Northwestern 1
11/3-4/18

A. Bloomberg

Localization and THH (Bloomberg-Mandell)

K-thy: $A \to B \to C$
stable categories

$C \simeq B/A$ Karoubi completion...

$K(A) \to K(B) \to K(C)$

(Thomason-Troubaugh)
(Neeman)

what makes it extend below zero
[nonconnective K-theory]

B w/notion of weak eq; wB

coarser equivalence vB

$$(B^v, w) \to (B, w) \to \cancel{(B, v)}\ vB$$

Examples A discrete valuation ring

$$K(k) \to K(A) \to K(K)$$

residue field

field of fractions

Hesselholt-Madsen: program of proving Lichtenbaum conjecture using trace methods
very beautiful paper in Ann Math

<u>Ex. 2</u> $K(\mathbb{Z}) \to K(k_U) \to K(KU)$

conjectured by Rognes; proved by Blumberg-M.

$K(\mathbb{Z})$: understand; $K(k_U)$: more or less, using trace methods...

$K(KU)$ hard. [Ausoni-Barthel-Gepner... trouble]

=

Suppose K connective K-thy.
To build nonconnective K-thy:

$$A \longrightarrow \mathrm{Ind}\, A \longrightarrow \mathrm{Ind}\, A/A$$

First thing in Thomason-Trobaugh: this is a localization sequence (up to smth in K_0 ...)

[Schlichting... also note: Calkin algebra...]

$$K(A) \to * \to K(\Sigma A) \ldots$$

$$\operatorname{colim}_n \Omega^n K(\Sigma^n A) \ldots$$

<u>What about</u> THH?

$\mathcal{C}$ some kind of stable category

$\mathcal{C}$ spectral category
dg category ...

$\mathrm{THH}(\mathcal{C})$

$$k \rightsquigarrow \bigvee \underbrace{\mathcal{C}(x_{k}, x_0)}_{\substack{\text{could be} \\ \text{bimod } M(x_k, x_0)}} \wedge \mathcal{C}(x_0, x_1) \wedge \ldots \wedge \mathcal{C}(x_{k-1}, x_k)$$

(Mitchell...)

$$M_1 \to M_2 \to M_3$$

$$\mathrm{THH}(\mathcal{C}, M_1) \to \mathrm{THH}(\mathcal{C}, M_2) \to \mathrm{THH}(\mathcal{C}, M_3)$$

$A \to B \to C$ an exact sequence of stable categories

$\overset{\Longrightarrow}{(?)}$

$$\mathrm{THH}(A) \to \mathrm{THH}(B) \to \mathrm{THH}(C)$$

$A \subseteq B$

full spectral
subcat (say)

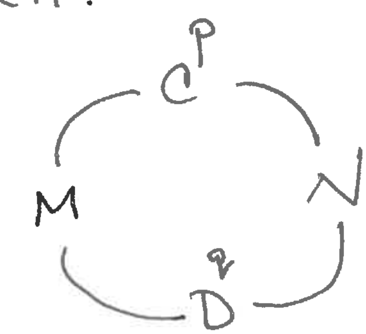
$$L^B_A(x,y) = \mathrm{Bar}\left(B(x,-),\, A,\, \mathrm{Bar}(-,y)\right)$$

$$THH(B, L^B_A) \to THH(B,B) \to THH\left(B, {}^{B}\!/L^B_A\right)$$

$$\star \quad THH(B, L^B_A) \simeq THH(A)$$

$$THH\left(B, {}^{B}\!/L^B_A\right) \simeq THH(B/A) \quad \star$$

Dennis-Waldhausen:



bisimplicial
object

$$THH\left(\mathcal{C}, B(N, D, M)\right)$$
$$\simeq$$
$$THH\left(D; B(M, \mathcal{C}, N)\right)$$

which explains $\star$.

This is <u>NOT</u> the correct localization seq.

<u>Why</u>?

$$\begin{tikzcd} K(\mathbb{Z}) \arrow[r] \arrow[d] & K(ku) \arrow[r] \arrow[d] & K(KU) \arrow[d] \\ ?? \arrow[r] & THH(ku) \arrow[r] & THH(KU) \end{tikzcd}$$

No devissage. Same for DVR.

The issue comes from connectivity.

$\mathcal{C}$ a Waldhausen category (Small?)

$w\mathcal{C} \subseteq \mathcal{C}$

distinguished family of pushouts.

<u>Assume</u>: $\mathcal{C}$ is simplicial; tensors w/ finite CW complexes.

$f: X \to Y$ is a weak equiv. $\iff$ $\mathcal{C}(Y,-) \to \mathcal{C}(X,-)$, $\mathcal{C}(-,X) \to \mathcal{C}(-,Y)$

are w.e. of spaces.

Build mapping spectra

$n \rightsquigarrow \mathcal{C}(X, \bigvee_n Y)$ Γ-space $\mathcal{C}^\Gamma$

$n \rightsquigarrow \mathcal{C}(X, \Sigma^n Y)$ symmetric spectrum $\mathcal{C}^{\Sigma}$.

Under good circumstances:

$\mathcal{C}^S$ is a connective cover of $\mathcal{C}^\Gamma$.

$\approx$

If $\mathcal{C}$ is a Waldhausen category:

$$\mathrm{THH}(N^w\mathcal{C}^S) \qquad N^w = \text{nerve of weak equivs}$$

$$\mathrm{THH}(N^w\mathcal{C}^\Gamma)$$

Inherits spectral enrichment from $\mathcal{C}^\Gamma$ or $\mathcal{C}^S$.

i.e. cyclic bar construction of a spectral cat.

[pushouts in the Waldhausen category are used to construct $\mathcal{C}^\Gamma$ and $\mathcal{C}^S$...]

$$w\mathcal{C} \subseteq v\mathcal{C}:$$

Thm * $\mathrm{THH}((N^w\mathcal{C}^\Gamma)^v) \to \mathrm{THH}(N^w\mathcal{C}^\Gamma)$

$$\downarrow$$

$$\mathrm{THH}(N^v\mathcal{C}^\Gamma)$$

$$\Uparrow$$

note: mapping spectra come from w!

Similar for S: (**)

(**) is "T-T" localization sequence.

In the known cases, $THH((N^w C^\Gamma)^\vee)$ satisfy devissage.

$$THH(k) \to THH(A) \to \underbrace{THH(A|k)}$$

terminology of Hesselholt - Madsen

$$THH(\mathbb{Z}) \longrightarrow THH(ku) \longrightarrow THH(ku|KU)$$

<u>What is</u> $THH(A|k)$?

H-M...

$TR(A|k)$?

log structure...

[Rogness - Sagave - Schlichtkroll...]

$$=$$

$$K(BP\langle n-1\rangle) \to K(BP\langle n\rangle) \to K(E(n))$$

$$=$$

$$K(HW_{p^n}[\![v_0, \dots, v_{n-1}]\!]) \to K(BP_n) \to K(E_n)$$

N. Rozenblyum

joint w/ Ayala & Mazel-Gee

NC stratifications and equivariant homotopy thy

Trying to understand Bökstedt-~~H~~-M...

genuine equivariant homotopy theory

X scheme $\qquad Z \xhookrightarrow{i} X \xhookleftarrow{j} U$

$$QCoh(X)$$

$$QCoh_Z(X) \xrightarrow{i_?} QCoh(X) \xrightarrow{j^*} QCoh(U)$$

"torsion modules"

$i_?$ has a right adjoint which itself preserves colims/has a right adjoint.

$$QCoh_Z(X) \simeq QCoh(\hat{X}_Z)$$

$\hat{i}^*$ is right adjoint to i?

(under this equivalence)

$\hat{i}_*$ is a right adjoint to $\hat{i}^*$

Can glue sheaves on X from sheaves on U and on $\hat{X}_Z$.

An object in $QCoh(X)$ can be described as: $\mathcal{F}_U \in QCoh(U)$ $\mathcal{F}_Z \in QCoh(\hat{X}_Z)$

$+ \quad \mathcal{F}_Z \to \hat{i}^* j_* \mathcal{F}_U$

Def Let $\mathcal{C}$ be a (presentable) stable ∞ category; a full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ closed under colims and admitting a colimit-preserving right adjoint

More general stratifications:

P poset. Recall: a P-stratified

scheme X consists of a map of posets

$$P \to \{\text{~~Subschemes~~ of } X\}$$
Closed subsets

$$p \longmapsto Z_p$$

s.t. for $p \not\leq q$ in P

$$X_p := Z_p \setminus \bigcup_{p'<p} Z_{p'} \subset X \setminus X_q$$

Dfn A stratification of $\mathcal{C}$ by P is a map of posets $\overset{?}{=}$ full subcats?..

$$P \mapsto \{\text{Closed subcats of } \mathcal{C}\}$$

$$p \longmapsto \mathcal{Z}_p$$

s.t. for

$$\mathcal{Z}_q \to \mathcal{C} \to \mathcal{X}_p = \mathcal{Z}_p / \langle \mathcal{Z}_{p'} \rangle_{p' \leq p}$$

the comp. is zero.

[in our ex.: $QCoh(X) / QCoh_Z(X) \simeq QCoh(U)$

For any $p \in \mathbb{P}$, we have

$$\mathcal{C}_p \quad \left(= Z_p \big/ \langle Z_{p'} \rangle_{p' \leq p} \right)$$

<u>$p < q$:</u>

$$\mathcal{C}_p \to \mathcal{Z}_p \to \mathcal{C} \to \mathcal{C}_q$$

using (second) right adjoint

the composite $\mathcal{C}_p \to \mathcal{C}_q$

is the generalization of $\cancel{\hat{i}^* j_*}$ above.

$j^* \hat{i}_*$

$$\mathcal{F}_U \to j^* \hat{i}_* \mathcal{F}_Z$$

Get (left) lax functor

$$\mathbb{P} \to \mathcal{C}at$$

($\Leftrightarrow$ locally coCartesian fibration...)

~~there~~ is an

<u>Thm</u> (Ayala-Mazel-Gee-R.)

equivalence

$$\left\{ \begin{matrix} \mathbb{P}\text{-stratified} \\ \text{categories} \end{matrix} \right\} \xleftarrow{\text{"right lax limit"}} \simeq \left\{ \begin{matrix} \text{(left) lax functors} \\ \mathbb{P} \to \mathcal{C}at \end{matrix} \right\}$$

(under the assumption that P is an artinian poset).

$=$

Take $\mathcal{C} = Sp^G$ category of genuine G-spectra

G a compact Lie group.

$$Spc^G = \text{"genuine } G\text{-spaces"} \quad \text{[the orbit category}$$
$$\| $$
$$Fun(\mathcal{O}_G, Spaces)$$

(Problem: Sp^G is not a stabilization of Spc^G ...)

$$\Sigma^\infty_G : Spc^G \to Sp^G$$

Sp^G is generated by $\Sigma^\infty_G ((G/H)_+)$

Let P_G be the poset of closed subgroups (up to conj).

Thm Sp^G is stratified by this poset.

$\mathcal{Z}_H =$ full subcategory gen. by $\langle \Sigma^\infty_G (G/K) \rangle_{K \leq H}$

Proof amounts to

$$\Phi^K\left(\Sigma^\infty_G (G/H)_+\right) = 0 \quad \text{if } K \not\leq H$$

$$\parallel$$

$$\Sigma^\infty (G/H)^K_+$$

[geom. fixed points...]

Associative graded categories:

$$(Sp^G)_H = Sp^{hW(H)} = \mathrm{Fun}(BW(H), Sp)$$

$$W(H) = {}^{W(H)}/_{H} = \mathrm{Aut}_{O(G)}(G/H)$$

the gluing functors are given by proper Tate cohomology.

<u>Proper Tate cohomology</u> G compact Lie group.

$$(\,)^{\tau G} : Sp^{hG} \to Sp$$

1) $$Sp^{hG} \to Sp^G \xrightarrow{\Phi^G} Sp$$

"Borel complete G-spectra"

Geom fixed pts

$$\begin{tikzcd} Sp^{hG} \arrow[r] & Sp^{hG} / \text{induced from proper closed subgroups} \arrow[d, "\mathrm{Hom}(\mathbb{S}^0, -)"] \\ & Sp \end{tikzcd}$$

<u>Examples</u> 1) $G = C_p$ $\quad P_G : * \, ; \, C_p$

two objects

$$* \rightsquigarrow Sp^{hG} = Sp^{hC_p}$$

$$C_p \rightsquigarrow Sp$$

<u>Upshot</u>: $Sp^{C_p} \simeq \begin{cases} X_0 \in Sp^{hC_p} \\ X_1 \in Sp \\ X_1 \to (X_0)^{\tau C_p} = (X_0)^{tC_p} \end{cases}$

(Greenlees - May)

$G = \mathbb{T}$

$Sp^{\mathbb{T}}$ gen. by $\Sigma_{\mathbb{T}} (\mathbb{T}/C_n)_+$

$n \geq 0$

stratified by the poset

$\mathbb{N}_{div}$ $(d \leq n$ iff $d \mid n)$

$$(n \in \mathbb{N}_{div}) \rightsquigarrow Sp^{h(\mathbb{T}/C_n)} \simeq Sp^{h\mathbb{T}}$$

gluing functors:

$$Sp^{h(\mathbb{T}/C_m)} \longrightarrow Sp^{h(\mathbb{T}/C_n)}$$

for $m \mid n$

$$X \longmapsto X^{\tau C_{n/m}}$$

D. Ayala Geometry of cyclotomic trace

$$
\begin{tikzcd}[column sep=huge, row sep=large]
K(\mathcal{C}) \arrow[rr, "\text{cyclotomic trace}", "\text{Dundas-Goodw.-McCarthy: local equiv}"'] \arrow[rrd, "\text{cyclic trace}", "\text{Goodwillie: } \mathbb{Q}\text{-equiv. in rel case}"'] \arrow[rrdd, bend right, "\text{Dennis trace}"'] & & TC(\mathcal{C}) \arrow[d, "\mathbb{Q}\text{-equiv}"] \\
& & THH^{-}(\mathcal{C}) \arrow[d] \\
& & THH(\mathcal{C}) \\
\Sigma^{\infty}_{+} \mathrm{Obj}(\mathcal{C}) \arrow[uuu] \arrow[rru, "tr"'] & &
\end{tikzcd}
$$

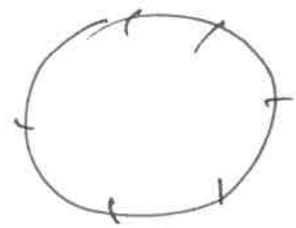
$$
\begin{tikzcd}
(-)^{hCyc} \simeq \Gamma^{hN^{\times}} & Cyc(Sp) \arrow[r, "\simeq", "Thm"'] \arrow[d] \arrow[ddl] & QC(\mathcal{B}_{<}\mathbb{T})^{hN^{\times}} \arrow[d] \arrow[r, phantom, "\ni"] & THH \arrow[d, mapsto] \\
& Sp^{ge\,\mathbb{T}} \arrow[r, "\simeq"] \arrow[d] & QC(\mathcal{B}_{<}\mathbb{T}) \arrow[d] \arrow[r, phantom, "\ni"] & THH \arrow[d, mapsto] \\
Sp & Sp^{h\mathbb{T}} \arrow[l] \arrow[r, "\simeq"] \arrow[d] & QC(B\mathbb{T}) \arrow[d] \arrow[r, phantom, "\ni"] & THH \arrow[d, mapsto] \\
& Sp \arrow[ul] \arrow[r, "\simeq"] & \cancel{Sp}\; QC(*) \arrow[r, phantom, "\ni"] & THH(\mathcal{C})
\end{tikzcd}
$$

Ex $\mathcal{C} = BA$, A ring (Spec)

Mod_A, Perf_A, $QC(X)$, $\mathrm{Perf}(X)$

Toward $THH(\mathcal{C})$ Let M be a 1-mfld

$\mathcal{D}(M)$ = a category of disk strats of M



$M = S^1$?

- For each $R \in \mathcal{D}(M)$,
  $\mathbb{C}(R)$ = free cat of the directed graph of R

- $\mathrm{Map}(\mathbb{C}(R), \mathcal{C}) \simeq$ Moduli spectrum of $\mathcal{C}$-reps of the graph

$$= \underset{(c_r) \in \underbrace{\mathrm{Obj}(\mathcal{C})^{R^0}}_{\text{moduli space of objects...}}}{\mathrm{colim}} \ \bigotimes_{e \in R} \hom_{\mathcal{C}}(s_e, t_e)$$

$$\int_M \mathcal{C} := \operatorname{colim}\left(\mathcal{D}(M) \to Sp\right)$$
$$R \mapsto \mathrm{Map}(\mathbb{D}(R), \mathcal{C})$$

$\approx$ Moduli spectrum of $\mathcal{C}$-labeled disc stratifications of M

Eg $\displaystyle\int_{S^1} BA \simeq THH(A)$

- Morita invariance, in particular,

$$\int_{S^1} \mathrm{Perf}(A) \simeq THH(A)$$

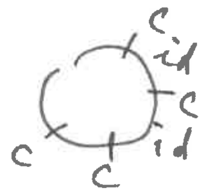
- $\displaystyle\int_{S^1} Sp^{fin} \simeq \mathbb{S}$

- $\displaystyle\int_M \mathrm{Perf}(X) \simeq \mathcal{O}(X^M)$ where X a derived scheme

Additivity: $\displaystyle\int_M$: Verdier Quotient sequence $\rightsquigarrow$ Short exact sequence

Notation $\int_{S^1} \mathcal{C} := THH(\mathcal{C})$ (agrees w/ Bloomberg-Mandell)

Obs 0.1 $\Sigma^{\infty}_{+} \mathrm{Obj}(\mathcal{C}) \longrightarrow THH(\mathcal{C})$

$c \longmapsto$  constant $\mathcal{C}$-labeled, for any disk strat

Eg $\mathrm{Obj}(\mathrm{Perf}(X)) \longrightarrow \mathcal{O}(X^{S^1})$

$$\begin{array}{c} E \\ \downarrow \\ X \end{array} \rightsquigarrow \left(S^1 \xrightarrow{\gamma} X\right) \mapsto$$

$$\mapsto \mathrm{Tr}\,(\underbrace{T_{\gamma^* E}}_{\substack{\text{trace of monodromy} \\ \text{of } \gamma^* E \\ \downarrow \\ S^1}}))$$

Cor (Bloomberg-Gepner-Tabuada)

$$K(\mathcal{C}) \to THH(\mathcal{C})$$

Obs. 1.1 For each $t \in S^1$,

$$\begin{tikzcd}
\gamma^*E|_0 \arrow[rr, "T_{\gamma^*E}", "SR"'] \arrow[dr, "R"'] & & \gamma^*E|_{2\pi} \arrow[dr] & \\
& \gamma^*E|_t \arrow[ur, "S"] \arrow[rr, "RS"'] & & \gamma^*E|_{t+2\pi}
\end{tikzcd}$$

$\mathrm{Tr}\left(T_{\gamma^*E} = SR\right) = \mathrm{Tr}(RS)$

what does that mean?

?

Cobordism hypothesis...

Obs. 1.2 $\mathbb{T}^\vee \curvearrowright S^1 \rightsquigarrow \mathbb{T}^\vee \curvearrowright \int_{S^1} \mathcal{C}$ homotopy

rotate

$\int_{S^1} \mathcal{C} = THH(\mathcal{C}) \in Sp_{12}^{h\mathbb{T}^\vee} = QC(B\mathbb{T}^\vee)$

Cor 1 $K(\mathcal{C}) \xrightarrow{\text{trace}} THH(\mathcal{C})^{h\mathbb{T}}$

Obs 2.1 For each fin sheeted cover, say

$S^1 \xrightarrow{\pi} S^1/C_r \overset{\gamma}{\rightarrow} X$

$$\mathrm{Tr}\left(T_{(\gamma\circ\pi_r)^*E}\right) = \mathrm{Tr}\left(T_{\gamma^*E}\right)$$

General Q | For $T \in \mathrm{End}_{\mathbb{k}}(V)$ ← perf $\mathbb{k}$-mod

what is the relationship between $\mathrm{Tr}(T)^r$ and $\mathrm{Tr}(T^r)$?

Idea: Postpone multiplying to observe symmetries.

Eg: $T = \begin{bmatrix} a_1 & & 0 \\ & \ddots & \\ 0 & & a_d \end{bmatrix}$ $r = 2$

$$\mathrm{Tr}(T)^{\otimes 2} - \mathrm{Tr}(T^{\otimes 2}) \quad \text{in } \mathbb{k}^{\otimes 2}$$

$$= \sum a_i \otimes a_j - \sum a_k \otimes a_k =$$

$$= \sum_{i \neq j} (a_i \otimes a_j + a_j \otimes a_i)$$

so it is in the image of the Norm map

$$\mathbb{k}^{\otimes 2} \xrightarrow{\text{Norm}} (\mathbb{k}^{\otimes 2})^{hC_2}$$

$$\left[\mathrm{Tr}(T)^{\otimes 2}\right] = \left[\mathrm{Tr}(T^{\otimes 2})\right] \in \Bbbk \simeq \left[\Bbbk^{\otimes 2}\right]^{tC_2}$$

$$[\ ]^{tC_2} = [\ ]^{hC_2} / \text{Norms}$$

$\underline{Ex}$ $T = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$ r bitrary:

$$\mathrm{Tr}(T)^{\otimes r} - \mathrm{Tr}(T^{\otimes r}) = (a_1 + a_2)^{\otimes r} - a_1^{\otimes r} - a_2^{\otimes r}$$

$$\in (\Bbbk^{\otimes r})^{\tau C_r}.$$

(when r = prime, $(\ )^{\tau C_r}$ = as above

if not, ...

$$\mathbb{T} \overset{ge}{\curvearrowright} THH(\mathcal{C}): \quad \mathbb{P}_{\leq \mathbb{T}} = \mathbb{N}^{\mathrm{div}}$$

$$\mathbb{T}/C_r \overset{\text{homotopy}}{\curvearrowright} THH(\mathcal{C})_{C_r} \simeq$$

$$\int_{S^1/C_r} \mathcal{C} = \left[\begin{array}{l} \simeq \text{Moduli spectrum of } \mathcal{C}\text{-labeled } C_r\text{-invariant disk stratifications of } S^1 \\ \simeq \text{Mod spec of } \mathcal{C}\text{-labeled disk-strats of } S^1/C_r \end{array}\right.$$

$$\mathrm{THH}(\mathcal{C})_{C_s} = \int_{S^1/C_s} \mathcal{C}$$

$$\Big\downarrow ?$$

$$\left(\int_{S^1/\mathcal{C}_r} \mathcal{C} \right)^{\tau C_{s/r}}$$

Obs 3.2 $\mathbb{N}^\times \circlearrowright S^1$ covers $\rightsquigarrow \mathbb{N}^\times \circlearrowright \mathrm{Sp}^{g_e \mathbb{T}} \ni V = \mathrm{THH}(\mathcal{C})$

via $(r \cdot V)_{C_k} := V_{C_{rk}}$

<u>In our case:</u>

$$\int_{S^1/C_{rk}} \mathcal{C} \ \underset{\text{reparam}}{\simeq} \ \int_{S^1/C_k} \mathcal{C} \rightsquigarrow \mathrm{THH}(\mathcal{C}) \in \left(\mathrm{Sp}^{g_e \mathbb{T}}\right)^{h\mathbb{N}^\times}$$

<u>Thm</u> (Ayala-Mazel-Gee-Rozenblyum)

$$\Big\downarrow$$

<u>Thm</u> (cont.)

$$\begin{tikzcd}
\mathrm{Cyc}(Sp) \arrow[r, "\sim"] \arrow[d, "(-)^{hCyc}"'] & (Sp^{g_e\mathbb{T}})^{h\mathbb{N}^\times} \simeq QC(B_\cdot\mathbb{T}) \arrow[dl] \arrow[d] \\
Sp & \mathrm{Fun}(sd(B\mathbb{N}^\times), Sp) \arrow[l]
\end{tikzcd}$$

<u>In general:</u>

$$V \in (Sp^{g_e\mathbb{T}})^{h\mathbb{N}^\times}$$

$$\begin{tikzcd}
& (V^{hC_r})^{h\mathbb{T}/C_r} \arrow[dr, "h \to \tau"] & \\
V^{h\mathbb{T}} \arrow[ur, "\sim"] \arrow[d, "\wr"', "\mathbb{N}^+_{1/2}"] & & (V^{\tau C_r})^{h\mathbb{T}/C_r} \\
(V_{C_r})^{h\mathbb{T}/C_r} \arrow[urr, "g_e \cdots ?"'] & &
\end{tikzcd}$$

D. Kaledin
(w. A. Fonarev)

Bökstedt vs Bott periodicity

A assoc. / k
M A-bimod

$THH(A, M)$ Eilenberg-MacLane spectrum

$k = \mathbb{F}_p$ $THH(k)$ is a Hopf algebra

Miracle: $THH_\bullet(\mathbb{F}_p) \simeq k[\sigma]$, $\deg \sigma = 2$

This implies lots of things. e.g.

$$\varepsilon \in THH_2(\mathbb{F}_p)^*$$

$$\langle \sigma^p, \varepsilon^p \rangle = 0$$

$$\Downarrow$$

$$\varepsilon^p = 0$$

A - dual Steenrod algebra

$$A = \mathbb{F}_p[\beta, P_i, Q_i]$$

$\deg \beta = 1$

$\deg P_i = 2(p^i - 1)$

$\deg Q_i = 2p^i - 1$

Fact $\left[\begin{array}{l} A \text{ is Koszul dual to} \\ THH_\bullet(\mathbb{F}_p)^* \end{array} \right.$

Can easily see standard properties of Steenrod alg.

Conversely:

$$\mathbb{F}_p[\sigma, A_i, B_i] \Rightarrow THH(\mathbb{F}_p)$$

$\deg \sigma = 2$ $\quad\deg A_i = 2p^i - 1$

$\deg B_i = 2p^i$

<u>Observation</u> Assume we know that

$\sigma^n \neq 0 \quad \forall n \quad$ in $THH(\mathbb{F}_p)$

Then A_i, B_i must die in the spec sequ.

$\sigma^p \neq 0 \Rightarrow \varepsilon^p = 0$ so we must have

$$\partial A_1 = \varepsilon^p$$

To see the spectral sequence:

$$THH(A, M) \underset{\text{Thm}}{=} HM_\bullet(A, M) \qquad \text{MacLane homology}$$

$$\| $$

$$HH_\bullet(\underbrace{Q_\bullet(A)}, M)$$

When $A = k$ then this is the dual Steenrod alg.

Idea: find a map

$$\mathrm{THH}_\bullet(\mathbb{F}_p) \longrightarrow ?$$
$$\sigma \longmapsto \sigma'$$

and $(\sigma')^n \neq 0$ in ?

Stabilization $\mathcal{E}$ additive, Karoubi-closed category.

$$F : \mathcal{E} \longrightarrow \Delta^o\,\mathrm{Sets}$$

$M \in \mathcal{E}$ **Def** $\mathrm{Stab}(F)(M) := \varinjlim_{n \to \infty} \Omega^n \left| F^{\Delta} DK(M[n]) \right|$

where $DK : \mathcal{C}_{\geq 0}(\mathcal{E}) \rightleftarrows \Delta^{op}\mathcal{E}$

Example $\mathcal{E} = k\text{-vect}$

$F(M) = M^{\otimes p}$ $\quad \mathrm{Stab}(F) = 0$ (?)

$|\cdot| =$ geom realization

$$F(M) = C^p(M) = (M^{\otimes p})^{\sigma}$$

σ the longest cyclic perm

$$\mathrm{Stab}(F)(M) = R.(M)$$

Then $R.(M) = \tau_{\leq 0}\, \check{C}^{\bullet}(\mathbb{Z}/p\mathbb{Z}, M^{\otimes p})$

[C^p = cyclic power, analogous to S^p etc.]

$$\overbrace{P_\bullet \xrightarrow{\sim} k \to 0}^{\widetilde{P}_\bullet}$$

$\uparrow$ proj $k[\mathbb{Z}/p]$ - resolution

Variant: $F(M) = (M^{\otimes p})^{\Sigma_p}$

$$\mathrm{Stab}(F)(M) \simeq \tau_{\leq 0}\, \check{C}^{\bullet}(\Sigma_p, M^{\otimes p})$$

Example 2 $\mathcal{C}$ small category

$$\mathcal{M} : \mathcal{C}^{\circ} \times \mathcal{C} \to \mathrm{Sets}$$

$N^{cy}(\mathcal{C}, \mathcal{M})$ cyclic nerve

$\mathrm{Stab}(N^{cy}(M)) = C_\bullet(N^{cy}(\mathcal{C}, M))$

A an algebra$/k$

$P(A) = \{\text{fin gen proj } A\text{-mods}\}$

Then $\boxed{THH_\bullet(A, M) \simeq \mathrm{Stab}(N(P(A)), M)}$

(Pirashvili et al)

Kind of NC version of the cyclic power σ?

$C^p(M)$:

$HH_0(A^{\otimes p}, M_\sigma^{\otimes p})^\sigma$ $\quad\left(= HH_0(A, M^{\otimes_A p})\right.$ or smth like.

where $M_\sigma^{\otimes p}$ is $A^{\otimes p}$ bimod, twisted on right by σ.

We have an obvious map

$\varphi : N(P(A), M) \to C^p(A, M)$

$$\longmapsto \quad \mathrm{Tr}(m^{\otimes p})$$

<u>Claim</u> $A = M = \mathbb{F}_p$ then

$$\mathrm{Stab}(\varphi)(\sigma) \neq 0 \in \check{H}^{\leq 0}(\mathbb{Z}/p, k)$$

More canonically, φ is:

$$\begin{tikzcd} N(P(A), M) \arrow[r, "\otimes^p"] & N(P(A^{\otimes p}), M_\sigma^{\otimes p}) \arrow[d, "\mathrm{Tr}"] \\ & HH_0(A^{\otimes p}, M_\sigma^{\otimes p})^{\otimes p} \end{tikzcd}$$

$$P(A) \longrightarrow P(A^{\otimes p})$$

$$P \longmapsto P^{\otimes p}$$

<u>Now</u> $A = M = k$;

take $V \in P(k)$ $\dim V \geq 2$.

$$\begin{tikzcd} N(BGL(V), \mathrm{End}(V) \otimes M) \arrow[d] \\ N(BGL(V), \mathrm{End}(V) \otimes M)^{\Sigma_p} \arrow[r] & \cdots \end{tikzcd}$$

<u>need</u> arbitrary M in order to do stabilization.

Stabilize:

$$H_\bullet\big(GL(V),\ M\otimes \mathrm{End}(V)\big)$$

$$\sigma\in H_2 \quad \underline{\text{and}} \quad H_1=0.$$
$$(\text{for } M=k)$$

$$\begin{tikzcd}
\overset{C_\bullet}{\cancel{H_\bullet}}\big(GL(V), M\otimes \mathrm{End}(V)\big) \arrow[r] & \overset{C_\bullet}{\cancel{H_\bullet}}\big(GL(V), R^{\Sigma}_\bullet(\mathrm{End}(V)\otimes M)\big) \arrow[d, bend right, "?"'] \arrow[dr] & \\
& C_\bullet\big(GL(V)\times\Sigma_p, (\mathrm{End}(V)\otimes M)^{\otimes p}\big) & \overset{C_\bullet}{\cancel{H_\bullet}}\big(GL(V), R_\bullet(\mathrm{End}(V))\otimes M\big) \arrow[d, "\mathrm{Tr}"] \\
& & R_\bullet(M) \arrow[d] \\
& & C_\bullet \cancel{H_\bullet}(\mathbb{Z}/p, M^{\otimes p}) \\
& & (?)
\end{tikzcd}$$

Some standard fact in invariant theory says something about

$$\mathcal{E}xt^{\bullet}_{GL(V)}(V,V)\ \ldots$$

$$0\to V\to C_p(V)\to C^p(V)\to V\to 0$$

$$\varepsilon\in H^2\big(GL(V),\ \mathrm{End}(V)\otimes M\big)\ \ldots$$

A. Mathew

K(1)-local K-theory

w/ B. Bhatt and D. Clausen

ring $R \rightsquigarrow K(R)$

No étale descend

Thomason: $K^\wedge(R)$ = p-adically complete K-thy

$(\frac{1}{p} \in R)$, R comm

After inverting Bott element, étale descent works.

Ex $R = \mathbb{C}$ $\quad K^\wedge(\mathbb{C}) \simeq KU_p^\wedge$ (Suslin)

$K^\wedge(\mathbb{C})_* = \mathbb{Z}[\beta] \quad |\beta| = 2$

Bott elt

Ex R a $\mathbb{Z}[\frac{1}{p}, \zeta_p]$-alg $[\zeta_p] \in K_1$ p-torsion

$\beta \in \pi_2(K(R)/p)$

β Bokshteins to $[\zeta_p]$

Thomason considers the opn

$$\left(K(R)^\wedge [1/\beta] \right)^\wedge$$

for $\mathbb{C}$-alg R over $\mathbb{Z}[1/p, \zeta_p]$

$$\longrightarrow \rightsquigarrow \left(K(R)/p \right)[\beta^{-1}]$$

More general construction:

$$K(1) = \text{mod } p \quad KU/p$$

Defines a functor

$$L_{K(1)} : Sp \to Sp$$

Bousfield localization wrt $K(1)$

<u>Thm</u> (Thomason) Let R be a $\mathbb{Z}[1/p]$-algebra with finiteness cond. The construction $L_{K(1)}K(-)$ satisfies étale descent and there is a spectral sequence

$$H^s_{\text{ét}}(\operatorname{Spec} R, \mathbb{Z}_p(t)) \Rightarrow \pi_{2t-s}\left(L_{K(1)}K(R)\right)$$

$s, t \in \mathbb{Z}$

Ex $R/\mathbb{C}$ fin gen $L_{K(1)}K(R) \simeq \widehat{KU}(\operatorname{Spec} R(\mathbb{C}))$

(Bloch-Kato)

$$K^\wedge(R) \to L_{K(1)} K(R)$$

equivalence in large degrees.

<u>Q</u> What if p is <u>not</u> inverted?

<u>Ex</u> $R = \mathbb{F}_p$ Quillen computes $K_*(\mathbb{F}_p)$

$$K^\wedge(\mathbb{F}_p) \simeq \mathbb{Z}_p \, ; \, L_{K(1)} K(\mathbb{F}_p) = 0.$$

More generally, if R is an $\mathbb{F}_p$-algebra, being an $L_{K(1)} K(\mathbb{F}_p)$-module,

$$L_{K(1)} K(R) = 0.$$

<u>Ex</u> $R/\mathbb{Z}$ smooth.

$$K(R/p) \to K(R) \to K(R[1/p])$$

fiber seq

$$\Rightarrow \quad L_{K(1)} K(R) \simeq L_{K(1)} K(R[1/p])$$

and that is known (Thomason)

<u>Ex</u> $R = \mathbb{Z}/p^2$ Don't know $L_{K(1)} K(\mathbb{Z}/p^2)$

Theorem (Bhatt, Clausen, M)

R any ring,

$$L_{K(1)} K(R) \simeq L_{K(1)} K(R[1/p])$$

Ex $L_{K(1)} K(\mathbb{Z}/p^n) = 0.$

everything reduces to this.

First observation:

Fact: $K(\mathbb{Z}/p^2)^\wedge_* \longrightarrow TC(\mathbb{Z}/p^2)^\wedge_*$ is an iso for $* \geq 0$

where $K(R) \to TC(R)$ is the cyclotomic tr.

Reason is D-G-M thm

$$K(\mathbb{F}_p)^\wedge_* \simeq TC(\mathbb{F}_p)^\wedge_* \quad * \geq 0$$

Why is $TC(\mathbb{Z}/p^2)$ hard?

For $\mathbb{F}_p$, easier: $THH(\mathbb{F}_p)_* = \mathbb{F}_p[\sigma]$, $|\sigma| = 2$

Fail for $\mathbb{Z}/p^2, \mathbb{Z}, \ldots$

<u>Step 1</u> Replace $\mathbb{Z}/p^n$ by a ring where evenness (of THH) returns.

<u>Thm</u> (Bhatt-Morrow-Scholze, Hesselholt)

$$R = \mathbb{Z}_p[\zeta_{p^\infty}] \to THH_*(R)^\wedge \simeq$$
$$\simeq R[\sigma] \quad |\sigma| = 2$$

The strategy is to replace $\mathbb{Z}/p^n$ by $\mathbb{Z}/p^n[\zeta_{p^\infty}]$ (still THH is even).

$$L_{K(1)} K(\mathbb{Z}[\zeta_{p^\infty}]/p^n) = 0 \quad (*)$$

<u>Thm</u> (Clausen, M, Naumann, Noel)

The construction $L_{K(1)}K(-)$ satisfies finite flat descent.

Adding p^{th} roots of unity is a finite flat cover.

Need to show $(*)$.

Firstly:

Firstly,

$$K^\wedge(\mathbb{Z}[\zeta_{p^\infty}]/p^n) \simeq TC^\wedge(\mathbb{Z}[\zeta_{p^\infty}]/p^n)$$

(deg ≥ 0)

<u>Step 2</u> Thm (Nikolaus - Scholze)

R a ring $\quad \underline{TC^\wedge(R)} \simeq eq\left(TC^-(R) \underset{\varphi}{\overset{can^\wedge}{\rightrightarrows}} TP(R)\right)$

$THH(R) \circlearrowleft S^1$

$$\begin{array}{ccc} THH(R)^{hS^1} & \longrightarrow & THH^{tS^1} \\ \parallel & & \parallel \\ TC^-(R) & & TP(R) \end{array}$$

Here can is canonical; φ uses the cyclotomic structure.

<u>Cor</u> $K(1)$-locally,

$L_{K(1)}TP(R)$ has an endomorphism φ;

$$L_{K(1)}TC = \left(L_{K(1)}TP(R)\right)^{\varphi=1}$$

<u>Upshot</u>: ETS (enough to show)

$$L_{K(1)}TP\left(\mathbb{Z}[\zeta_{p^\infty}]/p^n\right) = 0$$

This is where Bhatt - Morrow - Scholze come in...

$\underline{Ex}$ (of construction $TP(R) = THH(R)^{tS^1}$

$$TP(\mathbb{F}_p)_* \simeq \mathbb{Z}_p[x^{\pm 1}]$$

$\underline{\text{Step 3}}$ Thm (Bhatt - M - Sch):

$\underline{\text{Theorem}}$ $R = \mathbb{Z}_p[\zeta_{p^\infty}]$

$$TP(R)^\wedge = \mathbb{Z}_p[q^{1/p^\infty}]^\wedge_{(p, q-1)}[x^{\pm 1}]$$

$$|x| = 2$$

In general there is a thm for any perfectoid ring R ...

$$TP_0 = A_{inf}(R)$$

$\underline{Ex}$ $A_{inf}(\mathbb{F}_p) = \mathbb{Z}_p$

$$A_{inf}(\mathbb{Z}_p[\zeta_{p^\infty}]) =$$

$\beta = q - 1$ (x^{-1})

(so K-thy can be calculated)

Need to use an analog of this for $\mathbb{Z}_p[\zeta_{p^\infty}]/p^n$

Need to recall another notion.

<u>Def</u> (Joyal) A δ-ring is a comm rg R with an endomorphism

$$\psi^p : R \to R \quad \text{s.t. } \psi^p(x) = x^p \text{ mod } p.$$

this is the <u>defn</u> for R p-torsion free.

<u>In general</u>:

$$\psi^p(x) = x^p + p\delta(x)$$

and you impose identity on δ

$$\delta(x+y) = \delta(x) + \delta(y) - \sum_{i=1}^{p-1} \frac{1}{p}\binom{p}{i} x^i y^{p-i}$$

δ-rings show up as π_0 of $K(1)$-local E_∞ rings. $(p>2)$.

<u>Ex</u> R any ring:

$$THH(R)^{hS^1} \longrightarrow THH(R)^{tS^1}$$

in deg 0:

$$\varphi : TP_0(R) \to TP_0(R)$$

<u>Q</u> Is this part of a δ-structure?

<u>Thm</u> (BMS, Bhatt-Scholze):

$$R = \mathbb{Z}_p[\zeta_{p^\infty}]/(p^n)$$

$TP(R)_0 =$ free δ-ring on

$$\mathbb{Z}_p[q^{1/p^\infty}] \text{ w/ } v,$$

an element s.t. $$v\left(\frac{q^p-1}{q-1}\right) = p^n$$

& $(p, q-1)$-adically complete.

Equivalent to a calculation of

$$K(\mathbb{Z}/p^n)_* \simeq TC(\mathbb{Z}/p^n)_*$$

<u>Cor</u> (Clausen) R any p-complete ring

$$L_{K(1)} K(R[1/p]) \simeq L_{K(1)} TC(R)$$

A. Mazel-Gee

The secondary cyclotomic trace

(w/ Reuben Stern)

Def (Grothendieck) X scheme / $\mathbb{Z}$

$$K_0(X) = \frac{\mathbb{Z}\{\text{v. bdles}\}}{[B] = [A] + [C]}$$

$\forall$ short ex. sequ $A \to B \to C$

Quillen, Waldhausen

$K(\mathcal{C}) \in Sp$ $\qquad \pi_i K(\mathcal{C}) = K_i(\mathcal{C})$

e.g. $\mathcal{C} = \mathrm{Perf}_X$

<u>Dfn</u> A localization sequence in Stable Categs

$$\begin{tikzcd} \mathcal{A} \arrow[r, hook, "i"] & \mathcal{B} \arrow[l, dashed, bend left, "R"] \arrow[r, "L"] & \mathcal{C} \arrow[l, dashed, bend left, "j"] \end{tikzcd}$$

i fully faithful

j too

<u>exact seq</u>: $iR \xrightarrow{\varepsilon} id_{\mathcal{B}} \xrightarrow{\eta} Lj$

in $\mathrm{Fun}^{ex}(\mathcal{B}, \mathcal{B})$

Cons. $\forall b \in \mathcal{B}$ ex. seq $i(Rb) \to b \to L(jb)$

$\rightsquigarrow$ exact seq $K(A) \to K(B) \to K(C)$ of spectra

(Tabuada's thesis...)

<u>Idea</u>: $K : 1,\mathrm{St} \to \mathrm{Sp}$ <u>universal</u> w/ this property.

<u>Dfn</u> An additive invariant valued in $T \in \mathrm{St}$

$$E : 1,\mathrm{St} \to T$$

$$[\text{loc seqs}] \mapsto [\text{ex seqs}]$$

<u>Ex</u> $K, \mathrm{THH}, \mathrm{TC} \in \mathrm{Add}(\mathrm{Sp})$

<u>Universal ex.</u>

$$\begin{tikzcd} U : 1,\mathrm{St} \arrow[r, "\text{Yoneda}"] & \mathrm{Fun}(1,\mathrm{St}^{op}, \mathcal{S}) \arrow[d, "\Sigma^\infty_+ -"] \\ \mathrm{Mot} & \mathrm{Fun}(1,\mathrm{St}^{op}, \mathrm{Sp}) \arrow[l] \end{tikzcd}$$

additive NC motives

$$U^* : \mathrm{Fun}^{ex}(\mathrm{Mot}, T) \xrightarrow{\sim} \mathrm{Add}(T)$$

Amazing theorem (Blumberg-Gepner-Tabuada):

$$\hom_{\mathrm{Mot}}(\mathcal{U}(\mathcal{C}), \mathcal{U}(\mathcal{D})) \simeq K(\mathrm{Fun}^{ex}(\mathcal{C}, \mathcal{D}))$$

In particular,

$$\hom_{\mathrm{Mot}}(\mathcal{U}(Sp^{\omega}), \mathcal{U}(-)) \simeq K(-)$$

Cor $\exists$ canonical map $\Sigma^{\infty}_{+} \not{z}_{\cancel{iso}} \longrightarrow K$

Rmk Variant for Verdier $A \to B \to C$

e.g. $QC(X) \to QC(X)^{\omega} \longrightarrow$

$Z \hookrightarrow X \hookleftarrow U$ ...

replaces K-theory w/ nonconnective K-theory.

§2. $K^{(2)}$ [Toën-Vezzosi]

Chromatic height	typical thy	~~coh~~	Cocycles are families of	Category number
0	$H\mathbb{Q}^*$	nmbrs		0
1	KU^*	vect. sp.		1 obs of Vect
		(stable)	categories	2 obs of ~~St~~ 2-cat

<u>Also</u>: Azumaya algebras and Brauer groups: $[A] \in K^{(2)}$

$$\mathrm{Br} \to K_0^{(2)}$$

☆ Motivic measure $K(\mathrm{Var}_k) \to K^{(2)}(k)$

$$Z \subset X \hookleftarrow U$$

$$[X] = [Z] + [U]$$

"scissor relations"

the map is nontrivial (Bondal-Larsen-Lunts)

(meaning: they made a nontrivial statement about this map).

<u>Also</u>: point count / $\mathbb{F}_q$ ...

$$K(\mathrm{Var}_k) \to \mathbb{Z}$$

☆ home of invariants of shvs of stable categories...

<u>Defn</u> (M-G-S) a 2-cat $\mathcal{X}$ is <u>stable</u> if:

① enriched / $(\mathrm{St}, \otimes)$

② finite limits

$\rightsquigarrow \mathrm{St}_2$ a self-enriched 3-cat., hom's are "2-exact functors"

$\underline{\mathcal{E}x}$ X a scheme: $QC_{Cat}(X) :=$

$$:= \lim_{\mathrm{Spec}(R) \to X} \underbrace{\mathrm{St}_R}_{\text{Stable } R\text{-linear categories}}$$

$$\mathrm{St} \simeq QC_{cat}(\mathrm{Spec}\,\mathbb{S})$$

R an E_2-ring $\rightsquigarrow \mathrm{St}_R$

Source of examples of objs of $QC_{cats}(X)$

$$\boxed{\mathrm{Sch}/X \longrightarrow QC_{cat}(X):}$$

$$\begin{array}{c} Y \\ f \downarrow \\ X \end{array} \longmapsto \left(U \longmapsto QC(f^*U) \right)$$

Dfn A 1-localization seqn in

$\mathcal{X} \in St_2$ is

$A \hookrightarrow B \to C$ s.t. [same in St]

Construction

$\mathrm{Fun}(i, \mathcal{X}, Sp) \rightleftarrows \mathrm{Mot}(\mathcal{X}) \in St$

Dfn The secondary K-thy of $\mathcal{X} \in St_2$

$$K^{(2)}(\mathcal{X}) := K(\mathrm{Mot}(\mathcal{X})^{\omega})$$

$$K^{(2)} : i_1 St_2 \to Sp$$

Defn A 2-localization sequence in St_2 is

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook, "i"] & \mathcal{Y} \arrow[l, dashed, bend left, "R"] \arrow[r, "L"] & \mathcal{Z} \arrow[l, dashed, bend left, "j"] \end{tikzcd}$$

s.t. $iR \to id_{\mathcal{Y}} \to jL$ a 1-loc seq in $\mathrm{Fun}^{2\text{-ex}}(\mathcal{Y}, \mathcal{Y})$

Defn A 2-additive invariant valued in $\mathcal{T} \in St$ is $\mathcal{E} : i_1 St_2 \to \mathcal{T}$

[2-loc seqs] $\to$ [ex seqs]

Univ 2-loc syst

$$U_2 : {}_1St_2 \to Mot_2 \quad \leftarrow \text{2-additive NC 2-motives}$$

$U_2 \in Add_2(Mot_2)$:

$$U_2^* : Fun^{ex}(Mot_2, T) \xrightarrow{\sim} Add_2(T)$$

<u>Main thm</u>

$$\hom_{Mot_2}(U_2(Y), U_2(Z)) \simeq K^{(2)}(Fun^{2\text{-}ex}(Y, Z))$$

<u>Cor</u> $\Sigma^\infty_+ \iota_0 \to K^{(2)}$ initial map to a 2-add invt in Sp.

§3. Secondary cyclotomic trace

$$\begin{tikzcd}
Sp & \\
St \arrow[u] \arrow[r] \arrow[ur, phantom, "\Nearrow"] & \underbrace{Cat(Sp)}_{\text{enriched in } Sp} \arrow[ul, "THH = \int_{S^1}"'] \\
St_2 \arrow[u, "K^{(2,1)} = Mot(-)^\omega"] \arrow[r] \arrow[uu, bend left, "K^{(2)}"] & Cat(St) \simeq Cat_2(Sp) \arrow[u, shift left, "Cat(K)"] \arrow[u, shift right, "Cat(THH)"'] \arrow[uul, bend right, "\int_{\mathbb{T}^2}"']
\end{tikzcd}$$

D.A.G. picture:

$$K^{(2)}(X) \longrightarrow THH^{(1)}(X) = \int_{T^2} Q\mathrm{Cat}(X)^{\omega} \simeq \mathcal{O}(L^2 X)$$

$$tr^{(2)}\left((E_{ij} \circlearrowleft f_{ij}) \right)_{1 \le i,j \le d} = tr(tr(f_{ij}))$$

Closely related to 2-vector bundles of Bass-Dundas-Rognes.

<u>Q</u>: 2-cyclotomic trace?

<u>A</u>: 1d (w/ Ayala, Nick R.):

$$\begin{array}{c} S^1 \\ \downarrow \\ D^0 \end{array} + \text{symmetries} \rightsquigarrow \Sigma^{\infty}_{+} i_0 \mathcal{C}$$

$$\begin{array}{ccc} & & \Sigma^{\infty}_{+} i_0 \mathcal{C} \\ & & \| \\ \int_{D^0} \mathcal{C} & \rightarrow & \int_{S^1} \mathcal{C} = THH \\ \swarrow & & \uparrow \\ K(\mathcal{C}) & \xrightarrow[\text{[BGT]}]{} & \left[\int_{S^1} \mathcal{C}\right]^{hC_-} = TC \end{array}$$

A (2d):

$$\begin{array}{c} \mathbb{T}^2 \\ \downarrow \\ \mathbb{D}^0 \end{array} \text{ + symms } \rightsquigarrow \quad \mathcal{X} \in St_2$$

$$\begin{tikzcd} \Sigma^\infty_{+} \Omega_0 \mathcal{X} = \int_{\mathbb{D}^0} \mathcal{X} \arrow[r] \arrow[d] & \int_{\mathbb{T}^2} \mathcal{X} \\ K^{(2)}(\mathcal{X}) \arrow[r, "\text{Main Thm}"'] & \left(\int \mathcal{X}\right)^{h\,Cyc_2} \arrow[u] \end{tikzcd}$$

$$Cyc_2(Sp) \ni E := Aut^{fr}(T^2) \curvearrowright E$$

$$\forall\, T^2 \leftarrow T^2$$

$$E \longrightarrow E^{\tau G}$$

& fr cover of deck grp G

where

$$Aut^{fr}(T^2) = (\mathbb{T}^2 \cancel{\times} \mathbb{Z}) \rtimes \cancel{\mathbb{Z}}\, SL_2(\mathbb{Z})$$

(not compact Lie)

Ex. $K_0^{(2)}(\mathcal{X}) \simeq \dfrac{\mathbb{Z}[\pi_0(\iota\mathcal{X})]}{[\mathcal{B}] = [\mathcal{A}] + [\mathcal{C}]}$

Ben Antieau

The Beilinson t-structure and TC

$$HH(R/k) = \left|[n] \mapsto A^{\otimes_k n+1}\right|$$

I. History.

$$\begin{array}{ccc} \Omega^n_{R/k} & \longrightarrow & HH_n(R/k) \\ \wr\mid & & \\ \Lambda^n_R \Omega^1_{R/k} & & \\ a_0 da_1 \wedge \dots \wedge da_n & \longmapsto & \left[\sum (-1)^{|\sigma|} a_0 \otimes a_{\sigma 1} \otimes \dots \otimes a_{\sigma n}\right] \end{array}$$

Thm (Hochschild-Kostant-Rosenberg, 1961)

k comm reg

R sm comm k-alg

ε_n is an isom.

$$HC^-(R/k) = HH(R/k)^{hS^1}$$

$$HC(R/k) = HH(R/k)_{hS^1}$$

$$HP(R/k) = HH(R/k)^{tS^1}$$

Thm (Feigin-Tsygan) $\mathbb{Q} \subseteq k$:
Loday-Quillen; Connes

$$HC_n^-(R/k) \simeq Z^n(R/k) \times \prod_{i>0} H^{n+2i}(R/k)$$

$$HP_0(R/k) \simeq \prod_{i \in \mathbb{Z}} H^{2i}_{dR}(R/k)$$

Q (1) $\mathbb{Q} \not\subseteq k$
(2) what if R is not smooth? X scheme ...
(3) Functoriality

II HKR filtration
R/k smooth

$$F^\star_{HKR} HH(R/k) = \tau_{\geq \star} HH(R/k)$$

complete decreasing multiplicative filtrn

Filtered derived category:

$$DF(k) = \mathrm{Fun}(\mathbb{Z}^{op}, \mathcal{D}(k))$$

$$F^n_{HKR} \to F^{n-1}_{HKR}$$

df $X(\star)$ is complete if $\lim_n X(\star) = 0$.

Note: $H_n = H^{-n}$

$\tau_{\geq n} = \tau^{\leq -n}$

these are discrete simplicial algs which are $k[x_1, \dots, x_N]$

Now.

$$\begin{tikzcd}
\underbrace{(S)CAlg_k^{poly}}_{} \arrow[rr, "F^{\star}_{HKR} HH(-)"] \arrow[d, hook] & & DF(k)^{hS^1} \\
\underbrace{SCAlg_k}_{\text{simplicial comm rgs}} \arrow[urr, dotted, "F^{\star}_{HKR} HH(-/k)"'] & &
\end{tikzcd}$$

$$\cdots \; \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} \; \mathcal{R}_1 \; \underset{d_1}{\overset{d_0}{\rightrightarrows}} \; \mathcal{R}_0 \to S$$

resolution by polnml rings

Note $F^0_{HKR} HH(S/k) = HH(S/k)$

$gr^n_{HKR} HH(-/k)$ is the Kan ext of

$R \mapsto \Omega^n_{R/k} = \Lambda^n_R \Omega^1_{R/k}$

$$\mathrm{gr}^n_{HKR}\ HH(S/k) \\ \wr| \\ \mathbb{L}\Lambda^n L_{S/k}[n]$$

will write as $\Lambda^n L_{S/k}[n]$

<u>recently:</u> [Bhatt, BMS, Toën-Vezzosi...]

$$F^{\star}_{HKR}\, HC^-(R/k) \simeq \left(\Lambda^n L_{R/L}[n]\right)^{hS^1}$$

also: [Deligne...]

III Beilinson t-structure

$$DF(k)_{\geq 0} = \{X(\star) : \pi_i\, \mathrm{gr}^n X(\star) = 0,\ i < -n\}$$

$$DF(k)_{\leq 0} = \{X(\star) : \pi_i\, X(n) = 0,\ i > -n\}$$

<u>Thm</u> (Beilinson, BMS) This <u>is</u> a t-structure,

<u>and</u> $DF^{\heartsuit}(k) = Ch(k)$ as an Abelian category.

z

$\tau^B_{\geq n} X(\star)$ new filtered object.

$$\mathrm{gr}^i\left(\tau^B_{\geq n} X(\star)\right) = \tau_{\geq n-i}\left(\mathrm{gr}^i X(\star)\right)$$

$\pi_n^B X(\star)$ ~~iscards~~

$$gr^i \pi_n^B X(\star) = \pi_{n-i}(gr^i X(\star))$$

$\pi_0^B(X(\star))$ is :

$$\longrightarrow \pi_0 gr^0 X(\star) \longrightarrow \pi_{-1} gr^1 X(\star) \to \dots$$

· given by

$$F^1/F^2 \to F^0/F^2 \to F^0/F^1$$

V Beilinson filtration on HC^-, HP.

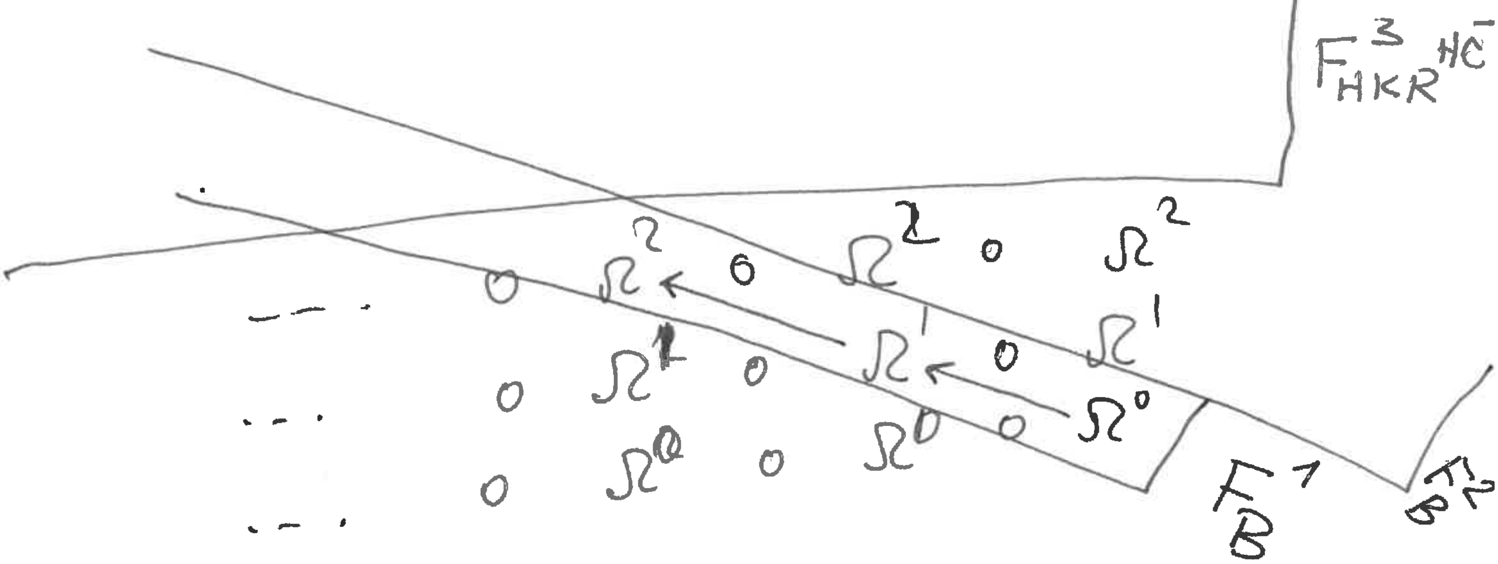
Say, R/k is smooth.

$$F_B^\star HC^-(R/k) = \tau^B_{\geq 2\star} F^\star_{HKR} HC^-(R/k)$$

Get a <u>chain complex</u>

$$gr_B^n HC^-(R/k)$$

$$E_2^{st} = H^s(BS^1, HH_t(R/k)) \Rightarrow HC^-_{t-s}(R/k)$$



with Hodge filtration.

Thm (BMS, A.)

Let $k \to R$ any commutative k-algebra.

Then there are filtrations $F_B^\star HC^-(R/k)$ and $F_B^\star HP(R/k)$

$$gr_B^n HC^-(R/k) \simeq \widehat{d\Omega}^{\geq n}[2n]$$

$$gr_B^n HP(R/k) \simeq \widehat{d\Omega}[2n]$$

V. TC. Thm (Hesselholt)

$$\pi_* TR(R) \simeq W\Omega_R$$

$TR(R) = \lim THH(R)^{fp\dots}$ where $R/\mathbb{F}_p$ smooth

Thm (A-Nikolaus) There is a t-structure on CycSp

$$\pi_i^{cyc} THH(R) \simeq \pi_i TR(R) \simeq W\Omega^i_R$$

$$\begin{array}{ccc} & & TR(R)^{tS^1} \\ & & \big\| \wr \\ TP(R) & = & THH(R)^{tS^1} \end{array}$$

$F_B^* TP(R)$ $\qquad gr_B^n TP(R) \simeq W\Omega_R^* [2n]$

$$gr^n_{BMS} TP(R) \simeq R\Gamma_{crys}(R/W)[2n]$$