

Gauss-Manin connection in noncommutative geometry, II

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ABSTRACT. This paper is a continuation of [?] where we reviewed results on the Gauss-Manin connection in noncommutative geometry, mostly from [13], [9], [27], [33]. The Gauss-Manin connection in periodic cyclic homology was introduced by Ezra Getzler in 1991, then generalized to a superconnection by the author in a joint work with Dolgushev and Tamarkin. The key to these constructions is the Cartan calculus in noncommutative geometry. In this paper, we prove explicit formulas for the Gauss-Manin connection which seem to be of interest because they strongly evoke physical formulas such as WKB expansion (where the role of the Planck constant $\hbar$ is played by the formal parameter u of degree two). One corollary is that the connection is p -adically integral, which allows to define noncommutative crystalline cohomology of algebras over a finite fields.

In loving memory of Yuri Ivanovich Manin

1. Introduction

2. Summary of results of [?]

2.1. The algebra $a(\mathfrak{g})$. Let k be a commutative unital ring where 2 is invertible. Let u be a formal parameter of degree two. For a graded Lie algebra $\mathfrak{g}$, let $a(\mathfrak{g})$ be the associative DG algebra over $k[u]$ generated by a subalgebra $U(\mathfrak{g})$ and by elements (X) , $X \in \mathfrak{g}$, $|(X)| = |X| + 1$, subject to relations

$$(2.1) \quad [X, (Y)] = (-1)^{|X|}([X, Y]); [(X), (Y)] = 0$$

with the differential ud where

$$(2.2) \quad ud(X) = uX; udX = 0$$

Alternatively, $a(\mathfrak{g}) \xrightarrow{\sim} (U_{k[u]}(\mathfrak{g}[u, \epsilon]), u\partial_\epsilon)$ via $(X) \mapsto \epsilon X$ where ϵ is of degree one and square zero. The isomorphism sends (X) to ϵX .

Let $\mathfrak{a}_n$ be the free graded Lie algebra generated by elements $\lambda_0, \dots, \lambda_n$ of degree one. By $\widehat{a}(\mathfrak{a}_n)$ we will denote the completion of $a(\mathfrak{a}_n)[u^{-1}]$ with respect to the filtration induced by the grading of $\mathfrak{a}_n$ (for these purposes, $|u| = 0$).

For an element of degree one of $\mathfrak{a}_n$, define an element of degree one in $\widehat{a}(\mathfrak{g})[u^{-1}]$ by

$$(2.3) \quad d_\lambda = \lambda - \frac{(\lambda^2)}{u}$$

Let k be of characteristic zero. Define an element of degree zero in $\widehat{a}(\mathfrak{a}_1)$ by

$$(2.4) \quad t(\lambda_0, \lambda_1) = \exp\left(\frac{(\lambda_1 - \lambda_0)}{u}\right)$$

LEMMA 2.1. (1)

$$ud(d_{\lambda_0}) + d_{\lambda_0}^2 = 0$$

in $\widehat{a}(\mathfrak{a}_0)$;

(2)

$$udt(\lambda_0, \lambda_1) + d_{\lambda_0}t(\lambda_0, \lambda_1) - t(\lambda_0, \lambda_1)d_{\lambda_1} = 0$$

in $\widehat{a}(\mathfrak{a}_1)$;

(3)

$$t(\lambda_0, \lambda_1)t(\lambda_1, \lambda_2) = t(\lambda_0, \lambda_2)$$

in $\widehat{a}(\mathfrak{a}_2)$.

PROOF.

$$ud(d_{\lambda_0}) + d_{\lambda_0}^2 = -\lambda_0^2 + \lambda_0^2 = 0;$$

if we denote $\lambda_1 - \lambda_0$ by α , then

$$\begin{aligned} ud\left(\frac{\alpha}{u}\right)^n &= \sum_{k=0}^{n-1} \left(\frac{\alpha}{u}\right)^{n-1-k} \alpha \left(\frac{\alpha}{u}\right)^k = \\ n\left(\frac{\alpha}{u}\right)^{n-1} \alpha - \sum_{k=0}^{n-1} \left(\frac{\alpha}{u}\right)^{n-1-k} \sum_{l=0}^{k-1} \left(\frac{\alpha}{u}\right)^{k-1-l} \left(\frac{[\alpha, \alpha]}{u}\right) \left(\frac{\alpha}{u}\right)^l &= \\ n\left(\frac{\alpha}{u}\right)^{n-1} \alpha - \sum_{l=0}^{n-2} \sum_{k=l+1}^{n-1} 1 \cdot \left(\frac{\alpha}{u}\right)^{n-2-l} \left(\frac{[\alpha, \alpha]}{u}\right) \left(\frac{\alpha}{u}\right)^l &= \\ n\left(\frac{\alpha}{u}\right)^{n-1} \alpha - \sum_{l=0}^{n-2} (n-l-1) \left(\frac{\alpha}{u}\right)^{n-2-l} \left(\frac{[\alpha, \alpha]}{u}\right) \left(\frac{\alpha}{u}\right)^l \end{aligned}$$

We will need this formula later; for now, we can use the fact that all (X) commute and obtain

$$(2.5) \quad ud\left(\frac{\alpha}{u}\right)^n = n\left(\frac{\alpha}{u}\right)^{n-1} \alpha - \frac{1}{2}n(n-1)\left(\frac{\alpha}{u}\right)^{n-2} \left(\frac{[\alpha, \alpha]}{u}\right)$$

Therefore

$$(2.6) \quad ud \exp\left(\left(\frac{\alpha}{u}\right)\right) = \exp\left(\left(\frac{\alpha}{u}\right)\right) \left(\alpha - \frac{(\alpha^2)}{u}\right)$$

But

$$\begin{aligned} udt(\lambda_0, \lambda_1) + d_{\lambda_0}t(\lambda_0, \lambda_1) - t(\lambda_0, \lambda_1)d_{\lambda_1} &= \\ ud \exp\left(\left(\frac{\alpha}{u}\right)\right) + \left(\lambda_0 - \frac{(\lambda_0^2)}{u}\right) \exp\left(\left(\frac{\alpha}{u}\right)\right) - \exp\left(\left(\frac{\alpha}{u}\right)\right) \left(\lambda_1 - \frac{(\lambda_1^2)}{u}\right) &= \\ \exp\left(\left(\frac{\alpha}{u}\right)\right) \left(\alpha - \frac{(\alpha^2)}{u}\right) + [\lambda_0, \exp\left(\left(\frac{\alpha}{u}\right)\right)] + \exp\left(\left(\frac{\alpha}{u}\right)\right) \left(-\alpha + \frac{(\lambda_1^2)}{u} - \frac{(\lambda_0^2)}{u}\right) &= \\ \exp\left(\left(\frac{\alpha}{u}\right)\right) \left(\frac{(\lambda_1^2)}{u} - \frac{(\lambda_0^2)}{u} - \frac{([\lambda_0, \alpha])}{u} - \frac{(\alpha^2)}{u}\right) = \exp\left(\left(\frac{\alpha}{u}\right)\right) \frac{(\lambda_1^2 - (\lambda_0 + \alpha)^2)}{u} &= 0 \end{aligned}$$

The last statement of the lemma is obvious. $\square$

2.2. Algebras $\mathcal{A}_0$ and $\mathcal{A}_1$.

DEFINITION 2.2. For any differential graded Lie algebra $\mathfrak{g}$, let $U^+(\mathfrak{g})$ be the kernel of the augmentation $U(\mathfrak{g}) \rightarrow k$. Let $\text{Cobar}(U^+(\mathfrak{g}))$ be the free associative algebra generated by $U^+(\mathfrak{g})[-1]$ (the degree shift by one). We denote the free generator corresponding to $x \in U^+(\mathfrak{g})$ by (x) . Define

$$(2.7) \quad \partial_{\text{Cobar}}(x) = \sum (-1)^{|x^{(1)}|} (x^{(1)})(x^{(2)})$$

where the comultiplication is defined by

$$\Delta x = \sum x^{(1)} \otimes x^{(2)}$$

In addition, the differential $d_{\mathfrak{g}}$ induces a differential on $\text{Cobar}(U^+(\mathfrak{g}))$. Now define the dg algebra as follows. It is an algebra over $k[u]$ generated by the DG subalgebra $(U(\mathfrak{g}), d_{\mathfrak{g}})$ and the subalgebra $\text{Cobar}(U^+(\mathfrak{g}))$. The only additional relations are

$$[X, (x)] = (\text{ad}_X(x)), \quad X \in \mathfrak{g}, x \in U^+(\mathfrak{g}).$$

The differential acts as follows:

$$(2.8) \quad x \mapsto d_{\mathfrak{g}}x, x \in U(\mathfrak{g}); (x) \mapsto (-d_{\mathfrak{g}}x) + \partial_{\text{Cobar}}(x) + ux, \quad x \in U^+(\mathfrak{g}).$$

Define also

$$(2.9) \quad \mathcal{A}_0(\mathfrak{g}) = U(\mathfrak{g}) \rtimes_0 \text{Cobar}(\text{Sym}^+(\mathfrak{g}))$$

in the same way as above, the differential being

$$(2.10) \quad x \mapsto d_{\mathfrak{g}}x, x \in U(\mathfrak{g}); (x) \mapsto (-d_{\mathfrak{g}}x) + \partial_{\text{Cobar}}(x) + uB_0(x)$$

where

$$B_0(x) = x, \quad x \in \text{Sym}^1(\mathfrak{g}); B_0(x) = 0, \quad x \in \text{Sym}^{>1}(\mathfrak{g})$$

LEMMA 2.3. *Let k contain the rationals. There is a natural isomorphism of DG algebras*

$$\mathcal{A}_0(\mathfrak{g}) \xrightarrow{\sim} \mathcal{A}_1(\mathfrak{g})$$

PROOF. Start with the case when $\mathfrak{g} = kD$ is a free k -module generated by an element D of degree zero. Let

$$(2.11) \quad \sum_{k,l \geq 0} a_{k,l} x^k y^l = y(y-x) \dots (y-(n-1)x).$$

The formula

$$(2.12) \quad (D^n) \mapsto [D^n] \stackrel{\text{def}}{=} \sum_{k+l=n} a_{k,l} D^k (D^l)$$

defines an isomorphism of DG algebras

$$\mathcal{A}_0(\mathfrak{g}) \rightarrow \mathcal{A}_1(\mathfrak{g})$$

In general, for $D_1, \dots, D_k$ in $\mathfrak{g}$ and $(D_1 \dots D_k \in \text{Sym}^k(\mathfrak{g}))$, define the image of $(D_1 \dots D_k)$ to be the coefficient at $t_1 \dots t_k$ in $\frac{1}{k!} [(t_1 D_1 + \dots + t_k D_k)^k]$. Here t_i are formal variables that are central and such that $|t_i| = -|D_i|$. $\square$

REMARK 2.4. Define the Stirling numbers as the coefficients of the power series

$$(2.13) \quad \sum_{k,l \geq 0} c_{k,l} x^k y^l = \sum_{n=1}^{\infty} \frac{1}{n!} y(y-x) \dots (y-(n-1)x).$$

Note that

$$(2.14) \quad \sum_{k,l \geq 0} u^{-(k+l)} c_{k,l} x^k y^l = \sum_{n=1}^{\infty} \frac{1}{u^n n!} y(y-x) \dots (y-(n-1)x) = \left(1 + \frac{x}{u}\right)^{\frac{y}{x}}$$

Then, formally (or in an appropriate completion),

$$(e^{\frac{1}{u}D} - 1) \mapsto 1 + \sum_{k,l \geq 1} \frac{1}{u^{k+l}} c_{k,l} D^k (D)^l$$

under the isomorphism from Lemma 2.3.

DEFINITION 2.5. Let $\widehat{\mathcal{A}}_0(\mathfrak{a}_n)$ be the completion of $\mathcal{A}_0(\mathfrak{a}_n)[u^{-1}]$ with respect to the increasing filtration induced by the grading of $\mathfrak{a}_n$. Define

$$D_{\lambda_j} = \lambda_j - (e^{\frac{1}{u}R_j} - 1) \in \widehat{\mathcal{A}}_0(\mathfrak{a}_n)^1$$

where $R_j = \lambda_j^2$.

LEMMA 2.6. *Let k be of characteristic zero.*

(1)

$$(\partial_{\text{Cobar}} + uB_0)D_{\lambda_0} + D_{\lambda_0}^2 = 0$$

in $\widehat{\mathcal{A}}_0(\mathfrak{a}_0)$

(2) *There exist elements $T(\lambda_0, \dots, \lambda_m)$ of degree $1 - m$ in $\widehat{\mathcal{A}}_0(\mathfrak{a}_m)$ for all $m > 1$ such that*

$$\begin{aligned} & (\partial_{\text{Cobar}} + uB_0)T(\lambda_0, \dots, \lambda_n) + D_{\lambda_0}T(\lambda_0, \dots, \lambda_n) + \\ & \sum_{j=1}^{n-1} (-1)^{j-1} T(\lambda_0, \dots, \lambda_j)T(\lambda_j, \dots, \lambda_n) + (-1)^n T(\lambda_0, \dots, \lambda_n)D_{\lambda_n} \\ & - \sum_{j=1}^{n-1} (-1)^{j-1} T(\lambda_0, \dots, \widehat{\lambda}_j, \dots, \lambda_n) = 0 \end{aligned}$$

3. Explicit formulas

3.1. Ordered exponential differences. Let I be a finite set. Let $\{y_i | i \in I\}$ be variables, $\{A_i | i \in I\}$ commuting operators on a k -module H , $\{\mathbf{X}_i | i \in I\}$ and $\mathbb{D}$ operators on H . Introduce a total ordering on I and use it to identify I with $\{1, \dots, n\}$. Put

(3.1)

$$\mathcal{E}_{\{\mathbf{X}_i\}, <}(\{A_i y_i\}) = \frac{e^{A_n y_n} - 1}{A_n} (\mathbf{X}_n + A_n \sum_{j=1}^{n-1} e^{A_{n-1} y_{n-1} + \dots + A_{j+1} y_{j+1}} \frac{e^{A_j y_j} - 1}{A_j} \mathbf{X}_j)$$

(We assume that the above is well-defined. For example, A_i could be pronilpotent operators on a completion associated to a filtration).

For two elements $i_0 \leq i_1$ of I , put

$$(3.2) \quad \mathcal{E}_{i_1 i_0} = \mathcal{E}_{\{\mathbf{X}_i | i_0 \leq i \leq i_1\}, <}(\{A_i y_i | i_0 \leq i \leq i_1\})$$

For example, if $\mathbf{X}_i = A_i \mathbf{X}$ for all i then

$$(3.3) \quad \mathcal{E}_{i_1 i_0} = (e^{A_{i_1} y_{i_1}} - 1) e^{\sum_{i_0 \leq j < i_1} A_j y_j} \mathbf{X}$$

3.2. Admissible words. We are going to encode (some) elements of $\mathcal{A}_0(\mathfrak{a}_n)$ by noncommutative polynomials. Set $\alpha_{ij} = \lambda_j - \lambda_i$. Let $\mathcal{B}_n$ be the algebra freely generated by $x_1, \dots, x_n$ and by k -modules $\text{Sym}(V_n)$ and $\text{Sym}(W_n)$, where

$$V_n = W_n = \mathfrak{a}_n^2$$

Here $\mathfrak{a}_n^2$ stands for the graded component of degree two in $\mathfrak{a}_n$. The k -module $\mathcal{B}_n$ is spanned by words on letters $x_1, \dots, x_n, f, g$ where $f \in \text{Sym}(V_n)$ and $g \in \text{Sym}(W_n)$. For such a word, let f_j , resp. g_j , be the j th letter of the type f , resp. g , counted from the left. We call a word admissible if:

- (1) There are exactly $n - 1$ letters of type g .
- (2) For every $j > 1$, if there are letters of type f between g_{j-1} and g_j , we denote by f^j the closest such letter to g_j . If there are letters of type f to the left of g_1 then we denote by f^1 the closest such letter to g_1 . Then
 - There are no letters x_k between f^j and g_j (or between g_{j-1} and g_j if there is no f^j) other than x_{n-j} and x_{n-j+1} .
 - There are no letters x_k between g_{j-1} and f^j other than x_{n-j+1} .
 - If there is no f^1 , there are no letters x_k to the left of g_1 other than x_n and x_{n-1} .
 - There are no letters x_k to the right of g_{n-1} other than x_1 .

We denote the span of admissible words by $\mathcal{H}_n$. For example:

$$x_2^4 \cdot f_1 \cdot x_2^2 \cdot f_2 \cdot x_1 x_2^2 x_1 \cdot g_1 \cdot x_1^3 \cdot f_3 \cdot x_1 \cdot f_4 \cdot f_5$$

is a word in $\mathcal{H}_2$; we have $f^1 = f_2$.

Note that $\mathcal{H}_1$ is just the free algebra generated by x_1 and by the k -module $k[\alpha_{01}^2, [\lambda_0, \alpha_{01}], \lambda_1^2]$.

REMARK 3.1. There are extra conditions of admissibility that all our expressions will automatically satisfy. If we put

$$V_{n,m} = W_{n,m+1} = \text{span}\{[\alpha_{0j}, \alpha_{0,n-m+1}] | j > n - m; [\lambda_0, \alpha_{0,1}], \lambda_n^2\}$$

then those conditions are:

- (1) All letters of type f between g_{j-1} and g_j (or left of g_j when $j = 1$, or right of g_{n-1} when $j = n$) denote elements of $\text{Sym}(V_{n,m})$.
- (2) For all m , $g_m \in \text{Sym}(W_{n,m})$.

Given an admissible word in $\mathcal{H}_n$, we associate to it an element of $\mathcal{A}_0(\mathfrak{a}_n)[u^{-1}]$ as follows. We write the same word, where

- x_j is replaced by $\alpha_{0,j}$;
- g_j is replaced by $(\frac{\alpha_{0,n-j}}{u} \frac{\alpha_{n-j,n-j+1}}{u} g_j)$;
- any f between g_{j-1} and g_j (or to the left of g_1 when $j = 1$, or to the right of g_{n-1} when $j = n$) is replaced by $(\frac{\alpha_{0,n-j+1}}{u} f)$.

The resulting map $\mathcal{H}_n \rightarrow \mathcal{A}_0(\mathfrak{a}_n)$ extends to

$$(3.4) \quad \widehat{\mathcal{H}}_n \rightarrow \widehat{\mathcal{A}}_0(\mathfrak{a}_n)$$

Here the left hand side is the completion of $\mathcal{H}_n[u, u^{-1}]$ with respect to the grading where $u=0$, $|x_j| = 1$, $|f|$ is the degree of f in $\text{Sym}(V_n)$, and $|g|$ is the degree of g in $\text{Sym}(W_n)$.

3.2.1. *The annihilation operators.* As in the commutative case, there are partial derivatives

$$\partial_{x_j} : \mathcal{H}_n \rightarrow \mathcal{H}_n$$

Such operator takes a word into the sum of all words obtained from it by removing any letter x_j from this word. (Or, alternatively, by replacing any fragment x_i^m by mx_i^{m-1}). But there are also *partial* partial derivatives. For example, we may demand that only letters x_j to the left or to the right of a certain position are deleted. For example:

DEFINITION 3.2. Let

$$\partial_{x_1}^- : \mathcal{H}_n \rightarrow \mathcal{H}_n$$

transform a word

$$\dots g_{n-1} \dots f_r \cdot x_1^{m_1} \cdot f_{r+1} \cdot x_1^{m_2}$$

into the sum of words obtained by replacing each fragment $\cdot x_1^m \cdot$ by $\cdot mx_1^{m-1} \cdot$, except for the two rightmost fragments $\cdot x_1^{m_1} \cdot$ and $\cdot x_1^{m_2} \cdot$;

let

$$\partial_{x_1}^- : \mathcal{H}_n \rightarrow \mathcal{H}_n$$

transform a word

$$\dots g_{n-1} \dots f_r \cdot x_1^{m_1}$$

into the sum of words obtained by replacing each fragment $\cdot x_1^m \cdot$ by $\cdot mx_1^{m-1} \cdot$, except for the one rightmost fragment $\cdot x_1^{m_1} \cdot$;

let

$$\partial_{x_1}^\uparrow : \mathcal{H}_n \rightarrow \mathcal{H}_n$$

transform a word

$$\dots g_{n-1} \dots f_r \cdot x_1^{m_1} \cdot f_{r+1} \cdot x_1^{m_2}$$

into

$$\dots g_{n-1} \dots f_r \cdot m_1 x_1^{m_1-1} \cdot f_{r+1} \cdot x_1^{m_2};$$

let

$$\partial_{x_1}^\uparrow : \mathcal{H}_n \rightarrow \mathcal{H}_n$$

transform a word

$$\dots g_{n-1} \dots f_r \cdot x_1^{m_1}$$

into

$$\dots g_{n-1} \dots f_r \cdot m_1 x_1^{m_1-1}$$

One has

$$\partial_{x_1} = \partial_{x_1}^- + \partial_{x_1}^\uparrow = \partial_{x_1}^- + \partial_{x_1}^\uparrow + \partial_{x_1}^\uparrow$$

3.2.2. *The creation operators.* The creation operators will always act on the rightmost letter of the type f or g , whichever it is. More precisely, we take the letter h such that it is not an x_k but all letters to its right are x_1 , and multiply it by $F \in V_n = W_n = \text{Sym}^+(\mathfrak{a}_n^2)$ (so that we obtain one new letter hF). We will denote this operator by

$$F : \mathcal{H}_n \rightarrow \mathcal{H}_n.$$

3.2.3. *The creation-annihilation operators.* There are two types of these operators. They decrease the degree in x_1 but create a new letter of type f or g .

1) Define

$$\mathbb{D} : \mathcal{H}_n \rightarrow \mathcal{H}_n$$

to be the operator transforming a word $\dots \cdot x_1^{m_1}$ into

$$\sum_{i+j=m-1} \dots \cdot x_1^i \cdot \mathbf{1} \cdot x_1^j$$

where $f = \mathbf{1}$ is the unit in $\text{Sym}(V)$.

2) Define $\Delta(x_2^m)$ as follows: in each monomial in the expansion of $(x_1 + x'_1)^m$, replace the rightmost letter x'_1 by $g = \mathbf{1} \in \text{Sym}(W_n)$; then replace all other x'_1 by $x_2 - x_1$. Now define

$$\mathbb{A} : \mathcal{H}_n \rightarrow \mathcal{H}_{n+1}$$

to be the operator transforming an admissible word into the following: replace all x_j by x_{j+1} ; replace all $f(\lambda_0, \lambda_1, \dots, \lambda_n)$ by $f(\lambda_0, \lambda_2, \dots, \lambda_{n+1})$; do the same for all g ; and then replace the rightmost fragment x_2^m by $\Delta(x_2^m)$.

3.3. Formulas for T . Using notation of 3.1, consider

$$(3.5) \quad \mathcal{E}_{\{\mathbf{X}_j | 1 \leq j \leq n; \mathbf{Y}, \mathbf{Z}\}, <} (\{A_j \frac{[\alpha_{0j}, \alpha_{01}]}{u} | 1 \leq j \leq n; B \frac{[\lambda_0, \alpha_{01}]}{u}; C \frac{\lambda_n^2}{u}\})$$

Here

$$(3.6) \quad A_j = -u \partial_{x_j} u \partial_{x_1}^\dagger, \quad \mathbf{X}_j = A_j \circ u \mathbb{D}$$

when $j \geq 2$;

$$(3.7) \quad A_1 = -u \partial_{x_1}^\leftarrow u \partial_{x_1}^\dagger - \frac{1}{2} (u \partial_{x_1}^\dagger)^2;$$

$$(3.8) \quad \mathbf{X}_1 = -u \mathbb{D} \circ \frac{1}{2} (u \partial_{x_1}^\dagger)^2 + \frac{1}{2} (u \partial_{x_1}^\dagger)^2 \circ u \mathbb{D};$$

$$(3.9) \quad B = -u \partial_{x_1}^\dagger, \quad \mathbf{Y} = B \circ u \mathbb{D}; \quad C = -\text{Id}, \quad \mathbf{Z} = -u \mathbb{D};$$

and we can choose any linear order $<$. We also need an ordered exponential difference related to creation/annihilation operators creating a new letter of type g . Consider

$$(3.10) \quad \mathcal{E}_{\{\mathbf{X}'_j | 1 \leq j \leq n; \mathbf{Y}', \mathbf{Z}'\}, <} (\{A_j \frac{[\alpha_{0j}, \alpha_{01}]}{u} | 1 \leq j \leq n; B \frac{[\lambda_0, \alpha_{01}]}{u}; C \frac{\lambda_n^2}{u}\})$$

where A_i, B, C are as above and

$$(3.11) \quad \mathbf{X}'_j = -A_j \circ u \partial_{x_1}^\dagger u \mathbb{A}; \quad \mathbf{Y}'_j = -B \circ u \partial_{x_1}^\dagger \mathbb{A}; \quad C = -u \partial_{x_1}^\dagger \mathbb{A}$$

Again, the choice of a linear ordering is arbitrary (but same as the choice above). We denote by i_0 the minimal element with respect to the order $<$. We recall the definition of $\mathcal{E}_{ij}$; in the case of (3.10), we use the notation $\mathcal{E}'_{ij}$.

THEOREM 3.3. *Consider the following elements $\mathbb{T}(\lambda_0, \dots, \lambda_n)$ of $\widehat{\mathcal{H}}_n$:*

$$\mathbb{T}(\lambda_0, \lambda_1) = (1 + \sum_{i_1 \geq i_0} \mathcal{E}_{i_1 i_0} + \sum_{i_2 \geq i_1 \geq i_0} \mathcal{E}_{i_2 i_1} \mathcal{E}_{i_1 i_0} + \dots) \exp\left(\frac{1}{u} x_1\right)$$

$$\mathbb{T}(\lambda_0, \dots, \lambda_{n+1}) = -\left(\sum_{k \geq 0} \sum_{i_k \geq \dots \geq i_0} \mathcal{E}_{i_k i_{k-1}} \dots \mathcal{E}_{i_2 i_1} \mathcal{E}'_{i_1 i_0}\right) \mathbb{T}(\lambda_0, \dots, \lambda_n)$$

for $n > 0$. Let $T(\lambda_0, \dots, \lambda_n)$ be elements of $\widehat{\mathcal{A}}_0(\mathfrak{a}_n)$ represented by $\mathbb{T}(\lambda_0, \dots, \lambda_n)$. Then they satisfy the equations from Lemma 11.4.

4. Proof of Theorem 3.3

4.1. The homotopy on the cobar complex.

4.1.1. *Divided powers.* Let V be a free graded k -module. Put

$$(4.1) \quad \text{Sym}_{\text{pd}}(V) = \text{Sym}_{\text{pd}}(V_{\text{even}}) \otimes \text{Sym}(V_{\text{odd}})$$

(This is independent of a choice of V). Choose a set of free generators $\{y_i | i \in I\}$ and a linear ordering on I . Then $\text{Sym}_{\text{pd}}(V)$ is the k -module freely generated by finite products

$$(4.2) \quad y^{[m]} = \prod_i y_i^{[m_i]}$$

where $m_i \geq 0$ for even y_i and $m = 0$ or 1 for odd m_i . Define

$$(4.3) \quad \prod_i y_i^{[m_i]} \cdot \prod_i y_i^{[n_i]} = \pm \prod_i \frac{(m_i + n_i)!}{m_i! n_i!} y_i^{[m_i + n_i]}$$

(with the obvious sign). By definition, $y_j^{[m_j]} = 0$ for $m_j > 1$.

There is unique graded coalgebra structure on $\text{Sym}_{\text{pd}}(V)$ compatible with multiplication and such that

$$(4.4) \quad \Delta(y_i^{m_i}) = \sum_{k+l=m_i} y_i^k \otimes y_i^l$$

We denote by $\text{Sym}_{\text{pd}}^+(V)$ the subcoalgebra linearly spanned by $y^{[m]}$, $m \neq 0$.

There is another product on $\text{Sym}_{\text{pd}}(V)$:

$$(4.5) \quad \prod_i y_i^{[m_i]} * \prod_i y_i^{[n_i]} = \pm \prod_i y_i^{[m_i + n_i]}$$

(same sign as in (4.3)).

4.1.2. *The homotopy.* Let V , y_i , $<$ be as above. Define the following operator on $\text{Cobar}(\text{Sym}_{\text{pd}}^+(V))$.

For a monomial in $\text{Cobar}(\text{Sym}_{\text{pd}}^+(V))$, let n be the maximal element of I for which y_n occurs in it. Assume that this word is of the form

$$(4.6) \quad w = w_0(u_0)(u_1) \dots (u_k)(y_n)w_1$$

where

$$w_1 = (y_{j_1}) \dots (y_{j_l})$$

for $j_1, \dots, j_l < n$;

$$w_0 \in \text{Cobar}(\text{Sym}_{\text{pd}}^+(V)); \quad u_1 = y_{i_1}, \dots, u_k = y_{j_k}$$

for $i_1, \dots, i_k < n$; and either $u_0 = y_n$ or $u_0 \in \text{Sym}_{\text{pd}}^{>1}(V)$. Put

$$h_0 w = \sum_{j=0}^k (-1)^{l_j} w_0(u_0) \dots (u_j * y_n)(u_{j+1}) \dots (u_k) w_1$$

where

$$l_j = |w_0| + (|y_n| + 1) \left(\sum_{j < i \leq k} (|u_i| + 1) + |u_j| \right)$$

LEMMA 4.1.

$$[h_0, \partial_{\text{Cobar}}] = \text{Id} - iP$$

Here $iP = 0$ on the two-sided ideal generated by (u) , $u \in \text{Sym}_{\text{pd}}^{\geq 1}(V)$, and

$$iP((y_{i_1}) \dots (y_{i_n})) = \pm (y_{j_1}) \dots (y_{j_n})$$

where the right hand side is the re-ordering of the left hand side such that $j_1 \geq \dots \geq j_n$ (the sign is obvious).

In other words: we have a deformation retract

$$(4.7) \quad i : \text{Sym}(V[-1]) \xrightarrow{\sim} \text{Cobar}(\text{Sym}_{\text{pd}}^+(V)) : P$$

with h_0 being the homotopy between Id and iP .

The proof of the lemma is straightforward. In particular, h_0 is a contracting homotopy for $\ker(P)$.

4.1.3. *The algebra $\mathcal{A}_0^{\text{pd}}(\mathfrak{g})$.* Now let k be arbitrary. It is easy to see that for a graded Lie algebra $\mathfrak{g}$, the coalgebra $\text{Sym}_{\text{pd}}(\mathfrak{g})$ is equivariant under the adjoint action of $\mathfrak{g}$. We define the DG algebra exactly as in Definition 2.2 but with free generators (x) for $x \in \text{Sym}_{\text{pd}}^+(\mathfrak{g})$. Now extend h_0 to $\text{Cobar}(\text{Sym}_{\text{pd}}^+(\mathfrak{g}))$ so that it commutes with the right action of $U(\mathfrak{g})$ and with u . We have a deformation retract

$$(4.8) \quad i : (\mathcal{A}_0(\mathfrak{g}), \partial_{\text{Cobar}}) \xrightarrow{\sim} \text{Cobar}(\text{Sym}_{\text{pd}}^+(\mathfrak{g}))U(\mathfrak{g})[u] : P$$

where, as before, $\partial_{\text{Cobar}} = \text{Id} - iP$.

The differential uB_0 on the left hand side gets intertwined by P with the Chevalley-Eilenberg differential on the standard chain complex $C_\bullet(\mathfrak{g}, U(\mathfrak{g}))$. The latter is acyclic when k contains the rationals. This is what was used in [?] to prove the existence of $T(\lambda_0, \dots, \lambda_n)$ in characteristic zero.

In general, the Chevalley-Eilenberg complex is not acyclic. For example: Let $\mathfrak{a}$ is an Abelian graded Lie algebra with one generator of α of degree one. Identify $(\alpha)^n \alpha^m$ with $x^n \epsilon^m$ where $n \geq 0$ and $m = 0$ or 1 . Then the Chevalley-Eilenberg differential becomes $\epsilon \frac{\partial}{\partial x}$ which of course has nontrivial homology. There is a divided power version of the Chevalley-Eilenberg complex which is acyclic. This makes it very plausible that $T(\lambda_0, \dots, \lambda_n)$ satisfy some integrality property. We will take a slightly different and more explicit route to show this.

4.2. Some properties of h_0 .

We have

$$(4.9) \quad B_0 = B' + B''$$

where

$$B'((u_1) \dots (u_n)) = \sum_{1 \leq j < k \leq n; u_j \in \mathfrak{g}} (-1)^{l_{jk}} (u_1) \dots \widehat{(u_j)} \dots ([u_j, u_k]) \dots (u_n)$$

$$B''((u_1) \dots (u_n)) = \sum_{1 \leq j \leq n; u_j \in \mathfrak{g}} (-1)^{l_j} (u_1) \dots \widehat{(u_j)} \dots (u_n) u_j$$

Here

$$l_{jk} = \sum_{i < j} (|u_i| + 1) + |u_j| \sum_{j < i < k} (|u_i| + 1) + |u_j|$$

$$l_j = \sum_{i < j} (|u_i| + 1) + |u_j| \sum_{j < i \leq n} (|u_i| + 1)$$

LEMMA 4.2. *Assume that the homotopy h_0 is constructed using the linear order on free generators of $\mathfrak{a}_n$ that is consistent with the grading.*

(1) Let α be an element of order one in $\mathfrak{a}_n$. Then

$$h_0 B' \left(\frac{\alpha}{u} \right)^n = \sum_{j=0}^{n-3} \frac{1}{2} (n(n-1) - (j+1)(j+2)) \left(\frac{\alpha}{u} \right)^j \frac{(\alpha[\alpha, \alpha])}{u^2} \left(\frac{\alpha}{u} \right)^{n-j-3}$$

(2) Let λ and α be two elements of degree one in $\mathfrak{a}_n$. Then

$$h_0 [\lambda, \left(\frac{\alpha}{u} \right)^n] = \sum_{j=0}^{n-2} (n-1-j) \left(\frac{\alpha}{u} \right)^j \frac{(\alpha[\lambda, \alpha])}{u^2} \left(\frac{\alpha}{u} \right)^{n-2-j}$$

(3) Let λ and α be as above. Then

$$h_0 \left(\left(\frac{\alpha}{u} \right)^n \frac{(\lambda^2)}{u} \right) = \sum_{j=0}^{n-1} \left(\frac{\alpha}{u} \right)^j \frac{(\alpha\lambda^2)}{u^2} \left(\frac{\alpha}{u} \right)^{n-1-j}$$

PROOF. The computation in the proof of Lemma 2.1 yields

$$B'' \left(\frac{\alpha}{u} \right)^n = n \left(\frac{\alpha}{u} \right)^{n-1} \alpha$$

and

$$B' \left(\frac{\alpha}{u} \right)^n = - \sum_{l=0}^{n-2} (l+1) \left(\frac{\alpha}{u} \right)^l \left(\frac{[\alpha, \alpha]}{u} \right) \left(\frac{\alpha}{u} \right)^{n-2-l}$$

Therefore $h_0 B'' \left(\frac{\alpha}{u} \right)^n = 0$ and

$$\begin{aligned} h_0 B' \left(\frac{\alpha}{u} \right)^n &= \sum_{l=0}^{n-2} \sum_{j=0}^{l-1} (l+1) \left(\frac{\alpha}{u} \right)^j \frac{(\alpha[\alpha, \alpha])}{u^2} \left(\frac{\alpha}{u} \right)^{n-j-3} = \\ &= \sum_{j=0}^{n-3} \sum_{l=j+1}^{n-2} (l+1) \left(\frac{\alpha}{u} \right)^j \frac{(\alpha[\alpha, \alpha])}{u^2} \left(\frac{\alpha}{u} \right)^{n-j-3} = \\ &= \sum_{j=0}^{n-3} \frac{1}{2} (n(n-1) - (j+1)(j+2)) \left(\frac{\alpha}{u} \right)^j \frac{(\alpha[\alpha, \alpha])}{u^2} \left(\frac{\alpha}{u} \right)^{n-j-3} \end{aligned}$$

Also:

$$\begin{aligned} h_0 [\lambda, \left(\frac{\alpha}{u} \right)^n] &= -h_0 \sum_{k=0}^{n-1} \left(\frac{\alpha}{u} \right)^k \frac{([\lambda, \alpha])}{u} \left(\frac{\alpha}{u} \right)^{n-1-k} = \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{k-1} \left(\frac{\alpha}{u} \right)^j \frac{(\alpha[\lambda, \alpha])}{u^2} \left(\frac{\alpha}{u} \right)^{n-2-j} = \\ &= \sum_{j=0}^{n-2} \sum_{k=j+1}^{n-1} \left(\frac{\alpha}{u} \right)^j \frac{(\alpha[\lambda, \alpha])}{u^2} \left(\frac{\alpha}{u} \right)^{n-2-j} = \\ &= \sum_{j=0}^{n-2} (n-1-j) \left(\frac{\alpha}{u} \right)^j \frac{(\alpha[\lambda, \alpha])}{u^2} \left(\frac{\alpha}{u} \right)^{n-2-j} \end{aligned}$$

Finally, the last statement in the lemma is straightforward. $\square$

LEMMA 4.3. For α and λ of degree one in $\mathfrak{a}_n$ and $f \in \text{Sym}^{>1}(\mathfrak{a}_n)$,

(1)

$$h_0 B' \left(f \left(\frac{\alpha}{u} \right)^n \right) = (-1)^{|f|+1} f h_0 B' \left(\frac{\alpha}{u} \right)^n + (f * \frac{[\alpha, \alpha]}{u}) \frac{1}{2} n(n-1) \left(\frac{\alpha}{u} \right)^{n-2}$$

(2)

$$h_0[\lambda, f(\frac{\alpha}{u})^n] = (-1)^{|f|+1} f h_0[\lambda, (\frac{\alpha}{u})^n] + (f * \frac{[\lambda, \alpha]}{u}) n (\frac{\alpha}{u})^{n-1}$$

(3)

$$h_0(f(\frac{\alpha}{u})^n (\frac{\lambda^2}{u})) = (-1)^{|f|+1} f h_0((\frac{\alpha}{u})^n (\frac{\lambda^2}{u})) + (f * \frac{\lambda^2}{u}) (\frac{\alpha}{u})^n$$

PROOF. We have

$$\begin{aligned} h_0 B'(f(\frac{\alpha}{u})^n) &= -(-1)^{|f|+1} f \sum_{l=0}^{n-2} (l+1) (\frac{\alpha}{u})^l (\frac{[\alpha, \alpha]}{u}) (\frac{\alpha}{u})^{n-2-l} = \\ &= (-1)^{|f|+1} (f h_0 B'(\frac{\alpha}{u})^n - \sum_{l=0}^{n-2} (l+1) (-1)^{|f|} (f * \frac{[\alpha, \alpha]}{u}) (\frac{\alpha}{u})^{n-2}) = \\ &= (-1)^{|f|+1} (f h_0 B'(\frac{\alpha}{u})^n - \frac{1}{2} n(n-1) (-1)^{|f|} (f * \frac{[\alpha, \alpha]}{u}) (\frac{\alpha}{u})^{n-2}) \\ &= (-1)^{|f|+1} f h_0 B'(\frac{\alpha}{u})^n + \frac{1}{2} n(n-1) (f * \frac{[\alpha, \alpha]}{u}) (\frac{\alpha}{u})^{n-2} \end{aligned}$$

and similarly for the other two statements. $\square$

LEMMA 4.4. (1)

$$h_0^2 = 0$$

(2) For any $w_0 \in \mathcal{A}_0(\mathfrak{a}_n)$, $f \in \text{Sym}^{>1}(\mathfrak{a}_n)$, and α of degree one in $\mathfrak{a}_n$,

$$h_0(w_0 f(\frac{\alpha}{u})^n) = (-1)^{|w_0|+|f|+1} w_0 h_0(f(\frac{\alpha}{u})^n)$$

(3) For α_i of degree one in $\mathfrak{a}_n$ and $f_i \in \text{Sym}^{>1}(\mathfrak{a}_n)$,

$$B'((\frac{\alpha_0}{u})^{n_0} (f_1) \dots (f_l) \frac{\alpha_l}{u})^{n_l} =$$

$$\sum_{j=0}^{l-1} (-1)^{\sum_0^{l-1} (|f_i|+1)} (\frac{\alpha_0}{u})^{n_0} \dots n_j (\frac{\alpha_j}{u})^{n_j-1} \dots (\frac{\alpha_{l-1}}{u})^{n_{l-1}} f_l (B'(\frac{\alpha_l}{u})^{n_l} + [\alpha_l, (\frac{\alpha_l}{u})^{n_l}])$$

4.3. The end of the proof of Theorem 3.3. First find $T = T(\lambda_0, \lambda_1)$.Because of Lemma 2.1, we look for $T = t + \bar{T}$ where $t = \exp((\frac{\alpha_0 1}{u}))$ and $\bar{T}$ is in $\ker(P)$. We have to solve

$$(\partial + u B_0)(t + \bar{T}) + D_{\lambda_0}(t + \bar{T}) - (t + \bar{T}) D_{\lambda_1} = 0$$

(we write $\partial = \partial_{\text{Cobar}}$). If

$$\phi = u B_0 t + D_{\lambda_0} t - t D_{\lambda_1}$$

then the above turns into

$$(4.10) \quad \phi + (\partial + u B_0) \bar{T} + D_{\lambda_0} \bar{T} - \bar{T} D_{\lambda_1} = 0$$

Put $\mathcal{D} = u B_0 T + D_{\lambda_0} T - T D_{\lambda_1}$. We have

$$\partial^2 = \partial \mathcal{D} + \mathcal{D}^2 = 0$$

We have seen that h_0 is a contracting homotopy for $(\ker(P), \partial)$. By the homotopy perturbation lemma,

$$(4.11) \quad h = (1 + [\mathcal{D}, h_0])^{-1} h_0$$

is a contracting homotopy for $\partial + \mathcal{D}$. Indeed:

$$\begin{aligned} [\partial + \mathcal{D}, h] &= [\partial + \mathcal{D}, (1 + [\mathcal{D}, h_0])^{-1} h_0] = -(1 + [\mathcal{D}, h_0])^{-1} [\partial + \mathcal{D}, [\mathcal{D}, h_0]] (1 + [\mathcal{D}, h_0])^{-1} \\ &\quad + (1 + [\mathcal{D}, h_0])^{-1} (1 + [\mathcal{D}, h_0]) = (1 + [\mathcal{D}, h_0])^{-1} [\partial \mathcal{D} + \mathcal{D}^2, h_0] (1 + [\mathcal{D}, h_0])^{-1} + 1 = 1 \end{aligned}$$

So the solution to (4.10) is

$$(4.12) \quad \bar{T} = -h\phi = -(1 + [\mathcal{D}, h_0])^{-1} h_0 \phi$$

In other words,

$$T = \sum_{k=0}^{\infty} T_k; \quad T_0 = \exp\left(\frac{\alpha_{01}}{u}\right); \quad T_{k+1} = -h_0(uB_0 T_k + D_{\lambda_0} T_k - T_k D_{\lambda_1})$$

Since T_k is in the image of h_0 for $k > 0$, and because of the properties of h_0 discussed above, this is the same as

$$T = \sum_{n=0}^{\infty} T_n; \quad T_0 = \exp\left(\frac{\alpha_{01}}{u}\right); \quad T_{k+1} = -h_0(uB' T_k + [\lambda_0, T_k] + T_k \left(\frac{\lambda_1^2}{u}\right))$$

Now the statement for $n = 1$ follows from Lemmas 4.2, 4.3, and 4.4, as well as from

$$y + Ay * y + a^2 y * y * y + \dots = \frac{e^{Ay} - 1}{A}$$

The proof for $n \geq 2$ is by induction in n . The argument is the same if we take ϕ to be the right hand side of the formula in Theorem 3.3. Choose the linear order on generators of degree one of $\mathfrak{a}_{n+1}$ so that $\alpha_{0,n+1} < \dots < \alpha_{01}$. (As always, our order is compatible with the grading). Then observe that all terms of ϕ are in the kernel of h_0 except for $-T(\lambda_0, \lambda_2, \dots, \lambda_{n+1})$. This completes the proof.

4.4. The integrality property. Optimistically, one would expect the expression $T(\lambda_0, \dots, \lambda_n)$ to be an infinite linear combination of terms $\frac{1}{m!} T(m)$ where $T(m)$ is of degree m (in the grading induced by the grading of $\mathfrak{a}_n$). A priori, it is an infinite linear combination of terms

$$(4.13) \quad \frac{1}{m!} \frac{1}{m_1!} (f_1) \dots \frac{1}{m_N!} (f_N)$$

in $T(m)$, where $m_j = \lfloor \frac{|f_j|}{2} \rfloor$ and $m = \sum_{j=1}^N |f_j|$. Actually, the integrability property is much stronger, although (unless we missed some cancellations) it is slightly weaker than the optimistic case. More precisely, T is a linear combination of monomials $\frac{1}{m!} (f_1) \dots (f_N)$ where each f_j contains no more than one factor $\frac{[\alpha_{0,i}, \alpha_{0,i}]^q}{q+1}$; there are no more denominators.

To see that, recall the notation of (3.5), (3.6), (3.7), (3.9). All A_j are of the form $A'_j \frac{\partial}{\partial x_i}$ for some i and some A'_j . Similarly, $B = B' \partial_{x_1}^\dagger$. Now, $\mathbf{X}_j = A_j \mathbb{D}$ for $j \geq 2$ and $\mathbf{Y} = B \mathbb{D}$; so contributions of those operators are divisible by $\frac{1}{q!} (\partial_{x_1}^\dagger)^q$. The terms with C also introduce no new denominators. (Example:

$$\frac{1}{n!} \left(\frac{\alpha_{01}}{u}\right)^{n-1} \left(\frac{\alpha_{01}}{u} \frac{\lambda_1^{2q}}{q! u^q}\right)$$

divides $(n + 2q)!$).

But the term corresponding to of A_j is

$$\frac{e^{A'_1 \partial_{x_1}^\dagger [\alpha_{01}, \alpha_{01}]} - 1}{A'_1 \partial_{x_1}^\dagger} \mathbf{X}_1$$

where $\mathbf{X}_1$ is not a multiple of $\partial_{x_1}^\dagger$. Its contributions are multiples of terms of the form $\frac{1}{(q+1)!}(\partial_{x_1}^\dagger)^q[\alpha_{01}, \alpha_{01}]^q$, and therefore of $\frac{1}{q+1}[\alpha_{01}, \alpha_{01}]^q$. Iterative steps push these factors to the left (and replace α_{01} by α_{0l}).

4.5. The formulas for $\mathcal{A}_1$.

4.5.1. *Special elements of $\widehat{\mathcal{A}}_1(\mathfrak{a}_n)$.* Let $\mathcal{A}_1^{(1)}$ be the linear span of elements $x(y)$, $x \in U(\mathfrak{a}_n)$, $y \in U^+(\mathfrak{a}_n)$. This is a k -submodule of $\widehat{\mathcal{A}}_1(\mathfrak{a}_n)$. It carries the product $\star$ which is defined as follows:

$$(4.14) \quad x \star y = xy; \quad x \star (y) - (-1)^{|x|(|y|+1)}(y) \star x = (-1)^{|x|}(\text{ad}_x(y)); \quad (x) \star (y) = (xy)$$

In other words, $\star$ is the cross product on $U(\mathfrak{a}_n) \ltimes U^+(\mathfrak{a}_n)$.

As usual, $\widehat{\mathcal{A}}_1^{(1)}$ is the completion of $\mathcal{A}_1^{(1)}[u, u^{-1}]$ with respect to the filtration induced by the grading of $\mathfrak{a}_n$.

Recall the isomorphism in Lemma 2.3. By definition, it sends $\exp(\frac{R}{u})$ to

$$(4.15) \quad \mathcal{E}(\frac{R}{u}) = (1 + \frac{R}{u})^{\star(\frac{R}{u})} \in \widehat{\mathcal{A}}_1^{(1)}$$

Next, for any $\mathfrak{a}$ and any $\alpha \in \mathfrak{a}^1$, $R \in \mathfrak{a}^2$, define

$$(4.16) \quad (\alpha D_R)(R^m(R^n)) = \sum_{k=0}^{m-1} R^k \alpha R^{n-1-k}(R^n) + \sum_{k=0}^{n-1} R^m(R^k \alpha R^{n-1-k})$$

Define

$$(4.17) \quad \mathcal{E}(\frac{\alpha}{u}, \frac{R}{u}) = (\alpha D_R)(\mathcal{E}(\frac{R}{u}))$$

In general, let $\mathfrak{a}_{n,1}$ be the free DG Lie algebra with generators $\alpha_1, \dots, \alpha_n$ of degree one and R of degree two. Let $\alpha_i D_R$ be its derivation of degree one that sends R to α_i and all other free generators to zero. Define

$$(4.18) \quad \mathcal{E}(\frac{\alpha_1}{u}, \dots, \frac{\alpha_n}{u}, \frac{R}{u}) = (\alpha_1 D_R) \dots (\alpha_n D_R) \mathcal{E}(R)$$

By universality, elements (4.18) are defined for α_i of degree one and for R of degree two in any $\mathfrak{a}$.

LEMMA 4.5. *Under the isomorphism $\widehat{\mathcal{A}}_0 \xrightarrow{\sim} \widehat{\mathcal{A}}_1$ as in Lemma 2.3,*

$$(\frac{\alpha_1}{u} \dots \frac{\alpha_n}{u} e^{\frac{R}{u}}) \mapsto \mathcal{E}(\frac{\alpha_1}{u}, \dots, \frac{\alpha_n}{u}, \frac{R}{u})$$

PROOF. We already know this for $n = 0$. For $n = 1$, by definition,

$$(\alpha R^n) \mapsto \frac{1}{n+1} \sum_{k+l=n+1} a_{k,l} (\alpha D_R)(R^k(R^l))$$

(as in (4.16)). Therefore

$$(\frac{\alpha}{u} e^{\frac{R}{u}}) \mapsto \sum_{n=0}^{\infty} \sum_{k+l=n+1} \frac{u^{-n-1} a_{k,l}}{(n+1)!} (\alpha D_R)(R^k(R^l)) = (\alpha D_R)(\mathcal{E}(\frac{R}{u})) = \mathcal{E}(\frac{\alpha}{u}, \frac{R}{u})$$

Similarly for all $n > 1$. □

4.5.2. *Admissible words, the case of $\mathcal{A}_1$.* Recall the k -module $\mathcal{H}_n$ from 3.2. We change the definition slightly and require letters f and g to be monomials in $\mathcal{A}_1^{(1)}$. The map $\mathcal{H}_n \rightarrow \mathcal{A}_0(\mathfrak{a}_n)$ is also different from the one we used before, and more straightforward: it sends x_j to $(\alpha_{0,j})$, and monomials f and g to their images in $\mathcal{A}_0(\mathfrak{a}_n)$; A concatenation of words is sent to the product of their images.

4.6. The new $\mathcal{E}_{ij}$. As in 3.1, let I be a finite set. Let $\{y_i | i \in I\}$ be elements of degree two in $\mathfrak{a}_n$. Also, let $\{A_i | i \in I\}$ be elements of some commutative ring Op of operators, such as the ring of linear differential operators in several variables with constant coefficients. (We assume, as it happens in our case, that Op has no zero divisors). Let $\mathcal{H}$ be a k -module on which Op acts, and let be operators on $\mathcal{H}$.

Choose a linear order on I and use it to identify I with $\{1, \dots, n\}$. Recall the expressions

$$(4.19) \quad \mathcal{E}_{\{\mathbf{x}_i\}, <}(\{A_i \frac{y_i}{u}\}) = \frac{e^{A_n y_n} - 1}{A_n} (\mathbf{X}_n + A_n \sum_{j=1}^{n-1} e^{A_{n-1} y_{n-1} + \dots + A_{j+1} y_{j+1}} \frac{e^{A_j y_j} - 1}{A_j} \mathbf{X}_j)$$

as in (3.1). They lie in the completed tensor product

$$(4.20) \quad \text{Sym}(\mathfrak{a}_n^2) \hat{\otimes} \sum_i \text{Op} \circ \mathbf{X}_i,$$

which is the completion of the usual tensor product with respect to the filtration induced by the grading on $\mathfrak{a}_n$.

Expressions (4.19) are linear combinations of the form

$$(4.21) \quad \mathcal{E}_{\{\mathbf{x}_i\}, <}(\{A_i \frac{y_i}{u}\}) = X_0 + \sum_{1 \leq j \leq k \leq n} \exp(\frac{1}{u} \sum_{i=j}^k A_i y_i) X_{jk}$$

where

$$X_0, X_{jk} \in \sum_i \text{Frac}(\text{Op}) \circ \mathbf{X}_i$$

($\text{Frac}(\text{Op})$ being the field of fractions of Op); in fact they lie in (4.20). (Example: $\frac{e^{\frac{1}{u} A y} - 1}{A}$).

Recall also

$$(4.22) \quad \mathcal{E}_{i_1 i_2} = \mathcal{E}_{\{\mathbf{x}_i | i_1 \leq i \leq i_2\}, <}(\{A_i \frac{y_i}{u} | i_1 \leq i \leq i_2\})$$

Now consider two partial examples (3.5), and (3.10). For the example (3.5),

$$(4.23) \quad \tilde{\mathcal{E}}_{\{\mathbf{x}_i\}, <}(\{A_i \frac{y_i}{u}\}) = X_0 + \sum_{1 \leq j \leq k \leq n} \mathcal{E}(\frac{1}{u} \alpha_{01}, \frac{1}{u} \sum_{i=j}^k A_i y_i) X_{jk}$$

For the example (3.10), we do the same but now

$$(4.24) \quad \tilde{\mathcal{E}}'_{\{\mathbf{x}_i\}, <}(\{A_i \frac{y_i}{u}\}) = X_0 + \sum_{1 \leq j \leq k \leq n} \mathcal{E}(\frac{1}{u} \alpha_{01}, \frac{1}{u} \alpha_{12}, \frac{1}{u} \sum_{i=j}^k A_i y_i) X_{jk}$$

We denote by $\tilde{\mathcal{E}}_{i_1 i_2}$ resp. $\tilde{\mathcal{E}}'_{i_1 i_2}$, the versions of $\tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}'$ with indices limited to $i_1 \leq i \leq i_2$.

4.6.1. *Explicit formulas.*

THEOREM 4.6. *Consider the following elements $\mathbb{T}(\lambda_0, \dots, \lambda_n)$ of $\widehat{\mathcal{H}}_n$:*

$$\mathbb{T}(\lambda_0, \lambda_1) = (1 + \sum_{i_1 \geq i_0} \tilde{\mathcal{E}}_{i_1 i_0} + \sum_{i_2 \geq i_1 \geq i_0} \tilde{\mathcal{E}}_{i_2 i_1} \tilde{\mathcal{E}}_{i_1 i_0} + \dots) \exp\left(\frac{1}{u} x_1\right)$$

$$\mathbb{T}(\lambda_0, \dots, \lambda_{n+1}) = -\left(\sum_{k \geq 0} \sum_{i_k \geq \dots \geq i_0} \tilde{\mathcal{E}}_{i_k i_{k-1}} \dots \tilde{\mathcal{E}}_{i_2 i_1} \tilde{\mathcal{E}}'_{i_1 i_0}\right) \mathbb{T}(\lambda_0, \dots, \lambda_n)$$

for $n > 0$. Let $T(\lambda_0, \dots, \lambda_n)$ be elements of $\widehat{\mathcal{A}}_0(\mathfrak{a}_n)$ represented by $\mathbb{T}(\lambda_0, \dots, \lambda_n)$. Then they satisfy the equations from Lemma 11.4.

Here i_0 is the minimal element of I .

4.7. The free Fock space description of T . Here we give another description of the operators T that we constructed above. For the sake of simplicity we only discuss the case $n = 1$, $\lambda_0 = 0$. The goal is to point out certain similarities with expressions arising in mathematical physics. The formal parameter u plays the role of the Planck constant $\hbar$.

4.7.1. *Formulas for $\mathcal{A}_0$.* Let

$$\widehat{\mathcal{H}} = k\left[\left[\frac{x_0}{u}, \frac{x_1}{u}, \frac{x_2}{u}, \dots; \frac{y_1}{u}, \frac{y_2}{u}, \dots\right]\right]$$

be the k -module of power series in infinitely many variables. We will be using the annihilation operators $p_j = u \frac{\partial}{\partial x_j}$ and the creation operators $\frac{y_j}{u}$, as well as creation/annihilation operators

$$(u\mathbb{D}_{x_j} f)(x_0, \dots) = \frac{u(f(x_0, \dots, x_{j-1}, x_{j+1}, x_{j+2}, \dots) - f(x_0, \dots, x_{j-1}, x_j, x_{j+2}, \dots))}{x_{j+1} - x_j}$$

Define

$$\mathbb{T} = \left(1 + \sum_{k=1}^{\infty} \frac{e^{A_k \frac{y_k}{u}} - 1}{A_k} \mathbf{X}_{k-1} \dots \frac{e^{A_1 \frac{y_1}{u}} - 1}{A_1} \mathbf{X}_0\right) \exp\left(\frac{1}{u} x_0\right)$$

where

$$(4.25) \quad \mathbf{X}_k = -u\mathbb{D}_{x_k} \circ \frac{1}{2} p_k^2 + \frac{1}{2} p_k^2 \circ u\mathbb{D}_{x_k} - u\mathbb{D}_{x_k}$$

$$(4.26) \quad A_k = -(p_1 + \dots + p_{k-1})p_k - \frac{1}{2} p_k^2 - 1$$

PROPOSITION 4.7. *Transform an element of $\widehat{\mathcal{H}}$ into an element of $\widehat{\mathcal{A}}_0(\mathfrak{a}_1)$ as follows: replace any monomial*

$$\left(\frac{x_0}{u}\right)^{n_0} \left(\frac{y_1}{u}\right)^{m_1} \left(\frac{x_1}{u}\right)^{n_1} \left(\frac{y_2}{u}\right)^{m_2} \dots$$

by

$$\left(\frac{\alpha}{u}\right)^{n_0} \left(\frac{\alpha[\alpha, \alpha]^{m_1}}{u^{m_1+1}}\right) \left(\frac{\alpha}{u}\right)^{n_1} \left(\frac{\alpha[\alpha, \alpha]^{m_2}}{u^{m_2+1}}\right) \dots$$

Then the element $\mathbb{T}$ gets transformed into an element $T(\alpha)$ that satisfies the equation from Theorem 3.3 when $n = 1$, $\lambda_0 = 0$, and $\lambda_1 = \alpha$.

4.7.2. *Formulas for $\mathcal{A}_0$.* Now let

$$\widehat{\mathcal{H}} = k\left[\frac{x_0}{u}, \frac{x_1}{u}, \frac{x_2}{u}, \dots; \frac{y_1}{u}, \frac{y_2}{u}, \dots; \frac{\eta_1}{u}, \frac{\eta_2}{u}, \dots; \frac{z_1}{u}, \frac{z_2}{u}, \dots; \frac{\zeta_1}{u}, \frac{\zeta_2}{u}, \dots\right]$$

where η_i, ζ_i are odd variables. Define

$$\mathbb{T} = (1 + \sum_{k=1}^{\infty} (F(y_k, z_k, \eta_k, \zeta_k, A_k) \mathbf{X}_{k-1} \dots F(y_1, z_1, \eta_1, \zeta_1, A_1) \mathbf{X}_0) \exp(\frac{1}{u} x_0))$$

where

$$(4.27) \quad F(y, z, \eta, \zeta, A) = (\eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z}) (1 + \frac{z}{u})^{\frac{y}{z}}$$

4.7.3. *Formulas for $\mathcal{A}_1$.* Now let

$$\widehat{\mathcal{H}} = k\left[\frac{x_0}{u}, \frac{x_1}{u}, \frac{x_2}{u}, \dots; \frac{y_1}{u}, \frac{y_2}{u}, \dots; \frac{\eta_1}{u}, \frac{\eta_2}{u}, \dots; \frac{z_1}{u}, \frac{z_2}{u}, \dots; \frac{\zeta_1}{u}, \frac{\zeta_2}{u}, \dots\right]$$

where η_i, ζ_i are odd variables. Define

$$\mathbb{T} = (1 + \sum_{k=1}^{\infty} (F(y_k, z_k, \eta_k, \zeta_k, A_k) \mathbf{X}_{k-1} \dots F(y_1, z_1, \eta_1, \zeta_1, A_1) \mathbf{X}_0) \exp(\frac{1}{u} x_0))$$

where

$$(4.28) \quad F(y, z, \eta, \zeta, A) = (\eta \frac{\partial}{\partial y} + \zeta \frac{\partial}{\partial z}) (1 + \frac{z}{u})^{\frac{y}{z}}$$

PROPOSITION 4.8. *Consider the map*

$$\widehat{\mathcal{H}} \rightarrow \widehat{\mathcal{A}}_0(\mathfrak{a}_1)$$

sending

$$(\frac{x_0}{u})^{n_0} \cdot (\frac{\eta_1}{u})^{a_1} (\frac{y_1}{u})^{p_1} (\frac{\zeta_1}{u})^{b_1} (\frac{z_1}{u})^{q_1} \cdot (\frac{x_1}{u})^{n_1} \dots$$

to

$$(\frac{\alpha}{u})^{n_0} \cdot \frac{\alpha^{a_1} [\alpha, \alpha]^{p_1}}{u^{p_1+1}} (\frac{\alpha^{b_1} [\alpha, \alpha]^{q_1}}{u^{q_1+1}}) (\frac{\alpha}{u})^{n_1} \dots$$

for $n_i, p_i, q_i \geq 0$ and $a_i, b_i \in \{0, 1\}$. Then the element $\mathbb{T}$ gets transformed into an element $T(\alpha)$ that satisfies the equation from Theorem 3.3 when $n = 1$, $\lambda_0 = 0$, and $\lambda_1 = \alpha$.

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