

GT3 $\quad S \in \mathrm{Cov}(U) \Rightarrow V \times_U S \in \mathrm{Cov}(V), \forall V \to U$

GT4 $\quad S_1, S_2 \in \mathrm{Cov}(U), S_1 \in \mathrm{Cov}(U), V \times_U S_2 \in \mathrm{Cov}(V)$ for $\forall V \in S_1$, then $S_2 \in \mathrm{Cov}(U)$.

GT4:



$\forall V$ in S_1

A site X: cat. $\mathcal{C}_X$ with admits finite prods, fiber products, with a Grothendieck topology.

$f^t: \mathcal{C}_Y \to \mathcal{C}_X$ continuous: preserves fiber products, covers.

(or: a morphism of sites $X \xrightarrow{f} Y$).

X: topological space

Site X: all open covers

X_f: open covers having a finite subcover.

X locally compact: Covers S of U s.t.:

- For every open U
- for any compact K

there is a finite subset of S that covers $K \cap U$

This is the topology X_{lf}

In particular: covers of <u>relatively compact</u> U must have a finite subcover.

Presheaf of k-modules: $\mathcal{C}_X^{op} \to \mathrm{Mod}(k)$

$$F(S) := \ker\Big(\prod_{V\in S} F(V) \rightrightarrows \prod_{V',V''\in S} F(V'\times_U V'')\Big)$$

(S a cover of U)

$$F(U) \to F(S)$$

F <u>separated</u> if $\hookrightarrow$; a <u>sheaf</u> if $\xrightarrow{\sim}$.

$\mathrm{Mod}(k_X) = \{\text{Sheaves of } k_X\text{-mods}\}$
(full subcat)

<u>Now</u>: start with X Hausdorff l.c.

T – family of opens in X

$T_c = \{$rel compacts in $T\}$

$T_0 = \{$connecteds in $T_c\}$

X_T: category T with the topology induced by X_{lf}

from what I underst.:

$V \subset U$ and $V \in T$

U_{XT}: $T \cap U$ with topology induced by X_T.

$$i_{U_{XT}} : U_{XT} \to X_T$$

$$V \cap U \longleftarrow\!\mapsto V \in T$$

$$j_{U_{XT}} : U_{XT} \leftarrow X_T$$

$$V \longmapsto V$$

U_T: also $T \cap U$, but w/ topology induced by U_{lf}.

The category $\mathrm{Coh}(T_c)$

a sheaf $\mathcal{F}$ on X (in usual topology) is T_c-coherent if it satisfies a) + b):

a) T_c-finite: $\bigoplus_{j=1}^{N} k_{U_j} \twoheadrightarrow \mathcal{F}$ $\quad U_j \in T_c$

b) T_c-pseudo-coherent:

$$\forall \varphi : \bigoplus_{i=1}^{M} k_{V_i} \longrightarrow \mathcal{F}, \quad V_i \in T_c,$$

$\ker(\varphi)$ is T_c-finite

Thm $\mathrm{Coh}(T_c)$ coherent.

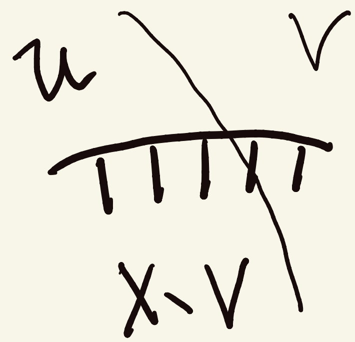
$\mathrm{Coh}(T_c) \hookrightarrow \mathrm{mod}(k_X)$ exact

PROOF

$$\begin{tikzcd}
 & & G \arrow[dl, bend right] \arrow[d, "\varphi"] \arrow[dr, bend left] & \\
0 \arrow[r] & \mathcal{F}' \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{F}''
\end{tikzcd}$$

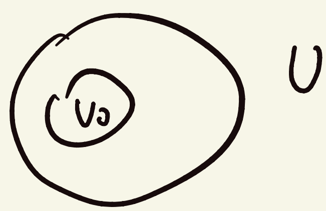
$\ker(\varphi)$
$\shortparallel$
ker of: $\ker(G \to \mathcal{F}'') \to \mathcal{F}'$

$\mathcal{F}', \mathcal{F}''$ pseudocoh $\Rightarrow$ $\mathcal{F}$ is.



$$\begin{tikzcd} k_U \arrow[r, "\mathbb{1}_{U,X\setminus V}"] \arrow[d] & k_{X\setminus V} \\ k_{U\cap(X\setminus V)} \arrow[ur] & \end{tikzcd}$$

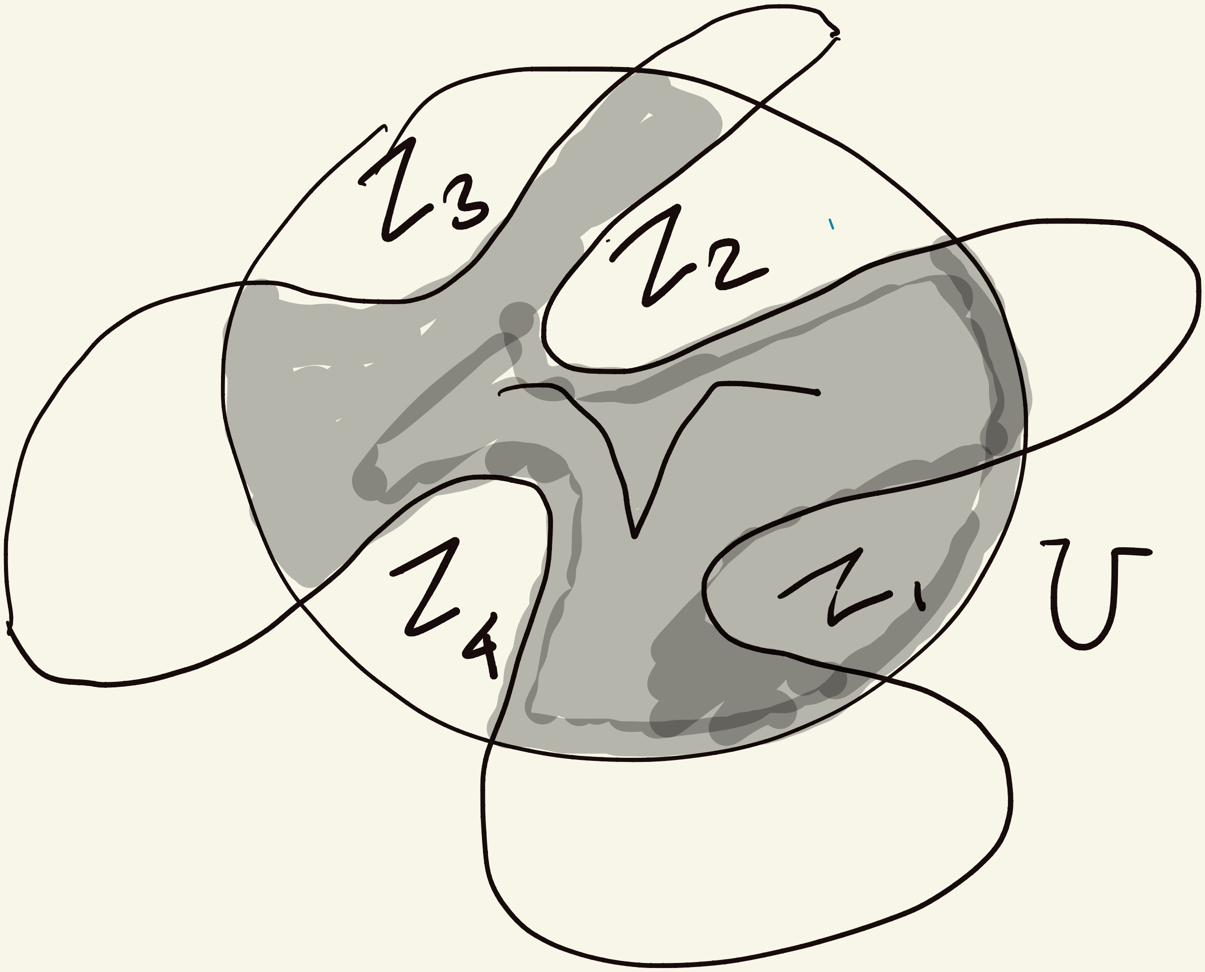
$$k_{U_0} \xrightarrow{\mathbb{1}_{U_0,U}} k_U$$



if $U, V \in T_c$: "All morphisms $k_U \to k_{X\setminus V}$ are locally $c \cdot \mathbb{1}_{U,X\setminus V}$"

$$\begin{tikzcd} \overset{\text{fin.}}{\bigoplus_i} k_{U_i \in T_c} \arrow[d] \arrow[dr, bend left] & \\ k_U \arrow[r, "\forall\varphi"] & k_{X\setminus V} \end{tikzcd}$$

$\varphi|k_{U_i} = c_i \cdot \mathbb{1}_{U_i, X\setminus V}$



$$U \cap (X \setminus V) = \bigcup_{\text{fin.}} Z_j$$

$$U_i = (U \cap V) \cup Z_i = U \setminus \Big(\underbrace{\bigcup_{\neq i} Z_j}_{\text{closed}}\Big)$$

$\Downarrow$ open

$$\begin{tikzcd}
k_{U_i} \arrow[r] \arrow[dr, dashed] & k_U \arrow[r, "\varphi"] & k_{X/V} \\
 & k_{Z_i} \arrow[ur, "\text{all proportional}"'] &
\end{tikzcd}$$

k_{Z_i} ⇐ connected

L. a T_c-finite subsheaf of k_U, $U \in T_c$, is $\simeq k_V$, $\underset{\substack{\uparrow \\ T_c}}{V} \subset U$.

Pf

$$\begin{tikzcd}
\bigoplus_i k_{U_i} \arrow[r, two heads] \arrow[dr] & \mathcal{F} \arrow[d] \\
\bigoplus a_i \cdot 1_{U_i \to U} & k_U
\end{tikzcd}$$

$U_i \in T_c$

$\mathcal{F} \simeq \bigoplus_{a_i \neq 0} k_{U_i}$

L. $V, W \in T_c \Rightarrow k_{V \setminus W}$ T_c-coh

Pf $k_V \twoheadrightarrow k_{V \setminus W}$ — T_c-finite ✓

$$\begin{tikzcd}
\forall \quad S = \bigoplus_i \underbrace{k_{U_i}} \arrow[r, "\varphi"] & k_{V \setminus W} \\
S' = \bigoplus_i \bigoplus k_{U_{ij}} \arrow[u, two heads] \arrow[ur, "\psi"] &
\end{tikzcd}$$

as in lemma above

enough: $\ker(\psi)$ T_c-finite

<u>Reduce to:</u>

$$\bigoplus k_{U_i} \xrightarrow{\Psi} k_{V\setminus W}$$

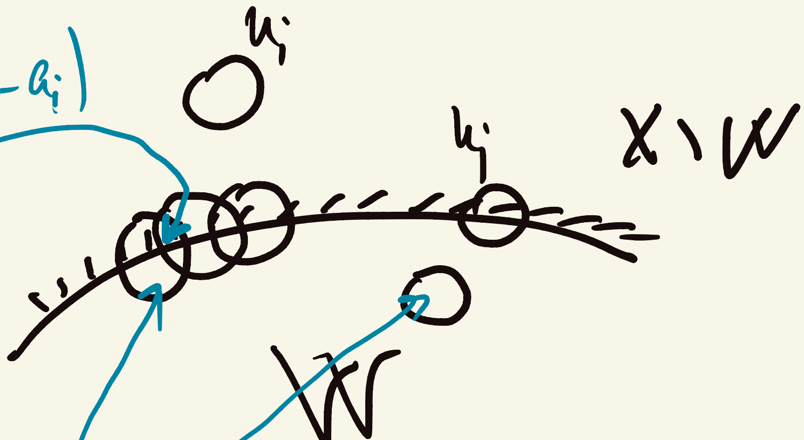
$$\sum a_i \mathbb{1}_{U_i} \to X\setminus W$$

$$\begin{tikzcd} \bigoplus k_{V_i} \arrow[r] \arrow[rr, bend left] & k_{V\setminus W} \arrow[r, hook] & k_{X\setminus W} \end{tikzcd}$$

is kernel T_c-finite?

epi from:

$(a_j - a_i)$



$\bigoplus_{i\neq j} k_{U_i\cap U_j}$

$\bigoplus_{i\,:\,a_i=0} k_{U_i}$

$\bigoplus k_{U_i\cap W}$

<u>Next</u>: T_c-coh closed under quotients

$$\underbrace{0 \to \mathcal{N} \to \mathcal{L}}_{T_c\text{-coh}} \to \mathcal{F} \to 0 \qquad \Longrightarrow \qquad \mathcal{F} \text{ is } T_c\text{-coh}$$

Pf

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{N} \arrow[r] & \mathcal{L} \arrow[r] & \mathcal{F} \arrow[r] & 0 \\
& \mathcal{N}_0 \arrow[u] \arrow[r] & \mathcal{L}_0 \arrow[u] \arrow[ur, bend left] & = \bigoplus \mathcal{L}_{n_i} &
\end{tikzcd}$$

$$\mathcal{N}_0 := \ker(\mathcal{L}_0 \longrightarrow) = \operatorname{im}\left(\text{of } \ker\left(\begin{matrix} \mathcal{N} \longrightarrow \mathcal{L} \\ \oplus \quad\ \ \uparrow \\ \qquad\ \mathcal{L}_0 \end{matrix}\right)\right) \text{ in } \mathcal{L}_0$$

since $\mathcal{L}$ is T_c-coh, $\mathcal{N}_0$ is T_c-finite $\Downarrow$ T_c-coh

So enough to prove:

$$\underbrace{0 \to \mathcal{N} \to \bigoplus_{i=1}^{N} \mathcal{L}_{n_i}}_{T_c\text{-coh}} \twoheadrightarrow \mathcal{F} \to 0 \qquad \Longrightarrow \qquad \mathcal{F} \text{ is } T_c\text{-coh}$$

$N=1$: By Lemma, $k_{v_1} \twoheadrightarrow \mathcal{F}$

$N = k_{v}, \; V \subset U_1.$

$N>1$: induction in N.

$$\begin{tikzcd}
 & N_1 \arrow[r, hook] \arrow[d, tail] & N \arrow[r, two heads] \arrow[d, tail] & N_2 \arrow[d, tail] & \\
k_{v_1} = & \mathcal{L}_1 \arrow[r] \arrow[d, two heads] & \mathcal{L} \arrow[r] \arrow[d, two heads] & \mathcal{L}_2 \arrow[d, two heads] & = \bigoplus_{2}^{N} k_{v_i} \\
 & \mathcal{F}_1 \arrow[r, hook] \arrow[d, equal] & \mathcal{F} \arrow[r] & \mathcal{F}_2 & \\
 & \text{image of } \mathcal{L}_1 & & &
\end{tikzcd}$$

Ψcoh from a Lemma above

End of proof of Thm

Next:

$$k_T\text{-mod} \underset{\rho_*}{\overset{\rho^{-1}}{\rightleftarrows}} k_X\text{-mod}$$

i.e. sheaves of k-mod in T-topology on X | i.e. sheaves of k-mod on X

ρ_* = restriction to open subsets that are in T.

Comes from a morphism of sites

$$X_T \xleftarrow{\rho} X$$

$$T \xrightarrow{\rho^t} Op_X$$

Prop For any $U \in T$:

$$\underbrace{k_{TU} \underset{\rho^{-1}}{\overset{\rho_*}{\leftrightarrows}} k_{XU}}_{\text{constant sheaves in resp. topologies}}$$

$$\begin{tikzcd}
k_{X\times T}\text{-mod} \arrow[r, bend left, "\rho^{-1}"] & k_X\text{-mod} \arrow[l, bend left, "\rho_*"] & T_c\text{-coh} \arrow[l, "i_c"'] \arrow[ll, bend left=40, "\rho_{*c}"]
\end{tikzcd}$$

Claim: comm. diagr.

claim: exact fully faithful

Claim: $\rho^{-1}\rho_{*c} = i_c$

ρ_{c*} exact: $\quad S \twoheadrightarrow \mathcal{F}$ T_c-coh sheaves on X

$s \in \mathcal{F}(U) = \Gamma(U, \rho_* \mathcal{F}) \quad U \in T_c$

$$\begin{tikzcd}
\bigoplus k_{XU_i} \arrow[r, two heads] & S' \arrow[r, two heads] \arrow[d] & k_{XU} \arrow[d, "s"] \\
 & S \arrow[r, two heads] & \mathcal{F}
\end{tikzcd}$$

$\exists$ fin

$\Uparrow$ also T_c-coh

(T_c coh closed under kers...)

$k_{XU_i} \longrightarrow k_{XU}$: $a_i \cdot 1_{U_i \to U}$

if $a_i \neq 0$: $s|_{U_i} \in \operatorname{im}(S|U_i| \to \mathcal{F}|U_i|)$

if $a_i = 0$: $s|_{U_i} = 0$, again in im.

(from commutativity of the diagram)

Fully faithful: ...

$\rho^{-1}\rho_{*c} = i_c$: true on $\mathcal{F} = k_{XU}$; otherwise by exactness of both. $\triangleleft$

From a T-sheaf to an ord sheaf

$F \in Mod(k_T)$

$I_0 = \{(U,s) \mid U \in T_c \,;\, s \in F(U)\}$

$$G_0 := \bigoplus_{(U,s)\in I_0} k_{TU} \overset{\varphi}{\twoheadrightarrow} F$$

$\varphi_{U,s} : k_{TU} \to F$

$$G_1 \longrightarrow G_0 \xrightarrow{\varphi} F$$

$$\shortparallel$$

$$\bigoplus_{(V,t)\in I_1} k_{TV} \twoheadrightarrow \ker\varphi$$

$$J = \{(J_1, J_0) \mid \text{both finite};\ \ J_1 \subset I_1,\ J_0 \subset I_0,\ \operatorname{im}(\varphi|_{G_{J_1}}) \subset G_{J_0} \}$$

$$\bigoplus k_{TV} \longrightarrow \bigoplus k_{TU}$$

$$\overset{\text{fin}}{\bigoplus_{J_1}} k_{TV} \rightrightarrows \bigoplus k_{TU}$$

fin but big enough

Therefore

any k_T-mod is filtered $\varinjlim_{i\in I} p_* \mathcal{F}_i$, $\mathcal{F}_i \in T_c$-coh

We have constructed

$$\lambda : I(\mathrm{Coh}(T_c)) \longrightarrow \mathrm{Mod}(k_T)$$

$$\text{``}\varinjlim_i\text{''} \mathcal{F}_i \longmapsto \varinjlim_i p_* \mathcal{F}_i$$

Thm λ is an equivalence of abelian categories

Pf essentially surj ✓

Fully faithful: ...

Recall:

$T \subset Op_X$ subset of opens in X (w/ nice properties)

$T_c = \{\text{relatively compacts in } T\}$

Groth Topology X_T: $C_T = T$; coverings: finiteness condition imposed.

$$X \xrightarrow{p} X_T$$

$$\begin{tikzcd} Coh(T_c) \arrow[r, "i_T"] & Mod(k_X) \arrow[r, shift right, "p_*"'] & Mod(k_{X_T}) \arrow[l, shift right, "p^{-1}"'] \end{tikzcd}$$

Roughly:

$\mathrm{coker}\left(\bigoplus_{i=1}^{N} k_{U_i} \to \bigoplus_{j=1}^{M} k_{V_j}\right)$ $\quad U_i, V_j \in T_c$

closure of that under ker, coker.

Under some assumptions:

$U \subset T$: $\quad \begin{tikzcd} k_{XU} \arrow[r, shift left, "p_*"] & k_{TU} \arrow[l, shift left, "p^{-1}"] \end{tikzcd}$

$U, V \in T \Leftrightarrow \begin{cases} U \cap V \in T \\ U \cup V \in T \end{cases}$

$\emptyset, X \in T$;

T is a covering of X

$U, V \in T_c \Rightarrow$ $U \setminus V$ has $< \infty$ conn. comps

$\Downarrow$

$$\begin{tikzcd} k_X\text{-mod} \arrow[r, shift left] & k_T\text{-mod} \arrow[l, shift left] \\ Coh(T_c) \arrow[u] & \end{tikzcd}$$

$\uparrow =$

If in addition T form a basis in the topology of X: $= \mathrm{id}$.

- Every $\mathcal{F} \in k_T$-mod is $\varinjlim_{I} \mathcal{F}_i$ $\qquad \mathcal{F}_i \in \mathrm{Coh}(T_c)$

  I — filtered
- $\mathrm{Coh}(T_c)$ is an Abelian subcategory of k_T-mod.

$$I\,\mathrm{Coh}(T_c) \xrightarrow[\lambda]{\sim} \mathrm{Mod}(k_T)$$

$$\text{“}\lim_{i}\text{”}\, \mathcal{F}_i \longmapsto \varinjlim_{i} \rho_{c*} \mathcal{F}_i$$

<u>Lemma</u> $\mathcal{C}$ Abelian category;

$$I\mathcal{C} \simeq \{A : \mathcal{C} \to Ab \mid A \text{ k-linear, left exact}\}$$

Pf Have to show: if A is k-lin, left exact:

$$\{(X,x) \mid x \in A(X)\} \qquad \begin{array}{ccc} (X,x) & \longrightarrow & (Y,y) \\ X & \longrightarrow & Y \\ x & \longleftarrow & y \end{array}$$

is filtered. i)

$$\begin{array}{ccc} (X_1,x_1) & & (X_2,x_2) \\ & {\scriptstyle i_1}\searrow \quad \swarrow{\scriptstyle i_2} & \\ & (X_1 \oplus X_2, ?) & \end{array} \qquad \begin{array}{c} A(X_1 \oplus X_2) \\ \cup\!\!\!\!\!\!\!\!\!\quad \\ ? = p_1^* x_1 + p_2^* x_2 \end{array}$$

ii)

$$\begin{array}{c} (X,x) \\ f \downarrow \downarrow g \\ (Y,y) \end{array} \qquad \begin{array}{c} X \\ f \downarrow \downarrow g \\ Y \end{array} \quad \begin{array}{c} x \\ \uparrow \uparrow \\ y \end{array} \qquad \begin{array}{c} X \\ \downarrow f-g \\ Y \\ \downarrow \\ Z = \mathrm{coker}(f-g) \end{array} \quad \begin{array}{c} 0 \\ \uparrow \\ y \end{array} \qquad \begin{array}{c} A(\ker(f-g)) \\ \uparrow \\ A(Y) \\ \text{exact} \uparrow \text{here} \\ A(Z) \end{array}$$

With this in mind: more functors

e.g.: In addition to "the embedding"

$$I_T : I(Coh(T_c)) \longrightarrow I(k_X)$$: we have

$$J_T : I(k_X) \longrightarrow I(Coh(T_c)) \; (\simeq Mod(k_T))$$

For $\mathcal{F} \in I(k_X)$ and $K \in Coh(T_c)$:

$$J_T \mathcal{F}(K) = Hom_{I(k_X)}(\underbrace{i_c K}_{\text{sheaf of } k_X\text{-mods}}, \mathcal{F})$$

$$\shortparallel$$

$$Hom_{I(Coh(T_c))}(K, J_T \mathcal{F})$$

or:

$$\Gamma(U, \underbrace{J_T(\mathcal{F})}_{\substack{\text{viewed in} \\ Mod(k_T)}}) = Hom_{I(k_X)}(k_{XU}, \mathcal{F})$$

Claim: J_T right adjoint to I_T.

$J_T I_T \simeq id$

$$Mod(k_T) \underset{J_T}{\overset{I_T}{\rightleftarrows}} I(k_X)$$

Also: in addition to

$$\begin{tikzcd} Mod(k_T) \arrow[r, "\rho^{-1}"] \arrow[r, bend left, "\text{also: } \rho_!"] & Mod(k_X) \arrow[l, bend left, "\rho_*"] \end{tikzcd}$$

Recall:

ρ_* : simple restriction ; ρ^{-1} its left adjoint;

$$\rho_* \mathcal{F}(U) = \varinjlim_{\substack{U \subset V \\ V \in T}} \mathcal{F}(V)$$

but there is also right adjoint $\rho_!$

$\rho_! \mathcal{F}(U) =$ Sheaf associated to the presheaf $U \mapsto \mathcal{F}(\overline{U})$
$\overset{\cap}{T}$.

Recall also β_X: (in general, for compactly generated $\mathcal{C}$:

$$\mathcal{C} \longrightarrow Ind(\mathcal{C})$$

compact obj $\mathcal{F}$ (e.g. k_U, $U \in X$) $\longmapsto$ constant ind system $\mathcal{F}$

In general: coker($\underset{I}{\bigoplus}$ compacts $\to$ $\bigoplus$ compacts)

same coker in $Ind\,\mathcal{C}$.

To summarize:

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
k_T\text{-mod} \arrow[r, "\rho^{-1}"] \arrow[d, phantom, "\wr"] & k_X\text{-mod} \arrow[l, bend right, "\rho_!"'] \arrow[l, bend left, "\rho_*"] \arrow[d, bend right, "\beta_X"'] \arrow[d, bend left, "\iota_X"] \\
\mathrm{ICoh}(T_c) \arrow[loop left, "\mathrm{id}"] \arrow[r, bend left, "I_T"] & I(k_X) \arrow[u, "\alpha_X"'] \arrow[l, bend left, "J_T"]
\end{tikzcd}
$$

Roughly: "identify" k_T-mod and $I(k_X)$ by means of $\wr$ and I_T, J_T. (although $J_T I_T = id$; $I_T J_T \neq id$). Then: the triple of adjoint functors $\rho_!, \rho^{-1}, \rho_*$ (constructed site-theoretically) "becomes" the triple $\beta_X, \alpha_X, \iota_X$ (constructed Ind-Sheaf-theoretically).

More precisely:

$$\begin{tikzcd} \mathrm{Mod}(k_T) \arrow[d, "\wr"'] & \mathrm{Mod}(k_X) \arrow[l, "\rho_*"'] \arrow[d, "\iota_X"] \\ {} \arrow[r, phantom, "\text{false}"] \arrow[r, "I_T"'] & I(k_X) \end{tikzcd}$$

Comm: ?!

$$\begin{tikzcd} \mathrm{Mod}(k_T) \arrow[d, phantom, "\wr"] & \mathrm{Mod}(k_X) \arrow[l, "\rho_*"'] \arrow[d, "\iota_X"] \\ {} \arrow[u] \arrow[r, phantom, "\text{true}"] & I(k_X) \arrow[l, "J_T"] \end{tikzcd}$$

i.e: J_T extends ρ_* (which is the re of sheaves from to T) to $I(k_X)$

$$\begin{tikzcd} \mathrm{Mod}(k_T) \arrow[r, "\rho^{-1}"] \arrow[d, "\wr"'] & \mathrm{Mod}(k_X) \\ {} \arrow[r, "I_T"] & I(k_X) \arrow[u, "\alpha_X"'] \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{Mod}(k_T) & \mathrm{Mod}(k_X) \arrow[l, "\rho_!"'] \arrow[d, "\beta_X"] \\ {} \arrow[u, "\wr"] & I(k_X) \arrow[l] \end{tikzcd}$$

$``\varinjlim" \mapsto \varinjlim$

Also: $\rho^{-1}\rho_* = \mathrm{id}$ if assume that T is a base of the topology.

ex. Compare: β_X, α_X, i_X to $\rho_!, \rho^{-1}, \rho_*$.

$U \Subset X$:

$$\begin{tikzcd}
\varinjlim\limits_{\substack{V \Subset U \\ T}} k_{TV}| \arrow[d, phantom, "\not\cong"] \arrow[rr, "\rho^{-1}"] & & k_{\bar U} \arrow[ll, "\rho_!"'] \arrow[lld, "\rho_*"] \arrow[dl, bend right, "\beta_X"'] \arrow[dr, bend left, "\alpha_X"] & \\
k_{TU}| \arrow[rr, "\rho^{-1}"'] & i_{U!!} k_{\bar U} \arrow[ur, "i_X"] & & i_{U!} k_U = k_U \arrow[ul, "i_X"']
\end{tikzcd}$$

since covers are finite, they differ, and both are sheaves on the site X_T.

$$i_{U!!} k_{\bar U} = \text{"}\varinjlim_{V \Subset U}\text{"} k_{\bar U} \not\cong \text{"}\varinjlim_{\text{const ind syst}}\text{"} k_{\bar U} = i_{U!} k_U$$

Subanalytic sites

$T = \{\text{Subanalytic subsets in } X\}$
(X real analytic)

Roughly: locally on $\mathbb{R}^n$: $\begin{cases} f_i \geq 0 \\ g_j > 0 \end{cases}$ close under projections along some coords

$$\begin{array}{ccc} I_{\mathbb{R}\text{-}c}(k_X) & = & \mathrm{Ind}\, \underbrace{\mathbb{R}\text{-}C^c(k_X)}_{\substack{\mathbb{R}\text{-constr. shvs w/ compact supp} \\ \wr\wr \\ \mathrm{Coh}(T_c)}} \\ \wr\| & & \\ \mathrm{Mod}(X_{sa}) & & \\ \| & & \\ X_T & & \end{array}$$

Classical Ind-sheaves:

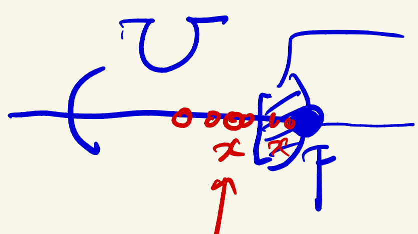
C^∞_X $\quad Db_X$ = Schwartz distributions

$$U \mapsto Db_X(U) = C^\infty_c(U)^{\text{dual}}$$

THIS ONE IS a SHEAF

$\mathcal{A}_X = C^\omega_X$ = real an fns ; $\Omega^\bullet_X$ real an forms

$f \in C^\infty_X$ has polynomial growth at $p \in X$:



$\exists K \ni p$ compact nbhd of p

$\exists N \geq 0$ $\quad \text{dist}(x, K \setminus U)^N \cdot |f(x)| < C$ on K

f tempered at p: f pol. growth w/all derivatives.

f tempered on U: tempered at all $p \in X$.

$C^{\infty,t}_X$ – presheaf of tempered C^∞ fns ; $U \mapsto C^{\infty,t}(U)$

No way they form a sheaf. BUT:

If U, V open subanalytic:

$$0 \to C^{\infty,t}(U \cap V) \to C^{\infty,t}(U) \oplus C^{\infty,t}(V) \to C^{\infty,t}(U \cup V) \to 0$$

Tempered distributions:

$$0 \to \Gamma_{X \setminus U}(X; Db_X) \to \Gamma(X, Db_X)$$
$$\downarrow$$
$$Db_X^t(U) \quad \text{def}$$
$$\downarrow$$
$$0$$

i.e.: distributions on all of X as linear functionals on $C_c^\infty(U)$.

$$0 \to Db_X^t(U \cup V) \to Db_X^t(U) \oplus Db_X^t(V) \to Db_X^t(U \cap V) \to 0$$

Prop: Db^t and $C^{\infty, t}$ are sheaves on the subanalytic site X_{sa} (only finite covers (or variation of that...) and only subanalytic open subsets)

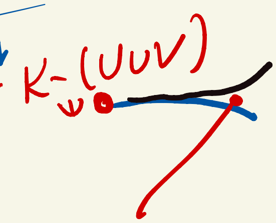
The proof uses Łojasiewicz inequality: for K compact in X, $\mathcal{U}$ and V open:

$\exists N > 0$:

$$\operatorname{dist}(x, K \setminus (U \cup V))^N \leq C \cdot (\operatorname{dist}(x, K \setminus U) + \operatorname{dist}(x, K \setminus V))$$

for any $x \in K$.

In C^∞ geom., there could be tangents of infinite order:



$\operatorname{dist}(x, K \setminus U)$
$\operatorname{dist}(x, K - V)$
decrease faster than any power of $\operatorname{dist}(x, K \setminus (U \cup V))$

In real analytic geometry: NO.

$\mathcal{D}$-modules on $\mathbb{C}$ with $\mathrm{Char}(\mathcal{M}) = \{\xi = 0\}$

1. The formal case: $\mathbb{C}[[z]][\partial]$-modules

<u>Flat connection</u> with pole at $z=0$

a) $\hat{\mathcal{O}} = \mathbb{C}[[z]]$ $\quad \hat{\mathcal{D}} = \mathbb{C}[[z]][\partial]$

$\hat{K} = \mathbb{C}[[z]][z^{-1}]$ $\quad \hat{\mathcal{D}}[z^{-1}]$

<u>Module:</u> $\hat{K}^m$ with connection $(\partial_z + A(z))\,dz$

<u>Lemma</u> There is a basis $e_1, \partial e_1, \dots, \partial^{m-1} e_1$

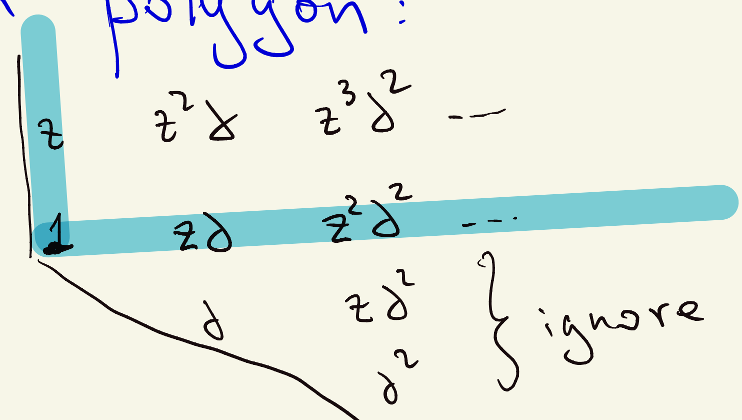
Therefore $\mathcal{M} \simeq \hat{\mathcal{D}}[z^{-1}] / \hat{\mathcal{D}}[z^{-1}] \cdot P$

$$P = \partial^m + a_{m-1}(z)\,\partial^{m-1} + \dots + a_0(z)$$

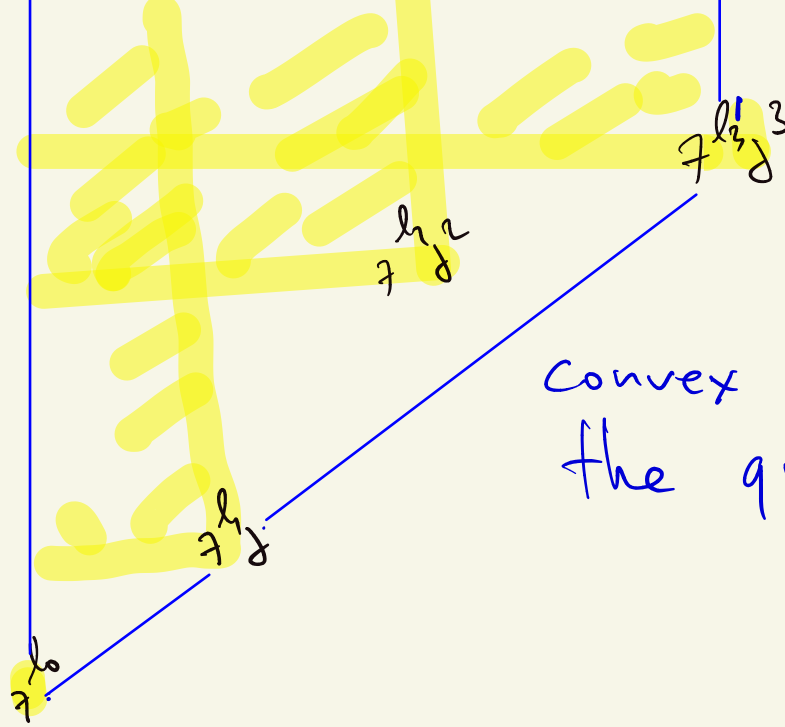
$$= \sum_{i=0}^{m} z^{l_i} \cdot u_i \cdot \partial^i \; ; \quad l_i \in \mathbb{Z}; \quad u_i \in \hat{\mathcal{O}}^\times.$$

$g \in \hat{K}$

The Newton polygon:



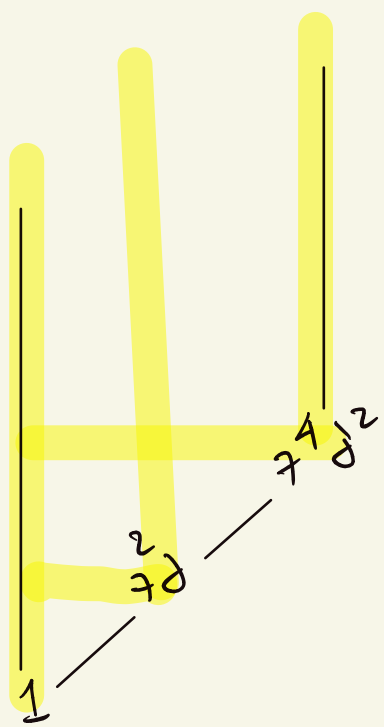
In each column, fix the vertex corr. to z^{l_i}

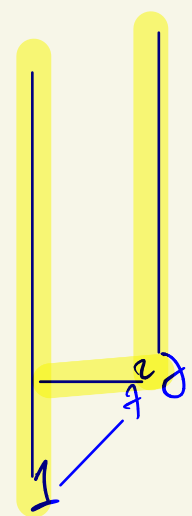


convex hull of the quadrants

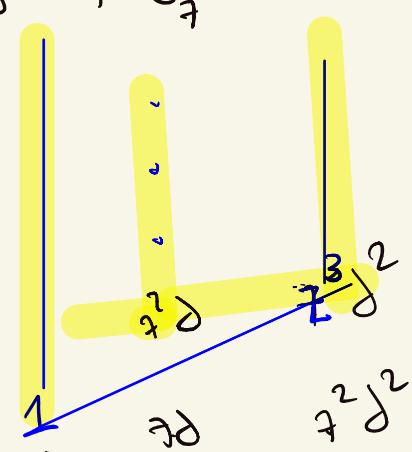
$1 \qquad z\partial \qquad z^2\partial^2 \qquad z^3\partial^3 \qquad \cdots$

Ex. $\quad z^2\partial - k$: $\quad k \neq 0$





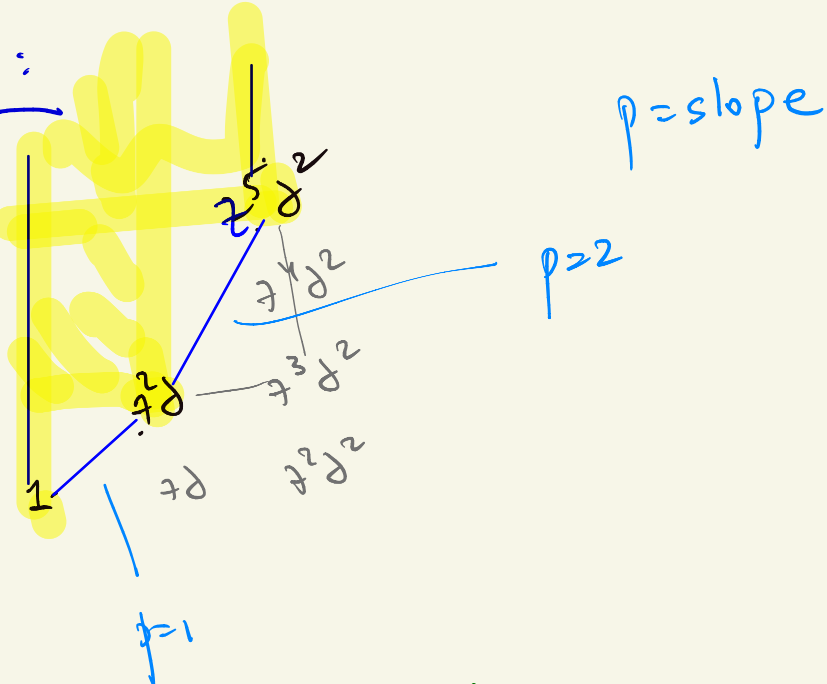
$z^3\partial^2 - z^2\partial_z + 1$:



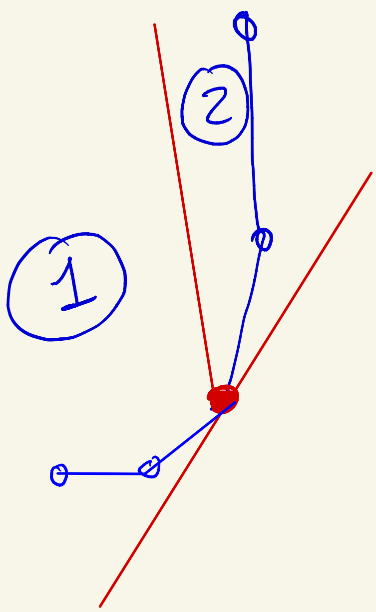
(all examples of single slope) $\quad a_i(0) \neq 0$

$1 + a_1(z)\cdot z^2\partial + a_2(z)\cdot z^4\partial^2$

Multiple slopes:



Reduction to the case of single slope:



Lemma If P has the Newton polygon, $P = QR$, where Q has Newton polygon 1 and R has Newton polygon 2.

$$\underset{M_Q}{\underbrace{\hat{D}[z^{-1}]/\hat{D}[z^{-1}]\cdot Q}} \xhookrightarrow{R} \underset{M_P}{\underbrace{\hat{D}[z^{-1}]/\hat{D}[z^{-1}]\cdot QR}} \to \underset{M_R}{\underbrace{\hat{D}[z^{-1}]/\hat{D}[z^{-1}]\cdot R}}$$

Also: R acts invertibly on M_Q, Q on M_R, therefore

$$\operatorname{Ext}^1_{\hat{D}}(M_Q, M_R) = \operatorname{Ext}^1_{\hat{D}}(M_R, M_Q) = 0$$

so the exact sequence splits.

Single slope:

Ex. $\quad z^2\partial_z + u_0 \qquad u_0 \in \mathbb{C}[[z]]^\times$

conj. by $e^{\lambda/z}$ $\quad z^2\left(\partial_z + \frac{\lambda}{z^2}\right) - u_0$

$z^2\partial_z + \lambda - u_0$; if $\lambda = -u_0(0)$, becomes regular, i.e. $z\partial - z(\dots)$

Ex.

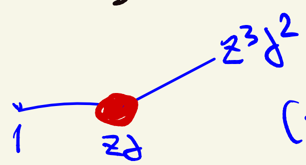
$$z^4\partial^2 + u_1 z^2\partial + u_0$$

$$z^4\left(\partial + \frac{\lambda}{z^2}\right)^2 + u_1 z^2\left(\partial + \frac{\lambda}{z^2}\right) + u_0$$

$$\begin{aligned} &\parallel \\ &z^4\partial^2 + 2\lambda z^2\partial + \lambda^2 \\ &\quad - 2\lambda z\partial + u_1\lambda \\ &\quad + u_1 z^2\partial + u_0 \end{aligned}$$

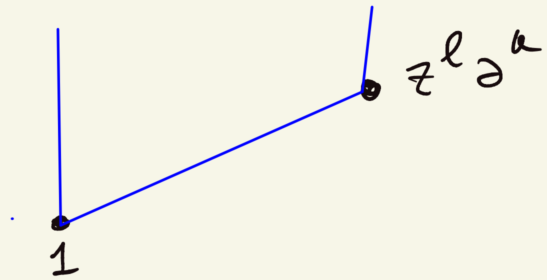
If $\lambda^2 + u_1(0)\lambda + u_0(0) = 0$: get the new operator

$$z\cdot\left(z^3\partial^2 + a(z)\cdot\partial + b(z)\right)$$



we reduced (max slope) + m

Ex. $z^\ell \partial^K + u_0, \quad \ell > K$



Slope

$$p = \frac{\ell - K}{K} \in \mathbb{Q}$$

Conjugate by λz^{-a} (look for a):

$$z^\ell (\partial - \lambda a z^{-1-a})^K + u_0 = z^\ell \partial^K + \ldots + (-\lambda)^K a^K z^{\ell + K(-1-a)} + u_0;$$

$$\ell + K(-1-a) = 0; \quad \ell - K = Ka; \quad a = \frac{\ell - K}{K} = p$$

$$(-\lambda p)^K + u_0 = 0$$

<u>In general</u>: There is classification if you allow coverings $z \mapsto z^p$ for $p \in \mathbb{Z}$.

And then:

$$\mathcal{M} \simeq \bigoplus L_{\varphi_j} \otimes \mathcal{M}_j \qquad \varphi_j \in \mathbb{C}[z^{-1/p}] z^{-1/p}$$

for some p;

$L_{\varphi_j} = \mathbb{C}[[z]][z^{-1}]$, ∂ via $\partial + \varphi_j'$;

$\mathcal{M}_j$ regular [and <u>those</u>: $\bigoplus N_k$; each N_k isomorphic to: ∂_z by $\frac{\partial}{\partial z} + A_\lambda \cdot z^{-1}$ Jordan block

Example (Airy equation)

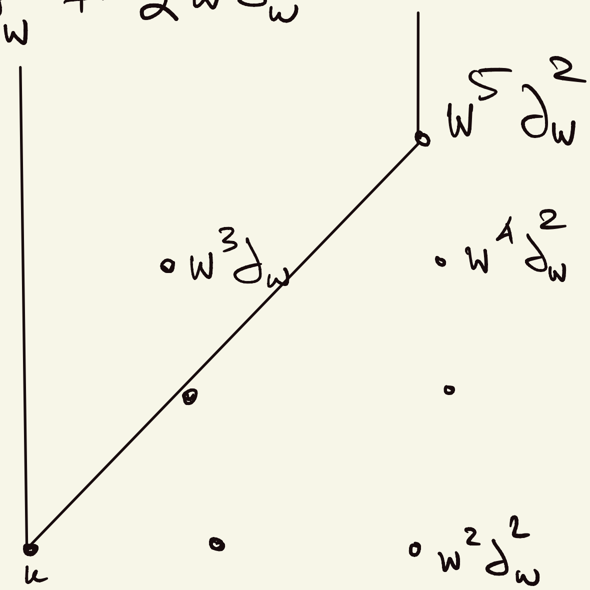
$$\partial_z^2 + u_0 z \qquad u \neq 0 \qquad \underline{\text{at } \infty}$$

$$w = 1/z \,; \qquad z\partial_z = - w\partial_w \,;$$

$$z^2\partial_z^2 + u z^3 = (z\partial_z)^2 - z\partial_z + u z^3 =$$

$$= (w\partial_w)^2 + (w\partial_w) + u w^{-3} \sim$$

$$w^5\partial_w^2 + 2w^3\partial_w + u_0$$



Single slope $\frac{3}{2}$

First gauge transformation: by $e^{\lambda w^{-3/2}}$

$$w^5\left(\partial - \lambda \cdot \tfrac{3}{2} \cdot w^{-\frac{5}{2}}\right)^2 + 2w^3\left(\partial - \lambda \cdot \tfrac{3}{2} \cdot w^{-5/2}\right) + u_0$$

$$= w^5\partial^2 - 3\lambda w^{\frac{5}{2}}\partial + \tfrac{15}{4}\lambda w^{\frac{3}{2}} + \boxed{\tfrac{9}{4}\lambda^2 + u_0}$$

$$+ 2w^3\partial - 3\lambda w^{1/2}$$

vanishes for $\lambda = \pm \frac{2}{3} u_0^{1/2}$

what to make of this?

$$x = w^{1/2}$$

$$x\partial_x = \text{const} \cdot w\partial_w$$

$$w^{5/2}\partial_w^2 \sim w^{1/2}(w\partial_w)^2 \sim x(x\partial_x)^2 \sim x^3\partial_x^2$$

$$\partial^2 - z \qquad \text{(say, } u_0 = -1\text{)}$$

$$e^{\mp\frac{2}{3}z^{3/2}} \cdot (\partial^2 - z) \cdot e^{\pm\frac{2}{3}z^{3/2}}$$

$$\parallel$$

$$(\partial \pm z^{1/2})^2 - z = \partial^2 \pm 2z^{1/2}\partial \pm \frac{1}{2}z^{-1/2}$$

$$\parallel$$

$$z^{-1/2}\left[z^{1/2}\partial^2 \pm 2z\partial_z \pm \frac{1}{2}\right]$$

$$\parallel$$

$$z^{-1/2}\left[z^{1/2-2}\cdot z^2\partial^2 \pm 2z\partial_z \pm \frac{1}{2}\right]$$

$$\parallel$$

$$z^{1/4}\cdot\left\{ z^{-1/2}\left[z^{-3/2}\cdot(z\partial)^2 - z^{-3/2}\cdot z\partial \pm 2z\partial \pm \frac{1}{2}\right]\right\}\cdot z^{-1/4}$$

$$\parallel$$

$$z^{-1/2}\left[z^{-3/2}\left(z\partial - \frac{1}{4}\right)^2 - z^{-3/2}\left(z\partial - \frac{1}{4}\right) \pm 2z\partial\right]$$

$$\parallel$$

$$z^{-1/2}\left[z^{-3/2}\left(z\partial - \frac{1}{4}\right)\left(z\partial - \frac{5}{4}\right) \pm 2z\partial\right]$$

$$\parallel$$

$$w^{1/2}\left[w^{3/2}\left(+w\partial + \frac{1}{4}\right)\left(+w\partial + \frac{5}{4}\right) \mp 2w\partial\right]$$

$$= x\left[x^3\cdot\left(\frac{1}{2}x\partial + \frac{1}{4}\right)\left(\frac{1}{2}x\partial + \frac{5}{4}\right) \mp x\partial\right] = x\left[\frac{x^3}{16}(2x\partial+1)(2x\partial+5) \mp x\partial\right]$$

$$\left[\frac{x^3}{16}(2x\partial+1)(2x\partial+5)\mp x\partial\right]u_\pm(x)=0$$

$$u_\pm(x)=1+\sum_{n=1}^{\infty}a_n x^n$$

$$\sum_{n=0}^{\infty}\frac{1}{16}(2n+1)(2n+5)a_n x^{n+3}\mp\cancel{\sum_{n=0}^{\infty} n a_n x^n}=0$$

$$\mp\; a_1 x+a_2x^2\mp\sum_{n=0}^{\infty}(n+3)a_{n+3}\,x^{n+3}=0$$

$$a_1=a_2=0$$

$$\mp(n+3)a_{n+3}+\frac{1}{16}(2n+1)(2n+5)a_n=0$$

$$3(k+1)a_{3(k+1)}=\pm\frac{1}{4^2}(6k+1)(6k+5)\,a_{3k}$$

$a_\ell=0,\ \ell\not\equiv 0\,(3)$; $$a_{3n}=(\pm1)^n\,\frac{\prod_{k=1}^{n-1}(6k-1)(6k-5)}{3^n\cdot4^{2n}\cdot n!}$$

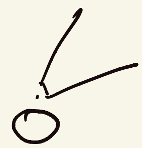
FORMAL GEN. SOL.

$$c_1\cdot e^{\frac{2}{3}x^{3/2}}x^{1/2}u_+(x)+c_2e^{-2/3}x^{1/2}u_-(x)$$

Very much divergent; convergence radius $=0$.

BUT: we are talking about series not converging to holomorphic fns but being their

asymptotic expansions.

On a sector $\alpha < \arg(x) < \beta$: 

let $f(x)$ be a holomorphic function.

$$f \sim \sum_{n=n_0}^{\infty} a_n x^n \qquad (n_0 \in \mathbb{Z}):$$

$$\forall p \quad \exists C_p > 0 : \quad \left| f(x) - \sum_{n=n_0}^{p} a_n x^n \right| \le C_p \, x^{p+1}$$

<u>L</u>: Given a sector (angle $< \pi$): any $\sum a_n x^n$ is an asymptotic expansion of <u>some</u> holo function.

Pf Say, sector is $\operatorname{Re}(x) > 0$:

$$f(x) = \sum \left(1 - e^{-\frac{1}{|a_n| x}}\right) \cdot a_n x^n$$

<u>Note</u>: $|1 - e^{-y}| \le y$ when $\operatorname{Re}(y) > 0$

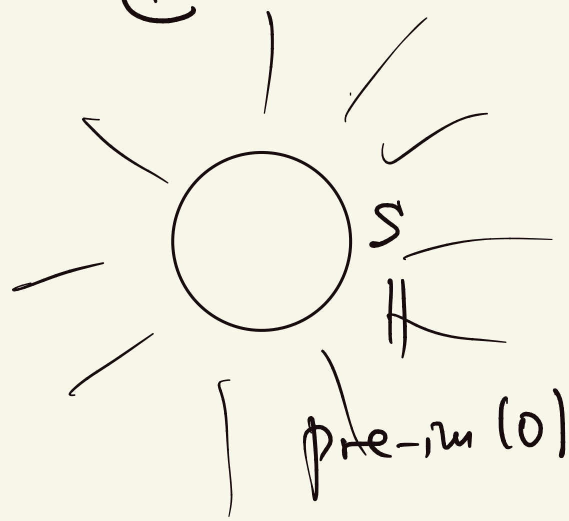
$$= \left| \int_0^y e^{-t} dt \right| \le$$

$|n^{th} \text{ term}| \le |x|^{n-1}$ ✓

The sheafs $\mathcal{A}$.

Blow up (really) $\mathbb{C}$ at 0;

$\tilde{\mathbb{C}}^*$

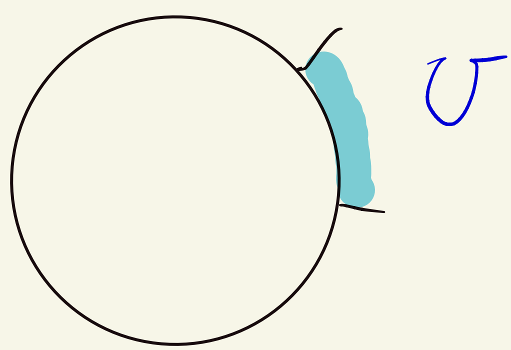


Sheaf on $\tilde{\mathbb{C}}^*$:

its restr. to S:

germs of holo fns having an asymptotic expansion

to $U \subset \mathbb{C}^*$: $\mathcal{O}_U$



germs of fns w/asympt exp 0

$$\mathcal{A}^{<0}\big|_S \hookrightarrow \mathcal{A}\big|_S \twoheadrightarrow \mathbb{C}((z))$$

constant sheaf on S

$$H^1(S, \mathcal{A}^{<0}|_S) \xrightarrow[0]{} H^1(S, \mathcal{A}|_S)$$

$$0 \to H^0(S, \mathcal{A}^{<0}|_S) \to H^0(S, \mathcal{A}|_S) \to H^0(S, \mathbb{C}((z))) \to H^1(S, \mathcal{A}^{<0}|_S) \to 0$$

$$H^0(S, \mathcal{A}^{<0}|_S) = 0, \quad H^0(S, \mathcal{A}|_S) = \mathbb{C}[z, z^{-1}], \quad H^0(S, \mathbb{C}((z))) = \mathbb{C}((z))$$

$$H^1(S, \mathcal{A}^{<0}|_S) \simeq \mathbb{C}((z)) / \mathbb{C}[z, z^{-1}]$$

<u>Now</u>: M meromorphic (at zero) conn.

$$\mathcal{A}^{<0}(M) = \mathcal{A}^{<0} \otimes_{\pi^{-1}(\mathcal{O})} \pi^{-1}(M)$$

<u>Thm</u>

$$\mathcal{A}^{<0}(M) \overset{\partial}{\twoheadrightarrow} \mathcal{A}^{<0}(M)$$

(i.e. $H^1_{dR} = 0$: a) away from 0
b) on fns w/ zero asymps

We get: 福原

Thm ($\overset{F}{\text{H}}$ukuhara-Turritin)

Any formal horizontal section of M (or $\hat{M}$) extends to an actual horizontal section of M (whose asymptotic expansion it is).

Pf

$$\begin{tikzcd}
0 \arrow[r] & A^{<0}(M) \arrow[r] \arrow[d, two heads, "\partial"] & A(M) \arrow[r] \arrow[d, "\partial"] & \hat{M} \arrow[r] \arrow[d, "\partial"] & 0 \\
0 \arrow[r] & A^{<0}(M) \arrow[r] & A(M) \arrow[r] & \hat{M} \arrow[r] & 0
\end{tikzcd}$$

exact

$$\ker\left(A(M) \xrightarrow{\partial} A(M)\right) \longrightarrow \ker\left(\hat{M} \xrightarrow{\partial} \hat{M}\right) \longrightarrow \operatorname{coker}(\dots) = 0$$

So:

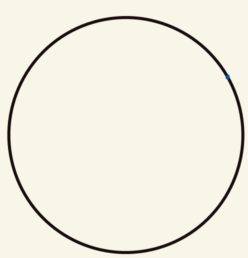
1) For any M, $\hat{M} \simeq \hat{M}_{standard}$

2) $\hat{M}_{stand}$ is <u>completion</u> of an <u>actual</u> $\mathcal{D}$-module

3) Apply Hukuhara–Turritin to $N = \underline{Hom}(M, M_{stand.})$:

$M \simeq M_{st.}$ <u>near</u> $z=0$ in any given sector

<u>Stokes phenomenon</u>



On each sector U:

$$M \simeq M_{standard} \underset{\phi_U}{\simeq} \bigoplus_{\omega} L_\omega \otimes M_\omega$$

near zero.

$\Phi_U \Phi_V^{-1} \in GL_m(A|_{U\cap V})^{hor}$ $\quad g_{UV} \equiv 1 \pmod{\mathcal{M}_{st}(A^{<0}|_{U\cap V})^{hor}}$

$\quad \| $

$\quad g_{UV}$

? By extension of $H^1(S, A^{<0}|_S) \xrightarrow{0} H^1(S, A|_S)$: this cocycle is trivial. From there:

? Near $z=0$, $\mathcal{M} \simeq \mathcal{M}_{standard}$

Extra data: we have a basis of solutions $u_1, \dots, u_m$ on U for every sector u; plus, partial order on every U:

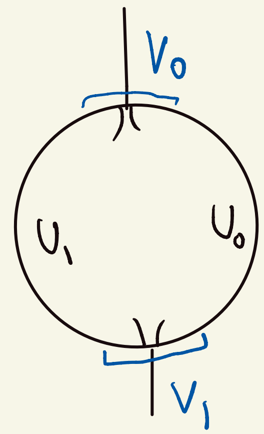
$$u_i \underset{U}{\prec} u_j \quad \text{if} \quad \begin{array}{l} u_i \cdot e^{-\varphi_i} \sim (\text{a Laurent series}); \\ u_j \cdot e^{-\varphi_j} \sim (\text{a Laurent series}); \end{array}$$

$e^{\varphi_i - \varphi_j}$ has moderate growth (i.e. $\sim z^{-N}$, $\exists N$)

$u_1 = e^{\frac{1}{z}}$ $\qquad u_2 = e^{-\frac{1}{z}}$

on V_0, V_1: incompatible

$\frac{1}{z} \underset{U_1}{<} -\frac{1}{z}$, $\qquad \frac{1}{z} \underset{U_0}{>} -\frac{1}{z}$



The above defines partial order $\underset{U}{\lesssim}$ on

$$\Omega = \bigcup_{k \geq 1} \mathbb{C}((z^{1/k}))/\mathbb{C}[[z^{1/k}]]$$

(which is a <u>local system</u> on S b/c $z^{1/k}$ are defined only locally)

An Ω-filtered local system V:

- A local system V on S
- <u>Subsheaves</u> $V^\alpha \subset V$, $\alpha \in \Omega$
- <u>there exist</u> locally defined decompositions

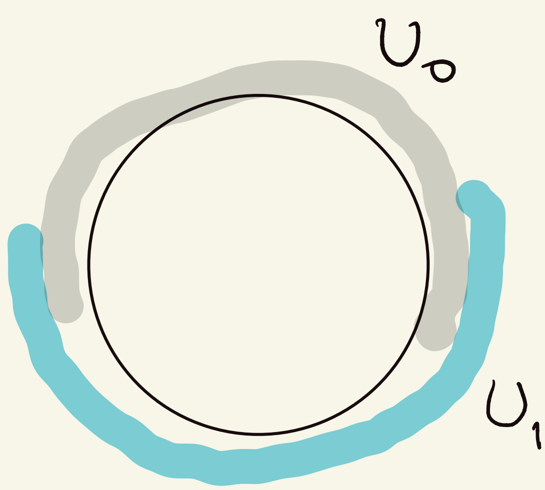
$$V = \bigoplus_{\alpha \in \Omega} V_\alpha \quad \underline{\text{on } U}$$

such that for every α,

$$V^\beta\big|_U = \bigoplus_{\alpha \underset{U}{<} \beta} V^\alpha \quad \text{on } U$$

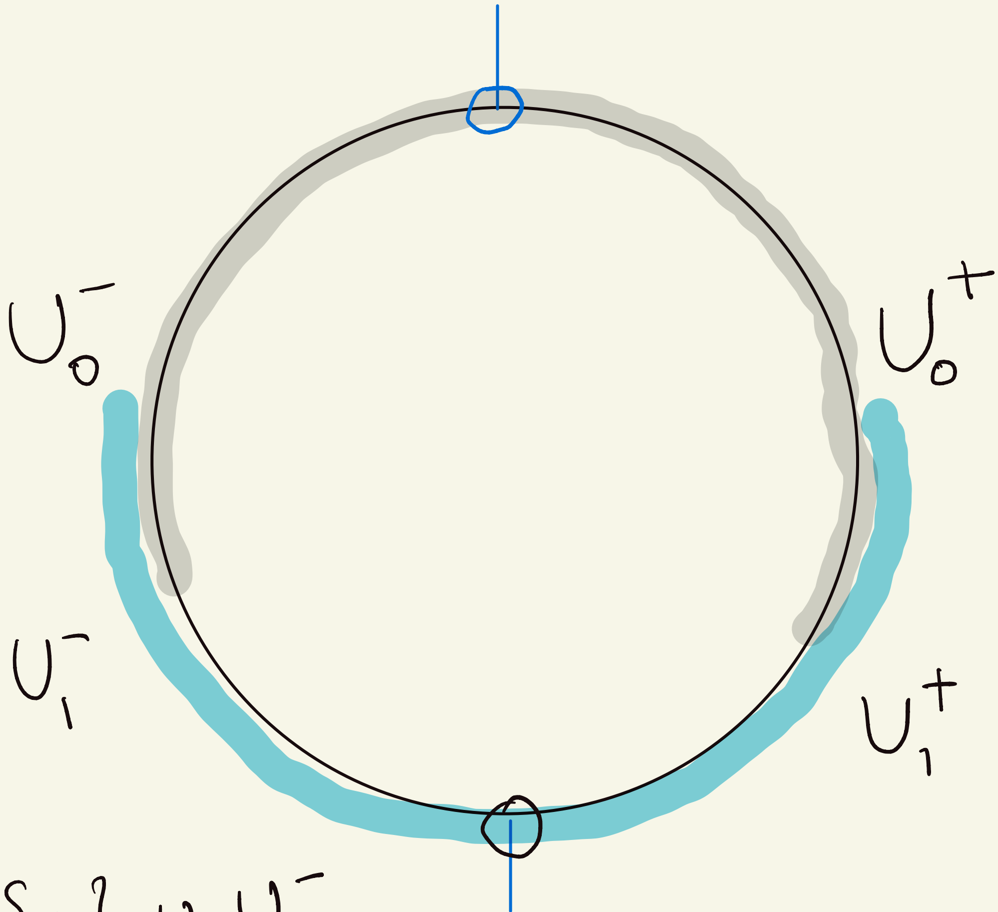
<u>Ex.</u> Say, there are two solutions:

$e^{\pm 1/x} u_\pm(x)$



Splitting:

$V_{1/x} \oplus V_{-1/x}$



$U_0 = U_0^+ \cup \{0\} \cup U_0^-$

$V^{1/x} = (V_{-1/x})_{U_0^+} \oplus V_{+1/x}$ on U_0

$(V_{-1/x})_{U_1^+} \oplus V_{+1/x}$ on U_1

$V^{-1/x} = (V_{1/x})_{U_0^-} \oplus V_{-1/x}$ on U_0^-;

$(V_{1/x})_{U_1^-} \oplus V_{-1/x}$ on U_1^-

Glueing isomorphisms on $U_0 \cap U_1$;

they only have to preserve the

sheaves $V^{\pm 1/x}$, not the decompositions

(the sheaves are part of the data; the decompositions are not, they only are supposed to exist locally). Therefore:

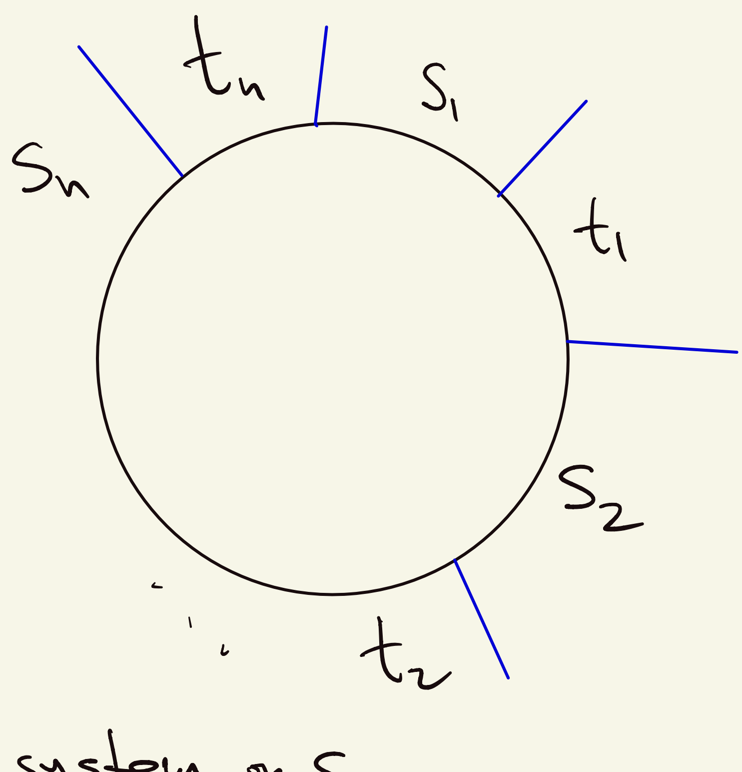
on $U_0^+ \cap U_1^+$: $\quad V_{1/x} \oplus V_{-1/x} \xrightarrow{\sim} V_{1/x} \oplus V_{-1/x}$

preserves $V_{-1/x}$

on $U_0^- \cap U_1^-$: preserves $V_{+1/x}$

So: lower triangular on $U_0^+ \cap U_0^-$, upper triangular on $U_0^- \cap U_1^-$.

More generally, Stokes data for $rk = 2$:



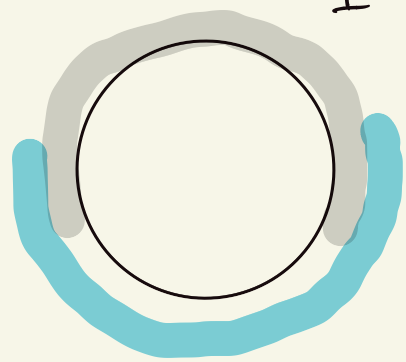
$$\left.\begin{array}{l} s_j \text{ upper} \\ t_j \text{ lower} \end{array}\right\} \text{triangular}$$

$$s_1 t_1 \dots s_n t_n = \underset{\uparrow}{g}$$

the automorphism defining the local system on S

This describes the phenomenon:

$$u_\pm^{(0)}(z) \sim e^{\pm 1/z} a_\pm^{(0)}(z)$$



$$u_\pm^{(1)}(z) \sim e^{\pm 1/z} a_\pm^{(1)}(z)$$

(or rather: $a_\pm^{(i)}(z)$ are asymptotic expansions of $e^{\mp 1/z} u_\pm^{(i)}(z)$)

<u>Now</u>, on $U_0^+ \cap U_1^+$, we have:

$u_-^{(0)}$ and $u_-^{(1)}$ are proportional

$u_+^{(0)}$ and $u_+^{(1)}$ are proportional

after adding $c \cdot u_-^{(1)}$

$$u_-^{(0)} = s_{--} u_-^{(1)}$$

$$u_-^{(0)} = s_{-+} u_+^{(1)} + s_{++} u_-^{(1)}$$

D'Agnolo and Kashiwara relate the Stokes description to the language of subanalytic/ ind- sheaves. (That looks plausible).

<u>Remarks</u> 1. There is an entire adjacent field when $\hbar$ plays the role of z, there are extra variables x, ξ, expressions $e^{\frac{1}{i\hbar}S(x)} a(x,\hbar)$ appear as solutions of the Schrödinger equation, and the Stokes phenomenon for them is studied.

2. As we have seen, asymptotic expansions of solutions tend to be $\sum a_n z^n$ where a_n grow like $n!$ For those, there is Borel resummation, analytically continued wherever possible. This idea, applied to 1 with $\hbar = z$, is the basis of resurgent analysis of Schrödinger equations. M.b. first source to look at: Kontsevich lectures on YouTube

③ As mentioned before: enhanced modules over DQ $\mathcal{O}^{\hbar}_{M}$ (for symplectic) M are formed by expressions

$$\sum e^{\frac{\varphi(x,\xi)}{i\hbar}} a(x,\xi,\hbar) \text{ mod}\_.$$

(appropriately understood)

with a filtration that could is essentially interpreted as the filtration by generalized eigenvalues of $\hbar^2 \partial_\hbar$. It would be interesting to see how this is related to the V-filtration (by generalized eigenvalues of $\hbar \partial_\hbar$ in a different version of enhanced $\mathcal{D}$-modules) and the Ω-filtration in the Stokes phenomenon, perhaps in a resurgent version of enhanced DQ modules.