Cyclic homology - a draft

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Contents

CHAPTER 1

Introduction

1. Motivation

Many geometric objects associated to a manifold M can be expressed in terms of an appropriate algebra A of functions on M (measurable, continuous, smooth, holomorphic, algebraic, ...). Very often those objects can be defined in a way that is applicable to any algebra A, commutative or not. Study of associative algebras by means of such constructions of geometric origin is the subject of noncommutative geometry [**115**]. If the invariants in question are of defferential geometric nature, the theory is called noncommutative differential geometry [**111**]. The Hochschild and cyclic (co)homology theory is the part of noncommutative differential geometry which generalizes the classical differential and integral calculus. The geometric objects being generalized to the noncommutative setting are differential forms, densities, multivector fields, etc.

An additional consideration: in classical algebraic geometry, for an algebra over an algebraicaly closed field k, the maximal spectrum of A is the space of morphisms from A to K, i.e. of one-dimensional representations of A. When A is noncommutative, all such representations descend to the Abelianization of A. One can instead consider the space of d-dimensional representations of A for any natural number d. This is naturally a scheme over k; we denote it by $\mathrm{Rep}_d(A)$. One may then look for constructions that reflect, if not completely recover, differential geometric objects on this scheme. ***Refs

Note that the idea to treat an element of a ring as an operator-valued function of a space of representations (or ideals) was pursued in various works, both in algebra (Dixmier, M. Artin) and analysis (Gelfand-Naimark, ...). This subject is farther away from the methods of this book. Of course considering only finite-dimensional representations is way too restrictive; many important noncommutative algebras just do not have any. This problem is dealt with not with including more general representations but with replacing the algebra by its resolution and passing to the derived representation scheme.

1.1. Functions and one-forms.

As a first example, an element of A can be viewed as a noncommutative analogue of a function. On the other hand, it does (tautologically) define a matrix-valued function on the scheme of representations of A. That gives a linear map

$$A \to M_n(\mathcal{O}(\mathrm{Rep}_d(A)));\ a \mapsto \widetilde{a} \tag{1.1}$$

To get an actual function, one can get the trace of this matrix-valued function. The map $a \mapsto \mathrm{tr}(\widetilde{a})$ descends to $A/[A, A]$ which is the quotient of A by the linear span of commutators (since the trace vanishes on commutators).

Now consider one-forms. An algebraic one-form on an algebraic variety is a formal combination

$$\alpha = \sum_{j=1}^{N} a_j \, db_j \tag{1.2}$$

where a_j, b_j are elements of the algebra A of functions. They satisfy the relations

$$ad(bc) = ca \cdot db + ab \cdot dc \tag{1.3}$$

and

$$a \cdot d1 = 0 \tag{1.4}$$

We denote this k-module by $\Omega^1_{A/k}$. This is the module of Kähler differentials of A over k.

1.1.1. *Noncommutative Kähler differentials.* Now observe that one can define the k-module of formal symbols (1.2) subject to relations (1.3) and (1.4) for any algebra A, commutative or not. Furthermore, each element of this module defines a one-form on the scheme $\mathrm{Rep}_d(A)$. Indeed, just take the form

$$\widetilde{\alpha} = \mathrm{tr} \sum_{j=1}^{N} \widetilde{a}_j \, d\widetilde{b}_j \tag{1.5}$$

(this is one way to test that the order of the factors in (1.3) is correct). We denote this k-module by $\Omega^1_{A,\natural}$.

We have

$$\Omega^1_{A,\natural} \xrightarrow{\sim} \mathrm{Coker}(b : A \otimes (A/k)^{\otimes 2} \to A \otimes (A/k));$$

$$b(a_0 \otimes a_1 \otimes a_2) = a_2 a_0 \otimes a_1 + a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 \tag{1.6}$$

(we identify k with $k \cdot 1$ inside A). Note also

$$A/[A, A] \xrightarrow{\sim} \mathrm{Coker}(b : A \otimes (A/k) \to A);$$

$$b(a_0 \otimes a_1) = a_0 a_1 - a_1 a_0 \tag{1.7}$$

One observes that the composition of (1.6) and (1.7) is zero. Therefore we have a *complex*

$$A \xleftarrow{b} A \otimes (A/k) \xleftarrow{b} A \otimes (A/k)^{\otimes 2} \tag{1.8}$$

We can consider a bigger version that projects onto it:

$$A \xleftarrow{b} A \otimes A \xleftarrow{b} A \otimes A^{\otimes 2} \tag{1.9}$$

We recognize the two as the beginning of the two versions of *Hochschild chain complex* from classical homological algebra:

$$C_0(A) \xleftarrow{b} C_1(A) \xleftarrow{b} C_2(A) \xleftarrow{b} \ldots \tag{1.10}$$

where, depending on a version, $C_n(A) = A \otimes (A/k)^{\otimes n}$ or $C_n(A) = A \otimes A^{\otimes n}$.

1.2. De Rham differential and the cyclic symmetry. Note that under the map

$$a_0 \otimes a_1 \mapsto a_0 da_1, \tag{1.11}$$

the expression $a_0 \otimes a_1 + a_1 \otimes a_0$ (in the middle term of (1.9) maps to the exact form $d(a_0 a_1)$. Therefore we have a map

$$A \otimes A/\mathrm{Im}(1-\tau) \to \Omega^1_{A/k}/d\Omega^0_{A/k} \tag{1.12}$$

for a commutative A, as well as

$$A \otimes A/\mathrm{Im}(1-\tau) \to \Omega^1_{\mathrm{Rep}_d(A)}/d\Omega^0_{\mathrm{Rep}_d(A)} \tag{1.13}$$

for any algebra A. Here $\tau(a_0 \otimes a_1) = a_1 \otimes a_0$. This marks the beginning of the relation between cyclic symmetry and noncommutative versions of the De Rham complex.

There is the De Rham differential $d : A \to \Omega^1_{A,\sharp}$ sending a to $da = 1 \cdot da$. One specific feature of the noncommutative case is that there is also the map b in the opposite direction sending adb to $[a, b]$. There composition is equal to zero and we get a two-periodic complex

$$\ldots \xrightarrow{b} A \xrightarrow{d} \Omega^1_{A,\sharp} \xrightarrow{b} A \xrightarrow{d} \Omega^1_{A,\sharp} \xrightarrow{b} \ldots \tag{1.14}$$

1.3. Vector fields. A noncommutative analogue of a vector field is a derivation of an algebra A. If we try to include derivations into a complex as we did in (1.8), we get the following:

$$A \xrightarrow{\delta} \mathrm{Hom}(A/k, A) \xrightarrow{\delta} \mathrm{Hom}((A/k)^{\otimes 2}, A) \tag{1.15}$$

where

$$\delta a(a_1) = a_1 a - a a_1; \ (\delta D)(a_1, a_2) = a_1 D(a_2) - D(a_1 a_2) + D(a_1) a_2 \tag{1.16}$$

As above, this is the beginning of the *Hochschild cochain complex*

$$C^0(A) \xrightarrow{\delta} C^1(A) \xrightarrow{\delta} C^2(A) \xrightarrow{\delta} \ldots \tag{1.17}$$

where, depending on a version, $C^n(A) = \mathrm{Hom}((A/k)^{\otimes n}, A)$ or $C^n(A) = \mathrm{Hom}(A^{\otimes n}, A)$.

1.4. Higher forms and multivector fields. The above opens the way to define noncommutative analogues of forms and multivector fields in terms of Hochschild complexes. In fact, Hochschild, Kostant and Rosenberg constructed morphisms of complexes

$$\mathrm{HKR} : (C_\bullet(A), b) \to (\Omega^\bullet_{A/k}, 0) \tag{1.18}$$

$$\mathrm{HKR} : (\wedge^\bullet T_{A/k}, 0) \to (C^\bullet(A), \delta) \tag{1.19}$$

for a commutative algebra A over a field k of characteristic zero. These morphisms are quasi-isomorphisms when A is regular. Soon after, Rinehart constructed the cyclic differential

$$B : C_n(A) \to C_{n+1}(A) \tag{1.20}$$

and showed that

$$b^2 = Bb + bB = B^2 = 0$$

and that the map HKR in (1.18) intertwines b with the De Rham differential d.

2. Definition and various versions of cyclic homology

2.1. The standard complexes. As we have seen, the action of the cyclic groups $C_{n+1} = \mathbb{Z}/(n+1)\mathbb{Z}$ on Hochschild n-chains $C_n(A) = A^{\otimes(n+1)}$ is related to noncommutative analogs of De Rham cohomology. Various versions of the cyclic complex are defined in terms of this action. The original definition was given in terms of the standard complex $C^{\lambda}_{\bullet}(A)$, or rather its linear dual, in [**111**] ***earlier ref? and [**565**].

In our exposition, the primary object is the negative cyclic complex

$$CC^{-}_{\bullet}(A) = (C_{\bullet}(A)[[u]], b + uB) \tag{2.1}$$

where u is a formal variable of homological degree -2. Other complexes, namely the Hochschild chain complex $C_{\bullet}(A)$ itself, the periodic cyclic complex $CC^{\mathrm{per}}_{\bullet}(A)$, and the cyclic complex $CC_{\bullet}(A)$, are defined as results of some natural procedure applied to $CC^{-}_{\bullet}(A)$. The cyclic homology is the homology of the cyclic complex $CC_{\bullet}(A)$ which in characteristic zero is quasi-isomorphic to $C^{\lambda}_{\bullet}(A)$. The study of this latter complex has a distinctly different flavor, mainly coming from the fact that it is related to the Lie algebra homology.

The above complexes are noncommutative versions of the space of differential forms (the Hochschild chain complex) and of the De Rham complex. One also defines the Hochschild cochain complex $C^{\bullet}(A, A)$ which is a noncommutative analogue of the space of multivector fields.

2.2. Nocommutative forms. Another approach is to generalize the construction $\Omega^1_{A,\sharp}$ in 1.1.1 from one-forms to higher forms. This approach, and its relation to the above, was studied extensively in the works of Connes, Cuntz-Quillen, Karoubi, and more recently Ginzburg-Schedler. One can indeed define the noncommutative De Rham complex

$$\Omega^0_{A,\sharp} \xrightarrow{d} \Omega^1_{A,\sharp} \xrightarrow{d} \Omega^2_{A,\sharp} \xrightarrow{d} \dots \tag{2.2}$$

and compare it to the (b, B) complex. Namely, we show (Theorem 2.3.2 and Corollary 2.3.3) that it is naturally quasi-isomorphic to the Beilinson truncation

$$\tau^{B}_{\leq 0} CC^{-}_{\bullet}(A)$$

of the negative cyclic complex by the Hodge filtration by powers of u. We present the theory of noncommutative forms in Chapter 15.

The Hochschild homology has an invariant meaning in terms of the Tor functors from classical homological algebra. Namely,

$$\mathrm{HH}_n(A) \xrightarrow{\sim} \mathrm{Tor}^{A\otimes A^{\mathrm{op}}}_n(A, A) \tag{2.3}$$

or

$$C_{\bullet}(A) \xrightarrow{\sim} A \otimes^{\mathbb{L}}_{A\otimes A^{\mathrm{op}}} A \tag{2.4}$$

in the derived category of A-bimodules. Here A^{op} is the algebra opposite to A. The above approaches to cyclic homology are rather in terms of explicit complexes. If one wants a more invariant definition, there are (at least) two ways to look for it. We discuss them next.

2.3. Cyclic objects. One can interpret the Hochschild and cyclic homology in terms of Connes' cyclic objects and their homology (chapter 8). This approach is well suited to developing analogies between this homology and De Rham cohomology in positive characteristic, and to eventually replacing rings by ring spectra or by algebras in stable infinity categories. It is also a good framework for studying *the circle action on the Hochschild complex* which is the major feature of cyclic theory.

2.4. Derived functors. Another important tool in Hochschild and cyclic theory is to replace the algebra A by its (semi-) free resolution, which is a differential graded (or simplicial, or more generally a derived ring). This is an application of Quillen's homotopical algebra. This approach was used by Feigin and the second author, and in another form by Cuntz and Quillen. For example, one shows that various versions of the standard complex are non-Abelian derived functors of various versions of the two-periodic De Rham complex (1.14). In characteristic zero, the *reduced* cyclic homology is the derived functor of $A \mapsto A/([A, A] + k)$. In other words:

Cyclic cohomology of an algebra is, up to some modification, the derived space of traces on this algebra.

2.5. More general notions of a noncommutative space. In more recent developments, starting roughly from the nineties, it became clear that the notion of a "noncommutative space" should be significantly more general than that of a ring. The first step is to include differential graded (DG) categories. Those can be viewed as DG algebras with many objects. The Hochschild complex of a small DG category $\mathcal{A}$ is

$$\bigoplus_{n\geq 0}\bigoplus_{x_0,\ldots,x_n\in \mathrm{Ob}(\mathcal{A})} \mathcal{A}(x_0, x_1) \otimes \mathcal{A}(x_1, x_2) \otimes \ldots \otimes \mathcal{A}(x_n, x_0)[n]$$

The differentials b and B are defined by the same formulas as for algebras, with correct signs; the differential $d_{\mathcal{A}}$ is part of the total differential.

The need to replace algebras with categories arises from the very beginning of noncommutative calculus. For example, as we will see in 5, if we want to compute the index of an operator acting from one space to another, it is natural to work with cyclic homology of the category of vector spaces rather than of the algebra of operators on one space. Relatedly, the very first attempts to construct characteristic classes (cf. 4) show that it is natural to work not with a ring, variety, etc. but rather with a category of modules, sheaves of modules, etc. over it.

More generally, working with geometry of a variety in terms of a category of sheaves of modules, so that the definitions and constructions work for any DG category, is the subject of *noncommutative algebraic geometry* ***Ref.

Once one is working with DG categories, it is natural and necessary to extend the theory to A_∞ categories. Roughly, this is because this is the structure that transfers well by quasi-isomorphisms. Noncommutative geometry of DG and A_∞ categories is treated in [**429**], [**423**], [**548**], [**394**], [**399**], and other works.

More general notions of a noncommutative space include ringed spectra, algebras in a symmetric monoidal stable infinity category, derived rings ****more refs***. The scope of this book is within linear algebra and does not reach beyond DG and A_∞ categories. We try, however, to align it with recent developments in more general contexts. Also, in the spirit of Loday's book and of Kaledin's words "a calculus, not a theory", we do not use triangulated categories, model categories, and infinity categories, as well as operads. (Nor are we mentioning Fréchet algebras or C^* algebras, for that matter). There are many excellent sources on those, and our exposition is often closely motivated by these concepts and aims at providing building blocks for statements involving them (as for example in Chapters 5, 8, 9, 22, 26).

3. Algebraic structure on Hochschild and cyclic complexes

In much of the book we study rather systematically various algebraic structures on the Hochschild and cyclic complexes. These structures are supposed to generalize the classical algebraic structures arising in calculus, namely: products on forms and multivectors; action of vector and, more generall, multi-vector fields on forms by Lie derivative and contraction; action of forms on multi-vectors by contraction.

3.1. The Cartan calculus. In classical calculus on manifolds, a vector field X acts on differential forms in two ways: by Lie derivative L_X and by contraction ι_X of degree -1.The following relations are satisfied:

$$[L_X, L_Y] = L_{[X,Y]};\ [L_X, \iota_Y] = \iota_{[X,Y]};\ \iota_X\iota_Y + \iota_Y\iota_X = 0 \tag{3.1}$$

and

$$d\iota_X + \iota_X d = L_X \tag{3.2}$$

("the Cartan magic formula"). Gelfand and Dorfman stressed the importance of these relations, and of the observation that *all* of them are written in terms of *graded commutators* $[a, b] = ab - (-1)^{|a||b|}ba$. They called a Lie algebra $\mathfrak{a}$ acting on a graded space Ω in a way described by (3.1), (3.2) an $(\mathfrak{a}, d)$ system.

The contraction operators can be defined for higher multi-vector fields (which form a graded Lie algebra with the Schouten bracket). The relations still hold, with a change in signs.

Much of the book deals with the analogue of the above relations in the noncommutative case. See for example Chapter 13. Let us outline the picture.

3.1.1. *Operations on cohomology.* First of all, the HKR isomorphism for regular commutative algebras suggests an answer: the noncommutative analogue of forms, resp. multi-vector fields, is the Hochschild *(co)homology.* Rinehart defined the contraction operators and proved the relations (3.1), (3.1) for the operators L_X and ι_X on $HH_\bullet(A)$ for any algebra A, X being a derivation of A. This was extended by Daletsky, Gelfand, and the second author [**248**] to the case when X is a Hochschild cohomology class of A. (The differential graded Lie algebra structure on the Hochschild cochain complex was discovered by Gerstenhaber).

So, in a more naive sense, even at the first level of noncommutative calculus we already achieve our goal: to generalize the basic definitions and structures of calculus to the noncommutative setting. But this generalization is not very interesting when the algebra becomes noncommutative. For example, for the algebra of differential operators on $\mathbb{R}^n$ both the spaces of "noncommutative forms" and "noncommutative multivectors" are one-dimensional.

3.1.2. *Operations on complexes.* To make noncommutative calculus useful for applications, one has to pass to the second level of complexity. Namely, one tries to reproduce more of the standard algebraic structure from the classical calculus not at the level of cohomology but rather at the level of complexes. Note that in noncommutative calculus we always have two different questions:

(1) Can an algebraic structure be generalized from classical to noncommutative calculus?
(2) When our algebra is commutative, does the noncommutative construction give the same as the classical one?

For Cartan calculus, we study these questions in Chapter 13. There, we already see a common feature in noncommutative calculus. Namely: the structures described by (3.1), (3.2) do exist on Hochschild (co)chains. But they exist in a (strongly) homotopical sense; they exist not in their most expected version, but with some corrections; one can get rid of these corrections, at a price of introducing nontrivial operators, not unlike the Todd class in the index theorem or the J factor in the Duflo theorem.

We give a positive answer to question 1 above, subject to all the caveats we mentioned. As to question 2, the answer is yes for the Lie algebra of derivations (and, importantly, for a bigger algebra of derivations of A extended by A). The answer to question 2 (in the smooth case) is provided by the formality theorem of Kontsevich and its extension to Hochschild chains. These structures are only part of what one could expect from the classical calculus. Their advantage is that they are defined by more or less explicit and canonical constructions (Chapters 7 and 13). They are also adequate for some applications, namely for the index theorems for symplectic deformations.

3.2. Products. All of the above was the "Lie part" of noncommutative calculus. We were dealing with various graded Lie algebra and Lie module structures on Hochschild (co)chains. Of course in classial calculus there is the wedge product on forms, as well as on multivectors. Our first experience with noncommutative forms tells us that the space of "zero-forms" is $HH_0(A) = A/[A, A]$, and it does not carry any natural product. Still, the noncommutative analogue of the *exterior* product

$$C_\bullet(A) \otimes C_\bullet(B) \to C_\bullet(A \otimes B) \tag{3.3}$$

exists. We study it in Chapter 4. The subtleties that we mention in 3.1.2 are all present here. Some of them, e.g. a more recent work of Moulinos, Robalo, and Toën [**456**], are outside the scope of this book.

On the other hand, the space of "zero-vectors" is $HH^0(A)$ which is the center of A. And indeed, Hochschild cochains form a differential graded algebra which is commutative at the level of cohomology. In classical calculus, the following relations are satisfied:

$$[a, bc] = [a, b]c + (-1)^{(|a|-1)|b|}b[a, c]; \;\; L_{ab} = L_a\iota_b + (-1)^{|a|}\iota_a L_b \tag{3.4}$$

for multi-vectors a and b.

We can define operations L_a and ι_b on the Hochschild chain complex for any Hochschild cochains a and b. All the relations (3.1), (3.2) (with correct signs) and (3.4) are true at the level of (co)homology. To what extent they are true at the level of complexes is a much more difficult question that we are not addressing in the book but briefly discuss below.

3.3. Operadic methods. In order to construct on Hochschild chains and cochains a richer algebraic structure that generalizes a fuller version of the classical calculus, one has to go to a different level of complexity. The work in this direction was started by Tamarkin in [**543**] and then continued in [**428**], [**190**], [**540**], [**541**], [**188**], [**593**], [**592**], and others. These methods are outside the scope of this book. They provide a considerable refinement of the results of Chapters 7 and 13 but there is no canonical and explicit construction anymore. The "noncomutative differential calculus" can be constructed using some inexplicit formulas; a choice of coefficients in these formulas depends on a choice of a Drinfeld associator [**543**], [**?**]. In particular, the Grothendieck-Teichmüller group acts on the space of all such calculi. This version of noncommutative calculus allows one to generalize the index theorem from symplectic deformations to arbitrary deformations.

Note that in [**115**], [**134**],[**133**],[**132**] a different, though perhaps related, version of noncommutative calculus is used. In particular, there the renormalization group appears as a hidden group of symmetries of the calculus, whereas in our construction the Grothendieck-Teichmüller group acts on the space of universal formulas defining a calculus. This group is closely related to the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. Note that finding a unified symmetry group incorporating both the renormalization group and the Galois group of $\mathbb{Q}$ is one of the important aims of Connes' noncommutative geometry program. In light of this, it seems to be an interesting problem to find a unified framework to the two approaches to noncommutative calculus (the second author would like to thank Alain Connes for his remarks on this subject).

3.4. What do categories form? The key to constructing noncommutative differential calculus is an answer to Drinfeld's question "What do DG categories form?" [**538**]. It is well-known that rings form not only a category but a two-category, bimodules playing the role of one-morphisms and morphisms of bimodules the role of two-morphisms. (In addition, tensor product of rings makes this a monoidal two-category, or a three-category with one object). The crucial point for us is being able to say this correctly for DG categories, in the derived context. Our construction here is intended to serve as a bridge between the theory of Lurie and the constructions that are more traditional in the operadic approach to formality theorems. It is, essentially, based on a specialized and simplified version of Lurie's definitions (chapter 19).

4. Characteristic classes and K-theory

A noncommutative analogue of a vector bundle is a finitely generated projective module over an algebra A. To describe characteristic classes of a vector bundle in terms that work in the noncommutative case, represent the module as the image of an idempotent e in $M_n(A)$. We get a class $[e]$ in $\mathrm{HH}_0(A)$ represented by the 0-cycle

$$\mathrm{tr}(e) = \sum e_{jj} \tag{4.1}$$

This class depends only on the isomorphism class of the module. Indeed, consider two idempotents e and f whose images are isomorphic. Then there are x and y such that

$$fx = ye;\ yxe = eyx = yx;\ xyf = fxy = xy.$$

Then the Hochschild differential b of the one-chain

$$\mathrm{tr}(yf \otimes x) = \sum (yf)_{jk} \otimes x_{kj} \tag{4.2}$$

is equal to $\mathrm{tr}(e) - \mathrm{tr}(f)$. We get a morphism

$$\mathrm{ch} : K_0(A) \to \mathrm{HH}_0(A) \tag{4.3}$$

When A is the algebra of functions on a manifold, $\mathrm{ch}(E) = \mathrm{rk}(E)$ which is a locally constant, therefore De Rham-closed, function. In the noncommutative case, one sees easily that $d\mathrm{tr}(e) = 0$ in $\Omega^1_{A,\sharp}$. which is the Hattori-Stallings trace map. We also get, for $e = f = 1 \in M_n(A)$ and $x = y = g \in \mathrm{GL}_n(A)$, the Hochschild one-cycle

$$\mathrm{ch}(g) = \mathrm{tr}(g^{-1} \otimes g) \tag{4.4}$$

If we use the language of noncommutative differential forms, we recognize the Chern character form

$$\mathrm{ch}(g) = \mathrm{tr}(g^{-1} dg) \tag{4.5}$$

An easy computation shows that $d\mathrm{ch}(g) = 0$ in $\Omega^2_{A,\sharp}$. This follows from

$$d\mathrm{tr}(g^{-1}dg) + \frac{1}{2}[\mathrm{tr}(g^{-1}dg), \mathrm{tr}(g^{-1}dg)] = 0 \tag{4.6}$$

in the universal algebra of noncommutative forms of A, as we will see in Chapter 15.

This, and the connection that we promised between noncommutative forms and cyclic theory, suggests that (4.3) and (4.4) extend to a Chern character with values in periodic cyclic homology. Indeed, Connes and Karoubi showed that the above extends to the Chern character

$$\mathrm{ch} : K_n(A) \to \mathrm{HC}^-_n(A) \tag{4.7}$$

This also extends the previously known Dennis trace map with values in $\mathrm{HH}_n(A)$.

A good way to see that (4.3) lifts from Hochschild to negative cyclic homology is to use the Morita invariance of both theories. Actually for any differential graded category one defines the Chern character of any object x by

$$\mathrm{ch}(x) = \mathbb{1}_x \in \mathcal{A}^0(x, x) \subset \mathrm{CC}^-_0(\mathcal{A}) \tag{4.8}$$

We will see that

$$\mathrm{HC}^-_\bullet(A) \xrightarrow{\sim} \mathrm{HC}^-_\bullet(\mathrm{Proj}(A))$$

where $\mathrm{Proj}(A)$ is the category of finitely generated projective modules. It still requires some work to show that the class of (4.8) is invariant under isomorphism, and of course to extend the Chern character to higher K theory.

4.0.1. *Higher Chern character for DG categories.* The Chern character (4.7) uses Quillen's definition of K_n in terms of the plus construction. To extend it to DG categories, one uses Waldhausen's S construction ***Ref.

4.1. Regulators. Note that there is a discrepancy, or a shift by one, between (4.7) and the discussion after (6.2). The reason is the following. In the language of noncommutative forms, the Chern character

$$K_1(A) \to \Omega^1_{A,\sharp}$$

involves $g^{-1}dg$ which is a) completely algebraic and b) well defined. A Chern character

$$K_1(A) \to \Omega^0_{A,\sharp}$$

would involve $\log(g)$ which is neither: it a) needs some topology to be defined, b) is defined up to $2\pi i\mathbb{Z}$, and c) is only defined in characteristic zero, at least if one does not make an extra effort. This suggests that a Chern chracter

$$\mathrm{ch}: K_n(A) \to \mathrm{HC}_{n-1}(A) \tag{4.9}$$

could be defined a) for a topological algebra A, b) only up to some discrete subgroup of the right hand side, and c) in characteristic zero. This is indeed the case, realized by the Karoubi regulator. The discrete subgroup, roughly speaking, is the image of the topological K theory of A.

When A is (pro)nilpotent, or more generally for relative K theory of a (pro)nilpotent ideal in characteristic zero, there is an actual regulator map $K_n \to \mathrm{HC}_{n-1}$ constructed by Goodwillie who showed that it is an isomorphism.

Over the p-adics, there is Beilinson's regulator map [**25**].

5. Relation to index theory

Index theory was one of the main motivations for cyclic (co)homology from the very beginning [**111**]. Let us explain the reason. Consider a Fredholm operator

$$A : H_+ \to H_- \tag{5.1}$$

Tautologically,

$$\mathrm{ind}(A) = \mathrm{Tr}(P_{\mathrm{Ker}(A)}) - \mathrm{Tr}(P_{\mathrm{Coker}(A)}) \tag{5.2}$$

(where P_V stands for a projection onto V. Now observe that Tr is a (periodic) cyclic zero-cocycle, and an idempotent P extends to a periodic zero-cycle $\mathrm{ch}(P)$ of the algebra of operators of finite rank (say, on $H_+ \oplus H_-$). We now upgrade (5.2) to

$$\mathrm{ind}(A) = \langle \mathrm{Tr}, \mathrm{ch}(P_{\mathrm{Ker}(A)})\rangle - \langle \mathrm{Tr}, \mathrm{ch}(P_{\mathrm{Coker}(A)})\rangle \tag{5.3}$$

Now we can try to replace Tr by a cohomologous cocycle, and the Chern character by a homologous cycle; we know that the answer will not change. Perhaps we will include the algebra of operators of finite rank into a bigger algebra of operators on which trace is still defined.

Relatedly, let $B : H_- \to H_+$ be an operator inverse to A modulo operators of finite rank (a parametrix of A). Then

$$\mathrm{ind}(A) = \mathrm{Tr}(1 - BA) - \mathrm{Tr}(1 - AB) \tag{5.4}$$

Note that $B \otimes A = \mathrm{ch}(A)$ is a Hochschild one-cycle of the quotient of the algebra $\mathcal{L}$ of all operators by the algebra $\mathcal{F}$ of operators of finite rank; we know from 4 that it extends to a periodic cyclic one-cycle. (Note: this discussion is literally true when $H_+ = H_-$; more generally, it is better to work with Hochschild and cyclic complexes of the category of vector spaces). Formula (5.4) gives an impression of the following:

$$\mathrm{ind}(A) = \langle \mathrm{Tr}, \partial \mathrm{ch}(A)\rangle \tag{5.5}$$

where $\partial : \mathrm{HC}_1^{\mathrm{per}}(\mathcal{L}/\mathcal{F}) \to \mathrm{HC}_0^{\mathrm{per}}(\mathcal{F})$ is a boundary map in a homological exact sequence. This is indeed the case because of Wodzicki's excision theorem (Section 3); so we can apply the same argument as above.

As an example, let us prove a simplified version of Noether's index theorem. Let $H = \mathbb{C}[t, t^{-1}]^N$; $H_+ = H_- = \mathbb{C}[t]^N$; $P : H \to H_+$ be the projection sending t^n to t^n for $n \geq 0$ and to zero otherwise. Let $g \in \mathrm{GL}_N(\mathbb{C})$. Then

THEOREM 5.0.1. *The operator*

$$T_g = PgP$$

is Fredholm, and

$$\mathrm{ind}(T_g) = -\mathrm{res}_{t=0}(\mathrm{tr}(g^{-1}dg))$$

PROOF. We observe that $Pg^{-1}P$ is a parametrix of PgP. By (5.4),

$$\mathrm{ind}(T_g) = \mathrm{Tr}(1 - Pg^{-1}PgP) - \mathrm{Tr}(1 - PgPg^{-1}P)$$

A simple calculation reduces that to

$$\mathrm{ind}(T_g) = -\mathrm{Tr}(g^{-1}[P, g])$$

(Note: we are using the fact that $\mathrm{Tr}([a, b])$ is zero if a or b is finite rank).

But in this expression, we can decouple g and g^{-1} and observe that

$$\phi(a_0, a_1) = \mathrm{Tr}(a_0[P, a_1]) \tag{5.6}$$

is a cyclic one-cocycle of the algebra $A = \mathrm{Matr}_n(\mathbb{C}[t, t^{-1}])$. Indeed, it vanishes on the image of both b and $1 - \tau$ as in 1.1.1. Alternatively, the map

$$a_0 da_1 \mapsto \phi(a_0, a_1)$$

is well defined on $\Omega^1_{A,\natural}/dA$. Moreover, it sends $g^{-1}dg$ to $-\mathrm{ind}(T_g)$. We have

$$\mathrm{ind}(T_g) = -\langle \phi, \mathrm{ch}(g)\rangle \tag{5.7}$$

But

$$\mathrm{HC}_1(A) \xrightarrow{\sim} \mathrm{HC}_1(\mathbb{C}[t, t^{-1}]) \xrightarrow{\sim} \mathbb{C},$$

the basis element being the class of $t^{-1} \otimes t$. This follows from the HKR theorem mentioned above, and from Morita invariance of cyclic homology. Therefore it is enough to check the formula in case $N = 1$ and $g = t$. □

5.1. Fredholm modules. Formula (5.6) gives rise to a more general algebraic method. Replace P by $F = 2P - 1$; then $F^2 = 1$. For a suitable operator F, there is a close relation between the algebraic properties of

$$\mathrm{Tr}(a_0[F, a_1] \dots [F, a_n]) \tag{5.8}$$

and

$$\phi(a_0 da_1 \dots da_n) \tag{5.9}$$

where

$$\phi : \Omega^n_{A,\natural}/d\Omega^{n-1}_{A,\natural} \to \mathbb{C}$$

(cf. (2.2)). This was used in [**111**] and became a principal motivation for cyclic homology.

In applications to index theorems, often the expression (5.8) is finite only when n is big enough. A variation on it gives a cyclic cocycle $\mathrm{ch}(F)$. One constructs such an F_D from an elliptic operator D and proves

$$\mathrm{ind}(D) = \langle \mathrm{ch}(F_D), 1\rangle \tag{5.10}$$

Instead of $\mathrm{ch}(F_D)$, one can use other cocycles that have better convergence properties, such as the JLO cocycle [**113**], [**326**], [**276**] or the cocycle in the Connes-Moscovici index formula [**132**], [**320**], as well as Perrot's cocycle [**479**].

5.2. Deforming the projection. The approach based on formula (5.2) was developed in [**206**], [**461**], [**465**], [**463**], [**62**]. Let us give a quick outline. We start with an operator between two Hilbert spaces $D : H_+ \to H_-$ and define a family of projections P_{tD} on $H_+ \oplus H_-$ so that

$$\lim_{t\to\infty} P_{tD} = \begin{bmatrix} P_{\mathrm{Ker}(D)} & 0 \\ 0 & 1 - P_{\mathrm{Ker}(D^*)} \end{bmatrix} \tag{5.11}$$

Since

$$P_{tD}\dot{P}_{tD}P_{tD} = (1 - P_{tD})\dot{P}_{tD}(1 - P_{tD}) = 0$$

and therefore

$$\frac{d}{dt}\mathrm{Tr}(P_{tD}) = 0,$$

we have

$$\mathrm{ind}(D) = \lim_{t\to 0}\mathrm{Tr}(P_{tD} - P_0) \tag{5.12}$$

where

$$P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

We therefore arrive at the following problem. (Let us switch our notation from t to $\hbar$ when we study asymptotics at $t = 0$).

5.2.1. *Trace of a projection in an algebra of asymptotic expressions.* Assume that we succeeded in defining an algebra $\mathcal{A}$ of asymptotic families $a_\hbar$ of operators, so that the product on $\mathcal{A}$ describes the asymptotic of composition of operators. Assume that $\mathcal{A}$ is defined over the algebra K of asymptotic families of scalars. Assume that $\mathcal{I}$ is an ideal of $\mathcal{A}$ and that there is a trace

$$\mathrm{Tr} : \mathcal{I}/[\mathcal{A}, \mathcal{I}] \to K \tag{5.13}$$

describing asymptotics of the operator trace. Given two idempotents P and Q in $\mathcal{A}$ such that $P - Q \in \mathcal{I}$: compute $\mathrm{Tr}(P - Q)$.

5.3. Index theory and deformation quantization. For pseudodifferential operators, an algebra $\mathcal{A}$ as above does exist; it is the matrix algebra over a *deformation quantization* of the algebra $C^\infty(T^*X)$. For such an algebra, the problem from 5.2.1 can be studied regardless of the index theoretical motivation. This is what we do in Chapter 12. We prove several closely related versions of the algebraic index theorem for deformation quantizations of symplectic manifolds, for example Theorem 6.5.5. The solution to the problem from 5.2.1 follows. This is the index theorem for deformation quantization first proven by Fedosov in [**225**].

Remark 5.3.1. Algebraic index theorem for deformation quantizations extends from the symplectic case to the general Poisson case. The proof uses the operadic methods that we discussed in 3.3, namely the results of [**?**], [**428**], [**541**], [**592**], [**593**], We briefly sketch the proof in 9

In particular, whenever we have an asymptotic calculus described by an algebra $\mathcal{A}$ as in 5.2.1, we get an index theorem. It would be interesting to see if index theorems for quantum groups from [**117**], [**118**] can be obtained by this approach.

5.4. Index theorems and noncommutative Cartan calculus. As we have mentioned earlier, the advantage of representing the index as the result of pairing between a cycle and a cocycle is that we now can replace the (co)cycle with a (co)homologous one. This is done by using some version of noncommutative Cartan calculus discussed in 3. For example, formula (5.8) explicitly involves the contraction operator by the derivation $\mathrm{ad}(F)$.

REMARK 5.4.1. A key motivation for Connes was the algebraic analogy between da and $[F, a]$. Manin's remark that perhaps the space of differentials should be of dimension more than one was one early motivation in developing noncommutative calculus.

5.5. Index theorem for elliptic pairs. A far reaching generalization of the Atiyah-Singer index theorem was proven in [**503**], [**504**], [**69**], [**67**]. We refer the reader to Section 11 of Chapter 12 for a quick review. Very broadly speaking, the proof follows the scheme outlined above. Namely, [**503**] establishes the finiteness (or Fredholmness) property; [**504**] reduces the index problem to a higher version of 5.2.1; [**69**], [**67**] finish the proof using algebraic index theorem for deformation quantization.

That said, the proof in [**503**] does not look too close to the methods of 5.1 and 5.2. It would be instructive to bring the two together. This could be probably carried out using the approach of Feigin-Losev-Schoikhet and Engeli-Felder (cf. [**?**] and [**219**]). The method used there, topological quantum mechanics, is based on a a physics-motivated homotopy perturbation formula of a special kind. The latter is an explicit formula for an A_∞-morphism

$$\mathcal{L}(\mathcal{M}^\bullet) \to \mathcal{L}(H^\bullet) \tag{5.14}$$

from linear operators on a complex to linear operators on its cohomology. In the spirit of (5.2), it involves an embedding of $H^\bullet$ to $\mathcal{M}^\bullet$ and a projection P onto the image. The formula involves integration over cubes of which the interval $[0, \infty]$ in 5.2 seems to be a first step. The method seems also to fit with Quillen's approach to the JLO cocycle ([**492**], cf. also 2.2.3).

6. Other topics

6.1. Relation to Lie algebra homology. It had been known since the late 60s that the expression

$$\psi(a, b) = \mathrm{res}_{t=0}\mathrm{tr}(a\,db) \tag{6.1}$$

(the Gelfand-Fuks cocycle) represents a two-cohomology class of the Lie algebra $\mathfrak{gl}_n(\mathbb{C}[t, t^{-1}])$ for any n. The fact that Lie algebra cohomology encodes the geometry of the circle was a principal motivation for [**565**]. It was shown there, and independently in [**443**], that, in characteristic zero,

$$H_\bullet(\mathfrak{gl}(A)) \xrightarrow{\sim} \mathrm{Sym}(\mathrm{HC}_{\bullet-1}(A)) \tag{6.2}$$

Here $\mathfrak{gl}(A)$ is the Lie algebra of infinite matrices over A that have finitely many nonzero entries.

This provides an analogy between cyclic homology and algebraic K theory. Indeed, if one replaces $\mathfrak{gl}$ by the group GL, over the rationals $\mathrm{HC}_{\bullet-1}$ gets replaced by $K_\bullet(A)_\mathbb{Q}$. This is the reason why cyclic homology (shifted by one) was called additive K theory in [**233**].

6.2. Positive characteristic. Much of the above requires the characteristic of the ground ring to be zero. However, Hochschild and cyclic theory of algebras over a field of characteristic $p > 0$ is a rich and subtle subject. Here are a few points that we address in the book.

6.2.1. *Homology over* $\mathbb{F}_p$. First option is just to repeat all the definitions, both classical and noncommutative, taking a field of characteristic p as our ground field. The HKR homomorphism does exist and is an isomorphism in the regular case; but now it identifies the space of forms with the Hochschild homology, not the Hochschild chain complex:

$$\mathrm{HKR}: \Omega^{\bullet}_{A/k} \to \mathrm{HH}_{\bullet}(A) \tag{6.3}$$

There is no natural HKR map at the level of complexes. In fact, it is shown in [**15**] that there is a smooth projective variety X for which the spectral sequence

$$E^2_{s,t} = H^{-s}(X, HH_t(\mathcal{O}_X)) \implies H^{s+t}(X, C_{-\bullet}(\mathcal{O}_X)) \tag{6.4}$$

does not degenerate in the E^2 term. Now, recall that De Rham cohomology in characteristic p behaves very differently from characteristic zero: in the smooth case, it is isomorphic to the forms themselves. For example, for $A = \mathbb{F}_p[t]$

$$\mathrm{Ker}(d: \Omega^0_{A/\mathbb{F}_p} \to \Omega^1_{A/\mathbb{F}_p}) = \mathbb{F}_p[t^p]; \tag{6.5}$$

$$\mathrm{Coker}(d: \Omega^0_{A/\mathbb{F}_p} \to \Omega^1_{A/\mathbb{F}_p}) \xrightarrow{\sim} t^{p-1}\mathbb{F}_p[t^p]dt \tag{6.6}$$

The inverse to this isomorphism is called the Cartier isomorphism. Its noncommutative version was constructed by Kaledin [**336**], [**?**]. The construction rests on the following. For a vector space V over $\mathbb{F}_p$, there is an additive isomorphism

$$V \xrightarrow{\sim} (V^{\otimes p})^{tC_p};\ v \mapsto v^{\otimes p} \tag{6.7}$$

Here for any representation M of the cyclic group C_p of order p with a generator σ,

$$M^{tC_p} = M^{C_p}/\mathrm{Im}(1 + \sigma + \ldots + \sigma^{p-1}) \tag{6.8}$$

Kaledin applies this to $M = A^{\otimes(n+1)}$ for any n. This maps the simplicial (actually cyclic) vector space computing $\mathrm{HH}_\bullet(A)$ to the Tate cohomology of C_p with coefficients in the simplicial space computing the Hochschild homology $\mathrm{HH}_\bullet(A^{\otimes p}, {}_\sigma A^{\otimes p})$ which is the twisted Hochschild homology of the algebra $A^{\otimes p}$. But the latter is isomorphic to $\mathrm{HH}_\bullet(A)$. This produces a noncommutative analogue of the Cartier isomorphism.

REMARK 6.2.1. The Frobenius map (6.8) is not easy to generalize from individual vector spaces to complexes. To resolve this difficulty, one turns to more sophisticated methods of dealing with cyclotomic objects, including a passage to ring spectra. ***Explain better

We discuss these topics in Chapters 23 and 22

6.2.2. *Homology over* $\mathbb{Z}_p$. For an algebraic variety over $\mathbb{F}_p$, apart from working over the ground field, one can also develop differential calculus and cohomology theory with coefficients in $\mathbb{Z}_p$. More generally, for a finitely generated commutative algebra A over a perfect field k of characteristic p, one constructs the De Rham-Witt complex over the ring $W(k)$ of Witt vectors of k.

To construct a noncommutative analogue of this, one works with various ways of lifting the algebra and its Hochschild and cyclic complexes from k to $W(k)$. This is accomplished by Kaledin's theory of noncommutative Witt vectors. Those are defined for a vector space over k rather than for an algebra (as in the classical case) in a functorial way; moreover, they define *a trace functor* from k-vector spaces to $W(k)$-modules. Apply this functor to the cyclic object associated to an algebra, then twist the cyclic structure using the trace functor; this is Kaledin's definition of Hochschld-Witt and cyclic Witt homology [**349**], [**348**]. When one deals with periodic cyclic homology, there are other approaches: one is due to Petrov, Vaintrob, and Vologodsky [**480**], [**?**]; another is based on the Gauss-Manin connection from Chapter 14.

The importance of the cyclic theory in positive characteristic was pointed out early on by Wodzicki. It would be interesting to apply the methods of this book to the examples he considered in [**611**], [**598**]. Also, we do not know of any work on the contents of section 3 for the invariants discussed here, in particular for Hochschild-Witt chains.

6.3. Smooth, proper, and CY DG categories. We now turn to discussing a program pursued by Kontsevich and his several groups of co-authors ***Refs. We present this program in the second half of Chapter 17.

Once we have established the principle that a noncommutative analogue of a variety is a differential graded category, we can ask what is the analogue of a smooth and/or proper algebraic variety. The answer lies in certain cohomological finiteness properties of Hochschild (co)homology. Furthermore, one can ask what is an analogue of a Calabi-Yau variety or, in the C^∞ case, of being a compact manifold with a non-vanishing volume form. Note that for such a manifold the algebraic structure on forms and multivectors becomes more extensive. It now includes integration of a function over the volume form, the star operator (identification of k-forms with $(n-k)$-multivectors) and the divergence operator of degree -1 on multivectors. The noncommutative amalogues of this extended algebraic structure are the subject of multiple works, including string topology of Chas and Sullivan. We review some of this work in present in the first half of Chapter 17. There are multiple nontrivial interactions between the contents of the two halves, as well as between the contents of each half.

REMARK 6.3.1. As we have seen in Section 1, there are two ways to build noncommutative geometry. One is to start with an algebra A and construct an object that, when A is the ring of functions on X, becomes a classical geometric object on X. Another is to start with A and construct an object that, for *any* A, produces a classical geometric object on the representation scheme $\mathrm{Rep}_d(A)$. When our classical object is, for example, a differential formthen the two ways are very much parallel, as we saw in 1.1. But with CY structures this is different. As we have seen, a volume form identifies forms and multivectors. But what it produces on the *derived* representation scheme is a (shifted) symplectic structure, i.e. an identification of one-forms and one-vectors. The reason is the following. To construct the derived representation scheme, we replace A by a semi-free resolution. But noncommutative one-forms of the resolution are a noncommutative analogue of *all* forms; this is because the Hochschild homology a) is the derived finctor of $\Omega^1_{-,\sharp}$ and b) is isomorphic by HKR to the space of all forms when A is smooth commutative. Similarly with Hochschild cohomology and one-vectors.

REFS

6.4. Matrix factorizations.

6.4.1. *Matrix factorizations and the singularity category.* Let k be of characteristic zero. For a commutative algebra A over k and an element W of A, a matrix factorization is a $\mathbb{Z}/2$-graded finitely generated A-module E with an odd A-linear endomorphism D such that $D^2 = W$. Assume that W is not a zero divisor. Note that, given two matrix factorizations E and F, their morphisms over A carry an odd operator of square zero:

$$\varphi \mapsto D_F\varphi - (-1)^{|\varphi|}\varphi D_E$$

Therefore matrix factorizations form a differential $\mathbb{Z}/2$-graded category.

There is a close link between matrix factorizations and A/WA-modules that are perfect as A-modules. ***Ref. Here is one reason. Replace A/WA by a quasi-isomorphic DG algebra $A[\xi], W\frac{\partial}{\partial\xi}$. Note that ξ is a free *commutative* variable, i.e. $\xi^2 = 0$. If (M, d_M) is a DG module over this DG algebra, then

$$(d_M + \xi)^2 = W \tag{6.9}$$

We can construct a complex of A-projective modules which is a DG module over $A[\xi], W\frac{\partial}{\partial\xi}$ which is quasi-isomorphic to M. This could for example be the standard bar resolution over this DG algebra. When M is A-perfect, i.e. quasi-isomorphic to a bounded complex E of finitely generated projective modules, we cannot quite transfer the module structure to E. We can in fact transfer the action of another, bigger algebra quasi-isomorphic to it, namely

$$k\langle\xi_1, \xi_2, \ldots\rangle;\ \ |\xi_n| = 2n - 1;\ d(\xi_1 + \xi_2 + \ldots) + (\xi_1 + \xi_2 + \ldots)^2 = W \tag{6.10}$$

The latter means:

$$d\xi_1 = W;\ d\xi_2 + \xi_1^2 = 0;\ d\xi_3 + \xi_1\xi_2 + \xi_2\xi_1 = 0; \ldots$$

Now we can put

$$D = d + \xi_1 + \xi_2 + \ldots$$

on E (the latter is viewed as a $\mathbb{Z}/2$-graded module).

Remark 6.4.1. In noncommutative Cartan calculus (cf. 3.1.2 and Chapter 13) the same relations appear, along with their twisted version

$$d(\xi_1 + \xi_2 + \ldots) + (\xi_1 + \xi_2 + \ldots)^2 = e^W - 1 \tag{6.11}$$

Here W is the Lie derivative L_D for a derivation D, ξ_1 is contraction by D, and ξ_n are higher contractions.

Conversely, given a matrix factorization E, start with its two-periodic $\mathbb{Z}$-graded version

$$\widetilde{E} = (\ldots \xrightarrow{D} E_0 \xrightarrow{D} E_1 \xrightarrow{D} E_0 \xrightarrow{D} \ldots)$$

and turn it into an acyclic complex

$$(\widetilde{E}[\xi], D + W\frac{\partial}{\partial\xi} - \xi)$$

Then truncate it from above at any place. We get an A-perfect bounded from above DG module over $(A[\xi], W\frac{\partial}{\partial\xi})$ defined up to DG modules that are perfect over $(A[\xi], W\frac{\partial}{\partial\xi})$.

6.4.2. *Cyclic homology of matrix factorizations.* The HKR theorem extends to matrix factorizations from the case $W = 0$. The role of the De Rham complex is played by the $\mathbb{Z}/2$-graded twisted De Rham complex $(\Omega^\bullet(X)[[u]], dW + ud)$. Various versions of this result are due to Efimov, Preygel, Shklyarov, [**204**], [**483**], [**510**]***More refs. A more general comparison between categorical invariants of the category of matrix factorizations and vanishing cycles of W can be found in [**40**] .

7. The contents of the book

In chapter 2 we give the main definitions, both of the standard chain and cochain complexes, in the generality of A_∞ algebras. We also introduce, following Wodzicki, the notion of an H-unital algebra and prove excision for H-unital ideals.

In chapter 3 we study the other definition of the cyclic homology, namely, via the complex $C^\lambda_\bullet$. It gives the same result as above when the ground ring contains $\mathbb{Q}$. We prove the theorem relating cyclic homology of A to Lie algebra homology of matrices over A.

In chapter 4 we start the study of operations on Hochschild and cyclic complexes. We define and study the Eilenberg Zilber product, the Alexander-Whitney coproduct, and the pairings between chains and cochains. All these are classical operations of homological algebra that are extended from Hochschild to cyclic chains when appropriate. We present first applications of operations, namely, to the simplest cases of Morita equivalence and of homotopy invariance of periodic cyclic homology.

In chapter 5 we explain how Hochschild and cyclic homology can be defined using Quillen's language of non-Abelian derived functors.

In chapter 6 we express the Hochschild and cyclic homology of an algebra in terms of its bar construction. We essentially follow Cuntz and Quillen and [**?**].

In chapter 7 we advance our study of operations. Roughly speaking, we introduce the algebra (let us call it $\mathcal{U}_A$ here) of operations on the negative cyclic complex and define by explicit formulas a pairing of *complexes* $\mathcal{U} \otimes \mathrm{CC}^-_\bullet(A) \to \mathrm{CC}^-_\bullet(A)$. Using this, we prove Goodwillie's rigidity theorem for periodic cyclic homology. We prove a more elaborate version later on. We also prove Cuntz and Quillen's excision theroem for periodic cyclic homology. In chapter 8 we give an exposition of Connes' theory of cyclic objects. We explain there relation to spaces with an S^1 action. Following Kaledin, we develop various tools needed for noncommutative geometry in positive characteristic, in particular the Frobenius morphism and an introductory version of the cyclotomic structure.

In chapter 9 we study the relation between cyclic objects and circle actions. The phenomenon was discovered by Connes and studied by Loday, Besser, and Drinfeld. More recently it was studied in a more general setting by Nikolaus-Scholze, Hoyois, Toën-Vezzosi, and others, in relation with the arithmetic applications ot topological Hochschild and cyclic homology. We construct a cyclic realisation of a cyclic object in topological spaces and show that it coincides with the geometric realization of the underlying simplicial topological space. This cyclic realisation carries an action of the circle. We give an analog of this construction for cyclic objects in more general categories; the action of the standard circle is replaced by the action of the simplicial group $B\mathbb{Z}$. We relate various versions of cyclic homology of algebras to homotpoy (co)limits and the Tate construction for the circle action on the cyclic realization.

In chapter 10, we provide various examples of computations of the Hochschild and cyclic homology. The examples that we choose in this presentation revolve mostly around several related classes of algebras: functions on manifolds or on algebraic varieties; operators on functions; deformed algebras of functions; group algebras. *** Not included yet)*** The second class of examples is relevant to representation theory of quivers and other topics (preprojective algebras, CY algebras). Many interesting examples, in particular the ones related to algebraic topology, are not considered here.

In chapter 11 we study characteristic classes in noncommutative geometry. We start with the Chern character of Connes and Karoubi and then define the Karoubi regulator for topological algebras. As a version of that, we get a Goodwillie morphism from relative K theory to relative cyclic homology of a nilpotent ideal over the rationals, as well as a version of a more refined Beilinson morphism over the p-adics. Then we extend the Chern character K_0 from projective modules over an algebra to perfect complexes over a sheaf of algebras

Some of the topics outside the scope of the book are: Chern character on K theory of DG categories, Karoubi regulator on DG categories over $\mathbb{C}$ using Blanc's topological K theory.

In chapter 12 we introduce an important generalization of a sheaf of algebras, namely, an algebroid stack. Algebroid stacks are concrete and explicit realizations of sheaves of categories. They are used, in particular, in deformation quantization. The constructions of cyclic and negative cyclic complexes, perfect complexes, and the Chern character generalize to this context.

In chapter 13 we use the results of 6 to advance our study of operations. We construct two A_∞ algebras and prove that they act on the negative cyclic complex. One has a motivation in classical calculus on manifolds. Namely, recall that multivector fields act on forms in two ways: by contraction ι_X and by Lie derivative $L_X = [d, \iota_X]$. if $\mathfrak{g}_M$ is the graded Lie algebra of multivector fields on a manifold M with the Schouten bracket, construct a new graded Lie algebra over $\mathbb{C}[[u]]$ generated by operators ι_X and L_X for $X \in \mathfrak{g}_M$. This algebra acts on $\Omega^\bullet_M[[u]]$. Take the universal enveloping algebra, and equip it with the differential induced by the commutator with ud_{DR}. The result is an associative algebra over $\mathbb{C}[[u]]$ that can be defined starting with any differential graded Lie algebra $\mathfrak{g}$. Apply this construction to $\mathfrak{g}_A$, the algebra of Hochschild cochains on A. This is our first algebra of operations on $\mathrm{CC}^-_\bullet(A)$. The other, larger A_∞ algebra of operations on the same complex is the negative cyclic complex of the *associative* differential graded algebra of Hochschild cochains. The fact that it is an algebra, and that it acts on $\mathrm{CC}^-_\bullet(A)$, is explained later in chapter

In chapter 14 we use the above results to prove the rigidity property of periodic cyclic homology and to construct the Gauss Manin connection on the periodic cyclic complex of a family of algebras. We generalize the theorems, respectively, of Goodwillie and Getzler. Our results are true over p-adic integers, not the rationals, and at the level of complexes, not homologies.

In chapter 15 we study the approach to cyclic homology via noncommutative differential forms. We follow Cuntz-Quillen, Karoubi, and Ginzburg-Schedler. In particular, we show that the standard HKR map, previously defined for commutative algebras, generalizes to any algebra if we use noncommutative forms. This HKR morphism maps Hochschild chains $C_\bullet(A)$ to noncommutative forms $\Omega^\bullet_A$. It intertwines the cyclic differential B with the De Rham differential d. and the Hochschild differential b with the Ginzburg-Schedler differential ι_Δ. We interpret ι_Δ as a homotopy between id and f^* for any homomorphism f, in case when $f = \mathrm{id}_A$. (In other words: noncommutative De Rham cohomology is trivial for *any* algebra A; in particular, any morphism of algebras acts trivially on this cohomology; construct a homotopy for it, and then evaluate it on id_A; we get a new differential that automatically commutes with d). In conclusion, we show how this can be used to generalize quantum moment map and quantum Hamiltonian reduction from Lie algebra actions on associative algebras to Hopf algebra actions.

In chapter 16 we systematically develop the theory of Hochschild and cyclic complexes for DG categories. The new elements, as compared to the case of DG algebras, are as follows. First of all, the notion of (weak) equivalence becomes more delicate. Second, there is a notion of a quotient by a full subcategory, due to Drinfeld. We prove Keller's excision theorem stating that a categorical quotient gives rise to a homotopy fibre sequence of Hochschild and cyclic complexes, as well as other invariance properties, such as invariance under week equivalence and a form of Morita invariance, also essentially due to Keller. We extend our constructions to A_∞ categories.

In chapter 17 we study Frobenius algebras and their generalizations. A Frobenius algebra is an algebra with a trace τ such that the pairing $\langle a, b\rangle = \tau(ab)$ is nondegenerate. Frobenius algebras have several interconnected generalizations in the context of DG categories and A_∞ categories. ***MORE*** Homotopy BV algebra; also oh Hochschild-Tate complexes, Rivera-Wang... relation to representation schemes...

In chapter 18 we compute the Hochschild and cyclic homology of the Drinfeld quotient of the DG category of perfect complexes by the full DG subcategory of acyclic complexes.

In chapter 19 we show that DG categories form a category up to homotopy in DG categories, in the sense of Leinster. The DG category $\mathbf{C}^\bullet(\mathcal{A}, \mathcal{B})$ for two DG categories $\mathcal{A}$ and $\mathcal{B}$ is defined already in chapter 16; the main ingredient in the definition is the brace structure on Hochschild cochains. Then we extend the structure of a category up to homotopy to Hochschild chains. We show that, taken together, Hochschild cochains and chains form a category up to homotopy with a trace functor. Trace functors are central to Kaledin's work on noncommutative generalization of Witt vectors and De Rham-Witt complexes.

REMARK 7.0.1. It looks like the correct answer to the question: What do DG categories form? should unify the construction above with the constructions in 8, as well as in 17. This should be a structure on all Hochschild chains and cochains of $A_1 \otimes \ldots \otimes A_n$ with coefficients in bimodules $B_1 \otimes \ldots \otimes B_n$; the bimodule structure is given by morphisms $f_j : A_j \to B_j$ and $g_j : A_j \to B_{j+1}$, $1 \geq j \geq n$ (and $B_{n+1} = B_1$). The constructions of chapter 19 should correspond to the case $n = 1$; the full structure should incorporate the Frobenius and the cyclotomic structure of 8, as well as the pre-CY structures of Kontsevich-Vlassopoulos discussed in 17.

In chapter 20 we study the link between Hochschild and cyclic homology of an algebra A and various versions of the representation scheme of A. Note that, similarly to defining the maximal spectrum of a commutative algebra over $\mathbb{C}$ as the space of its one-dimensional representations, one can develop parts of noncommutative geometry by studying spaces of representations of a noncommutative algebra A. Cyclic homology theory initially took a different road, namely it defined various invariants as complexes of forms on an imaginary non-existent space that could be thought of as a noncommutative spectrum of our algebra A. It was later that connections were established between these invariants and actual functions and forms on the algebraic variety of finite dimensional representations of A. The approach with noncommutative forms (cf. 15) is related to these developments. We study both the usual and derived versions of representation varieties.

In chapter 21 we discuss noncommutative Hodge theory. The idea, due to Kontsevich-Soibelman and Katzarkov-Kontsevich-Pantev, is as follows. De Rham cohomology of a smooth and proper algebraic variety carries a Hodge structure. Noncommutative analogues of smooth proper varieties are smooth proper DG categories (chapter 17). A noncommutative analogue of De Rham cohomology is periodic cyclic homology. What is a noncommutative analogue of a Hodge structure? A possible answer: the definition of a Hodge structure contains two parts: the integrl structure and the Hodge filtration. As in the classical case, a candidate for a rational lattice is the image of some version of K theory under the Chern character. The Hodge filtration is the filtration by eigenvalues of a version of the Gauss-Manin connection.

In chapter 22 we study cyclic homology of algebras in characteristic $p > 0$. The first observation is that the theory becomes more adequate if we work over the p-adics; that is, we pass to a DG resolution of or $\mathbb{F}_p$-algebra which is flat over $\mathbb{Z}_p$. We compute the basic example $A = \mathbb{F}_p$ and observe that the result is still not quite satisfactory. There are several ideas to improve the construction. One was already exhibited in chapter 14. Another approach is Kaledin's theory of noncommutative Witt vectors. Combined with his notion of a trace functor, it yields a noncommutative version of De Rham-Witt cohomology.

In chapter 23 we discuss Kaledin's coperiodic cyclic homology. This construction is suitable for algebras in positive characteristic. This homology the conjugate filtration which is a noncommutative analogue of a key construction in differential calculus in characteristic p. Chapters 22 and 23, together with chapter 8, provide prerequisites for Kaledin's degeneration theorem.

A more powerful tool for noncommutative differential geometry in positive and mixed characteristic is topological Hochschild and cyclic homology. We hope that our book gives a gateway to this theory.

In chapter 24 we discuss Hochschild and cyclic (co)homology of the second kind. The notion is due to Possitselsky and Polishchuk. We have already encountered it once, in the dual context of coalgebras (chapter 6). Cyclic cohomology of the second kind of the *coalgebra* $U(\mathfrak{g})$ where $\mathfrak{g}$ is a DG Lie algebra is central in our studies of operations on the negative cyclic complex. ***MORE***

In chapter 25 we study Hochschild and cyclic homology of the DG category which is the Drinfeld quotient of the category of (bounded from above) complexes of A-modules by the full subcategory of perfect complexes. We establish that it is (weakly) equivalent to the category of matrix factorizations, and then prove Efimov's theorem about their Hochschild and cyclic homology. Complexes of the second kind are used.

In chapter 26 we study the category of singularities and its relation to matrix factorizations. In the commutative case, the category of singularities was studied by Eisenbud, Buchweitz, and Orlov. More recently, Efimov, Keller, and others put it into the framework of noncommutative geometry of DG categories.

Many key aspects of cyclic theory did not make it into this book. One is analytic, local, or entire cyclic cohomology of Connes, Mayer, and Puschnigg. Another is topological Hochschild and cylic homology of Böckstedt and Madsen. The third is cyclic homology of Hopf algebras developed by Connes and Moscovici. As we have already mentioned, the operadic aspect of cyclic theory is missing, in particular the works of Costello, Kontsevich-Soibelman, Tamarkin, and Willwacher.

7.1. Conventions and notation. An algebra, DG category, etc. A over a commutative unital ring k is always assumed flat over k.

7.2. The current state of the book. *Most sections are about ninety per cent finished. References are far from finished. Assumptions in sections may be missing/incomplete.*

7.3. Acknowledgements. We are grateful to Anton Alekseev, Ben Antieau, Sasha Beilinson, Bhargav Bhatt, Paul Bressler, Guillermo Cortiñas, Joachim Cuntz, Sasha Efimov, Ezra Getzler, Victor Ginzburg, Paul Goerss, Sasha Gorokhovsky, Lars Hesselholt, Nigel Higson, Marc Hoyois, Mamuka Jibladze, Dima Kaledin, Max Karoubi, Ralph Kaufmann, Bernhard Keller, Masoud Khalkhali, Maxim Kontsevich, Kirill Magidson, Zhouhang Mao, Nikita Markarian, Ralph Mayer, Florian Naef, Irakli Pachkoria, Sasha Petrov, Leonid Positselski, Michael Puschnigg, Manuel Rivera, Nick Rozenblyum, Alex Takeda, Dima Tamarkin, Vadim Vologodsky, Nathalie Wahl, Ben Ward, Mariusz Wodzicki, Waikit Yeung for many fruitful discussions and for their kind and patient explanations of many aspects of cyclic theory and related topics.

CHAPTER 2

Hochschild and cyclic homology of algebras

1. Basic homological complexes

Let k denote a commutative unital ring and let A be a flat k-algebra with unit, not necessarily commutative. Let $\overline{A} = A/k \cdot 1$, and let

DEFINITION 1.0.1.

$$\widetilde{C}_p(A) \overset{\mathrm{def}}{=} A \otimes_k A^{\otimes_k p}.$$
$$C_p(A) \overset{\mathrm{def}}{=} A \otimes_k \overline{A}^{\otimes_k p}.$$

We call elements of $\widetilde{C}_\bullet$ *non-normalized and the elements of* $C_\bullet$ *normalized Hochschild chains of* A.

DEFINITION 1.0.2. *Define*

$$(1.1) \qquad \begin{aligned} b : A \otimes A^{\otimes p} &\to A \otimes A^{\otimes p-1} \\ a_0 \otimes \cdots \otimes a_p &\mapsto (-1)^p a_p a_0 \otimes \cdots \otimes a_{p-1} + \\ &\quad \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \end{aligned}$$

$$(1.2) \qquad \begin{aligned} B : A \otimes \overline{A}^{\otimes p} &\to A \otimes \overline{A}^{\otimes p+1} \\ a_0 \otimes \cdots \otimes a_p &\mapsto \sum_{i=0}^{p} (-1)^{pi} 1 \otimes a_i \otimes \cdots \otimes a_p \otimes a_0 \otimes \cdots \otimes a_{i-1} \end{aligned}$$

LEMMA 1.0.3. *The map* b *descends to the map*

$$b : C_\bullet(A) \to C_{\bullet-1}(A)$$

PROPOSITION 1.0.4. *The maps*

$$b : C_\bullet(A) \to C_{\bullet-1}(A) \text{ and } B : C_\bullet(A) \to C_{*+1}(A).$$

satisfy the identities $b^2 = B^2 = bB + Bb = 0$

PROOF. We will leave the proof of this claim to the reader. ****OR...**** □

DEFINITION 1.0.5. The complex $(C_\bullet(A), b)$ is called *the (normalized) standard Hochschild complex of* A and its homology is denoted by $H_\bullet(A, A)$ or by $HH_\bullet(A)$. We sometimes write $C_\bullet(A, A)$ instead of $C_\bullet(A)$.

REMARK 1.0.6. For any algebra A we will use A^{op} to denote the opposite algebra, i. e.

$$(1.3) \qquad \begin{aligned} &A^{op} = \{a^\circ \mid a \in A\} \text{ as a } k\text{-module} \\ &a^\circ b^\circ = (ba)^\circ. \end{aligned}$$

We will set

$$A^e = A \otimes A^{op}. \tag{1.4}$$

In particular, an A^e-module is the same as an A-bimodule. Suppose that A is unital. The Hochschild complex $(C_\bullet(A), b)$ is just the tensor product $A \otimes_{A^e} \mathcal{B}_\bullet(A)$, where

$$\mathcal{B}_\bullet(A) = A^e \otimes_k \overline{A}^{\otimes \bullet} \tag{1.5}$$

is the standard free resolution of A as an A^e-module. In particular, $H_\bullet(A, A)$ is the same as the the left derived tensor product $A \otimes^{\mathbb{L}}_{A \otimes A^{op}} A$ in the category of A-bimodules. More precisely, if we identify P_n with $A \otimes \overline{A}^{\otimes n} \otimes A$ via

$$(a_0 \otimes a^\circ_{n+1}) \otimes a_1 \ldots \otimes a_n \mapsto a_0 \otimes \ldots \otimes a_{n+1},$$

then the differential is as follows:

$$b'(a_0 \otimes \ldots \otimes a_{n+1}) = \sum_{j=0}^{n} (-1)^j a_0 \otimes \ldots \otimes a_j a_{j+1} \otimes \ldots a_{n+1} \tag{1.6}$$

We have

$$H_\bullet(A, A) = \mathrm{Tor}_\bullet^{A \otimes A^{op}}(A, A).$$

The identity $Bb + bB$ means that the map B induces a morphism of complexes

$$B : (C_\bullet(A), b) \to (C_\bullet(A)[-1], -b)\ .$$

Lemma 1.0.7. *The morphism of complexes*

$$(A \otimes A^{\otimes \bullet}, b) \to (A \otimes \overline{A}^{\otimes \bullet}, b)$$

induces an isomorphism on homology.

Proof. Let $(\widetilde{\mathcal{B}}(A), b)$ be the free resolution of A given by

$$\widetilde{\mathcal{B}}_\bullet(A) = A^e \otimes A^{\otimes \bullet},$$

where b is given by the formula 1.1. Then the quotient map $A \to A/k1$ induces a morphism of resolutions of A:

$$(\widetilde{\mathcal{B}}_\bullet(A), b) \to (\mathcal{B}_\bullet(A), b).$$

In particular, the induced map

$$A \otimes_{A^e} \widetilde{\mathcal{B}}_\bullet(A) \to A \otimes_{A^e} \mathcal{B}_\bullet(A)$$

induces an isomorphism in homology. □

Definition 1.0.8. *For* $i, j, p \in \mathbb{Z}$ *let*

$$CC^-_p(A) = \prod_{\substack{i \geq p \\ i = p \mod 2}} C_i(A)$$

$$CC^{per}_p(A) = \prod_{i = p \mod 2} C_i(A)$$

$$CC_p(A) = \bigoplus_{\substack{i \leq p \\ i = p \mod 2}} C_i(A)$$

The associated complexes are:

(1) *the negative cyclic complex* $(CC^-_\bullet(A), B + b)$*;*

(2) *the periodic cyclic complex* $(CC^{per}_\bullet(A), B + b)$ *and*
(3) *the cyclic complex* $(CC_\bullet(A), b + B)$

The homology of these complexes is denoted by $HC^-_\bullet(A)$*, respectively by* $HC^{per}_\bullet(A)$*, respectively by* $HC_\bullet(A)$*.*

In what follows we will use the notation of Getzler and Jones ([**270**]). Let u denote a variable of degree -2. Then the negative and periodic cyclic complexes are described by the following formulas:

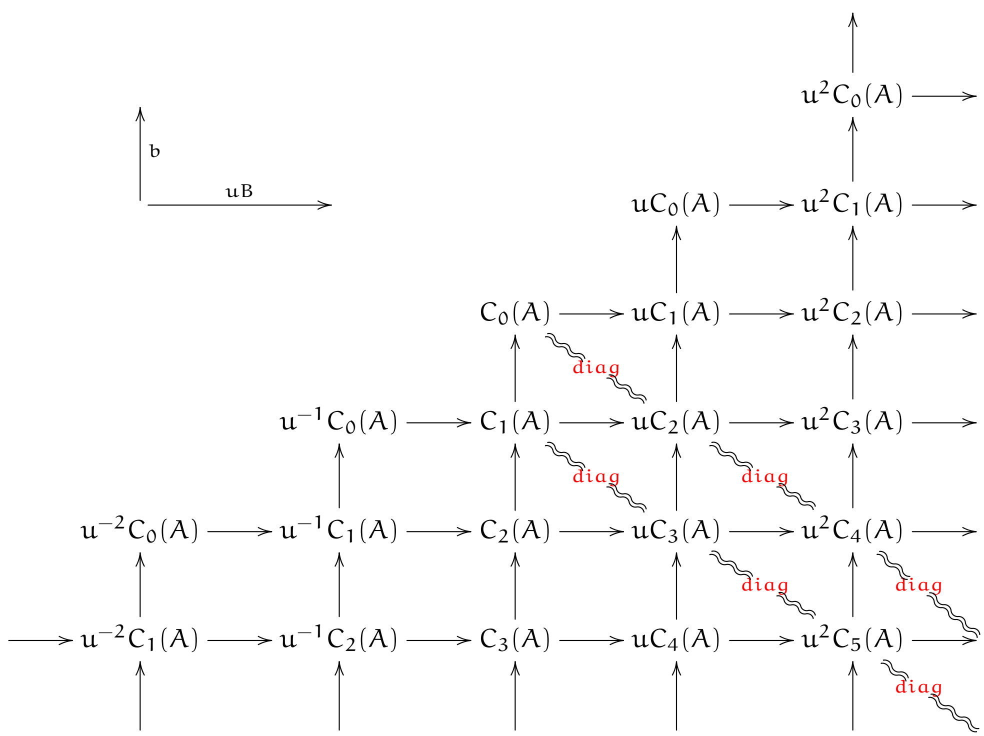
$$CC^-_\bullet(A) = (C_\bullet(A)[[u]], b + uB) \tag{1.7}$$

$$CC^{per}_\bullet(A) = (C_\bullet(A)[[u, u^{-1}], b + uB) \tag{1.8}$$

$$CC_\bullet(A) = (C_\bullet(A)[[u, u^{-1}]/uC_\bullet(A)[[u]], b + uB) \tag{1.9}$$

REMARK 1.0.9. Here and in the future we will always consider the algebra of formal power series $k[[u]]$ in its u-adic topology.

The following is a good picture to keep in mind:



As immediately seen from the picture, there are inclusions of complexes

$$CC^-_\bullet(A)[-2] \hookrightarrow CC^-_\bullet(A) \hookrightarrow CC^{per}_\bullet(A) \tag{1.10}$$

and short exact sequences:

$$0 \to CC^-_\bullet(A)[-2] \xrightarrow{S} CC^-_\bullet(A) \to C_\bullet(A) \to 0 \tag{1.11}$$

$$0 \to C_\bullet(A) \to CC_\bullet(A) \xrightarrow{S} CC_\bullet(A)[2] \to 0 \tag{1.12}$$

and

$$0 \to CC^-_\bullet(A) \to CC^{per}_\bullet(A) \to CC_\bullet(A)[2] \to 0. \tag{1.13}$$

The periodicity map map S is just the multiplication by u.

REMARK 1.0.10. The long exact sequence of homology induced by the short exact sequence of complexes 1.12 has the form

$$\dots \longrightarrow H_k(A,A) \xrightarrow{I} HC_k(A) \xrightarrow{S} HC_{k-2}(A) \xrightarrow{B} H_{k-1}(A,A) \xrightarrow{I} \dots$$

and is sometimes called the *Connes-Gysin exact sequence*. More generally, let $\mathcal{F}_p$ be the to horisontal filtration of the double complex $CC_\bullet(A)$:

$$\mathcal{F}_p(CC_\bullet(A)) = \bigoplus_{l-k=p} u^{-k} C_l(A).$$

The associated spectral sequence has the E^2-term

$$E^2_{pq} = H_{p-q}(A,A) \tag{1.14}$$

and converges to $HC_{p+q}(A)$.

PROPOSITION 1.0.11. *The quotient map* $CC^{per}_n(A) \to CC_n(A)$ *induces a short exact sequence*

$$0 \to \varprojlim{}^1 HC_\bullet(A) \to HC^{per}_\bullet(A) \to \varprojlim HC_\bullet(A)$$

PROOF. The claim follows immediately from the fact that

$$CC^{per}_\bullet(A) = \varprojlim CC_\bullet(A).$$

□

REMARK 1.0.12.
All the three complexes can be just as well thought of as covariant functors from the category of unital algebras over k to the category of complexes of vector spaces over k.

EXAMPLE 1.0.13. Suppose that $A = k$. Then

$$C_n(k) = \begin{cases} k & n = 0 \\ 0 & n > 0 \end{cases}$$

and hence

$$HC_\bullet(k) = k[u^{-1}];\ HC^{per}_\bullet(k) = k[u^{-1}, u]] \text{ and } HC^-_\bullet(k) = k[[u]].$$

with the grading given by $|u| = -2$.

PROPOSITION 1.0.14. *Suppose that* A_1 *and* A_2 *are unital algebras over* k*. Then the inclusion*

$$(C_\bullet(A_1), b) \oplus (C_\bullet(A_2), b) \hookrightarrow (C_\bullet(A_1 \oplus A_2), b)$$

is a quasiisomorphism of complexes.

PROOF. Since A_1 and A_2 are unital, $A_1^e \oplus A_2^e$ is a projective $(A_1 \oplus A_2)^e$-module (see 1.4 for the notation) and hence

$$\mathcal{B}_\bullet = (A_1^{\otimes \bullet} \oplus A_2^{\otimes \bullet}) \otimes_k (A_1^e \oplus A_2^e)$$

is a subcomplex of $(\mathcal{B}_\bullet(A_1 \oplus A_2)^e, b)$ contractible in positive degrees and such that each term is again a projective $(A_1 \oplus A_2)^e$-module. Hence the inclusion

$$(\mathcal{B}_\bullet, b) \hookrightarrow (\mathcal{B}_\bullet(A_1 \oplus A_2), b)$$

is a quasiisomorphism. As a corollary, the inclusion ι of complexes

$$\begin{aligned} C_\bullet(A_1) \oplus C_\bullet(A_2) = \mathcal{B}_\bullet \otimes_{(A_1 \oplus A_2)^e} (A_1 \oplus A_2) \hookrightarrow \\ (\mathcal{B}_\bullet(A_1 \oplus A_2) \otimes_{(A_1 \oplus A_2)^e} (A_1 \oplus A_2) = C_\bullet(A_1 \oplus A_2) \end{aligned} \tag{1.15}$$

is a quasiisomorphism. □

COROLLARY 1.0.15. *Hochschild, cyclic, negative cyclic and periodic homologies are additive, i.e.* $HC_\bullet^{\#}(A_1 \oplus A_2) = HC_\bullet^{\#}(A_1) \oplus HC_\bullet^{\#}(A_2)$ *whenever* A *and* B *are unital algebras (where* $\#$ *stands for cyclic, negative and resp. periodic homology).*

PROOF. The part of the claim about the Hochschild homology follows from the above proposition.

Let e_1 denote the unit of A_1 and e_2 denote the unit if A_2. Set

$$C'_n(A_1) = \begin{cases} A_1, \text{ for } n = 0 \\ (A_1 \oplus ke_2) \otimes A_1^{\otimes n} \text{ for } n > 0. \end{cases}$$

and similarly for $C'_\bullet(A_2)$. Then

$$C'_\bullet(A_1) \oplus C'_\bullet(A_2)$$

is a subcomplex of $C_\bullet(A)$ invariant under B. The summands $ke_2 \otimes A_1^{\otimes \bullet}$ and $ke_1 \otimes A_2^{\otimes \bullet}$ are both contractible, the contracting homotopies given by

$$e_2 \otimes a_1 \otimes \ldots \otimes a_n \to e_2 \otimes e_1 \otimes a_1 \otimes \ldots \otimes a_n$$

and

$$e_1 \otimes a_1 \otimes \ldots \otimes a_n \to e_2 \otimes e_2 \otimes a_1 \otimes \ldots \otimes a_n$$

respectively. Together with the lemma 1.0.7 this implies that, say,

$$C'_\bullet(A_1)[u^{-1}, u]] \oplus C'_\bullet(A_2)[u^{-1}, u]], b + uB) \to (CC^{per}(A), b + uB)$$

is a morphism of double complexes which induces quasiisomorphism on the columns and hence is a quasiiomorphism of double complexes. This proves the claimed result for the periodic cyclic homology. The other two versions of the claim follow from the same argument (replacing Laurent series in u^{-1} by polynomials in u^{-1} and formal power series in u respectively). □

2. The $(b, b', 1-\tau, N)$ double complex

Let A be any algebra, not necessarily unital. Let $\tau = \tau_p$ denote the endomorphism of $A^{\otimes_k (p+1)}$ given by the formula

$$\tau(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p \otimes a_0 \cdots \otimes a_{p-1} \tag{2.1}$$

Define

$$N : C_p(A) \to C_{p-1}(A)$$
$$N = \mathrm{id} + \tau + \ldots \tau^{p-1}$$

One has

$$b(\mathrm{id}-\tau) = (\mathrm{id}-\tau)b'; \quad b'N = Nb, \tag{2.2}$$

where $b' : A^{\otimes\bullet} \to A^{\otimes\bullet-1}$ is given by

$$b'(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = \sum_{k=0}^{n-1} (-1)^k a_0 \otimes \ldots \otimes a_{k-1} \otimes a_k a_{k+1} \otimes \ldots \otimes a_n.$$

Suppose now that A is unital and set

$$B_0(a_1 \otimes \ldots \otimes a_n) = 1 \otimes a_1 \otimes \ldots \otimes a_n. \tag{2.3}$$

Note that

$$[B_0, b'] = \mathrm{id} \tag{2.4}$$

and therefore

LEMMA 2.0.1. *For a unital algebra A, the complex $(A^{\otimes(\bullet+1)}, b')$ is acyclic.*

DEFINITION 2.0.2. *Let A be an associative algebra. Define $\mathbf{CC}_\bullet(A)$ to be the total complex of the double complex*

$$\begin{array}{ccccccccccc}
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b \\
\ldots & \xrightarrow{1-\tau} & A^{\otimes 3} & \xrightarrow{N} & A^{\otimes 3} & \xrightarrow{1-\tau} & A^{\otimes 3} & \xrightarrow{N} & A^{\otimes 3} & \xrightarrow{1-\tau} & A^{\otimes 3} \\
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b \\
\ldots & \xrightarrow{1-\tau} & A^{\otimes 2} & \xrightarrow{N} & A^{\otimes 2} & \xrightarrow{1-\tau} & A^{\otimes 2} & \xrightarrow{N} & A^{\otimes 2} & \xrightarrow{1-\tau} & A^{2} \\
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b \\
\ldots & \xrightarrow{1-\tau} & A & \xrightarrow{N} & A & \xrightarrow{1-\tau} & A & \xrightarrow{N} & A & \xrightarrow{1-\tau} & A
\end{array}$$

Put also

$$\mathbf{C}_\bullet(A) = \mathrm{Cone}((A^{\otimes(\bullet+1)}, b') \xrightarrow{1-\tau} ((A^{\otimes(\bullet+1)}, b'))$$

(the total complex of the double complex consisting of the right two columns).

For any associative algebra A set $A^+ = A + k \cdot \mathbb{1}$, where both A and $k \cdot \mathbb{1}$ are subalgebras and

$$\forall a \in A \;\; a \cdot \mathbb{1} = a = \mathbb{1}\dot{a}$$

LEMMA 2.0.3. *There are isomorphisms of complexes*

$$\mathbf{C}_\bullet(A) \xrightarrow{\sim} \mathrm{Ker}(C_\bullet(A^+) \to C_\bullet(A))$$

$$\mathbf{CC}_\bullet(A) \xrightarrow{\sim} \mathrm{Ker}(CC_\bullet(A^+) \to CC_\bullet(A))$$

PROOF. The inverse map acts as follows. On the b columns,

$$a_0 \otimes \ldots \otimes a_n \mapsto a_0 \otimes \ldots \otimes a_n;$$

on the b' columns,

$$a_0 \otimes \ldots \otimes a_n \mapsto \mathbb{1} \otimes a_0 \otimes \ldots \otimes a_n$$

□

LEMMA 2.0.4. *Let* A *be unital. There is a natural quasi-isomorphism*

$$\mathbf{C}_\bullet(A) \to C_\bullet(A)$$

$$\mathbf{CC}_\bullet(A) \to CC_\bullet(A)$$

PROOF. The morphism is induced by the morphism of algebras $A^+ \to A$ which sends A to itself and $\mathbb{1}$ to the unit of A. It is a quasi-isomorphism because the b' columns are acyclic. □

Therefore we can define the Hochschild and cyclic homology of any algebra, unital or not, using the complexes $\mathbf{C}_\bullet$ and $\mathbf{CC}_\bullet$. Similarly for negative and periodic cyclic homology.

3. H-unitality and excision

Suppose that A is a non-unital algebra. We set

$$b' : C_p(A) \to C_{p-1}(A) \tag{3.1}$$

$$a_0 \otimes \cdots \otimes a_p \mapsto \sum_{i=0}^{p-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_p \tag{3.2}$$

DEFINITION 3.0.1. A *is H-unital if the complex* $(C_\bullet(A), b')$ *is acyclic.*

THEOREM 3.0.2 (Excision in Hochschild homology). *Given a short exact sequence*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0,$$

where I *is H-unital, there exists a long exact sequence*
(3.3)

$$\ldots \xrightarrow{\partial} H_k(I) \xrightarrow{H(\iota)} H_k(A) \xrightarrow{\pi} H_k(A/I) \xrightarrow{\partial} H_{k-1}(I) \xrightarrow{H(\iota)} H_{k-1}(A) \xrightarrow{\pi} \ldots$$

$$\xrightarrow{\partial} H_0(I) \xrightarrow{H(\iota)} H_0(A) \xrightarrow{\pi} H_0(A/I) \longrightarrow 0$$

SKETCH OF THE PROOF. The standard way of proving this kind of result consists of proving that the map

$$C_\bullet(I) \to \text{ Ker } (H_\bullet(\pi))$$

induced by ι is a quasiisomorphism. Instead we will sketch the construction of the boundary map ∂. A complete proof of the exactness of the sequence (3.3) follows the same pattern.

Construction of the boundary map

Let B_0 be the contracting homotopy for the complex $(C_\bullet(I), b')$. The boundary map is given by the following recipe.

Let $x = \sum a_0 \otimes \ldots a_n$ *be a* b*-cycle in* $C_n(A/I)$*. Let* $\tilde{x} = \sum \tilde{a}_0 \otimes \ldots \tilde{a}_n$ *be its lift to a chain in* $C_n(A)$*. Then, provided that we can choose* $\tilde{x}$ *so that* $b\tilde{x} \in C_{n-1}(I)$*,*

$$\partial(x) = b\tilde{x}.$$

So suppose that we have an $x \in C_n(A/I)$ satisfying $bx = 0$ and let $\tilde{x}$ be a lift of x to $C_c(A)$. Since $bx = 0$,

$$b\tilde{x} \in \bigoplus_{k+l=n-1} A^{\otimes k} \otimes I \otimes A^{\otimes l}.$$

Using B_0 on the I factor, we get an element

$$X_1 \in \left(\bigoplus_{k+l=n-1} A^{\otimes k} \otimes I \otimes I \otimes A^{\otimes l} \right) \oplus \left(I \otimes A^{\otimes n-1} \otimes I \right)$$

such that, if we set $\tilde{x}_1 = \tilde{x} - X_1$,

$$\pi(\tilde{x}_1) = x \text{ and } b(\tilde{x}_1) \in \left(\bigoplus_{k+l=n-2} A^{\otimes k} \otimes I^{\otimes 2} \otimes A^{\otimes l} \right) \oplus \left(I \otimes A^{\otimes n-2} \otimes I \right).$$

One checks readily that b'_I, i. e. b' used on the $I^{\otimes 2}$ factor, kills $b(\tilde{x}_1)$ where, in the last summand, we will order the I-factors as $i_1 \otimes a_1 \otimes \ldots a_{n-2} \otimes i_0$. It implies that, if we again use B_0 on the $I^{\otimes 2}$ factor in $b\tilde{x}_1$, we get an X_2 in

$$\bigoplus_{k+l=n-1} A^{\otimes k} \otimes I^{\otimes 3} \otimes A^{\otimes l}$$

and such that $\tilde{x}_2 = \tilde{x}_1 - X_2$ satisfies

$$\pi(\tilde{x}_2) = x \text{ and } b(\tilde{x}_2) \in \bigoplus_{k+l=n-3} A^{\otimes k} \otimes I^{\otimes 3} \otimes A^{\otimes l}.$$

An obvious induction on the number of succesive I factors on $b\tilde{x}_\bullet$ completes the construction.

For the details of the proof we will refer the reader to the original paper [**605**]. □

COROLLARY 3.0.3. *Given a short exact sequence*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0,$$

where I *is H-unital, there exists a long exact sequence*

(3.4)

$$\ldots \xrightarrow{\partial} HC^{\#}_k(I) \xrightarrow{HC^{\#}(\iota)} HC^{\#}_k(A) \xrightarrow{HC^{\#}(\pi)} HC^{\#}_k(A/I) \xrightarrow{\partial} HC^{\#}_{k-1}(I) \xrightarrow{HC^{\#}(\iota)} HC^{\#}_{k-1}(A) \xrightarrow{HC^{\#}(\pi)} \ldots$$

$$\xrightarrow{\partial} HC^{\#}_0(I) \xrightarrow{HC^{\#}(\iota)} HC^{\#}_0(A) \xrightarrow{HC^{\#}(\pi)} HC^{\#}_0(A/I) \longrightarrow 0$$

where $HC^{\#}$ *stands for cyclic and negative cyclic homology. The corresponding two periodic version in cyclic periodic homology has the form of an exact triangle*

(3.5)
$$\begin{array}{ccc} HC^{per}_\bullet(I) & \longrightarrow & HC^{per}_\bullet(A) \\ & {\scriptstyle [1]}\nwarrow \quad \swarrow & \\ & HC^{per}_\bullet(A/I) & \end{array}.$$

SKETCH OF THE PROOF. Follows essentially from the act that, according to the above theorem, the inclusion of complexes

$$(C_\bullet(I), b) \to (\operatorname{Ker} H_\bullet(\pi), b)$$

is a quasiisomorphism, hence the same holds for the inclusion of the double complexes computing cyclic homologies. □

REMARK 3.0.4. We will see later that the excision in periodic cyclic homology holds without the H-unitality assumption on the ideal.

4. Homology of differential graded algebras

One can easily generalize all the above constructions to the case when A is a differential graded algebra (DGA). For future reference we will recall the definition.

DEFINITION 4.0.1. *A differential graded algebra (DGA) is a pair* (A, d)*, where* A *is a* $\mathbb{Z}$*-graded algebra and* d *is a derivation of degree* 1 *such that* $d^2 = 0$.

So suppose that (A, d) is a DGA. The action of d extends to an action on Hochschild chains of A by the Leibniz rule:

$$d(a_0 \otimes \cdots \otimes a_p) = \sum_{i=1}^{p} (-1)^{\sum_{k<i}(|a_k|+1)+1}(a_0 \otimes \cdots \otimes \delta a_i \otimes \cdots \otimes a_p)$$

The maps b and B are modified to include signs:

$$b(a_0 \otimes \ldots \otimes a_p) = \sum_{k=0}^{p-1} (-1)^{\sum_{i=0}^{k}(|a_i|+1)+1} a_0 \ldots \otimes a_k a_{k+1} \otimes \ldots a_p \tag{4.1}$$

$$+(-1)^{|a_p|+(|a_p|+1)\sum_{i=0}^{p-1}(|a_i|+1)} a_p a_0 \otimes \ldots \otimes a_{p-1}$$

$$B(a_0 \otimes \ldots \otimes a_p) = \sum_{k=0}^{p} (-1)^{\sum_{i\le k}(|a_i|+1)\sum_{i\ge k}(|a_i|+1)} 1 \otimes a_{k+1} \otimes \ldots \otimes a_p \otimes \tag{4.2}$$

$$\otimes a_0 \otimes \ldots \otimes a_k$$

The complex $C_\bullet(A)$ now becomes the total complex of the double complex with the differential $b + d$. In other words:

$$\widetilde{C}_\bullet(A) = (\bigoplus_{n\ge 0} A \otimes A^{\otimes n}, d + b);\ C_\bullet(A) = (\bigoplus_{n\ge 0} A \otimes \overline{A}^{\otimes n}, d + b) \tag{4.3}$$

The cyclic, negative cyclic, and the periodic cyclic complexes are defined as before using the new definition of $C_\bullet(A)$.

5. Cyclic cohomology

. *****Maybe a bit more***** The definitions of cyclic, negative cyclic and cyclic periodic cohomology follow the usual pattern of replacing the associated complexes with their linear duals and the boundary maps b and B with their transpose.

Note however that, since

$$\mathrm{Hom}_k(k[[u]], k) \simeq k[u^{-1}],$$

the cocycles are given by *finite sums* of cochains. So, for example, the complex computing periodic cyclic cohomology of a unital algebra A becomes the complex of continuos cochains, i. e.

$$(\mathrm{Hom}_k(CC_\bullet(A), k)[u^{-1}, u], b^t + u^{-1}B^t).$$

6. The Hochschild cochain complex

As usual, for any graded k-module E, $E[1]^p = E^{p+1}$ for all p; for any two graded k-modules E and F,

$$\underline{\mathrm{Hom}}^p(E, F) = \prod_{n\in\mathbb{Z}} \mathrm{Hom}_k(E^n, F^{n+p}) \tag{6.1}$$

$$(E \otimes F)^p = \bigoplus_{n\in\mathbb{Z}} E^n \otimes_k F^{p-n} \tag{6.2}$$

DEFINITION 6.0.1. *Let* $A = \oplus_{n\in\mathbb{Z}} A^n$ *be a graded module over a commutative unital ring* k. *The* k*-module of (non-normalized) Hochschild cochains of* A *is by definition*

$$\widetilde{C}^\bullet(A,\, A) = \prod_{n\geq 0} \underline{\mathrm{Hom}}(A[1]^{\otimes n}, A)$$

If 1 *is a chosen element of* A^0 *then the* k*-module of (normalized) Hochschild cochains of* A *is*

$$C^\bullet(A, A) = \prod_{n\geq 0} \underline{\mathrm{Hom}}(\overline{A}[1]^{\otimes n}, A)$$

where $\overline{A} = A/k \cdot 1$.

We will often shorten the notation and write $\widetilde{C}^\bullet(A)$ or $C^\bullet(A)$.

DEFINITION 6.0.2. *Suppose that* D *and* E *are homogeneous cochains on* A. *Set*

$$D\circ E)(a_1, \ldots, a_{d+e-1}) = \sum_{j\geq 0} (-1)^{(|E|+1)\sum_{i=1}^{j}(|a_i|+1)} D(a_1, \ldots, a_j, E(a_{j+1}, \ldots, a_{j+e}), \ldots);$$

and

$$[D,\ E] = D \circ E - (-1)^{(|D|+1)(|E|+1)} E \circ D.$$

The above bracket is called the Gerstenhaber bracket.

PROPOSITION 6.0.3. *Let* $A = \oplus_{n\in\mathbb{Z}} A_n$ *be a graded module over a commutative unital ring* k. *Then*

$$(C^\bullet(A)[1], [\ ,\ ])$$

is a graded Lie algebra.

Suppose moreover that A is a differential graded algebra. Define a non-normalized Hochschild 2-cochain m by

$$m = m_1 + m_2;\ m_1(a_1) = da_1;\ m_2(a_1, a_2) = (-1)^{|a_1|} a_1 a_2;$$

m_p vanishes on $A[1]^{\otimes q}$ with $n = 1$, 2 and $q \neq p$.

LEMMA 6.0.4. *One has*

$$m \circ m = 0$$

LEMMA 6.0.5. *The maps*

$$d : \widetilde{C}^\bullet(A) \to \widetilde{C}^{\bullet+1}(A);\ dD = [m_1, D]$$

and

$$\delta : \widetilde{C}^\bullet(A) \to \widetilde{C}^{\bullet+1}(A);\ \delta D = [m_2, D]$$

descend to the k*-module of normalized cochains* $C^\bullet(A)$.

LEMMA 6.0.6.
$(C^\bullet(A)[1], [\ ,\ ], d+\delta)$ *is a differential graded Lie algebra.*

DEFINITION 6.0.7. The cohomology of the complex $(C^\bullet(A), d+\delta)$ is called the Hochschild cohomology of the differential graded algebra A with coefficients in the A-bimodule A and will be denoted by $H^\bullet(A, A)$.

REMARK 6.0.8. Explicitly, one has

$$(\delta D)(a_1, \dots, a_{d+1}) = (-1)^{|a_1||D|+|D|+1} a_1 D(a_2, \dots, a_{d+1}) + \\ + \sum_{j=1}^{d} (-1)^{|D|+1+\sum_{i=1}^{j}(|a_i|+1)} D(a_1, \dots, a_j a_{j+1}, \dots, a_{d+1}) \\ +(-1)^{|D|\sum_{i=1}^{d}(|a_i|+1)} D(a_1, \dots, a_d) a_{d+1}$$

and

$$(dD)(a_1, \dots, a_d) = dD(a_1, \dots, a_d) - \sum_{j=1}^{p} \epsilon_j D(a_1, \dots, da_j, \dots, a_d)$$

where $\epsilon_j = (-1)^{\sum_{p<j}(|a_p|+1)}$.

In the case when A is an ordinary unital algebra, $H^\bullet(A, A)$ coincides with $\mathrm{Ext}^\bullet_{A\otimes A^{op}}(A, A)$.

DEFINITION 6.0.9. *Suppose that* A *is a graded associative algebra. For homogeneous cochains* D *and* E *from* $C^\bullet(A, A)$ *we set*

$$(D \smile E)(a_1, \dots, a_{d+e}) = (-1)^{|E|\sum_{i\le d}(|a_i|+1)} D(a_1, \dots, a_d) E(a_{d+1}, \dots, a_{d+e}).$$

Extending this by linearity to all of $C^\bullet(A)$ *we get the cup product*

$$\smile : C^i(A) \times C^j(A) \to C^{i+j}(A)$$

PROPOSITION 6.0.10. *Let* A *be a graded associative algebra. Then* $(C^\bullet(A, A)\ , \smile\ , d+\delta)$ *is a differential graded associative algebra.*

PROOF. Since the proof is a pure bookkeeping just like in the case of the previous proposition, we will refer the reader to the standard references [**101**] and [**249**]). □

REMARK 6.0.11. Under the isomorphism $H^\bullet(A, A) \simeq \mathrm{Ext}^\bullet_{A\otimes A^0}(A, A)$, the cup product induces the Yoneda product on Hochshild cohomology.

7. Braces

The following definition is essentially due to Gerstenhaber (see [**249**],[**262**]).

DEFINITION 7.0.1 (**Braces**). *Suppose that* A *is graded* k*-module and*

$$D_i,\ i = 0, \dots, m$$

are Hochschild cochains on A*. The following formula defines a new Hochschild cochain on* A*:*

$$D_0\{D_1, \dots, D_m\}(a_1, \dots, a_n) = \\ \sum_{i_1,\dots,i_m} \epsilon_{i_1,\dots,i_m} D_0(a_1, \dots, a_{i_1}, D_1(a_{i_1+1}, \dots), \dots, D_m(a_{i_m+1}, \dots), \dots, a_n)$$

where the sign is given by

$$\epsilon_{i_1,\dots,i_m} = (-1)^{\sum_p \sum_{k\le i_p}(|a_k|+1)(|D_p|+1)}$$

PROPOSITION 7.0.2. *One has*

$$(D\{E_1, \ldots, E_k\})\{F_1, \ldots, F_l\} = \sum (-1)^{\sum_{q \le i_p}(|E_p|+1)(|F_q|+1)} \times$$
$$\times D\{F_1, \ldots, E_1\{F_{i_1+1}, \ldots,\}, \ldots, E_k\{F_{i_k+1}, \ldots,\}, \ldots,\}$$

PROOF. The proof of the statement reduces immediately to the question of bookkeeping and is left as an exercise to the reader. ***OR: in terms of NC diff ops...*** □

The above proposition can be restated as follows.

PROPOSITION 7.0.3. *Suppose that A is an associative algebra and endow both $C^\bullet(A)$ and $C^\bullet(C^\bullet(A))$ with the differential graded algebra structure induced by the cup product. For a cochain D on A let $D^{(k)}$ be the following k-cochain on $C^\bullet(A)$:*

$$D^{(k)}(D_1, \ldots, D_k) = D\{D_1, \ldots, D_k\}$$

Then the map

$$C^\bullet(A) \to C^\bullet(C^\bullet(A))$$

given by

$$D \mapsto \sum_{k \ge 0} D^{(k)}$$

is a morphism of differential graded algebras.

7.1. Hochschild cochains as coderivations. Let V be a ($\mathbb{Z}$-)graded k-module. The tensor algebra

$$TV = \oplus_{n \ge 1} V^{\otimes n}$$

has the structure of the universal counital coalgebra (co-)generated by V. We give $T^c V$ the standard grading, i. e.

$$|v_1 \otimes \ldots \otimes v_n| = \sum_k |v_k|.$$

The reduced tensor algebra

$$T^c V = \oplus_{n \ge 1} V^{\otimes n}$$

is the quotient of TV by the image of the counit.

For any graded coalgebra B we denote by Coder(B) the graded Lie algebra of its coderivations.

LEMMA 7.1.1. *For a graded k-module A there is an isomorphism of differential graded Lie algebras*

$$\widetilde{C}^\bullet(A)[1] \xrightarrow{\sim} \mathrm{Coder}(T(A[1]))$$

PROOF. Recall that the universal coalgebra generated by a vector space V is a coalgebra $C(V)$ together with a linear map $\pi : C(V) \to V$ such that, given any coalgebra C, every linear map $\phi : C \to V$ has a unique extension to a coalgebra morphism $\tilde{\phi} : C \to C(V)$ such that $\phi = \pi \circ \tilde{\phi}$. By universality, a coderivation D of $T(A[1])$ is uniquely determined by the composition

$$m : T(A[1]) \xrightarrow{D} T(A[1)] \to A[1],$$

where the second map is the projection of the tensor coalgebra on its first direct summand. It is straightforward that the Gerstenhaber Lie bracket from Definition 6.0.2 corresponds to commutator of coderivations. □

8. A_∞ algebras and their Hochschild complexes

DEFINITION 8.0.1. *An A_∞-algebra structure on A is a degree 1 coderivation D of $T^c(A[1])$ satisfying $D^2 = 0$.*

DEFINITION 8.0.2. *For an A_∞ algebra A, the DG coalgebra $(T(A), D)$ is called the bar construction of A and denoted by* $\mathrm{Bar}(A)$.

Let us record the following alternative definition.

LEMMA 8.0.3. *An A_∞ structure on a graded k-module V is given by a Hochschild cochain m on V of degree 2 satisfying the identity*

$$m \circ m = 0.$$

PROOF. Follows from Lemma 7.1.1. A little bit more precisely, the $D^2 = 0$ condition is easily seen to be equivalent to the associativity condition $m \circ m = 0$. □

REMARK 8.0.4. A Hochschild cochain as in the lemma above has the form of infinite sum

$$m = m_1 + m_2 + m_3 + \dots,$$

where

$$m_p \in \mathrm{Hom}^1_k(V[1]^{\otimes p}, V[1]) = \prod_{n_1+\dots+n_p-n=p-2} \mathrm{Hom}(V^{n_1} \otimes \dots V^{n_p}, V^n)$$

We set

$$d = m_1$$

and, for homogeneous elements a_1 and a_2 of V,

$$m(a_1, a_2) = (-1)^{|a_i|} m_2(a_1, a_2).$$

Then

- d is a differential of degree one on V;
- m is a graded bilinear product on V which is associative up to homotopy determined by m_3 and such that $[d, m] = 0$;
- m_3 satisfies, up to homotopy m_4, the pentagonal identity

$$m_2(m_3(a_1, a_2, a_3), a_4) \pm m_2(a_1, m_3(a_2, a_3, a_4)) =$$

$$m_3(m_2(a_1, a_2), a_3, a_4) \pm m_3(a_1, m_2(a_2, a_3), a_4) \pm m_3(a_1, a_2, m_2(a_3, a_4)) = 0,$$

etc.

In particular, the following holds:

PROPOSITION 8.0.5 (Quillen). *****Are we sure it is Quillen?**** An A_∞-structure on a graded vector space V of the form $m = m_1 + m_2$ is the same as the structure of a differential graded algebra (V, m, d) (in the notation above).*

Once we have description of an A_∞ structure in the terms of the lemma 8.0.3, the following definition is quite natural.

DEFINITION 8.0.6. *An A_∞-module over an A_∞ algebra (A, m_A) is a graded k-module M and a degree one element*

$$m_M \in \mathrm{Hom}(M[1] \otimes T^c(A[1]), M[1])$$

satisfying

$$m_M \circ m = 0,$$

where the right hand m *stands for* m_A *or* m_M *depending on whether its arguments include an element of* M.

Before continuing we need a bit of notation.

DEFINITION 8.0.7. *Let* A *be a graded vector space and* D *a Hochschild cochain on* A. *We set*

$$\begin{aligned} L_D(a_0 \otimes \ldots \otimes a_n) = D(a_0 \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n + \\ \textstyle\sum_{k=0}^{n-d} \epsilon_k a_0 \otimes \ldots \otimes D(a_{k+1}, \ldots, a_{k+d}) \otimes \ldots \otimes a_n + \\ \textstyle\sum_{k=n+1-d}^{n} \eta_k D(a_{k+1}, \ldots, a_n, a_0, \ldots) \otimes \ldots \otimes a_k, \end{aligned}$$

where the second sum in the above formula is taken over all cyclic permutations such that a_0 *is inside* D. *The signs are given by*

$$\epsilon_k = (-1)^{(|D|+1)\sum_{i=0}^{k}(|a_i|+1)}$$

and

$$\eta_k = (-1)^{|D|+1+\sum_{i\leq k}(|a_i|+1)\sum_{i\geq k}(|a_i|+1)}$$

PROPOSITION 8.0.8.

$$[L_D, L_E] = L_{[D,E]} \text{ and } [L_D, B] = 0.$$

PROOF. We will leave the proof as an exercise for the reader. □

DEFINITION 8.0.9. *Suppose that* (A, m) *is an* A_∞*-algebra. Then*

(1) *The non-normalized Hochschild chain complex of* A *is*

$$(\widetilde{C}_\bullet(A), L_m),$$

(2) *The non-normalized Hochschild cochain complex of* A *is*

$$(\widetilde{C}^\bullet(A), [m,\ ]).$$

DEFINITION 8.0.10. Let (A, m) is a unital A_∞ algebra, i.e. assume there is an element $1 \in A$ satisfying $m_2(1, a) = (-1)^{|a|} m_2(a, 1) = a$ for all homogeneous $a \in A$ and $m_k(\ldots, 1, \ldots) = 0$. Then the differential $[m,\ ]$ descends to $C_\bullet(A)$. We define the (normalized) Hochschild cochain, resp. chain, complex of A to be

$$(C^\bullet(A), [m,\ ]), \text{resp.} (C_\bullet(A), L_m).$$

Let u be an element of degree -2. Then $[L_m, B] = 0$ and the negative cyclic complex of A is defined by

$$CC_*^-(A) = (C_*(A)[[u]], L_m + uB)$$

and similarly for the periodic cyclic and cyclic complexes.

A simple modification (using full and not reduced Hochschild complexes) can be given for non-unital A_∞ algebras.

8.1. A_∞ morphisms. Given two A_∞ algebras A and B, an A_∞ morphism $T : A \to B$ is a morphism of differential graded coalgebras

$$(T^c(A[1]), D_A) \to (T^c(B[1]), D_B)$$

As in the proof of Lemma 7.1.1, any morphism of graded coalgebras is determined by its composition with the projection $T^c B[1] \to B[1]$ which amounts to a collection of

$$F_n : A^{\otimes n} \to A \tag{8.1}$$

of degree $1 - n$, $n \geq 1$. Intertwining m_A with m_B is equivalent to the relation

(8.2)

$$\sum_{j,k} \pm \epsilon_{jk} F_{n-k}(a_1, \ldots, m(a_{j+1}, \ldots, a_{j+k}), \ldots, a_n) +$$

$$\sum_{p \geq 1; n_1, \ldots, n_{p-1}} \eta_{n_1, \ldots, n_{p-1}} m_p(F_{n_1}(a_1, \ldots, a_{n_1}), \ldots, F_{n_p}(a_{n_{p-1}+1}, \ldots, a_n)) = 0$$

where the signs are ****** Clearly, for two A_∞ morphisms $A \to B \to C$, their composition $A \to C$ is defined.

8.1.1. A_∞ *morphisms acting on Hochschild and cyclic complexes.* For an A_∞ morphism $F : A \to B$, define

$$F_* : a_0 \otimes \ldots \otimes a_n \mapsto \sum \pm F_{n_0}(A_0) \otimes \ldots \otimes F_{n_p}(A_p) \tag{8.3}$$

where $(A_0, \ldots, A_p)$ run through all subdivisions of some cyclic permutation $(a_{j+1}, \ldots, a_j)$ into $p + 1$ segments so that A_0 contains a_0. The sign is

$$(-1)^{\sum_{k>j}(|a_k|+1) \sum_{k \leq j}(|a_k|+1)}.$$

PROPOSITION 8.1.1. *Formula* (8.3) *defines a morphism of Hochschild complexes*

$$F_* : \widetilde{C}_\bullet(A) \to \widetilde{C}_\bullet(B)$$

commuting with the cyclic differential B.

PROOF. This can be done by direct computation or using the interpretation of the Hochschild complexes given in 6. □

PROPOSITION 8.1.2. *Assume that* F *is an* A_∞ *morphism such that* F_1 *is a quasi-isomorphism. Then* F_* *is a quasi-isomorphism.*

PROOF. Indeed, F_* preserves the filtration

$$\mathcal{F}_n = \bigoplus_{m \leq n} A \otimes A^{\otimes m} \tag{8.4}$$

and induces a quasi-isomorphism on differential graded quotients. □

If A and B are A_∞ algebras with unit then an A_∞ morphism F is called unital if $F_1(1) = 1$ and $F_n(\ldots, 1, \ldots) = 0$ for $n \geq 2$. It is easy to see that in this case F_* descends to a morphism

$$F_* : C_\bullet(A) \to C_\bullet(B) \tag{8.5}$$

An analogue of Proposition 8.1.2 is true in this case.

REMARK 8.1.3. The projection $\widetilde{C}_\bullet(A) \to C_\bullet(A)$ is a quasi-isomorphism. Indeed, consider the spectral sequence associated to the filtration (8.4). Its E_1 term is $\widetilde{C}_\bullet(H^*(A))$, resp.

$$\bigoplus_{n\geq 0} H^*(A) \otimes H^*(\overline{A})^{\otimes n}$$

If the above were $C_\bullet(H^*(A))$, the projection would be an isomorphism of the E_2 terms and we would be done, but...********

8.2. The bialgebra structure on $\mathrm{Bar}(C^\bullet(A, A))$**.** Let us first recall the product on the bar construction $\mathrm{Bar}(C^\bullet(A, A))$ where $C^\bullet(A, A)$ is the algebra of Hochschild cochains of an algebra A with coefficients in A (cf. [**269**], [**260**]). For cochains D_i and E_j, define

$$(D_1|\ldots|D_m) \bullet (E_1|\ldots|E_n) = \sum \pm(\ldots|D_1\{\ldots\}|\ldots|D_m\{\ldots\}|\ldots)$$

Here the space denoned by $\ldots$ inside the braces contains $E_{j+1}, \ldots, E_k$; outside the braces, it contains $E_{j+1}|\ldots|E_k$. The factor $D_i\{E_{j+1}, \ldots, E_k\}$ is the brace operation as in Section 7. The sum is taken over all possible combinations for which the natural order of E_j's is preserved. The signs are computed as follows: a transposition of D_i and E_j introduces a sign $(-1)^{(|D_i|+1)(|E_j|+1)}$. In other words, the right hand side is the sum over all tensor products of $D_i\{E_{j+1}, \ldots, E_k\}$, $k \geq j$, and E_p, so that the natural orders of D_i's and of E_j's are preserved. For example,

$$(D) \bullet (E) = (D|E) + (-1)^{(|D|+1)(|E|+1)}(E|D) + D\{E\}$$

PROPOSITION 8.2.1. *The product $\bullet$ together with the comultiplication Δ make* $\mathrm{Bar}(C^\bullet(A, A))$ *an associative bialgebra.*

9. Homotopy and homotopy equivalence

9.1. The case of DG algebras. Let $C^\bullet(\Delta^1)$ be the algebra of (nondegenerate) cochains with coefficients in k of the one-simplex Δ^1 with the standard triangulation. Explicitly, this algebra has the basis e_0, e_1, ξ where

$$|e_0| = |e_1| = 0;\ |\xi| = 1;\ de_0 = \xi = -de_1;\ e_0^2 = e_0;\ e_1^2 = e_1;$$

$$e_0e_1 = e_1e_0 = 0;\ e_0\xi = \xi e_1 = \xi; e_1\xi = \xi e_0 = 0$$

There are two DGA morphisms

$$\mathrm{ev}_0,\ \mathrm{ev}_1 : C^\bullet(\Delta^1) \to k$$

given by $\mathrm{ev}_i(e_j) = \delta_i^j$ and $\mathrm{ev}_i(\xi) = 0$.

DEFINITION 9.1.1. *Two morphisms of* DG *algebras* f_0, $f_1 : A \to B$ *are homotopic if there exists a morphism* $f : A \to B \otimes C^\bullet(\Delta^1)$ *such that* $\mathrm{ev}_j \circ f = f_j$ *for* $j = 0, 1$. *A homotopy equivalence between* A *and* B *is a pair of morphisms* $f : A \to B$ *and* $g : B \to A$ *such that* gf *is homotopic to* id_A *and* fg *is homotopic to* id_B.

Explicitly, a homotopy between A and B is a k-linear map $A \to B$ of degree one satisfying

$$D(ab) = D(a)g(b) + (-1)^{|a|}f(a)D(b); \tag{9.1}$$

$$[d, D] = f_0 - f_1. \tag{9.2}$$

Indeed, D satisfies the two equalities above if and only if

$$f(a) = f_0(a)e_0 + f_1(a)e_1 + D(a)\xi,$$

is a DGA morphism.

9.2. The case of A_∞ algebras. We start by rewriting the definition of two homotopic morphisms of DG algebras from 9.1 in a way that works for A_∞ morphisms of $A_i nfty$ algebras. First note that a pair of A_∞ morphisms $f, g : A \to B$ turn B into an A_∞ bimodule ${}_fB_g$. If B is unital, then one has a zero-cochain $\mathbf{1}$ in $C^\bullet(A,{}_f B_g)$ given by

$$ {}_f\mathbf{1}_g = 1 \in B^0 \tag{9.3}$$

One has

$$\delta_m({}_f\mathbf{1}_g) = f - g \tag{9.4}$$

Definition 9.2.1. *A homotopy between two A_∞ morphisms f_0, $f_1 : A \to B$ is a cochain D of total degree one in $C^\bullet(A,{}_{f_0}\mathcal{B}_{f_1})$ that is supported on the product of terms with $n \geq 1$ (9.1) and satisfies*

$$\delta_m D = f_0 - f_1 \tag{9.5}$$

In other words, if B is unital, a homotopy between f_0 and f_1 is an extension ${}_{f_0}\mathbb{I}_{f_1}$ of ${}_{f_0}\mathbf{1}_{f_1}$ to a Hochschild cocycle.

Lemma 9.2.2. *Being homotopic is an equivalence relation on A_∞ morphisms $A \to B$.*

Proof. Start with B being a unital DG algebra. In terms of Definition 9.2.1 we can define

$${}_{f_0}\mathbb{I}_{f_0} = {}_{f_0}\mathbf{1}_{f_0};\ {}_{f_0}\mathbb{I}_{f_2} = {}_{f_0}\mathbb{I}_{f_1} \cup {}_{f_1}\mathbb{I}_{f_2};\ {}_{f_1}\mathbb{I}_{f_0} = ({}_{f_0}\mathbb{I}_{f_1})^{-1}$$

(the inverse is with respect to the cup product). If there is no unit then we can formally attach it. If B is A_∞ then nothing changes: the cup product is still defined exactly as before in terms of m_2 (now it is not associative but is a part of an A_∞ structure, in particular a morphism of complexes which is enough for us). □

Definition 9.2.3. *An A_∞ homotopy equivalence between A and B is a pair $F : A \to B$ and $G : B \to A$ such that gf is homotopic to id_A and fg is homotopic to id_B.*

We also say that each f_0 and f_1 is an A_∞ homotopy equivalence.

Lemma 9.2.4. *Being homotopy equivalent is an equivalence relation.*

Proof. The relation is obviously reflexive and symmetric. As for transitivity, observe that, given

$$A \xrightarrow{f} B \underset{g_1}{\overset{g_0}{\rightrightarrows}} C \xrightarrow{h} D$$

We claim that if g_0 is homotopic to g_1 then hg_0f is homotopic to hg_1f. Indeed, we can put

$${}_{hg_0f}\mathbb{I}_{hg_0f} = h_*f^*{}_{g_0}\mathbb{I}_{g_1}$$

Now assume that we are given

$$A \underset{f_1}{\overset{f_0}{\rightleftarrows}} B \underset{g_1}{\overset{g_0}{\rightleftarrows}} C$$

such that $f_1 f_0$ is homotopic to Id_A and $g_1 g_0$ is homotopic to Id_B. Then $f_1 g_1 g_0 f_0$ is homotopic to $f_1 f_0$ which is homotopic to Id_A. Similarly in the opposite direction. □

If we use the $\bullet$ product ***REF*** we can prove more:

LEMMA 9.2.5. *For A_∞ algebras and A_∞ morphisms as shown on the diagram below, assume that f_0 is homotopic to f_1 and g_0 is homotopic to g_1. Then $g_0 f_0$ is homotopic to $g_1 f_1$.*

$$A \underset{f_1}{\overset{f_0}{\rightrightarrows}} B \underset{g_1}{\overset{g_0}{\rightrightarrows}} C$$

PROOF. In fact, we can put

$${}_{g_0 f_0}\Vert_{g_1 f_1} = {}_{f_0}\Vert_{f_1} \bullet {}_{g_0}\Vert_{g_1}$$

□

CHAPTER 3

The cyclic complex $C^\lambda_\bullet$

In this chapter we assume $\mathbb{Q} \subset k$.

1. Introduction

2. Definition

Recall the original definition of the cyclic complex from [**111**], [**?**]. As in (2.1), put

$$\tau(a_0 \otimes \cdots \otimes a_p) = (-1)^p a_p \otimes a_0 \cdots \otimes a_{p-1}$$

Let

$$C^\lambda_p(A) = A^{\otimes_k (p+1)}/\mathrm{Im}(\mathrm{id} - \tau) \ . \tag{2.1}$$

Because of formulas (2.2), the differential b descends to a map

$$b : C^\lambda_p(A) \to C^\lambda_{p-1}(A) \ .$$

PROPOSITION 2.0.1. *The complex $C^\lambda_\bullet(A)$ is quasi-isomorphic to the complex* $\mathbf{CC}_\bullet(A)$ *(Definition 2.0.2.*

SKETCH OF THE PROOF. Consider the diagram

$$\begin{array}{ccccccccccccc}
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b \\
\cdots & \xrightarrow{1-\tau} & A^{\otimes 3} & \xrightarrow{N} & A^{\otimes 3} & \xrightarrow{1-\tau} & A^{\otimes 3} & \xrightarrow{N} & A^{\otimes 3} & \xrightarrow{1-\tau} & A^{\otimes 3} & \Rightarrow & C^\lambda_2(A) \\
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b \\
\cdots & \xrightarrow{1-\tau} & A^{\otimes 2} & \xrightarrow{N} & A^{\otimes 2} & \xrightarrow{1-\tau} & A^{\otimes 2} & \xrightarrow{N} & A^{\otimes 2} & \xrightarrow{1-\tau} & A^{\otimes 2} & \Rightarrow & C^\lambda_1(A) \\
 & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b' & & \downarrow b & & \downarrow b \\
\cdots & \xrightarrow{1-\tau} & A & \xrightarrow{N} & A & \xrightarrow{1-\tau} & A & \xrightarrow{N} & A & \xrightarrow{1-\tau} & A & \Rightarrow & C^\lambda_0(A)
\end{array}$$

Since the rows are acyclic in positive dimensions, the stipled arrows give a quasi-isomorphism from the total complex to the $(C^\lambda_\bullet(A), b)$ complex.

□

3. The reduced cyclic complex

Now let A be unital.
Let $\overline{A} = A/k$ and let

$$\overline{C}^\lambda_p(A) = \overline{A}^{\otimes p+1}/\mathrm{Im}(\mathrm{id} - \tau) \ . \tag{3.1}$$

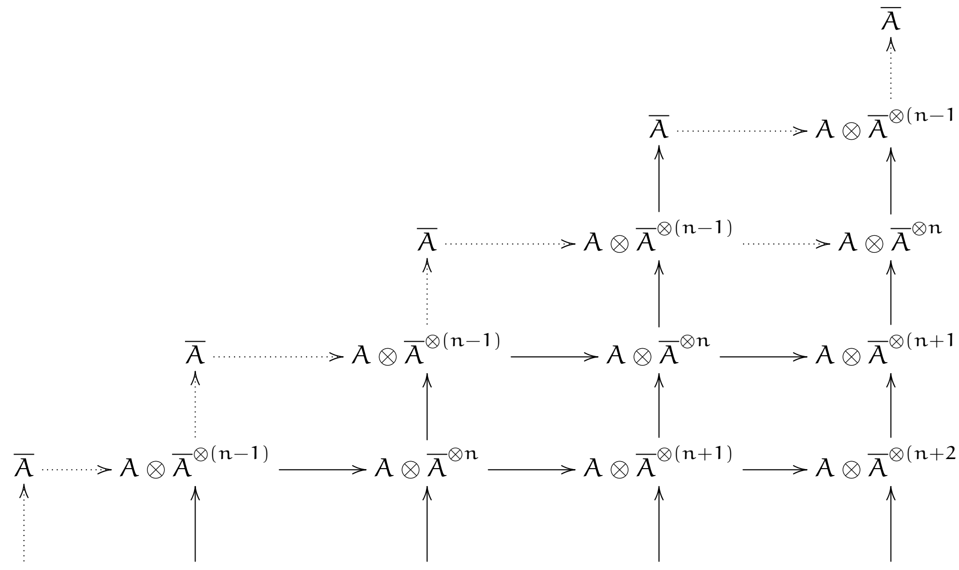
It is easy to see that the diferential b descends to $\overline{C}^{\lambda}_{\bullet}(A)$. We denote the homology of the complex $\overline{C}^{\lambda}_{\bullet}(A)$ by $\overline{HC}_{\bullet}(A)$.

The following is the reduced analogue of the proposition 2.0.1.

PROPOSITION 3.0.1. *The complex* $(\overline{C}^{\lambda}_{\bullet}(A), b)$ *is quasiisomorphic to the complex* $\overline{CC}_{\bullet}(A)$ *given by the cokernel of the inclusion of complexes*

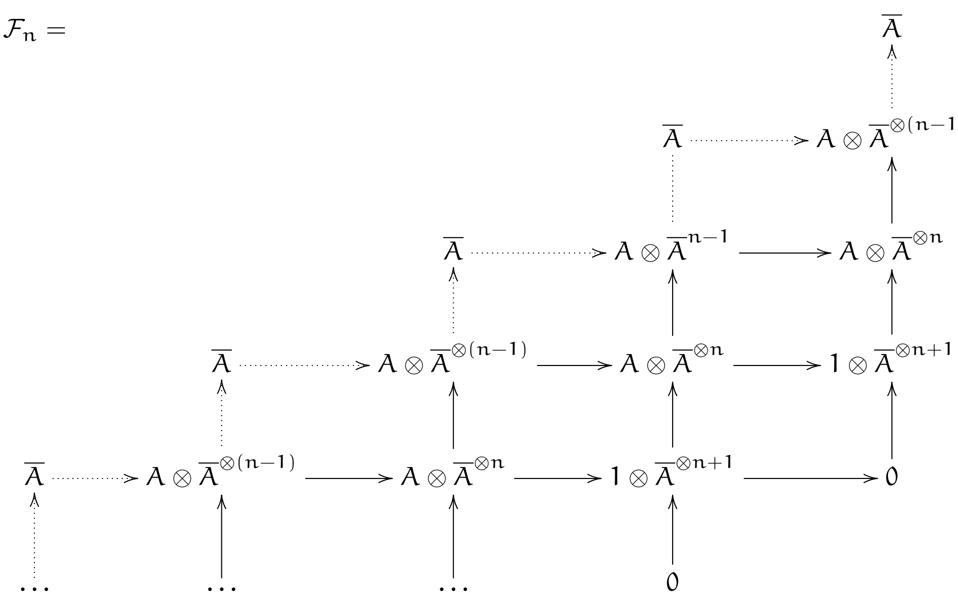
$$CC_{\bullet}(k) \to CC_{\bullet}(A).$$

PROOF. $\overline{CC}_{\bullet}(A)$ has the form



where, for simplicity of the grafical representation, we did not include the powers of u. We will filter it by subcomplexes of the form

$\mathcal{F}_n =$



The corresponding spectral sequence collapses so that $E^2 = E^\infty$ and, as immediately seen, $\bigoplus_{p+q=n} E^2_{p,q} = \overline{C}^\lambda_n(A)$ and $d_2 = b$. □

Proposition 3.0.2. *There is an exact triangle*

$$C^\lambda_\bullet(k) \to C^\lambda_\bullet(A) \to \overline{C}^\lambda_\bullet(A) \to C^\lambda_\bullet(k)[1] \ . \tag{3.2}$$

Proof. Since by the proposition 3.0.1, C^λ and $\overline{CC}$ complexes are quasiisomorphic, the claim is just the formulation of the fact that, by the definition of $\overline{CC}$, we have the short exact sequence of complexes

$$0 \to CC_\bullet(k) \to CC_\bullet(A) \to \overline{CC}_\bullet(A) \to 0$$

□

Remark 3.0.3. The above proposition could also be deduced from the Hochschild-Serre spectral sequence associated to the inclusion $\mathfrak{gl}(k) \to \mathfrak{gl}(A)$ with the E^2-term

$$E^2_{pq} = H_p(\mathfrak{gl}(A), \mathfrak{gl}(k)) \otimes H_q(\mathfrak{gl}(k))$$

which converges to $H_{p+q}(\mathfrak{gl}(A))$ and from theorem 4.0.2).

4. Relation to Lie algebra homology

Let us start by recalling some standard notions from Lie algebra (co-)homology.

Definition 4.0.1. *Let $(\mathfrak{g}, d)$ be a DGLA. The standard Chevalley-Eilenberg complex of chains of $\mathfrak{g}$ with coefficients in the trivial module k has the form*

$$C_\bullet(\mathfrak{g}) = (\Lambda^\bullet \mathfrak{g}[1], \partial^{Lie} + d),$$

where the Lie boundary operator is defines by

$$\partial^{Lie}(X_1 \wedge \ldots \wedge X_n) = \sum_{i<j} c_{i,j}[X_i, X_j] \wedge X_1 \wedge \hat{X_i} \wedge \ldots \wedge \hat{X_j} \wedge \ldots \wedge X_n.$$

Here $\hat{\cdot}$ means that the corresponding element of the product should be omitted, and the sign rule is

$$c_{ij} = (-1)^{(|X_i|+1)\sum_{k<i}(|X_k|+1)+(|X_j|+1)\sum_{k<j,k\neq i}(|X_k|+1)}$$

Let $\mathfrak{h}$ be a DG-subalgebra of $\mathfrak{g}$ acting reductively on $\mathfrak{g}$. The complx of coinvariants of $C_\bullet(\mathfrak{g})$ with respect to the adjoint action of $\mathfrak{h}$ will be denoted by

$$C_\bullet(\mathfrak{g})_\mathfrak{h}$$

and

$$C_\bullet(\mathfrak{g}, \mathfrak{h}) = \Lambda(\mathfrak{g}/\mathfrak{h})^\mathfrak{h}$$

denotes the complex of relative chains.

For any DG algebra (A, δ) over k let $\mathfrak{gl}(A) = \varinjlim_n \mathfrak{gl}_n(A)$, where the imbeddings $\mathfrak{gl}_n(A) \hookrightarrow \mathfrak{gl}_{n+1}(A)$ are of the form

$$X \longrightarrow \begin{pmatrix} X & 0 \\ 0 & 0 \end{pmatrix}.$$

$\mathfrak{gl}(A)$ can be thought of as the Lie algebra of $\mathbb{N} \times \mathbb{N}$-matrices $M_\infty(A)$ with finitely many non-zero coefficients in A. Note that $\mathfrak{gl}(k)$ is a DG Lie subalgebra of $\mathfrak{gl}(A)$. Theorem 4.0.2 below identifies the cyclic complex of A, resp. the relative cyclic complex of A, with the subcomplex of primitive elements of the DG coalgebra of coinvariants $C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}$, resp. $C_\bullet(\mathfrak{gl}(A),\mathfrak{gl}(k))$. Note that these complexes have naturally the structure of Hopf algebras, with the product induced by the diagonal block inclusion

$$\mathfrak{gl}(A)\times\mathfrak{gl}(A)\to\mathfrak{gl}(A)\oplus\mathfrak{gl}(A)\hookrightarrow\mathfrak{gl}(A).$$

It is associative since we work with coinvariants (resp. relative complex). The coproduct is induced by the diagonal map

$$\Delta:\mathfrak{gl}(A)\ni x\to x\oplus x\in\mathfrak{gl}(A)\oplus\mathfrak{gl}(A)$$

using the canonical identification

$$\Lambda^\bullet(\mathfrak{gl}(A)\oplus\mathfrak{gl}(A))\simeq\Lambda^\bullet(\mathfrak{gl}(A))\otimes\Lambda^\bullet(\mathfrak{gl}(A)).$$

Let E^a_{pq} denote the elementary matrix with $\left(E^a_{pq}\right)_{pq}=a$ and other entries equal to zero.

THEOREM 4.0.2. *The map*

$$\begin{array}{rcl} A^{\otimes p+1} & \to & \bigwedge^{p+1}\mathfrak{gl}(A) \\ a_0\otimes\cdots\otimes a_p & \mapsto & E^{a_0}_{01}\wedge E^{a_1}_{12}\wedge\cdots\wedge E^{a_p}_{p-1,0} \end{array}$$

induces isomorphisms of complexes

$$\begin{array}{rcl} C^\lambda_\bullet(A) & \to & \mathrm{Prim}\, C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}[1] \\ \overline{C}^\lambda_\bullet(A) & \to & \mathrm{Prim}\, C_\bullet(\mathfrak{gl}(A),\mathfrak{gl}(k))[1] \end{array}$$

SKETCH OF THE PROOF. The basic part of the proof is the identification of the primitive part of the, say, complex

$$C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}.$$

$\mathfrak{gl}(A)$ acts reductively on $\mathfrak{gl}(A)$ and, by basic invariant theory,

$$(M_\infty(k)^{\otimes n})_{\mathfrak{gl}(k)}=k[\Sigma_n].$$

Let σ denote the sign representation of the symmetric group Σ_n. Then

$$\Lambda^n(\mathfrak{gl}(k)\otimes A)_{\mathfrak{gl}(k)}=((\mathfrak{gl}(k)\otimes A)^n\otimes_{\Sigma_n}\sigma)_{\mathfrak{gl}(k)}=((\mathfrak{gl}(k)^n_{\mathfrak{gl}(k)}\otimes A)^n\otimes_{\Sigma_n}\sigma)=(k[\Sigma_n]\otimes A^{\otimes n})\otimes_{k[\Sigma_n]}\sigma.$$

In the left-most term, the symmetric group acts on itself by conjugation. It is now an exercise to check that action of the coproduct on the terms on the right hand side translates into an expression of the form

$$\pi\otimes(a_1\otimes a_2\otimes\dots a_n)\to\sum(\pi|_I\otimes a_I)\otimes(\pi|_J\otimes a_J),$$

where the sum is over all partitions $\{1,2,\dots,n\}=I\cup J$ which are invariant under the action of the permutation π and, if

$$I=\{i_1,\dots i_k\}\subset\{1,2,\dots,n\}$$

then

$$a_I=a_{\pi(i_1)}\otimes\dots\otimes a_{\pi(k)}.$$

In particular, the primitive part of

$$(k[\Sigma_n]\otimes A^{\otimes n})\otimes_{k[\Sigma_n]}\sigma$$

is given by the conjugacy class of the cyclic permutation $\tau\in\Sigma_n$, i.e.

$$\mathrm{Prim}\, C_\bullet(\mathfrak{gl}(A))_{\mathfrak{gl}(k)}[1]=A^{\otimes n}/(1-\tau)=C^\lambda_{n-1}$$

Let # denote the trace map

$$(M_\infty(k) \otimes A)^{\otimes \bullet} \to A^{\otimes \bullet}$$

given by

$$(T_1 \otimes a_1) \otimes (T_2 \otimes a_2) \otimes \ldots \otimes (T_n \otimes a_n) \mapsto \mathrm{Tr}(T_1 T_2 \ldots T_n) a_1 \otimes a_2 \otimes \ldots \otimes a_n.$$

One checks that # implements the above identification and intertwines the boundary maps completing the proof. □

5. The connecting morphism

Here we give an explicit formula for the connecting morphism ∂ of the exact triangle from Proposition 3.0.2. Suppose that (A, δ) is a DGA. Let j be a k-linear map $A \to k$ satisfying $j(1) = 1$. The splitting j induces the splitting

$$\rho : \mathfrak{gl}(A) \to \mathfrak{gl}(k)$$

of the inclusion $\mathfrak{gl}(k) \hookrightarrow \mathfrak{gl}(A)$. We set *the curvature of* ρ to be equal to

(5.1)

$$R(\rho) = (\partial^{\mathrm{Lie}} + \delta)\rho + \frac{1}{2}[\rho, \rho] \in \mathrm{Hom}_k(\mathfrak{gl}(A)[1], \mathfrak{gl}(k)) \oplus \mathrm{Hom}_k(\Lambda^2 \mathfrak{gl}(A)[1], \mathfrak{gl}(k))$$

Let P_n denote the invariant polynomial $X \mapsto \dfrac{1}{n!}\mathrm{tr}(X^n)$ on $\mathfrak{gl}(k)$. Set

$$c_n = P_n(R(\rho)).$$

Then $c_n \in C^{2n}_{\mathrm{Lie}}(\mathfrak{gl}(A); \mathfrak{gl}(k)))$ is a relative Lie algebra cocycle and, by the theorem 4.0.2, defines a linear map

$$\mathrm{ch}_n(\rho) : \overline{C}^{\lambda}_{2n+1}(A) \to k \tag{5.2}$$

which descends to homology.We will set

$$1^{(n+1)} = n!(n+1)! \cdot 1^{\otimes 2n+1} \in C^{\lambda}_{2n}(k) \tag{5.3}$$

Proposition 5.0.1. *The morphism* $\mathrm{Br}^A : \overline{C}^{\lambda}_{\bullet}(A) \to C^{\lambda}_{\bullet}(k)[1]$ *given by*

$$\mathrm{Br}^A = \sum_n \mathrm{ch}_n(\rho) 1^{(n+1)}$$

represents the connecting morphism in the triangle (3.2).

5.1. Explicit formula for the product $HC_p \otimes HC_q \to HC_{p+q+1}$. Here we give an explicit formula for the product $\times$ from Theorem 1.0.5 in one of the realizations of the cyclic complex. Note first that, because of (2.2), the map N induces an isomorphism

$$C^{\lambda}(A) \simeq (\mathrm{Ker}(\mathrm{id} - \tau), b') \tag{5.4}$$

where in the right hand side $\mathrm{id} - \tau$ is considered as an operator on $A^{\otimes(\bullet+1)}$.

Proposition 5.1.1. *Suppose that* A *and* C *are unital algebras. We will identify them as aubalgebras of* $A \otimes C$ *using the imbeddings*

$$A \ni a \to a \otimes 1 \in A \otimes C \text{ and } C \ni c \to 1 \otimes c \in A \otimes C.$$

After identifiyng $C^{\lambda}_{\bullet}$ *with the right hand side of (5.4), then the shuffle product*

$$(a_0 \otimes \ldots \otimes a_p) \times (c_0 \otimes \ldots \otimes c_q) = \mathrm{sh}_{p+1,q+1}(a_0, \ldots a_p, c_0, \ldots, c_q)$$

is a morphism of complexes

$$\times : C^{\lambda}_{\bullet}(A) \otimes C^{\lambda}_{\bullet}(C) \to C^{\lambda}_{\bullet+1}(A \otimes C) \tag{5.5}$$

which induces on homology the $\times$ product from Theorem 1.0.5.

By the same construction one defines the product

$$\times : \overline{C}^{\lambda}_{\bullet}(A) \otimes \overline{C}^{\lambda}_{\bullet}(C) \to \overline{C}^{\lambda}_{\bullet}(A \otimes C)$$

on the reduced cyclic homology.

6. Adams operations

6.1. Euler decomposition. Suppose that $(\mathcal{H}, \mu, \Delta)$ is a Hopf algebra over a field k of characteristic zero, with the product μ, coproduct Δ, unit ϵ and the counit μ. The linear space

$$\mathrm{Hom}_k(\mathcal{H}, \mathcal{H})$$

has an associative product given by the convolution:

$$f * g = \mu(f \otimes g)\Delta. \tag{6.1}$$

Assume now that $\mathcal{H}$ is graded commutative, with $\mathcal{H}_k = 0$ for $k < 0$ and $\mathcal{H}_0 = k$. For any

$$f \in \mathrm{Hom}_k(\mathcal{H}, \mathcal{H}),\ |f| = 0,\ f(1) = 0,$$

f^{*s} vanishes on $\mathcal{H}_n$ for $n < s$. Moreover the composition $\epsilon \circ \mu$ is the unit of the graded ring $(\mathrm{Hom}_k(\mathcal{H}, \mathcal{H}), *)$ and hence the series

$$e^{(1)}(f) = \log(\epsilon \circ \mu + f) = \sum_{n>0} (-1)^{n+1} \frac{1}{n} f^{*n}$$

makes sense and is a degree zero endomorphism of $\mathcal{H}$.

DEFINITION 6.1.1. *Suppose that $\mathcal{H}$ is a graded commutative Hopf algebra as above. We set*

$$e^{(k)} = \frac{1}{k!}(e^{(1)}(\mathrm{Id} - \epsilon \circ \mu))^{*k}$$

and

$$e^{(k)}_n = e^{(k)}|_{\mathcal{H}_n}.$$

The following is the basic fact about the $e's$.

PROPOSITION 6.1.2.

(1) $e^{(k)}_n$, $k = 1, \dots, n$ *are pairwise othogonal idempotents;*
(2) $\mathrm{Id}^{*k}|_{\mathcal{H}_n} = \sum_{i=1}^{n} k^i e^{(i)}_n$.

ABOUT THE PROOF. The statement reduces to relatively straightforward identities relating the exponential and logarithmic power series and we refere for it to the original papers of Gerstenhaber, Schack and Loday.

□

6.2. λ operations. If A is a vector space, the (non-connected) tensor algebra

$$TA = \oplus_{n\geq 0} A^{\otimes n}$$

has a structure of a graded commutative Hopf algebra, with the product

$$\mu(a_1 \otimes \ldots \otimes a_p, a_{p+1} \otimes \ldots \otimes a_{p+q}) = \sum_{\text{pq shuffles } \sigma} sgn(\sigma) a_{\sigma(1)} \otimes \ldots \otimes a_{\sigma(p+q)}$$

and the coproduct

$$\Delta(a_1 \otimes \ldots \otimes a_n) = \sum_{i=0}^{n} a_1 \otimes \ldots \otimes a_i \otimes a_{i+1} \otimes \ldots \otimes a_n.$$

DEFINITION 6.2.1. *The λ_n operations on $A^{\otimes n}$ are defined by*

$$\lambda_n^k = (-1)^{k-1} \mathrm{Id}^{*k}|_{A^{\otimes n}}.$$

The Adams operations are given by

$$\psi_n^k = (-1)^{k-1} k \lambda_n^k.$$

THEOREM 6.2.2. *Suppose that A is a commutative unital algebra. The λ operations descend to Hochschild homology.*

PROOF. Note first that

$$\mathrm{Id}^{*k} = \mu^k \circ \Delta^k$$

where we, as usual, we use the notation

$$\Delta^k = (\Delta \otimes id^{\otimes k-1}) \circ (\Delta \otimes id^{\otimes k-2}) \circ \ldots \circ \Delta$$

and the similar (dual) definition of μ^k. The identity (0.5) now implies easily that

$$\lambda^k b = b \lambda^k.$$

□

The behaviour of cyclic complexes under the λ operations is controlled by the following result of combinatorial nature.

PROPOSITION 6.2.3. *Suppose that A is a commutative unital algebra. Then*

$$\lambda_n^k B = k B \lambda_{n-1}^k$$

and

$$B e_n^{(k)} = e_{n+1}^{(k+1)} B,$$

where $e_n^{(k)}$ are the Euler idempotents (see the definition (6.1.1)).

Since the proof is entirely combinatorial, we will omit it here (see f. ex.[**?**]). The following is the main (and easy) corollary.

THEOREM 6.2.4. *Suppose that A is a commutative unital algebra. The cyclic (negative) complex $CC_\bullet$ of A splits into a direct sum of subcomplexes:*

$$CC_n(A)^{(k)} = \bigoplus_i u^{-i} e_n^{(k-2i)} C_{n-2i}(A).$$

In particular, the cyclic homology of A has the decomposition

$$HC_n(A) = \bigoplus_k HC_n(A)^{(k)}$$

which is called the λ-decomposition.

7. Bibliographical notes

[?], [?], [?];

CHAPTER 4

Operations on Hochschild and cyclic complexes, I

We start our analysis of operations with the classical Eilenberg-Zilber and Alexander-Whitney exterior products and coproducts that we extend from simplicial to cyclic situation. We will later show that both the product and (dual version of) the coproduct are parts of a more general operation.

0.1. The Eilenberg-Zilber exterior product on Hochschild complexes.

DEFINITION 0.1.1. *For two algebras A_1 and A_2 define the shuffle product*

$$\mathrm{sh} : C_p(A_1) \otimes C_q(A_2) \to C_{p+q}(A_1 \otimes A_2) \tag{0.1}$$

as follows.

(0.2)

$$(a_0^{(1)} \otimes \ldots \otimes a_p^{(1)}) \otimes (a_0^{(2)} \otimes \ldots \otimes_q^{(2)}) \mapsto a_0^{(1)} a_0^{(2)} \otimes \mathrm{sh}_{pq}(a_1^{(1)}, \ldots, a_p^{(1)}, \; a_1^{(2)} \; \ldots, a_q^{(2)})$$

where

$$\mathrm{sh}_{pq}(x_1, \ldots, x_{p+q}) = \sum_{\sigma \in \mathrm{Sh}(p,q)} \mathrm{sgn}(\sigma) x_{\sigma^{-1}1} \otimes \ldots \otimes x_{\sigma^{-1}(p+q)} \tag{0.3}$$

and

$$\mathrm{Sh}(p, q) = \{\sigma \in \Sigma_{p+q} \, | \sigma 1 < \ldots < \sigma p; \; \sigma(p+1) < \ldots < \sigma(p+q)\}$$

(We identify $a_j^{(1)}$ with $a_j^{(1)} \otimes 1$ and $a_j^{(2)}$ with $1 \otimes a_j^{(2)}$).
In the graded case, $\mathrm{sgn}(\sigma)$ *gets replaced by the sign computed by the following rule: in all transpositions, the parity of a_i is equal to $|a_i| + 1$ if $i > 0$, and similarly for c_i. A transposition contributes a product of parities.*

Put also

$$m_{\mathrm{EZ}}(c_1, c_2) = (-1)^{|c_1|} \mathrm{sh}(c_1 \otimes c_2)$$

THEOREM 0.1.2. *For two unital algebras A_1 and A_2*

$$m_{\mathrm{EZ}} : C_\bullet(A_1) \otimes C_\bullet(A_2) \to C_\bullet(A_1 \otimes A_2)$$

is a quasi-isomorphism.

SKETCH OF THE PROOF.

Recall the free bimodule resolution $\mathcal{B}_\bullet(A) \to A$ of an algebra A as an A-bimodule given by (1.5). Let us recall their construction from [**101**]. For any algebra C, let $\mathcal{B}_\bullet(C)$ be the bar resolution for C. We use the notation

$$c_0 \otimes \ldots \otimes c_{p+1} = c_0[c_1 \ldots c_p] c_{p+1}$$

For any two algebras A and B, define

$$\mathrm{EZ} : \mathcal{B}_\bullet(A_1) \otimes \mathcal{B}_\bullet(A_2) \to \mathcal{B}_\bullet(A_1 \otimes A_2) \tag{0.4}$$

to be the $A_1 \otimes A_2$-bimodule morphism such that

$[a_1^{(1)}|\dots|a_p^{(1)}] \otimes [a_1^{(2)}|\dots|a_q^{(2)}] \mapsto \mathrm{sh}_{p,q}(a_1^{(1)} \otimes 1, \dots, a_p^{(1)} \otimes 1, 1 \otimes a_1^{(2)}, \dots, 1 \otimes a_q^{(2)})$

This gives a quasi-isomorphism of complexes of free $A_1 \otimes A_2$-bimodules

$$\bigoplus_{k+l=\bullet} \mathcal{B}_k(A_1) \otimes \mathcal{B}_l(A_2) \to \mathcal{B}_\bullet(A_1 \otimes A_2)$$

Both sides are free resolutions of the bimodule $A_1 \otimes A_2$. In particular, after tensoring with $A_1 \otimes A_2$, we get a quasiisomorphism of complexes

$$\bigoplus_{k+l=\bullet} (\mathcal{B}_k(A_1) \otimes \mathcal{B}_l(A_2)) \otimes_{A_1^e \otimes A_2^e} A_1 \otimes A_2 \to \mathcal{B}_\bullet(A_1 \otimes A_2) \otimes_{A_1^e \otimes A_2^e} A_1 \otimes A_2$$

The right hand side computes Hochschild homology of $A_1 \otimes A_2$. The obvious spectral sequence identifies the homology of the left hand side complex with

$$C_\bullet(A_1) \otimes C_\bullet(A_2) = \bigoplus_{k+l=\bullet} C_k(A_1) \otimes C_l(A_2)$$

and, in particular, we get a quasi-isomorphism

$$\bigoplus_{k+l=\bullet} C_k(A_1) \otimes C_l(A_2) \to C_\bullet(A_1 \otimes A_2).$$

We leave it to the reader to check that the shuffle product satisfies

$$b(sh(x \times y)) = sh(bx \times y) + (-1)^{|x|} sh(x \times by) \tag{0.5}$$

Proof. We leave this to the reader. □

and implements the quasi-isomorphism in question. □

Lemma 0.1.3. *The shuffle product is associative.*

1. The Hood-Jones exterior product on negative cyclic complexes

For any n unital algebras $A_1, \dots, A_n$, $n \geq 2$, we will construct a $k[[u]]$-linear map of degree $n-2$

$$m(A_1, \dots, A_N)\colon CC_\bullet^-(A_1) \otimes_{k[[u]]} \dots \otimes_{k[[u]]} CC_\bullet^-(A_N) \to CC_\bullet^-(A_1 \otimes \dots \otimes A_N) \tag{1.1}$$

such that $m(A_1) = b + uB$ and the following A_∞ relation holds:

$$\sum_{k \geq 1, k+l \leq n} \pm m(A_1, \dots, A_{k+1} \otimes \dots \otimes A_{k+l}, \dots, A_n) \circ m(A_{k+1}, \dots, A_{k+l}) = 0 \tag{1.2}$$

(compare to **??**). In particular, for a commutative algebra A, $CC_\bullet^-(A)$ is an A_∞ algebra over $k[[u]]$. We will later substantially enlarge the class of algebras A for which this is the case.

Definition 1.0.1. *Let A be an algebra. The map*

$$\mathrm{sh}'_n : C_{p_1}(A_1) \otimes \dots \otimes C_{p_n}(A_n) \to C_{p_1+\dots+p_n+n}(A_1 \otimes \dots \otimes A_n) \tag{1.3}$$

as follows. Consider the embeddings

$$i_j : A_j \to A_1 \otimes \dots \otimes A_n,\ a \mapsto 1 \otimes \dots \otimes a_j \otimes \dots \otimes 1.$$

Identify algebras A_j with their images under these embeddings. Denote

$$(x_1, \dots, x_{n+\sum p_j}) = (a_0^{(1)}, \dots, a_{p_1}^{(1)}, \dots, a_0^{(n)}, \dots, a_{p_n}^{(n)})$$

For $c_j = a_0^{(j)} \otimes \ldots \otimes a_{p_1}^{(1)}$, $1 \le j \le n$, *set*

$$\mathrm{sh}'_n(c_1, \ldots, c_n) = 1 \otimes \sum \mathrm{sgn}(\sigma) x_{\sigma^{-1}(1)} \otimes \ldots \otimes x_{\sigma^{-1}(\sum p_j + n)}$$

where σ *runs through the set* $\mathrm{Sh}'(p_1 + 1, \ldots, p_n + 1)$ *of all permutations such that:*

a) the cyclic order of every group $(a_0^{(j)}, \ldots, a_{p_j}^{(j)})$ *is preserved;*

b) if $j < k$ *then* $a_0^{(j)}$ *appears to the left of* $a_0^{(k)}$.

In the graded case, the sign rule is as follows: any $a_i^{(j)}$ *has parity* $|a_i^{(j)}| + 1$.

DEFINITION 1.0.2.

$$m_1 = b + uB;$$
$$m_2(c_1, c_2) = (-1)^{|c_1|}(\mathrm{sh}(c_1, c_2) + u\mathrm{sh}'(c_1, c_2));$$
$$m_n(c_1, c_2) = (-1)^{***}u\mathrm{sh}'(c_1, \ldots, c_n)),\ n > 2.$$

THEOREM 1.0.3. *The above* m_n *satisfy the* A_∞ *relations* (1.2).

PROOF. □

THEOREM 1.0.4. *The map* $\mathrm{sh} + u\mathrm{sh}'$ *defines a* $k[[u]]$*-linear,* (u)*-adically continuous quasi-isomorphisms of complexes*

$$C_\bullet(A_1) \otimes C_\bullet(C_2))[[u]] \to CC_\bullet^-(A_1 \otimes A_2),$$
$$(C_\bullet(A_1) \otimes C_\bullet(A_2))[u^{-1}, u]] \to CC_\bullet^{\mathrm{per}}(A_1 \otimes A_2)$$

and

$$(C_\bullet(A_1) \otimes C_\bullet(A_2))[u^{-1}, u]]/u(C_\bullet(A_1) \otimes C_\bullet(A_2))[[u]] \to CC_\bullet(A_1 \otimes A_2).$$

The differentials on the left hand sides are equal to

$$b \otimes 1 + 1 \otimes b + u(B \otimes 1 + 1 \otimes B).$$

SKETCH OF THE PROOF. We already know that $sh + ush'$ is (up to a sign) a morphism of total complexes. If we think of the left hand side as a double complex with the vertical boundary map $b \otimes 1 + 1 \otimes b$, Theorem 0.1.2 implies that all three morphisms of double complexes are quasiisomorphisms on the columns and hence are quasiisomorphisms on the total complexes. □

As a corollary we get the following Künneth formula for the cyclic homology.

THEOREM 1.0.5 (Künneth Theorem). ****Under the assumption*** There is a long exact sequence*

$$\cdots \xrightarrow{\times} HC_n(A \otimes C) \xrightarrow{\Delta} \bigoplus_{p+q=n} HC_p(A_1) \otimes HC_q(A_2) \xrightarrow{S\otimes 1 - 1 \otimes S}$$

$$\longrightarrow \bigoplus_{p+q=n-2} HC_p(A_1) \otimes HC_q(A_2) \xrightarrow{\times} HC_{n-1}(A_1 \otimes A_2) \xrightarrow{\Delta} \cdots$$

where Δ *is induced by the diagonal embedding*

$$u^{-p} c \otimes c' \mapsto (u^{-1} \otimes 1 + 1 \otimes u^{-1})^p c \otimes c'.$$

SKETCH OF THE PROOF. One checks that Δ is an embedding whose cokernel is the kernel of the multiplication by $u \otimes 1 - 1 \otimes u$ which, in turn, is the same as the kernel of $S \otimes 1 - 1 \otimes S$ (S is as in (1.12)). □

2. The Alexander-Whitney exterior coproduct on the Hochschild complex

For two algebras A_1 and A_2 define

$$\Delta_{\mathrm{AW}} : C_\bullet(A_1 \otimes A_2) \to C_\bullet(A_1) \otimes C_\bullet(A_2) \tag{2.1}$$

$$a_0^{(1)} a_0^{(2)} \otimes \ldots \otimes a_n^{(1)} a_n^{(2)} \mapsto$$

$$\sum_{j=0}^{n} (a_0^{(1)} \ldots a_j^{(1)} \otimes a_{j+1}^{(1)} \otimes \ldots \otimes a_n^{(1)}) \otimes (a_{j+1}^{(2)} \ldots a_n^{(2)} a_0^{(2)} \otimes a_1^{(2)} \otimes \ldots \otimes a_j^{(2)})$$

Similarly to EZ, the morphism AW is induced by a morphism of bar resolutions. Namely, define

$$[a_1^{(1)} a_1^{(2)} | \ldots | a_m^{(1)}\ a_m^{(2)}] \mapsto \sum_{j=0}^{m} [a_1^{(1)} | \ldots | a_j^{(1)}] a_{j+1}^{(1)} \ldots a_m^{(1)} \otimes a_1^{(2)} \ldots a_j^{(2)} [a_{j+1}^{(2)} | \ldots | a_m^{(2)}]$$

This gives a morphism

$$\mathcal{B}_\bullet(A_1 \otimes A_2) \to \mathcal{B}_\bullet(A_1) \otimes \mathcal{B}_\bullet(A_2) \tag{2.2}$$

THEOREM 2.0.1. Δ_{AW} *is a quasi-isomorphism of complexes. It is homotopy inverse to* m_{EZ} *from Theorem 0.1.2.*

PROOF. One checks that EZ ∘ AW = id.

LEMMA 2.0.2. *Let*

$$t([a_1^{(1)} a_1^{(2)} | \ldots | a_n^{(1)} a_n^{(2)}]) = \sum_j \sum_{k>j} \pm [a_1^{(1)} a_1^{(2)} | \ldots | a_j^{(1)} a_j^{(2)} | a_{j+1}^{(1)} \ldots a_k^{(1)} | C_{jk}] a_{k+1}^{(2)} \ldots a_n^{(2)}$$

where

$$C_{jk} = \mathrm{sh}([a_{k+1}^{(1)} | \ldots | a_n^{(1)}], [a_{j+1}^{(2)} | \ldots | a_k^{(2)}]).$$

Then t *is a homotopy between* id *and* AW ∘ EZ.

The proof is a direct computation and we leave it to the reader ***OR NOT?*** The homotopy t is the one constructed by Eilenberg and Zilber in [**?**]. □

3. Exterior coproduct on $CC_\bullet^-$

THEOREM 3.0.1. *For any* n *algebras* $A_1, \ldots, A_n$, $n \geq 2$, *there is a natural* $k[[u]]$*-linear map of degree* $n-2$

$$\Delta(A_1, \ldots, A_N) \colon CC_\bullet^-(A_1 \otimes \ldots \otimes A_N) \to CC_\bullet^-(A_1) \otimes_{k[[u]]} \ldots \otimes_{k[[u]]} CC_\bullet^-(A_N) \tag{3.1}$$

such that

a)

$$\Delta(A_1) = b + uB;\ \Delta(A_1, A_2) = \eta \Delta_{\mathrm{AW}} \bmod u$$

where $\eta(a^{(1)} \otimes a^{(2)}) = (-1)^{|a^{(1)}|} a^{(1)} \otimes a^{(2)}$;

b) the following dual A_∞ *relation is satisfied*

$$\sum_{k \geq 1, k+l \leq n} \pm \Delta(A_{k+1}, \ldots, A_{k+l}) \circ \Delta(A_1, \ldots, A_{k+1} \otimes \ldots \otimes A_{k+l}, \ldots, A_n) = 0$$

(again, compare to **??***). In particular, for a bialgebra* A, $CC_\bullet^-(A)$ *is an* A_∞ *coalgebra over* $k[[u]]$.

PROOF. Denote by $\Lambda([m],[n])$ the set of all natural operations $A^{\otimes(m+1)} \to A^{\otimes(n+1)}$ for a unital monoid A that are compositions of:

a) $a_0 \otimes \ldots \otimes a_n \mapsto a_0 a_1 \otimes \ldots \otimes a_n$;

b) $a_0 \otimes \ldots \otimes a_n \mapsto 1 \otimes a_0 \otimes \ldots \otimes a_n$;

c) $a_0 \otimes \ldots \otimes a_n \mapsto a_1 \otimes \ldots \otimes a_n \otimes a_0$.

For any such operations

$$\lambda(j) : A^{\otimes(m+1)} \to A^{\otimes(n_j+1)}$$

and using notation

$$\lambda(j)(a_0^{(j)} \otimes \ldots \otimes a_m^{(j)}) = c_0^{(j)} \otimes \ldots \otimes c_{n_j}^{(j)},$$

define an (odometer-like) operation

$$\text{Op}(\lambda(1), \ldots, \lambda(N)) : C_m(A_1 \otimes \ldots \otimes A_N) \to C_{n_1}(A_1) \otimes \ldots \otimes C_{n_N}(A_N); \tag{3.2}$$

$$a_0^{(1)} \ldots a_0^{(N)} \otimes \ldots \otimes a_m^{(1)} \ldots a_m^{(N)} \mapsto (c_0^{(1)} \otimes \ldots \otimes c_{n_1}^{(1)}) \otimes \ldots \otimes (c_0^{(N)} \otimes \ldots \otimes c_{n_N}^{(N)}) \tag{3.3}$$

Denote the k-linear span of these operations by $\mathcal{P}([m];[n_1],\ldots,[n_N])$. Put

$$\mathcal{P}_\bullet^{(N)}([m]) = (\bigoplus_{n_1+\ldots+n_N=\bullet} \mathcal{P}([m];[n_1],\ldots,[n_N]),\ b = (\sum_{j=1}^{N} b_{A_j}) \circ \_) \tag{3.4}$$

We also denote $\mathcal{P}_\bullet^{(1)}([m])$ by $\mathcal{P}_\bullet([m])$.

We claim (cf. also ??) that the homology of $(\mathcal{P}_\bullet, b)$ is concentrated in degrees zero and one only, and is of rank 1 over k. It is easy to write the morphisms of complexes

$$\mathcal{P}_\bullet[m] \underset{i}{\overset{p}{\rightleftarrows}} (k \xrightarrow{0} k) \tag{3.5}$$

together with $s : \mathcal{P}_\bullet[m] \to \mathcal{P}_{\bullet+1}([m])$ such that $pi = \text{id}$, $ip = [s, b]$. Explicilly:

Denote the generators of degree zero and one of $k \xrightarrow{0} k$ by α_0 and α_1 respectively. Then

$$i(\alpha_0) = a_0 \ldots a_m;\ i(\alpha_1) = \sum_{j=0}^{m} a_{j+1} \ldots a_{j-1} \otimes a_j \tag{3.6}$$

$$p(a_{j+1} \ldots a_j) = \alpha_0;\ p(a_{j+1} \ldots a_k \otimes a_{k+1} \ldots a_j) = \alpha_1 \tag{3.7}$$

if a_0 is a factor of $a_{k+1} \ldots a_j$ and zero otherwise; $p = 0$ on $\mathcal{P}_n([m])$ for $n \geq 2$;

$$s(a_{j+1} \ldots a_j) = \sum_{k=j+1}^{n} a_{k+1} \ldots a_{k-1} \otimes a_k; \tag{3.8}$$

for $n > 0$,

$$s(r_0 \otimes r_1 \otimes \ldots \otimes r_n) = \sum_{p=j}^{k} r_0 a_j \ldots a_{p-1} \otimes a_p \otimes a_{p+1} \ldots a_{k+1} \otimes \ldots r_2 \ldots r_n \tag{3.9}$$

where r_i are monomials, i.e. products of consecutive a_l in the cyclic order, and $r_1 = a_j \ldots a_{k+1}$. (Note that r_1 is also understood as a product in the cyclic order, i.e. it may contain a_0 as a factor).

For every N there are homotopy equivalences

$$\mathcal{P}_\bullet^{(N)}[m] \underset{i^{\otimes N}}{\overset{p^{\otimes N}}{\rightleftarrows}} (k \xrightarrow{0} k)^{\otimes N} \tag{3.10}$$

One has

$$p^{\otimes N} i^{\otimes N} = \mathrm{id};\ \mathrm{id} - i^{\otimes N} p^{\otimes N} = [b, s^{(N)}]$$

where

$$s^{(N)} = \sum_{j-1}^{N} (-1)^j p^{\otimes (j-1)} \otimes s \otimes \mathrm{id}^{\otimes (N-j)} \tag{3.11}$$

□

We will write Δ_N instead of $\Delta(A_1, \ldots, A_N)$. We want to construct

$$\Delta_N = \sum_{k=0}^{\infty} u^k \Delta_N^{(k)}$$

Start with $\Delta_2^{(0)}$. Define it as the AW coproduct from 2. Now compute $[B, \Delta_2^{(0)}]$. It is equal to zero on $\mathcal{P}_0$:

$$\Delta_2^{(0)} B(a_0^{(1)} a_0^{(2)}) = \Delta_2^{(0)}(1 \otimes a_0^{(1)} a_0^{(2)}) =$$
$$(a_0^{(1)}) \otimes (1 \otimes a_0^{(2)}) + (1 \otimes a_0^{(1)}) \otimes (a_0^{(2)}) = B\Delta_2^{(0)}(a_0^{(1)} a_0^{(2)})$$

Furthermore, it sends $\mathcal{P}_n$, $n > 0$, to $\ker(p \otimes p)$. Indeed, this is enough to check for the component $\mathcal{P}_1 \to \mathcal{P}_1 \otimes \mathcal{P}_1$, in which case there are four terms of $[B, \Delta_2^{(0)}]$:

$$(a_0^{(1)} \otimes a_1^{(1)}) \otimes (a_1^{(2)} \otimes a_0^{(2)});\ (a_1^{(1)} \otimes a_0^{(1)}) \otimes (a_0^{(2)} \otimes a_1^{(2)});$$
$$(1 \otimes a_0^{(1)} a_1^{(1)}) \otimes (a_0^{(2)} \otimes a_1^{(2)});\ (a_0^{(1)} \otimes a_1^{(1)}) \otimes (1 \otimes a_1^{(2)} a_0^{(2)})$$

But p only detects terms in C_1 that have a factor a_0 on the right, and none of the four terms have that in both tensor factors.

Therefore there is $\Delta_2^{(1)}$ such that $[b, \Delta_2^{(1)}] + [B, \Delta_2^{(0)}] = 0$. By degree considerations, its commutator with B lands in $\ker(p \otimes p)$, etc. We construct Δ_2 by recursion. Now we have to construct $\Delta_3^{(j)}$ starting with $j = 1$ (because $\Delta_2^{(0)}$ is coassociative). We proceed by recursion, on every step finding $\Delta_N^{(j)}$ from the condition that its commutator with b is a given morphism $\mathcal{P}_\bullet \to \ker(p^{\otimes N})$.

LEMMA 3.0.2. *There are no nonzero components of $\Delta^{(k)}(A_1, \ldots, A_N)$ of the form*

$$C_m(A_1 \otimes \ldots \otimes A_N) \to C_{n_1}(A_1) \otimes \ldots \otimes C_{n_N}(A_N)$$

with any of the n_j being equal to zero, unless $N = 2$ and $k = 0$.

PROOF. Let $\Delta_N^{(k)}(m)$ be the restriction of $\Delta_N^{(k)}(A_1, \ldots, A_N)$ to $C_m(A_1 \otimes \ldots \otimes A_N)$. Recall how the construction of $\Delta_N^{(k)}(m)$ goes. We assume that $\Delta_{N'}^{(k')}(m')$ are constructed for

a) all $N' < N$ and all k', m';
b) $N' = N$, $k' < k$, all m';
c) $N' = N$, $k' = k$, $m' < m$.

Then one constructs a particular linear combination of terms of the form

1) $\Delta_{N'}^{k'}(m') \circ \Delta_{N-N'+1}^{k-k'}(m)$;

2) $\Delta_N^{(k)}(m-1)\circ b$;
3) $\Delta_N^{(k-1)}(m+1)\circ B$;
4) $B\circ\Delta_N^{(k-1)}(m)$.

We obtain $\Delta_N^{(k)}(m)$ by applying to this expression the homotopy $s^{(N)}$. By the induction hypothesis, among all these terms, only 1) may contain terms with some $n_j=0$, and only when $k'=0$, $N'=2$ or $k-k'=0$, $N-N'+1=2$.

When $k'=0$ and $m'=2$, we get the operation $\Delta^{(k)}(A_1,\dots,A_j\otimes A_{j+1},\dots,A_N)$

$$C_m(A_1\otimes\dots\otimes A_N)\to C_{n_1}(A_1)\otimes\dots\otimes C_{n_{j,j+1}}(A_j\otimes A_{j+1})\otimes\dots\otimes C_{n_N}(A_N)$$

followed by

$$\Delta^{(0)}(A_j,A_{j+1}):C_{n_{j,j+1}}(A_j\otimes A_{j+1})\to C_{n_j}(A_j)\otimes C_{n_{j+1}}(A_{j+1}).$$

Consider the terms with either $n_j=0$ or $n_{j+1}=0$ (by the inductive hypothesis it cannot be both). When we apply $s^{(N)}$, these terms can be hit by either s or p. But s increases the degree n_j. And applying p transforms such a term into

$$a_0^{(j')}\dots a_m^{(j')} \tag{3.12}$$

where $j'=j$ or $j+1$. So applying p to the position j' is the same as computing

$$\Delta^{(k)}(A_1,\dots,\widehat{A_{j'}},\dots,A_N) \tag{3.13}$$

and then inserting the tensor factor (3.12) in the j'th position. Applying to this the terms of $s^{(N)}$ that hit $C_{n_j'}$ with p becomes the same as applying $s^{(N-1)}$ to (3.12) and then inserting the tensor factor. But (3.12) is itself in the image of $s^{(N-1)}$, therefore the result is zero.

Now consider the case $k-k'=0$ and $N-N'+1=2$. Then we get the composition of $\Delta_2^{(0)}(A_1\otimes\dots\otimes A_{N-1},A_N)$

$$C_m(A_1\otimes\dots\otimes A_N)\to C_n(A_1\otimes\dots\otimes A_{N-1})\otimes C_{n_N}(A_N)$$

composed with $\Delta^{(k)}(A_1,\dots,A_{N-1})$. By the induction hypothesis, the only possibly nonzero terms with $n_j=0$ occur when $j=N$. Applying $s^{(N)}$ to them is the same as applying $s^{(N-1)}$ to $\Delta^{(k)}(A_1,\dots,A_{N-1})$ and then tensoring by $a_0^N\dots a_m^{(N)}$. This is zero because $\Delta^{(k)}(A_1,\dots,A_{N-1})$ is in the image of $s^{(N-1)}$ Similarly for A_1 and $A_2\otimes\dots\otimes A_N$. □

REMARK 3.0.3. We chose to avoid mentioning cyclic modules in the above proof, not only to make it self-sufficient but because we needed an extra degree of explicitness. Here we would like to relate our construction to Chapter 8. Recall the complex of cocyclic k-modules

$$\mathcal{P}_\bullet([m])=\Lambda([m],[\bullet]);\ \lambda\mapsto b\lambda;\ b=\sum_{j=0}^{n}(-1)^j d_j\ \text{ on}\mathcal{P}_n \tag{3.14}$$

from (4.17). For any $N\geq 1$, define the complexes

$$\mathcal{P}_\bullet^{\otimes N}[[u]]([m])=(\mathcal{P}_\bullet([m])^{\otimes N}[[u]],b+uB) \tag{3.15}$$

(We use the tensor product of complexes combined with the diagonal tensor product of cocyclic modules). Note that, when tensored by $k[[u,u^{-1}]/k[[u]]$ over $k[[u]]$, all of them become resolutions of the constant cocyclic module $k_\sharp$. In particular there is a chain map between them over $k_\sharp$. What is a little more subtle is the question of its linearity over $k[[u]]$. What we have proven is that there are $k[[u]]$-linear morphisms of complexes of cocyclic modules

$$\Delta_N \colon \mathcal{P}_\bullet[[u]] \to \mathcal{P}_\bullet^{\otimes N}[[u]] \tag{3.16}$$

from which the above follows, and that the following is true:

$$[b + uB, \Delta_N] = \sum_{k,l} \pm(\mathrm{id}^{\otimes k} \otimes \Delta_l \otimes \mathrm{id}^{\otimes(n-k-l)}) \circ \Delta_{N-l+1} \tag{3.17}$$

We identify morpisms in

$$\mathrm{Hom}_{\Lambda^{\mathrm{op}}}(\mathcal{P}_n, \mathcal{P}_{n_1} \otimes \ldots \otimes \mathcal{P}_{n_N}) \tag{3.18}$$

with k-linear natural maps

$$C_n(A_1 \otimes \ldots \otimes A_N) \to C_{n_1}(A_1) \otimes \ldots \otimes C_{n_N}(A_N) \tag{3.19}$$

For example, when $N = 2$, the map from (3.19)

$$a_0 b_0 \otimes a_1 b_1 \otimes a_2 b_2 \mapsto (a_2 a_0 \otimes a_1) \otimes (b_0 \otimes 1 \otimes b_1 b_2)$$

corresponds to the only morphism $\mathcal{P}_2 \to \mathcal{P}_1 \otimes \mathcal{P}_2$ for which

$$(\mathrm{id} \in \Lambda([2],[2])) \mapsto d_0 \otimes s_1 d_1 \in \Lambda([2],[1]) \otimes \Lambda([2],[3]).$$

4. Multiplication on cochains of a coalgebra

Now let $C_1, \ldots, C_n$ be coalgebras. Let $C^\bullet(C_j)$ be the Hochschild cochain complex. Dually to

$$m(C_1, \ldots, C_N) \colon C^\bullet(C_1) \otimes \ldots \otimes C^\bullet(C_N) \to C\bullet(C_1 \otimes \ldots \otimes C_N)[[u]] \tag{4.1}$$

such that $m(C_1) = b + uB$ and the A_∞ relation (1.2) holds.

For a bialgebra H the compositions of the above maps (when $C_1 = \ldots = C_N = H$) with the morphism of complexes induced by the product on H define a $k[[u]]$-linear continuous A_∞ algebra structure on $C^\bullet(H)[[u]]$. Modulo u, it is an associative graded algebra with the product

(4.2)

$$(x_0 \otimes \ldots \otimes x_p) \otimes (y_0 \otimes \ldots \otimes y_q) = x_0^{(0)} y_0^{(p+1)} \otimes x_0^{(1)} y_1 \ldots \otimes x_0^{(q)} y_q \otimes x_1 y_0^{(1)} \otimes \ldots \otimes x_p y_0^{(p)}$$

where we use the notation

$$\Delta^q x_0 = \sum x_0^{(0)} \otimes \ldots \otimes x_0^{(q)};\ \Delta^p y_0 = \sum y_0^{(1)} \otimes \ldots \otimes y_0^{(p+1)}$$

In general, the A_∞ structure involves the following operations.

Consider all operations $H^{\otimes(p+1)} \to H^{\otimes(q+1)}$ that are compositions of:

a) $x_0 \otimes \ldots \otimes x_p \mapsto \sum x_0^{(1)} \otimes x_0^{(2)} \otimes x_1 \otimes \ldots \otimes x_p$;

b) $x_0 \otimes \ldots \otimes x_p \mapsto \epsilon(x_0) x_1 \otimes \ldots \otimes x_p$;

c) $x_0 \otimes \ldots \otimes x_p \mapsto x_1 \otimes \ldots \otimes x_p \otimes x_0$.

Consider a collection of elements

$$x(j) = x_0(j) \otimes \ldots \otimes x_{p_j}(j) \in C^{p_j}(H),\ j = 1, \ldots, N.$$

For any collection of operations

$$\lambda(j) : H^{\otimes(p_j+1)} \to H^{\otimes(q+1)}$$

as above, and using notation

$$\lambda(j)(x_0(j) \otimes \ldots \otimes x_{p_j}(j)) = y_0(j) \otimes \ldots \otimes y_q(j),$$

define

$$\mathrm{Op}(\lambda(1),\ldots,\lambda(N)) : C^{p_1}(H)\otimes\ldots\otimes C^{p_N}(H)\to C^q(H); \tag{4.3}$$

$$x(1)\otimes\ldots\otimes x(N)\mapsto y_0(1)\ldots y_0(N)\otimes\ldots\otimes y_q(1)\ldots y_q(N) \tag{4.4}$$

The construction in 3 implies that the A_∞ operations are linear combinations of (4.4). We get

LEMMA 4.0.1. *Let H be a cocommutative bialgebra. Then*
1) The A_∞ operations are linear combinations of (4.4).
2) $H = C^0(H)$ is a DG subalgebra with respect to m_2.
3) The m_2 multiplication by H from left and right is the standard left and right action of H on tensor powers of H via comultiplication.
4) The A_∞ operations m_N are H-bimodule maps

$$m_N : C^\bullet(H)\otimes_H\ldots\otimes_H C^\bullet(H)\mapsto C^\bullet(H)[[u]].$$

5) Substituting $x\in H$ into m_N gives 0 when $N\geq 3$.

PROOF. 2) is true because of the formulas (4.2) for the product and because $b: C^0\to C^1$ is the cocommutator. 3) follows from (4.2). 4) follows from (4.4), and 5) from Lemma 3.0.2. □

4.1. $CC^\bullet(H)$ for cocommutative Hopf algebras.

Let H be a cocommutative Hopf algebra. Let

$$\overline{H} = \ker(\epsilon) \tag{4.5}$$

Use the normalized Hochschild complex

$$C^n(H) = H\otimes \overline{H}^{\otimes n}$$

It is an A_∞ algebra (clearly the structure on the full complex descends to it). Also, the embedding into the full complex is a quasi-isomorphism.

4.2. The DG algebra $H\ltimes \mathrm{Cobar}(\overline{H})$.

For a bialgebra H and an algebra A, an action of H on A is a linear map $H\otimes A\to A$, $x\otimes a\mapsto \rho(x)a$, such that

$$\rho(xy)=\rho(x)\rho(y);\ \rho(x)(ab)=\sum \rho(x^{(1)})(a)\rho(x^{(2)})(b)$$

If H is a Hopf algebra acting on A then one can define a cross product

$$H\ltimes A = A\otimes H; (a\otimes x)(b\otimes y) = a\rho(x^{(1)})b\otimes Sx^{(2)}y \tag{4.6}$$

Let $A=\mathrm{Cobar}(\overline{H})$. Put

$$\rho(x)(x_1|\ldots|x_n)=\sum (x^{(1)}x_1S(x^{(n+1)})|\ldots|x^{(n)}x_nS(x^{(2n)})) \tag{4.7}$$

where S is the antipode. The action commutes with the differential on $\mathrm{Cobar}(\overline{H})$ (which we denote by b), and we get a DG algebra $H\ltimes \mathrm{Cobar}(H)$.

REMARK 4.2.1. Note that the comultiplication on $\overline{H}$ is given by

$$\Delta x=\sum x^{(1)}\otimes x^{(2)} - 1\otimes x - x\otimes 1$$

In other words, $H \ltimes \mathrm{Cobar}(\overline{H})$ is the DG algebra generated by a subalgebra H and by elements (x), linear in $x \in \overline{H}[1]$, subject to

$$x \cdot (y) = \sum (x^{(1)} y S(x^{(2)})) \cdot x^{(3)};\ bx = 0;\ b(x) = \sum (x^{(1)})(x^{(2)}) \tag{4.8}$$

This DG algebra admits a derivation B determined by

$$Bx = 0,\ x \in H;\ B(x) = x,\ x \in \overline{H}[1].$$

It is easy to see that B is well defined and commutes with b. Of course, if H is a DG Hopf algebra, then its differential d induces an extra differential on $H \ltimes \mathrm{Cobar}(\overline{H})$.

We state the next result in the generality that we will need later. Recall that the Hochschild complex of the second kind $C^\bullet_{II}(H)$ of a DG coalgebra (H, d) is defined as the totalization of the (b, d) double complex where one uses direct sums, not products. For an ordinary coalgebra this is just the usual Hochschild complex. The cyclic complex of the second kind is defined by

$$CC^\bullet_{II}(H) = (C^\bullet_{II}(H), b + uB) \tag{4.9}$$

PROPOSITION 4.2.2. *For a cocommutative DG Hopf algebra H,*
1) there is an isomorphism of DG algebras

$$C^\bullet_{II}(H) \xrightarrow{\sim} (H \ltimes \mathrm{Cobar}(\overline{H}), b + d);$$

*2) there are natural $k[[u]]$-linear (u)-adically continuous A_∞ ***ISO? morphism*

$$CC^\bullet_{II}(H) \xrightarrow{\sim} ((H \ltimes \mathrm{Cobar}(\overline{H}))[[u]], b + d + uB)$$

PROOF. Let us prove 1). Note that the product on Hochschild cochains is as follows:

$$(1 \otimes x_1 \otimes \ldots \otimes x_n)(1 \otimes y_1 \otimes \ldots \otimes y_n) = \pm 1 \otimes y_1 \otimes \ldots \otimes y_n \otimes x_1 \otimes \ldots \otimes x_n \tag{4.10}$$

Therefore, if we denote $1 \otimes x$ by (x), the k-submodule generated by $1 \otimes x_1 \otimes \ldots \otimes x_n$ for $x \in \overline{H}$ is a DG subalgebra isomorphic to $\mathrm{Cobar}(\overline{H})$. Combining this with the formulas for left and right multiplication by H, we get 1).

Now let us prove 2). Let us start with an observation: while $B(x) = x$ which is compatible with 2), the action of B on $\mathrm{Cobar}(\overline{H})$ in general does not agree for the algebras in the two sides of 2). In fact, on the right hand side we have

$$B(1 \otimes x_1 \otimes \ldots \otimes x_n) = \pm B((x_n) \ldots (x_1)) = \sum \pm (x_n) \ldots (x_{j+1}) x_j (x_{j-1}) \ldots (x_1)) =$$
$$\sum \pm (1 \otimes x_{j+1} \otimes \ldots \otimes x_n)(x_j^{(0)} \otimes x_j^{(1)} x_1 \otimes \ldots \otimes x_j^{(j-1)} x_{j-1}) =$$
$$\sum x_j^{(0)} \otimes x_j^{(1)} x_1 \otimes \ldots \otimes x_j^{(j-1)} x_{j-1} \otimes x_{j+1} x_j^{(j)} \otimes \ldots \otimes x_n x_j^{(n-1)}$$

whereas on the left hand side we have the usual

$$B(1 \otimes x_1 \otimes \ldots \otimes x_n) = \sum \pm x_j \otimes \ldots \otimes x_{j-1}$$

For $n = 1$ the two expressions are equal. For $n = 2$ they are not but the difference is cohomologous to zero. In fact, it is equal to the value of the map

$$1 \otimes x_1 \otimes x_2 \otimes x_3 \mapsto x_2 \otimes x_3 x_1$$

at $b(1 \otimes x_1 \otimes x_2)$. Our aim is to extend this calculation.

More precisely, we will prove 2) by constructing a universal A_∞ morphism comprised of the following expressions. Let us write

$$x[m] = \sum x^{(1)} \ldots x^{(m)} \tag{4.11}$$

for $x \in H$ and $m \geq 0$ (in particular, $x[0] = 1$). Consider maps $\mathrm{Cobar}^n(\overline{H}) \to C^q_{II}(H)$ of the form

$$F_{(m_j^{(k)})} : (y_1|\dots|y_n) \mapsto y_n[m_n^{(0)}]\dots y_1[m_1^{(0)}] \otimes \dots \otimes y_n[m_n^{(q)}]\dots y_1[m_1^{(q)}]$$

defined for any collection $(m_j^{(k)})$, $m_j^{(k)} \geq 0$, such that $\sum_k m_j^{(k)} > 0$ for any j and $\sum_j m_j^{(k)} > 0$ for any k. Now define the composition

$$\mathrm{Cobar}^{n_1}(\overline{H}) \otimes \dots \otimes \mathrm{Cobar}^{n_p}(\overline{H}) \to \mathrm{Cobar}^{n_1+\dots+n_p}(\overline{H}) \to C^q_{II}(H)$$

where the first map is the product *in the opposite order* and the second is $F_{(m_j^{(k)})}$ for some collection as above.

Extend these maps to H-bi-invariant linear maps

$$(H \ltimes \mathrm{Cobar}(\overline{H})) \otimes_H \dots \otimes_H (H \ltimes \mathrm{Cobar}(\overline{H})) \to C^\bullet_{II}(H)$$

Because of 1), they and their linear combinations can be viewed as Hochschild cochains of $H \ltimes \mathrm{Cobar}(\overline{H})$. We will see next that they form a subcomplex, meaning, the differential preserves this class of cochains.

Every (homogeneous) cochain gives rise to three numbers: q, $-p$, and $-n$ where $n = n_1 + \dots + n_p$. The total differential is the sum of three differentials:

a) the differential $\_ \circ b'$ (induced by the differential in $\mathrm{Cobar}(\overline{H})$) is of tri-degree $(0, 0, 1)$;

b) the differential $\partial_{\mathrm{Cobar}}$ (determined by the multiplication in $\mathrm{Cobar}(\overline{H})$) is the sum of components of tri-degree $(k, 1, -k)$ for $k \geq 0$;

c) the differential $b \circ \_$ (induced by the differential in $C^\bullet(H)$) is of degree $(1, 0, 0)$.

The component of the total complex that has total degree m is the direct product of components of tri-degree $q, -p, -n$ with $q - p - n = m$. The spectral sequence that converges to the cohomology of the total complex starts with the double complex where q is fixed. For every fixed n, the cohomology of this double complex can be computed by the spectral sequence where the cohomology of $\partial_{\mathrm{Cobar}}$ is computed first. For any fixed n, the $\partial_{\mathrm{Cobar}}$ complex is the product of the linear span of all operations corresponding to $(m_j^{(k)}), 1 \leq j \leq n,\ 1 \leq k \leq q$, and the complex whose basis is formed by partitions $(n_1, \dots, n_p)$, $n = n_1 + \dots + n_p$, $n_i > 0$, with the differential

$$(n_1, \dots, n_p) \mapsto \sum_{i=1}^{p-1} (-1)^i (n_1, \dots, n_i + n_{i+1}, \dots, n_p).$$

The cohomology of the latter complex is zero when $n > 1$. So the first term of the spectral sequence has the basis formed by operations corresponding to $(m^{(k)}), 0 \leq k \leq q$, $m^{(k)} > 0$ for $k > 0$. Note that $\Delta x[m] = x[m] \otimes x[m]$. Therefore the differential $b \circ \_$ computes the Hochschild cohomology of the coalgebra has basis $\{[m],\ m > 0\}$ over k, subject to $\Delta[m] = [m] \otimes [m]$. The basis of the cohomology is $\{[m],\ m > 0\}$. (In other words, cohomology vanishes for $q > 0$).

For $m > 0$, consider the composition

$$\mathrm{Cobar}(\overline{H}) \to \overline{H} \to C^0(H)$$

where the first map is the projection and the second is given by $(x) \mapsto x[m]$. This map is a restriction to to $\mathrm{Cobar}^1(\overline{H})$ of a cocycle of the total complex that is defined uniquely up to a coboundary. We conclude that the total complex has cohomology whose basis are such cocycles. For $m = 1$ we get the derivation B. A FEW MORE WORDS □

5. Pairings between chains and cochains

DEFINITION 5.0.1. *Let a* A *be a graded algebra. For* $D \in C^d(A, A)$ *we set*

$$\iota_D(a_0 \otimes \ldots \otimes a_n) = (-1)^{|D| \sum_{i \le d}(|a_i|+1)} a_0 D(a_1, \ldots, a_d) \otimes a_{d+1} \otimes \ldots \otimes a_n \quad (5.1)$$

The following identities are straightforward.

PROPOSITION 5.0.2.

$$[b, \iota_D] = i_{\delta D}$$

$$\iota_D \iota_E = (-1)^{|D||E|} \iota_{E \smile D}$$

Recall also the L-operations as defined in (8.0.7). The following holds

PROPOSITION 5.0.3.

$$[L_D, L_E] = L_{[D,E]};\ [b, L_D] + L_{\delta D} = 0 \text{ and } [L_D, B] = 0. \quad (5.2)$$

Now let us extend the above operations to the cyclic complex. Define

$$S_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j \ge 0;\ k \ge j+d} \epsilon_{jk} 1 \otimes a_{k+1} \otimes \ldots a_0 \otimes \ldots \otimes \quad (5.3)$$

$$D(a_{j+1}, \ldots, a_{j+d}) \otimes \ldots \otimes a_k$$

(The sum is taken over all cyclic permutations; a_0 appears to the left of D). The signs are as follows:

$$\epsilon_{jk} = (-1)^{|D|(|a_0| + \sum_{i=1}^n (|a_i|+1)) + (|D|+1) \sum_{j+1}^k (|a_i|+1) + \sum_{i \le k}(|a_i|+1) \sum_{i \ge k}(|a_i|+1)}$$

As we will see later, all the above operations are partial cases of a unified algebraic structure for chains and cochains; the sign rule for this unified construction will be explained in 4.2.

PROPOSITION 5.0.4. *([**?**])*

$$[b + uB, \iota_D + uS_D] - i_{\delta D} - uS_{\delta D} = uL_D$$

LEMMA 5.0.5.

$$[L_D, \iota_E + uS_E] = \iota_{[D,E]} + uS_{[D,E]}$$

if $D \in C^{\le 1}(A, A)$.

For a general cochain D the above is true up to homotopy:

PROPOSITION 5.0.6. *([**248**]) There exists a linear transformation* $T(D, E)$ *of the Hochschild chain complex, bilinear in* D, $E \in C^\bullet(A, A)$*, such that*

$$[b + uB, T(D, E)] - T(\delta D, E) - (-1)^{|D|} T(D, \delta E) =$$

$$= [L_D, \iota_E + uS_E] - (-1)^{|D|+1}(\iota_{[D,E]} + uS_{[D,E]})$$

We use the notation

$$I_D = \iota_D + uS_D \quad (5.4)$$

6. Basic invariance properties of Hochschild and cyclic homology

6.1. Morita invariance.

THEOREM 6.1.1. *The trace map*

$$\# : C_\bullet(M_N(k) \otimes A) \to C_\bullet(A)$$

given by

$$(T_1 \otimes a_1) \otimes (T_2 \otimes a_2) \otimes \ldots \otimes (T_n \otimes a_n) \mapsto Tr(T_1 T_2 \ldots T_n) a_1 \otimes a_2 \otimes \ldots \otimes a_n.$$

descends is a quasiisomorphism of cyclic(resp. periodic and negative periodic) complexes.

PROOF. By the Künneth formula it is sufficient to check the claim for $A = k$, and we will leave as an exercise for the reader. □

6.2. Homotopy invariance.

THEOREM 6.2.1. *Supppose that*

$$t \to \phi(t) : A \to B$$

is a one parameter family of homomorphisms depending polynomially on $t \in \mathbb{R}$. *Then the induced family of morphisms of complexes* $\phi(t)_* : CC^{per}_\bullet(A) \to CC^{per}_\bullet(B)$ *is constant up to homotopy.*

PROOF. $\phi(t)$ induces a homomorphism of algebras

$$A \to B \otimes k[t]$$

and, by Künneth formula, it is sufficient to show that the evaluation homomorphism

$$k[t] \ni P \mapsto P(a) \in k$$

induces a map on periodic cyclic homology which is independent of the choice of a. We will see later that the map

$$C_n(k[t]) \ni f_0 \otimes \ldots \otimes f_n \to f_0 df_1 \ldots df_n \in \Omega^n(\mathbb{R})$$

induces a quasiisomorphism of the periodic cyclic complex of $k[t]$ with the de Rham complex of $\mathbb{R}$ with coefficients in $k[t]$ and the Poincare lemma finishes the proof.

An alternative proof can e given using the Cartan formula from the proposition 5.0.4 in the next section applied to the operator L_{∂_t} acting on the cyclic periodic complex of $B[01]$. □

7. Bibliographical notes

CHAPTER 5

Hochschild and cyclic homology as non-Abelian derived functors

1. Homology of free algebras

Let V be a free k-module and $A = T(V)$ the free algebra over k generated by V.

PROPOSITION 1.0.1. *The embedding of the subcomplex*

$$T(V) \otimes V \xrightarrow{b} T(V) \tag{1.1}$$

located in degrees 1 *and* 0 *into* $C_\bullet(V)$ *is a homotopy equivalence.*

PROOF. Indeed, the subcomplex $T(V) \otimes V \otimes T(V) \to T(V) \otimes T(V)$ of the bar resolution $\mathcal{B}_\bullet(T(V)) = T(V) \otimes \overline{T}(V)^{\otimes \bullet} \otimes T(V)$ is a free bimodule resolution of V. The proof follows from applying the functor $\otimes_{T(V) \otimes T(V)^{op}} T(V)$ to this resolution. $\square$

The subcomplex V can be defined more invariantly as a quotient rather that a subcomplex: for any algebra A, put

$$C_\bullet(A)^{sh} = (C_1(A)/bC_2(A) \xrightarrow{b} C_0(A)) \tag{1.2}$$

To see that (1.1) and (1.2) are the same for $A = T(V)$, observe that the former maps to the latter; denote this map by i. Now construct the map P in the opposite direction as follows: in degree zero it is the identity; in degree one,

$$r \otimes v_1 \dots v_n \mapsto \sum_{j=1}^{n} v_{j+1} \dots v_n r v_1 \dots v_{j-1} \otimes v_j \tag{1.3}$$

for $r \in T(V)$ and $v_i \in V$. We have $P \circ i = \mathrm{id}$ whereas

$$(\mathrm{id} - i \circ P)(r \otimes v_1 \dots v_n) = b \sum_{j=1}^{n-1} v_1 \dots v_{j-1} \otimes v_j \otimes v_{j+1} \dots v_n \tag{1.4}$$

COROLLARY 1.0.2. *For* $A = T(V)$, *the projection* $C_\bullet(A) \to C_\bullet(A)^{sh}$ *is a homotopy equivalence.*

PROOF. It is immediate that for $A = T(V)$ the above projection comes from a map of bimodule resolutions, and the statement follows from standard homological algebra. It is easy to write an explicit homotopy, and we will do so in order to use it later in various cases, instead of referring to more general statements of homological algebra. In fact our homotopy directly generalizes (1.4).

Put

(1.5)

$$h(r_0 \otimes v_1 \ldots v_n \otimes r_2 \otimes \ldots \otimes r_m) = r_0 \sum_{j=1}^{n-1} v_1 \ldots v_{j-1} \otimes v_j \otimes v_{j+1} \ldots v_n \otimes r_2 \otimes \ldots \otimes r_m$$

for $r_k \in T(V)$ and $v_i \in V$.

LEMMA 1.0.3. *Let* $P : C_\bullet(T(V)) \to C_\bullet(T(V))^{\mathrm{sh}}$ *be the projection; let* $i : C_\bullet(T(V))^{\mathrm{sh}} \to C_\bullet(T(V))$ *be the embedding equal to the identity on* $T(V)$ *and sending* $r \otimes v$ *to* $r \otimes v$ *for* $r \in T(V)$ *and* $v \in V$. *Then*

$$P \circ i = \mathrm{id};\ \mathrm{id} - i \circ P = [b, h]$$

where h *is as in* (1.5).

The proof is straightforward. □

REMARK 1.0.4. Formula (1.5) for h can be generalized to the graded case. The sign of the jth term in the sum becomes $(-1)^{|r_0| + \sum_{p<j} |v_p|}$.

1.1. Cyclic complexes of a free algebra. For any algebra we have well-defined

$$C_0(A)^{\mathrm{sh}} \xrightarrow{B} C_1(A)^{\mathrm{sh}} \xrightarrow{b} C_0(A)^{\mathrm{sh}} \tag{1.6}$$

satisfying $bB = 0$; $Bb = 0$. We can therefore form short versions of the negative and other cyclic complexes; for example, put

$$\mathrm{CC}^-_\bullet(A)^{\mathrm{sh}} = (C_\bullet(A)^{\mathrm{sh}}[[u]], b + uB) \tag{1.7}$$

PROPOSITION 1.1.1. *The projection* $\mathrm{CC}^-_\bullet(T(V)) \to \mathrm{CC}^-_\bullet(T(V))^{\mathrm{sh}}$ *is a homotopy equivalence, and similarly for* $\mathrm{CC}_\bullet$ *and for* $\mathrm{CC}^{\mathrm{per}}_\bullet$.

PROOF. Follows from the fact that the projection preserves the filtration by powers of u and is a homotopy equivalence on associated graded quotients. □

REMARK 1.1.2. For any algebra A one has $C_1(A)^{\mathrm{sh}} \xrightarrow{\sim} \Omega^1(A)/[A, \Omega^1(A)] = \mathrm{DR}^1(A)$ in the language of 15. Under this identification, (1.6) becomes

$$A \xrightarrow{d} \mathrm{DR}^1(A) \xrightarrow{b} A \tag{1.8}$$

This justifies calling the short cyclic complexes like (1.7) *two-periodic De Rham complexes* of A.

COROLLARY 1.1.3. *The reduced cyclic complex* $\mathrm{CC}_\bullet(T(V))/\mathrm{CC}_\bullet(k)$ *is homotopy equivalent to*

$$\bigoplus_{n \geq 1} (\ldots \xrightarrow{N} V^{\otimes n} \xrightarrow{1-t} \ldots \xrightarrow{N} V^{\otimes n} \xrightarrow{1-t} V^{\otimes n})$$

where t *is the cyclic permutation of cyclic factors and* $N = 1 + t + \ldots + t^{n-1}$.

When k *contains* $\mathbb{Q}$, *then*

$$\overline{\mathrm{HC}}_m(T(V)) \xrightarrow{\sim} 0$$

for $m > 0$*;*

$$\overline{\mathrm{HC}}_0(T(V)) \xrightarrow{\sim} T(V)/([T(V), T(V)] + k).$$

PROOF. The embedding i identifies both $C_0(T(V))^{\mathrm{sh}}$ and $C_1(T(V))^{\mathrm{sh}}$ with $\oplus_{n \geq 1} V^{\otimes n}$. Under this identification, b becomes $1 - t$ and B becomes N. □

2. Semi-free algebras

DEFINITION 2.0.1. *A differential graded algebra* R *is semi-free over* k *if*

(1) *as a graded algebra, it is equal to* $T(V)$ *where* V *is a free graded* k*-module;*
(2) *V has a filtration* $0 = V_{-1} \subset V_0 \subset V_1 \subset \ldots \subset V_n \subset \ldots$ *such that* dV_n *is contained in the subalgebra generated by* V_{n-1} *for all* n.

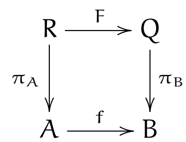
If R is concentrated in non-positive degrees then the second condition is redundant as we can take $V_n = \oplus_{j \le n} V^{-j}$.

PROPOSITION 2.0.2. *For any DG algebra* A *there exists a semi-free DG algebra* R *together with a surjective quasi-isomorphism* $R \to A$.

PROOF. Choose a k-module V_0 that generates A as an algebra. Let $R_0 = TV_0$ with zero differential. Consider the epimorphism ********** □

A DG algebra R such as in Proposition 2.0.2 is called a semi-free resolution of A.

PROPOSITION 2.0.3. *Let* R *be a semi-free resolution of* $R \xrightarrow{\pi_A} A$ *and* $S \xrightarrow{\pi_B} B$ *a semi-free resolution of* B*. For a morphism* $f : A \to B$ *there exists a morphism* $F : R \to S$ *such that* $\pi_B F = f\pi_A$*. Any two such morphisms* F *are homotopic.*



Any two semi-free resolutions of a DG algebra A *are homotopy equivalent.*

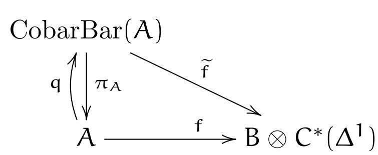
PROOF. We construct F, as well as D, on V_n inductively in n. ***A bit more?*** □

LEMMA 2.0.4. *Let* A *be semi-free. Then being homotopic is an equivalence relation on morphisms* $A \to B$.

PROOF. As shown in 9, being homotopic is an equivalence relation on A_∞ morphisms $A \to B$. But such an A_∞ morphism is a DG algebra morphism

$$\mathrm{CobarBar}(A) \to B.$$

Let π_A be the projection of CobarBar(A) to A. If A is semi-free then there is a morphism of DG algebras q suct tat $q\pi_A = \mathrm{id}_A$. For any $f_0, f_1 : A \to B$, a homotopy $\widetilde{f}$ between $f_0\pi_A$ and $f_1\pi_A$ leads to a homotopy $f = \widetilde{f}q$ between f_0 and f_1.



□

REMARK 2.0.5. If we replace morphisms of DG algebras by morphisms of complexes then we arrive at the usual definition of chain homotopic maps.

3. Hochschild and cyclic homology and semi-free resolutions

3.1. Hochschild and cyclic complexes of semi-free algebras. Let us start by observing that complexes $C_\bullet(A)^{\mathrm{sh}}$ are defined for DG algebras (now they are double complexes with two columns). Also, the complexes $CC^-_\bullet(R)^{\mathrm{sh}}$, etc. are defined.

LEMMA 3.1.1. *For a semi-free DG algebra* R*, the projections*

$$C_\bullet(R) \to C_\bullet(R)^{\mathrm{sh}};\ CC^-_\bullet(R) \to CC^-_\bullet(R)^{\mathrm{sh}};$$

$$CC_\bullet(R) \to CC_\bullet(R)^{\mathrm{sh}};\ CC^{\mathrm{per}}_\bullet(R) \to CC^{\mathrm{per}}_\bullet(R)^{\mathrm{sh}}$$

are homotopy equivalences of complexes.

PROOF. Let h be the homotopy as in Remark 1.0.4. Define

$$H = h + \sum_{n\geq 1}(-1)^n(hd)^n h;\ I = \sum_{n\geq 0}(-1)^n(hd)^n i \tag{3.1}$$

These are infinite sums but $[h, d]$ is locally nilpotent because the algebra is semi-free. We have

$$P \circ I = \mathrm{id};\ \mathrm{id} - I \circ P = [b + d, H] \tag{3.2}$$

Indeed,

$$[b, (hd)^n h] = (hd)^n + (dh)^n - (hd)^n iP$$

(which follows from $Ph = 0$);

$$[d, (hd)^n h] = (hd)^{n+1} + (dh)^{n+1}$$

Formulas (3.2) show that the projection of the long Hochschild complex to the short is a homotopy equivalence. Therefore the same is true for all the cyclic complexes. Explicitly, one can modify H and I from (3.1) replacing d by $d+uB$. □

PROPOSITION 3.1.2. *Let* R *be a semi-free resolution of a DG algebra* A*. Then*

(1) *the complex* $C_\bullet(R)^{\mathrm{sh}}$ *computes the Hochschild homology of* A;
(2) *the complex* $CC^-_\bullet(R)^{\mathrm{sh}}$ *computes the negative cyclic homology of* A;
(3) *similarly for cyclic and periodic cyclic homologies.*

PROOF. In fact both morphisms

$$C_\bullet(R)^{\mathrm{sh}} \longleftarrow C_\bullet(R) \longrightarrow C_\bullet(A) \tag{3.3}$$

are quasi-isomorphisms. Same for cyclic complexes of all types. □

PROPOSITION 3.1.3. *Let* k *contain* $\mathbb{Q}$*. Let* R *be a semi-free resolution of* A*. The complex* $R/([R,R]+k)$ *computes the reduced cyclic homology* $\overline{HC}_\bullet(A)$.

PROOF. Indeed, both morphisms

$$R/([R,R]+k) \longleftarrow \overline{CC}_\bullet(R) \longrightarrow \overline{CC}_\bullet(A) \tag{3.4}$$

are quasi-isomorphisms. □

REMARK 3.1.4. Because of Remark 2.0.5 , it is clear that all complexes defined above in terms of a quasi-free resolution are well-defined up to chain homotopy equivalence of complexes.

3.2. The relative version. Consider two DG algebras A and R. We say that R is semi-free over A if R is freely generated over A as a graded algebra, V has a filtration $0 = V_{-1} \subset V_0 \subset V_1 \subset \ldots$, dV_n is inside the subalgebra generated by A and V_{n-1}, and $d|A$ is the differential of the DGA A.

Let $A \xrightarrow{f} B$ is a morphism of DG algebras. Let R be a DG algebra semi-free over A. A morphism $R \to B$ is a morphism over A if its restriction to A is f. A homotopy between two such morphisms is a homotopy over A if its restriction to A is the composition $A \to B \to B \otimes C^*(\Delta^1)$.

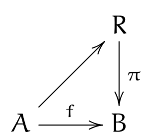
LEMMA 3.2.1. *Being homotopic over* A *is an equivalence relation on morphisms* $R \to A$ *iver* A.

PROOF. The proof is exactly as in the absolute case, except we use the relative Hochschild cochain complex

$$\widetilde{C}^\bullet(R/A, B) = \prod_{n=0}^{\infty} \underline{\mathrm{Hom}}_{A \otimes A^{\mathrm{op}}}(R \otimes_A \ldots \otimes_A R, B) \tag{3.5}$$

□

Let $f : A \to B$ be a morphism of DG algebras. A semi-free resolution of B over A is a semi-free DG algebra over A together with a surjective quasi-isomorphism $\pi : R \to B$ whose restriction to A is f.



Any two such resolutions of the same B are homotopy equivalent over A.

Now define $\Omega^1_{R/A}$ to be the DG bimodule generated by symbols dr, $r \in R$, that are k-linear in r and of degree $j+1$ for $r \in R^j$, subject to relations

$$d(r_1 r_2) = dr_1 r_2 + (-1)^{|r_1|} r_1 dr_2;\ da = 0, a \in A. \tag{3.6}$$

Put

$$\mathrm{DR}^1(R/A) = \Omega^1_{R/A}/[R, \Omega^1_{R/A}]; \tag{3.7}$$

define

$$\mathrm{DR}^1(R/A) \xrightarrow{b} (R/A)/[A, R/A] \xrightarrow{B} \mathrm{DR}^1(R/A) \tag{3.8}$$

by

$$b(r_0 dr_1 r_2) = (-1)^{|r_0|}(|r_1| + |r_2| + 1)[r_1, r_2 r_0];\ Br = dr$$

PROPOSITION 3.2.2. *Let* $f : A \to B$ *be a morphism of DGA. Let* R *be a resolution of* B *which is quasi-free over* A. *Then the complex*

$$\mathrm{DR}^1(R/A) \xrightarrow{b} (R/A)/[A, R/A] \tag{3.9}$$

is quasi-isomorphic to $\mathrm{Cone}(C_\bullet(A) \xrightarrow{f} C_\bullet(B))$; *the complex*

$$\ldots \xrightarrow{B} \mathrm{DR}^1(R/A) \xrightarrow{b} (R/A)/[A, R/A] \xrightarrow{B} \mathrm{DR}^1(R/A) \xrightarrow{b} (R/A)/[A, R/A]$$

is quasi-isomorphic to $\mathrm{Cone}(CC_\bullet(A) \xrightarrow{f} CC_\bullet(B))$; *and similarly for the negative and periodic cyclic complexes.*

PROOF. Consider resolutions Q of A and $\mathbf{R}$ of B such that the following diagram is commutative and $\mathbf{R}$ is semi-free over Q.

$$\begin{array}{ccc} Q & \xrightarrow{\widetilde{f}} & \mathbf{R} \\ \downarrow{\scriptstyle \pi_A} & & \downarrow{\scriptstyle \pi_B} \\ A & \xrightarrow{f} & B \end{array}$$

LEMMA 3.2.3. *Proposition 3.2.2 is true with* (3.9) *replaced by*

$$\mathrm{DR}^1(\mathbf{R}/Q) \xrightarrow{b} (\mathbf{R}/Q)/[Q, \mathbf{R}/Q] \tag{3.10}$$

where Q *and* $\mathbf{R}$ *are as above.*

PROOF. For a DG algebra D and a DG bimodule M we will write

$$M_{\sharp,D} = M/[D, M] = M \otimes_{D\otimes D^{op}} D \tag{3.11}$$

Consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\Omega^1_Q \otimes_Q \mathbf{R})_{\sharp,Q} & \longrightarrow & (\Omega^1_{\mathbf{R}})_{\sharp,\mathbf{R}} & \longrightarrow & (\Omega^1_{\mathbf{R}/Q})_{\sharp,\mathbf{R}} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle b} & & \\ 0 & \longrightarrow & Q + [Q, \mathbf{R}] & \longrightarrow & \mathbf{R} & \longrightarrow & (\mathbf{R}/Q)_{\sharp,Q} & \longrightarrow & 0 \end{array}$$

We observe that its rows are exact. Now, the left column of this diagram fits into its own diagram with short exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (\Omega^1_Q)_{\sharp,Q} & \longrightarrow & (\Omega^1_Q \otimes_Q \mathbf{R})_{\sharp,Q} & \longrightarrow & (\Omega^1_Q \otimes_Q (\mathbf{R}/Q))_{\sharp,Q} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Q & \longrightarrow & Q + [Q, \mathbf{R}] & \longrightarrow & [Q, \mathbf{R}/Q] & \longrightarrow & 0 \end{array}$$

We claim that the right column is an acyclic complex. In fact, the complex

$$(\Omega^1_Q \otimes_Q (R/Q))_{\sharp,Q} \to R$$

is quasi-isomorphic to the Hochschild complex $C_\bullet(Q, R/Q)$. Because R/Q is a semi-free bimodule over Q, the projection

$$C_\bullet(Q, R/Q) \to (R/Q)/[Q, R/Q]$$

is a quasi-isomorphism, whence the claim. □

Now consider a diagram

$$\begin{array}{ccc} Q & \longrightarrow & \mathbf{R} \\ \downarrow{\scriptstyle \pi_A} & & \downarrow \\ A & \longrightarrow & R \\ & \searrow^{f} & \downarrow{\scriptstyle \pi_B} \\ & & B \end{array}$$

It remains to compare the column complexes

$$\begin{array}{ccc} (\Omega^1_{\mathbf{R}/Q})_{\natural,\mathbf{R}} & \longrightarrow & (\Omega^1_{R/A})_{\natural,R} \\ \downarrow & & \downarrow \\ (\mathbf{R}/Q)_{\natural,Q} & \longrightarrow & (R/A)_{\natural,A} \end{array} \tag{3.12}$$

We claim that the horizontal maps induce their quasi-isomorphism. Indeed, for a morphism of algebras $D \to E$ and for an E-bimodule M, define the relative Hochschild complex $C_\bullet(E/D, M)$ by

$$C_n(E/D, M) = (M \otimes_D (E/D) \otimes_D \ldots \otimes_D (E/D))_{\natural,D}$$

where there are n factors E/D; the Hochschild differential b is given by the usual formula. We claim that the projections of $C_\bullet(R/A, R)$ to the left column and of $C_\bullet(\mathbf{R}/Q, \mathbf{R})$ to the left column of (3.12) are quasi-isomorphisms. This is easily seen, for example, by observing that Lemma 1.0.3 holds for a D-free algebra $E = D * T(V)$ for any algebra D, with the identical proof. Consequently, Proposition 3.1.1 also admits generalization to the case of semi-free DGA over a DGA D. Finally, observe that

$$C_\bullet(\mathbf{R}/Q, \mathbf{R}) \to C_\bullet(R/A, R)$$

is a quasi-isomorphism. □

CHAPTER 6

Hochschild and cyclic homology and the bar construction

1. Hochschild and cyclic (co)homology of coalgebras

Just as we constructed the Hochschild and cyclic complexes for differential graded algebras, we can in a dual way construct analogous complexes for differential graded coalgebras. For a DG coalgebra C with coproduct Δ, differential d and counit ϵ, let

$$\widetilde{C}^{\bullet}(C) = \prod_{n \geq 0} C \otimes C^{\otimes n} \tag{1.1}$$

$$C^{\bullet}(C) = \prod_{n \geq 0} C \otimes \overline{C}^{\otimes n} \tag{1.2}$$

where $\overline{C} = \ker \epsilon$. The following is a construction dual to the one for algebras. Put

$$d^j(c_0 \otimes \dots c_n) = (-1)^{\sum_{p<j} |c_p| + |c_j^{(1)}|} c_0 \otimes \dots \otimes c_{j-1} \otimes c_j^{(1)} \otimes c_j^{(2)} \otimes \dots \otimes c_n,$$

$0 \leq j < n+1$;

$$d^{n+1}(c_0 \otimes \dots \otimes c_n) = (-1)^{(\sum_{p=1}^{n} |c_p| + |c^{(1)}|)|c_0^{(1)}} c_0^{(2)} \otimes c_1 \otimes \dots \otimes c_n \otimes c_0^{(1)};$$

$$s^j(c_0 \otimes \dots \otimes c_n) = (-1)^{\sum_{p \leq j} |c_j|} c_0 \otimes \dots c_j \epsilon(c_{j+1}) \otimes c_{j+2} \otimes \dots \otimes c_n,$$

$-1 \leq j \leq n$;

$$t(c_0 \otimes \dots \otimes c_n) = (-1)^{|c_0| \sum_{p>0} |c_p|} c_1 \otimes \dots \otimes c_n \otimes c_0$$

Note that d^j and s^j with $j \geq 0$ define a cosimplicial structure module structure $[n] \mapsto C^{\otimes(n+1)}$; together with t they define a cocyclic module structure. We put

$$b = \sum_{j=0}^{n+1} (-1)^j d^j;\ \tau = (-1)^n t;\ N = \sum_{j=0}^{n} \tau^j \tag{1.3}$$

on $C \otimes C^{\otimes n}$;

$$B = N s_{-1}(1 - \tau) \tag{1.4}$$

As in the case of algebras, b and B descend to $C^{\bullet}(B)$ and satisfy

$$b^2 = B^2 = bB + Bb = 0$$

Now define the Hochschild complex of C to be

$$(C^{\bullet}(C), d + b) \tag{1.5}$$

Define also

$$\mathrm{CC}^\bullet(C) = (C^\bullet(C)[[u]], b + d + uB) \tag{1.6}$$

where u is a formal parameter of cohomological degree -2;

$$\mathrm{CC}^\bullet_{\mathrm{per}}(C) = (C^\bullet(C)[[u, u^{-1}], b + d + uB) \tag{1.7}$$

and

$$\mathrm{CC}^\bullet_-(C) = (C^\bullet(C)[[u, u^{-1}]/uC^\bullet(C)[[u]], b + d + uB) \tag{1.8}$$

1.1. Complexes of the second kind. The above definition is dual to the one we used for DG algebras. An important feature of both is that they are invariant with respect to quasi-isomorphisms of DG (co)algebras. In this chapter, however, we are going to consider the example $C = \mathrm{Bar}(A)$ where A is a DG algebra. Since C is contractible when A has a unit, we cannot get anything meaningful using the complexes above. We can, however, define *the Hochschild complex of the second kind*

$$C^\bullet_{II}(C) = \bigoplus_{n \geq 0} C \otimes \overline{C}^{\otimes n} \tag{1.9}$$

with the differential $b + d$. Define also

$$\mathrm{CC}^\bullet_{II}(C) = (C^\bullet_{II}(C)[[u]], b + d + uB) \tag{1.10}$$

and similarly for the negative and periodic complexes as in (1.7), (1.8).

1.2. Two-periodic De Rham complex. Define for a DG counital coalgebra C

$$\mathrm{DR}_1(C) = \ker(b : C \otimes \overline{C} \to C \otimes \overline{C}^{\otimes 2}); \tag{1.11}$$

Let $C^\bullet(C)_{\mathrm{sh}}$ be the total complex

$$C \xrightarrow{b} \mathrm{DR}_1(C) \tag{1.12}$$

(with the differential $d + b$). Define also

$$\mathrm{CC}^\bullet(C)_{\mathrm{sh}} = (C^\bullet(C)_{\mathrm{sh}}[[u]], b + uB) \tag{1.13}$$

where v is a formal parameter of cohomological degree -2;

$$\mathrm{CC}^\bullet_{\mathrm{per}}(C)_{\mathrm{sh}} = (C^\bullet(C)_{\mathrm{sh}}[[u, v^{-1}], b + uB) \tag{1.14}$$

and

$$\mathrm{CC}^\bullet_-(C)_{\mathrm{sh}} = (C^\bullet(C)_{\mathrm{sh}}[[v, v^{-1}]/vC^\bullet(C)_{\mathrm{sh}}[[u]], b + uB) \tag{1.15}$$

PROPOSITION 1.2.1. *For any DG algebra A, the embedding*

$$C^\bullet(\mathrm{Bar}(A))_{\mathrm{sh}} \longrightarrow C^\bullet_{II}(\mathrm{Bar}(A))$$

is a homotopy equivalence. Same if one replaces $C^\bullet$ by $\mathrm{CC}^\bullet$, $\mathrm{CC}^\bullet_-$, or $\mathrm{CC}^\bullet_{\mathrm{per}}$.

PROOF. Since $\mathrm{Bar}(A)$ is cofree as a graded coalgebra, we cal construct P, I, and H by formulas dual to (3.1). $\square$

2. Homology of an algebra in terms of the homology of its bar construction

We will now apply the above to the DG coalgebra $C = \mathrm{Bar}(A)$ for a DG algebra A. To avoid confusion, we will use boldface for $C \xrightarrow{\mathbf{B}} \mathrm{DR}_1(C) \xrightarrow{\mathbf{b}} C$, while reserving the symbols b and B for the differentials on the Hochschild complex of A.

PROPOSITION 2.0.1. *There are isomorphisms of complexes*

$$\mathrm{Bar}(A)) \xrightarrow{\sim} (A^{\otimes \bullet}, b'); \ \mathrm{DR}_1(\mathrm{Bar}(A)) \xrightarrow{\sim} C_\bullet(A, A)$$

These isomorphisms intertwine $\mathbf{B}$ *with* N *and* $\mathbf{b}$ *with* $\mathrm{id} - \tau$.

PROOF. ****** □

Therefore we can express the $(b, b', \mathrm{id} - \tau, N)$ double complex computing the cyclic homology of A as

$$\ldots \xrightarrow{\mathbf{B}} \mathrm{Bar}(A) \xrightarrow{\mathbf{b}} \mathrm{DR}_1(\mathrm{Bar}(A)) \tag{2.1}$$

Because of Proposition 1.2.1, the above computes the *negative* cyclic homology of $\mathrm{Bar}(A)$. Similarly, the negative cyclic complex of A gets identified with

$$\mathrm{Bar}(A) \xrightarrow{B} \mathrm{DR}_1(\mathrm{Bar}(A)) \xrightarrow{\mathbf{b}} \ldots \tag{2.2}$$

which computes the *cyclic* homology of $\mathrm{Bar}(A)$. We obtain

THEOREM 2.0.2.

$$\mathrm{HC}_\bullet(A) \xrightarrow{\sim} \mathrm{HC}^{-\bullet}_{-,II}(\mathrm{Bar}(A)); \ \mathrm{HC}^-_\bullet(A) \xrightarrow{\sim} \mathrm{HC}^{-\bullet}_{II}(\mathrm{Bar}(A))$$

PROOF. □

2.1. Action of A_∞ morphisms on Hochschild and cyclic complexes. Since an A_∞ morphism is by definition a morphism of DG coalgebras $\mathrm{Bar}(A) \to \mathrm{Bar}(B)$ and the short complexes $C^\bullet(C)_{\mathrm{sh}}$, $\mathrm{CC}^\bullet(C)_{\mathrm{sh}}$, etc. are functorial in C, Theorem 2.0.2 implies that an A_∞ morphism induces morphisms of Hochschild and cyclic complexes. It is easy tho see that these are the same morphisms as in (8.3).

3. Bibliographical notes

Quillen, Cuntz-Quillen

CHAPTER 7

Operations on Hochschild and cyclic complexes, II

1. Introduction

Our motivation is the following. Recall that for an algebra A we denote by $\mathfrak{g}_A^\bullet$ the DG algebra $C^{\bullet+1}(A)$ with the Gerstenhaber bracket. When $A = C^\infty(M)$ then $HC_\bullet^-(A)$ is isomorphic to the cohomology of the complex $(\Omega^\bullet(M)[[u]], ud)$; the cohomology of $\mathfrak{g}_A^\bullet$ is the graded Lie algebra $\mathfrak{g}_M^\bullet$ of multivector fields on M; one can define an action of $\mathfrak{g}_M^\bullet[\epsilon][u]$ on $HC_\bullet^-(A)$: for two multivector fields X, Y the action of $X + \epsilon Y$ is given by $L_X + \iota_Y$ where ι_Y is the contraction operator and $L_X = [d, \iota_X]$.

We would like to have a noncommutative analog of the above action. In fact, because of Theorem 4.1.1, we know that there is an L_∞ action of $\mathfrak{g}_A^\bullet[\epsilon][u]$ on $CC_\bullet^-(A)$, i.e., a DGLA which is quasi-isomorphic to the former acts on a complex quasi-isomorphic to the latter. However, the proof of Theorem 4.1.1 is inexplicit, and we need explicit operations for applications. The earliest and easiest such formulas express not the operations themselves but their compositions with a trace on our algebra (recall that such a trace is a (periodic, negative) cyclic cocycle). We discuss this in section 2. Later ***where? we provide an explicit formula for an action of the *complex* $U(\mathfrak{g}_A^\bullet[\epsilon])[u]$ on $CC_\bullet^-(A)$. It is still relatively explicit, unlike the action of the *algebra* provided by Theorem 4.1.1.

The difficulty is due to the following. In the classical situation, the multiplicative structure (wedge product on multi-vectors and the action on forms by contraction) and the Lie structure (the Schouten bracket on multi-vectors and their action on forms by Lie derivative) are compatible. In the noncommutative case this turns out to be true but is way harder to prove (and is beyond the scope of this book).

2. Operations composed with traces

We use the pairing (5.1) and Proposition 5.0.4 to carry out our first constructions in the spirit outlined in the introduction 1. More precisely, we observe that, if we apply the pairing (5.1) to a Hochschild cochain obtained by cup product from Hochschild cochains of degree ≤ 1 and follow this with a trace, the result will be a cochain with good properties. We give two versions of this construction. The first is due to Quillen ***Ref, the second is from ***NT ref

2.1. The characteristic map. Let $\mathcal{A}$ be a graded associative algebra. $\mathcal{A}[1] \rtimes \mathrm{Der}(\mathcal{A})$ is a DG Lie subalgebra of $\mathfrak{g}_A^\bullet$. Let $\mathcal{L}$ be a DG Lie subalgebra of $\mathcal{A}[1] \rtimes \mathrm{Der}(\mathcal{A})$ and K a graded space on which $\mathcal{L}$ acts so that elements of $\mathcal{A}[1]$ act by zero. Let $\mathrm{tr} : \mathcal{A} \to K$ is an $\mathcal{L}$-equivariant trace. We extend it by zero to the entire Hochschild complex $C_{-\bullet}(\mathcal{A})$.

Given the $X_1, \ldots, X_n \in \mathcal{L}$, and for a Hochschild chain c, we set

$$\chi(X_1, \ldots, X_n)(c) = \frac{1}{n!} \sum_{\sigma \in S_n} \pm \mathrm{tr}(\iota_{X_{\sigma(1)}} \ldots \iota_{X_{\sigma(n)}} c); \tag{2.1}$$

The operations ι_D are the ones from the definition 5.0.1 and the sign is computed as follows: a permutation of X_i and X_j introduces a sign $(-1)^{(|X_i|+1)(|X_j|+1)}$.

Proposition 2.1.1. *(cf. [**461**], [**463**]).* χ *defines a cocycle of the complex*

$$C^{\bullet}(\mathcal{L}, \mathrm{Hom}(C_{-\bullet}(A), K)[[u]]$$

with the differential $b + uB + \delta + u\partial_{\mathrm{Lie}}$*; the action of* $\mathcal{L}$ *on* $\mathrm{Hom}(C_{-\bullet}, K)$ *is induced by the action on* K*. In other words,*

$$\chi(X_1, \ldots, X_n)((b + uB)(c)) = \frac{1}{n!}(\sum \pm\chi(X_1, \ldots, \delta X_i, \ldots, X_n) +$$

$$u \sum \pm\chi([X_i, X_j], \ldots, \widehat{X_i}, \ldots, \widehat{X_j}, \ldots, X_n) +$$

$$u \sum \pm X_i \chi(X_1 \ldots, \widehat{X_i}, \ldots, X_n))(c)$$

.

Proof. □

Explicitly, χ is defined as follows. Let $c = a_0 \otimes \ldots \otimes a_n$. Let D be an odd derivation and x an even element of $\mathcal{A}$. Then

$$\chi(D^n x^N)(c) = \frac{n!N!}{(N+n)!} \sum_{N_0+N_1+\ldots+N_n=N} \mathrm{tr}(a_0 x^{N_0} D(a_1) x^{N_1} \ldots D(a_n) x^{N_n})$$

Assume now that λ is an element of total degree one in $\mathcal{L}$. Define an *improper* (i.e. infinite) periodic cyclic cochain of $\mathcal{A}$, or more precisely an element of

$$\mathrm{Hom}_{k[u]}(C_{\bullet}(A)[u], K[u, u^{-1}]), \tag{2.2}$$

by

$$\chi_\lambda(c) = \mathrm{tr}(\exp(\frac{\iota_\lambda}{u})(c)) \tag{2.3}$$

Recall that a Maurer-Cartan (MC) element of a DG Lie algebra $(\mathcal{L}, [,], \delta)$ is an element λ of degree one such that

$$\delta\lambda + \frac{1}{2}[\lambda, \lambda] = 0 \tag{2.4}$$

Lemma 2.1.2. *Let* λ *be an MC element of* $\mathcal{L}$*.*

$$\chi_\lambda((b + uB)c) = \lambda(\chi_\lambda(c))$$

Proof. Follows from Proposition 2.1.1. □

2.1.1. *JLO cocycle via the characteristic map.* Let A be a graded algebra and let D be an element of A of degree one. Put

$$\lambda_D = \delta D - D^2 \tag{2.5}$$

This is an element of total degree two in $C^\bullet(A, A)$: the first summand is a one-cochain of degree one and the second a zero-cochain of degree two. We have

$$\delta\lambda_D + \frac{1}{2}[\lambda_D, \lambda_D] = 0 \tag{2.6}$$

Given a trace tr on A, put

$$\phi = \chi(\exp(\lambda_D)) \tag{2.7}$$

Using the Duhamel formula for the exponential, we compute the components of the cocycle $\phi(\mathrm{tr}, \theta_D)$:

$$\phi_{2n}(a_0, \ldots, a_{2n}) = \tag{2.8}$$
$$\int_{\Delta^{2n}} \mathrm{tr}(a_0 e^{-t_0 D^2}[D, a_1]e^{-t_1 D^2} \ldots [D, a_{2n}]e^{-t_{2n} D^2})dt_1 \ldots dt_{2n}.$$

where Δ^k is the standard simplex

$$\Delta^k = \{(t_0, \ldots, t_k) \mid t_0 + \ldots + t_k = 1; \, t_i \geq 0, i = 0, \ldots, k.\}$$

It follows from Lemma 2.1.2 that

$$b\phi_{2n} + B\phi_{2n+2} = 0 \tag{2.9}$$

n applications, the exponential factors $e^{-t_j D^2}$ regularize the expression under the integral sign so that the total cocycle can be evaluated on non-trivial classes in the periodic cyclic homology of A

REMARK 2.1.3. The MC equation (2.6) can be formally written as

$$(\delta + \lambda_D)^2 = 0 \tag{2.10}$$

Therefore λ_D plays the role of a flat connection. In fact it gauge equivalent to the trivial flat connection, namely:

$$\delta + \lambda_D = \exp(\mathrm{ad}(D))(\delta) \tag{2.11}$$

where D is viewed as a zero-cochain of degree one.

2.2. Quillen's cochain construction.

2.2.1. *Quillen's infinite periodic cyclic cycle.* Let $(\mathcal{A}, \partial)$ be a DG algebra and let θ be an element of $\mathcal{A}^1$. Put

$$\Omega = \partial\theta + \theta^2 \tag{2.12}$$

One has the Bianchi identity

$$\partial\Omega + [\theta, \Omega] = 0 \tag{2.13}$$

Define the element of $C^{sh}_\bullet(\mathcal{A})((u^{-1}))$

$$(1 + d\theta)\exp(\frac{\Omega}{u}) = (1 + d\theta)\sum_{n=0}^{\infty} \frac{\Omega^n}{u^n n!} \tag{2.14}$$

LEMMA 2.2.1.

$$(\partial + b + uB)((1 + d\theta)\exp(\frac{\Omega}{u})) = 0$$

PROOF. Follows from:

$$\partial(\frac{\Omega^n}{n!}) = -[\theta, \frac{\Omega^n}{n!}]; \tag{2.15}$$

$$d(\frac{\Omega}{n!}) = d\Omega\frac{\Omega^{n-1}}{(n-1)!}; \tag{2.16}$$

$$\partial(d\theta\frac{\Omega^{n-1}}{(n-1)!}) = -d\Omega\frac{\Omega^{n-1}}{(n-1)!} \tag{2.17}$$

$$b(d\theta\frac{\Omega^{n-1}}{(n-1)!}) = [\theta, \frac{\Omega^{n-1}}{(n-1)!}] \tag{2.18}$$

□

Next, given algebras $\mathcal{A}$ and L together with a trace $\mathrm{tr} : L \to K$, consider the morphism

$$\mathrm{tr}_\sharp : C^{sh}_\bullet(\mathcal{A} \otimes L) \to C^{sh}_\bullet(\mathcal{A}) \otimes K \tag{2.19}$$

defines as follows:

$$a \otimes \ell \mapsto a \otimes \mathrm{tr}(\ell);$$

$$(a \otimes \ell)db \mapsto adb \otimes \mathrm{tr}(\ell);$$

$$(a \otimes \ell)d\ell_1 \mapsto 0$$

for $a, b \in \mathcal{A}$, $\ell, \ell_1 \in L$. It is easy to see that (2.19) is well defined and commutes with b and with B. Let us (slightly) generalize this construction as follows: let $\mathcal{B}$ be a DG coalgebra, L an algebra, $\mathrm{tr} : L \to K$ a trace on L. Then there is a morphism

$$\mathrm{tr}_\sharp : C^{sh}_\bullet(\mathrm{Hom}(\mathcal{B}, L)) \to \mathrm{Hom}(C^\bullet_{sh}(\mathcal{B}), K) \tag{2.20}$$

commuting with b and B.

Now let A and L be two algebras. Take $\mathcal{B}$ to be $\mathrm{Bar}(A)$. We write

$$C^\bullet(A, L) = \mathrm{Hom}(\mathcal{B}, L) \tag{2.21}$$

By Proposition 2.0.1, given a K-valued trace on L and an element θ of degree one in $C^\bullet(A, L)$, we get an *improper* (or infinite) periodic cyclic cocycle

$$\mathrm{tr}_\sharp((1 + d\theta)\exp(\frac{\Omega}{u})) \tag{2.22}$$

of A with values in K. More precisely, we get a formal series

$$\phi(\mathrm{tr}, \theta) = \sum_{n=0}^{\infty} \phi_n; \;\; \phi_n : A^{\otimes(n+1)} \to K; \tag{2.23}$$

$$b\phi_{2n} + N\phi_{2n+1} = 0; \;\; b'\phi_{2n+1} + (1-\tau)\phi_{2n+2} = 0 \tag{2.24}$$

More generally, we may allow L to be a graded algebra and assume that tr is of degree zero.

We consider the following examples.

2.2.2. *Quillen's definition of Connes' Chern character of Fredholm modules.*

2.2.3. *Quillen's JLO cocycle.* Let $\rho : A \to L$ be a morphism of algebras. We have

$$\rho \in C^1(A,L);\ \partial\rho + \rho^2 = 0 \tag{2.25}$$

For any $D \in L^1$, let

$$\theta_D = \rho + D \tag{2.26}$$

This is an element of total degree one in $C^\bullet(A,L)$.

As in **??** by the Duhamel formula for the exponential, we have

$$\phi_{2n}(a_0,\ldots,a_{2n}) = \int_{\Delta^{2n}} \mathrm{tr}(\rho(a_0)e^{-t_0 D^2}[D,\rho(a_1)]e^{-t_1 D^2}\ldots[D,\rho(a_{2n}]))e^{-t_{2n} D^2} dt_1\ldots dt_{2n}. \tag{2.27}$$

$$\phi_{2n+1}(a_0,\ldots,a_{2n+1}) = \int_{\Delta^{2n+1}} \mathrm{tr}(e^{-t_0 D^2}[D,\rho(a_0)]e^{-t_1 D^2}\ldots[D,\rho(a_{2n}]))e^{-t_{2n+1} D^2} dt_1\ldots dt_{2n+1}. \tag{2.28}$$

In fact more than (2.24) is true, namely

$$b\phi_{2n} + B\phi_{2n+2} = 0 \tag{2.29}$$

To see that, ***FINISH; do the odd case***

REMARK 2.2.2. We have constructed (an algebraic version of) the JLO cochain using MC the same element (2.26) that plays two different roles. Here we viewed it as an element of degree two and treated it as the curvature of a connection a connection $\partial + \theta$ where θ is as in (2.26). In 2.1.1, we treated it as a flat connection and exhibited the gauge transformation that makes it trivial. It would be interesting to understand this better. One is reminded of the relation between differential forms on a manifold and on ils loop space.

3. Action of the Lie algebra cochain complex of $C^{\bullet+1}(A;A)$

DEFINITION 3.0.1. *Set*
$\mathfrak{g}_A$ *denotes the differential graded Lie algebra* $(C^{\bullet+1}(A),[\ ,\ ],\delta)$.

$$\mathfrak{g}_A[\epsilon] = \mathfrak{g}_A + \mathfrak{g}_A\epsilon,\ \epsilon^2 = 0,\ |\epsilon| = 1.$$

Let $U(\mathfrak{g}_A[\epsilon])$ *denote the universal enveloping algebra of* $\mathfrak{g}_A[\epsilon]$ *and let* u *be a formal parameter of degree 2. We will give*

$$U(\mathfrak{g}_A[\epsilon])[u]$$

a differential graded algebra structure with the differential

$$u \cdot \frac{\partial}{\partial\epsilon} + \delta,$$

where δ *denotes the total differential in* $\mathfrak{g}_A$.

THEOREM 3.0.2. *Consider the action of* $U(\mathfrak{g}_A)$ *on* $CC^-(A)$ *where* $D \in \mathfrak{g}_A$ *acts as the* L_D *operation (see the formula* (8.0.7)*).*

Then there is a morphism of complexes of $k[[u]]$*-modules that extends this action:*

$$U(\mathfrak{g}_A[\epsilon])[[u]] \otimes_{U(\mathfrak{g}_A)[[u]]} CC^-_\bullet(A) \quad\to\quad CC^-_\bullet(A)$$
$$U(\mathfrak{g}_A[\epsilon])[u^{-1},u]] \otimes_{U(\mathfrak{g}_A)[u,u^{-1}]} CC^{per}_\bullet(A) \quad\to\quad CC^{per}_\bullet(A)$$

The signs ϵ_σ *are computed according to the rule under which the parity of any* D_i *is* $|D_i|+1$. *We will also use the notation* $I(D_1,\ldots,D_m)\alpha$ *for the left hand side of the above equation.*

3.0.1. *Proof of Theorem 3.0.2.* Let us start by introducing some notation.

NOTATION 3.0.3. *Let* A *be an associative unital algebra.*

- $\mathcal{E}_A$ *denotes the differential graded algebra* $(C^\bullet(A),\cup,\delta)$*;*

Recall that, for a differential graded algebra A, we constructed in the subsection **??** the following structures.

An A_∞ *structure on* $C_\bullet(\mathcal{E}_A)[[u]]$

$$m_n^{(1)}+um_n^{(2)},\ n\in\mathbb{N}, \tag{3.1}$$

where $m_1^{(1)}+um_1^{(2)}=b+\delta+uB$.

An A_∞*-module structure over* $C_\bullet(\mathcal{E}_A)[[u]]$ *on* $C_\bullet(A)[[u]]$

$$\mu_n^{(1)}+u\mu_n^{(2)},\ n\in\mathbb{N}. \tag{3.2}$$

DEFINITION 3.0.4.
Let A *be a unital associative (differential graded) algebra.*

(1) $\star: C_\bullet(\mathcal{E}_A)[[u]]\times C_\bullet(\mathcal{E}_A)[[u]]\to C_\bullet(\mathcal{E}_A)[[u]]$

denotes the binary operation (product) on $C_\bullet(\mathcal{E}_A)[[u]]$ *given by restricting the corresponding binary operation*

$$a\star b=(-1)^{|a|}(m_2^{(1)}(a,b)+um_2^{(2)}(a,b))$$

constructed on $C_\bullet(\mathcal{E}_A))[[u]]$ *in the theorem 4.2.1 to the subspace*

$$C_\bullet(\mathcal{E}_A)[[u]]=C_\bullet(C^0(\mathcal{E}_A))[[u]]\subset C_\bullet(C^\bullet(\mathcal{E}_A))[[u]].$$

(2) $\diamond: C_\bullet(A)[[u]]\times C_\bullet(\mathcal{E}_A)[[u]]\to CC_\bullet(A)[[u]]$

denotes the binary pairing part of the A_∞*-module structure of* $C_\bullet(\mathcal{E}_A)[[u]]$ *over* $C_\bullet(C^\bullet(\mathcal{E}_A))[[u]]$, $\mu_2^{(1)}+u\mu_2^{(2)}$, *constructed in the theorem 4.2.2, restricted to*

$$C_\bullet(A)[[u]]\times C_\bullet(\mathcal{E}_A)[[u]]=C_\bullet(\mathcal{E}_A{}^0)[[u]]\times C_\bullet(C^0(\mathcal{E}_A))[[u]].$$

Now, the formula

$$(\epsilon D_1\cdots\cdot\epsilon D_m)\bullet\alpha=(-1)^{|\alpha|\sum_{i=1}^m(|D_i|+1)}\frac{1}{m!}\sum_{\sigma\in\Sigma_m}\epsilon_\sigma\alpha\diamond(D_{\sigma_1}\star(D_{\sigma_2}\star(\ldots\star D_{\sigma_m}))\ldots)$$

gives the morphism in the statement of Theorem 3.0.2. The proof follows immediately from the fact that, in the A_∞ structures constructed in the theorems 4.2.1 and 4.2.2, the total boundary map commutes with the total binary product structure and

$$[L_D,I(D_1,\ldots,D_m)]=\sum_i(-1)^{(|D|+1)(\sum_{k<i}|D_k|+1)}I(D_1,\ldots,[D,D_i],D_{i+1},\ldots,D_m).$$

***Move/reconcile ***The following observation will be useful later

COROLLARY 3.0.5. *Let* τ *be a tracial functional on an algebra* A *and suppose that*

$$\delta_1, \dots, \delta_n$$

is a family of commuting derivations of A *satisfying*

$$\tau \circ \delta_i = 0, \; i = 1, \dots, n$$

Then

$$a_0 \otimes \dots \otimes a_n \mapsto \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sgn(\sigma) \tau(a_0 \delta_{\sigma(1)}(a_1) \dots \delta_{\sigma(n)}(a_n))$$

defines a cyclic cocycle on A.

PROOF. Let U_+ denote the ideal generated by the derivations $\{\delta_i\}_{i=1,\dots,n}$ in $U(\mathfrak{g}_A[\epsilon])[[u]]$. Under our assumptions,

$$[I(\delta_1, \dots, \delta_n), b + uB] \in U_+,$$

hence, since τ vanishes on U_+, $\tau \circ I(\delta_1, \dots, \delta_n)$ is a cyclic cocycle. It is easy to check that

$$I(\delta_1, \dots, \delta_n)(a_0 \otimes \dots \otimes a_n) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} sgn(\sigma) a_0 \delta_{\sigma(1)}(a_1) \dots \delta_{\sigma(n)}(a_n),$$

hence the claimed result holds. □

****

COROLLARY 3.0.6. *Suppose that* λ *is an odd element of* $\mathfrak{g}_A^\bullet$ *satisfying the Maurer-Cartan equation*

$$\delta\lambda + \frac{1}{2}[\lambda, \lambda] = 0. \tag{3.3}$$

Set

$$\chi(\lambda) = I(\exp(\frac{\lambda\epsilon}{u})) : C_\bullet(A)[u^{-1}, u] \to C_\bullet(A)[u^{-1}, u]]. \tag{3.4}$$

Then

$$[b + uB, \chi(\lambda)] = L_\lambda \chi(\lambda). \tag{3.5}$$

PROOF. By Theorem 3.0.2, given $D_1, \dots, D_n$ in $\mathfrak{g}_A$,

$$[b + uB, I(D_1\epsilon \wedge \dots \wedge D_n\epsilon] = \sum_i \pm L_{D_i} I(D_1\epsilon \wedge \dots \wedge \hat{D_i} \wedge \dots \wedge D_n) +$$
$$\sum_{i<j} \pm I([D_i, D_j]\epsilon \wedge D_1\epsilon \dots \hat{D_i}\epsilon \dots \hat{D_j}\epsilon \dots \wedge D_n\epsilon)$$

where the hat $\hat{D}$ means that the corresponding term is omitted from the argument of I. The signs are explained in the statement of the theorem 4.2.1. The claimed identity follows immediately. □

The complex $(C_\bullet(A[\eta])[u^{-1}, u], b + uB)$ is contractible but, in many situations, $\chi(\lambda)$ extends to a large subcomplex of the periodic cyclic complex of A and gives interesting maps.

3.1. The characteristic map, II. Here we restrict the action of $U(\mathfrak{g}_A[\epsilon, u]$ to the subalgebra of cochains of degree ≤ 1. We obtain an action of the cyclic complex on the negative cyclic complex. *** *We do not use this anymore for the index theorem*

Let u be a formal parameter of degree two. Consider the differential graded algebra $A[\eta] = A + A\eta$, $\deg\eta = -1$, $\eta^2 = 0$ with the differential $\frac{\partial}{\partial\eta}$. Consider the complex $\overline{C}^{\lambda}_{\bullet}(A[\eta])[u]$ with the differential $\frac{\partial}{\partial\eta} + u \cdot b$.

THEOREM 3.1.1. *There exist natural pairings of* $k[[u]]$*-modules*

$$\bullet : \overline{C}^{\lambda}_{\bullet-1}(A[\eta])[[u]] \otimes CC^{-}_{-\bullet}(A) \to CC^{-}_{-\bullet}(A)$$

$$\bullet : \overline{C}^{\lambda}_{\bullet-1}(A[\eta])[[u, u^{-1}] \otimes CC^{per}_{-\bullet}(A) \to CC^{per}_{-\bullet}(A)$$

such that:

(1) $\eta^{\otimes m} \bullet ? = \frac{1}{(m-1)!} Id$ *for* $m > 0$.
(2) *For* $x_i \in A$ *the operation* $(x_1 \otimes \cdots \otimes x_p) \bullet ?$ *sends* $C_N(A)$ *to* $\sum_{i,j\geq 0} C_{N-p+i}(A)u^j$.
(3) *The component of* $(x_1 \otimes \cdots \otimes x_p) \bullet (a_0 \otimes \cdots \otimes a_N)$ *in* $C_{N-p}[[u]]$ *is equal to*

$$\frac{1}{p!}\sum_{i=1}^{p}(-1)^{i(p-1)} a_0[x_{i+1}, a_1][x_{i+2}, a_2] \cdots [x_i, a_p] \otimes a_{p+1} \otimes \cdots \otimes a_N$$

(4) $(x_1 \otimes \cdots \otimes x_p) \bullet 1 = \sum_{i=1}^{p} (-1)^{i(p-1)} 1 \otimes x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_i$.

PROOF. First note that we have a natural the morphism of DGLA:

$$\mathfrak{gl}(A[\eta]) \to \mathfrak{gl}(A)\eta \oplus \mathfrak{gl}(A)/k \hookrightarrow C^{\bullet}(M_{\infty}(A)),$$

where $\mathfrak{gl}(A)\eta$ is identified with $C^0(M_\infty(A))$ and

$$\mathfrak{gl}(A)/k = \mathrm{Im}(\delta|_{C^0(M_\infty(A))}) \subset C^1(M_\infty(A)).$$

The theorem 3.0.2 provides us with a morphism of complexes:

$$\bullet : U(\mathfrak{gl}(A[\eta, \epsilon]))[[u]] \otimes_{U(\mathfrak{gl}(k))} CC^{-}_{\bullet}(M_\infty(A)) \to CC^{-}_{\bullet}(M_\infty(A)) \tag{3.6}$$

(in the negative cyclic case). Let $M(A)$ denote the algebra of $\mathbb{N} \times \mathbb{N}$-matrices with entries in A and only finitely non-zero diagonals. Set

$$\iota : A \ni a \to a \cdot 1 \in M(A)$$

and

$$\# : M_\infty(A) = M_\infty(\mathbb{C}) \otimes A \ni T \otimes a \to \mathrm{Tr}(T)a \in A. \tag{3.7}$$

It is easy to check that the composition

$$\# \bullet (\mathrm{id} \otimes \iota)$$

is well defined and induces a morphism of complexes

$$k \otimes_{\mathfrak{gl}(k[\eta])} U(\mathfrak{gl}(A[\eta, \epsilon]))[[u]] \otimes_{\mathfrak{gl}(k[\epsilon])} k \to \mathrm{End}(CC^{-}_{\bullet}(A)).$$

A composition of morphisms of complexes:

$$\overline{C}^{\lambda}_{\bullet}(A[\eta])[[u]] \to C^{Lie}_{\bullet}(\mathfrak{gl}(A[\eta]), \mathfrak{gl}(k); k)[[u]] \to k \otimes_{\mathfrak{gl}(k[\eta])} U(\mathfrak{gl}(A[\eta, \epsilon]))[[u]] \otimes_{\mathfrak{gl}(k[\epsilon])} k$$

completes the costruction. Here the first morphism comes from the theorem 4.0.2 while the second one can be constructed using the observation that the quotient morphism

$$k \otimes_{\mathfrak{gl}(k[\eta])} U(\mathfrak{gl}(A[\eta,\epsilon])) \otimes_{\mathfrak{gl}(k[\epsilon])} k \to k \otimes_{\mathfrak{gl}(A[\eta])} U(\mathfrak{gl}(A[\eta,\epsilon])) \otimes_{\mathfrak{gl}(k[\epsilon])} k$$

is in fact a quasiisomorphism (this follows from the fact that $k[\eta]/k \to A[\eta]/k$ is a quis). □

Remark 3.1.2. Note that the formula (4) above defines the map from $\overline{C}^{\lambda}_{\bullet}(A)$ to $Ker(B : C_{\bullet}(A) \to C_{\bullet+1}(A))$ (one can show that this map is a quasi-isomorphism). Clearly, the kernel above embeds into $CC^{-}_{\bullet}(A)$. The above theorem shows that this embedding extends to a pairing $\bullet$.

The complex $\overline{C}^{\lambda}_{\bullet-1}(A[\eta])[[u]]$ is very simple at the level of homology; it is quasi-isomorphic to $k[[u]]$. Therefore the pairing $\bullet$ does not define any new homological operations. It is, however, very important at the level of chains, as one sees in [**69**]. To give some applications, let us suppose that τ is a trace on the algebra A and hence the operations on the cyclic periodic complex constructed in the theorem 3.1.1 produce a map $\#\tau = (\tau \otimes id) \circ \bullet$:

$$\#\tau : \overline{C}^{\lambda}_{\bullet}(A[\eta])[[u]] \to HC^{\bullet}_{per}(A).$$

A few of the examples:

(1) $\#\tau(\eta^{\otimes m}) = \frac{1}{(m-1)!}\tau$;
(2) Suppose that $\sum x_1 \otimes \cdots \otimes x_p$ is a reduced cyclic cycle. then

$$\#\tau(\sum x_1 \otimes \cdots \otimes x_p)(a_0, \dots, a_p) = \frac{1}{p!} \sum \sum_{i=1}^{p} (-1)^{i(p-1)} \tau(a_0[x_{i+1}, a_1][x_{i+2}, a_2] \cdots [x_i, a_p])$$

is a cyclic periodic cocycle in the same class a a multiple of τ determined by the class of $\sum x_1 \otimes \cdots \otimes x_p$.
(3) Suppose that F is an odd element of A satisfying $F^2 = 1$. Then $F^{\wedge(n+1)}$ is a reduced cyclic cycle and

$$\#\tau(F^{\wedge(n+1)})(a_0, \dots, a_n) = \tau(Fa_0[F, a_1] \dots [F, a_n])$$

is a cyclic cocycle representing τ in the periodic cyclic cohomology.

4. Rigidity of periodic cyclic homology

4.1. Nilpotent extensions.

Theorem 4.1.1. *(Goodwillie) Let (A, m) be an associative algebra over a ring k of characteristic zero and let I be a two-sided nilpotent ideal of A. The natural map $CC^{per}_{\bullet}(A) \to CC^{per}_{\bullet}(A/I)$ is a quasi-isomorphism.*

Proof. Using exact sequences

$$0 \longrightarrow I^n/I^{n-1} \longrightarrow A/I^{n-1} \longrightarrow A/I^n \longrightarrow 0$$

the claim reduces to the case $I^2 = 0$. Fix a k-linear isomorphism

$$\phi : A \simeq A/I \oplus I$$

which reduces to identity modulo I. The pull back of the product from $A/I \oplus I$ by ϕ defines on A an associative product, say m_1. Set $\lambda = m - m_1$. This is a Maurer Cartan element of the Hochschild cohomological complex of (A, m)). since $I^2 = 0$, the infinite series $\chi(\lambda)$ converges an, by the corollary 3.0.6, provides an isomorphism of the cyclic periodic complexes of (A, m) and (A, m_1). But, by additivity of cyclic periodic homology,

$$CC_\bullet^{per}(A) \simeq CC_\bullet^{per}(A/I) \oplus CC_\bullet^{per}(I) \simeq CC_\bullet^{per}(A/I).$$

□

4.2. Completed Hochschild and cyclic complexes. Let A be an algebra and I an ideal in A. Each tensor power of A has a filtration

$$F_I^N(A^{\otimes(p+1)}) = \sum_{n_0+\ldots n_p \geq N+1} I^{n_0} \otimes \ldots \otimes I^{n_p}$$

The differential b preserves the filtration. We denote the induced filtration on Hochschild chains by $F_I^N C_p(A)$. Put

$$\widehat{C}_\bullet(A)_I = \varprojlim C_\bullet(A)/F_I^N C_\bullet(A)$$

$$\widehat{CC}_\bullet^{per}(A)_I = (\widehat{C}_\bullet(A)_I((u)), b + uB)$$

THEOREM 4.2.1. *Suppose that m_1 and m_2 are two associative products on A with the same unit. Suppose moreover that* I *is an ideal with respect to m_1 and $m_1(a_1, a_2) - m_2(a_1, a_2) \in I$ for all a_1, a_2 in A. Then there is a natural isomorphism of complexes*

$$\widehat{CC}_\bullet^{per}(A, m_1)_I \simeq \widehat{CC}_\bullet^{per}(A, m_2)_I$$

.

PROOF. Set $\lambda = m_1 - m_2$. Then $\chi(\lambda)$ the infinite series converges $\chi(\lambda)$ and produces a quasiisomorphism of the respective cyclic periodic complexes. □

THEOREM 4.2.2. *The projection*

$$\widehat{CC}_\bullet^{per}(A)_I \to CC_\bullet^{per}(A/I)$$

is a quasi-isomorphism.

PROOF. Choose a linear section $A/I \to A$ of the projection. This allows to identify A with $A/I \times I$ as k-modules. Consider two products on A : the original one and the one coming from this identification, with the product on A/I being the product in the quotient algebra and the product on I being zero. These two products satisfy the conditions of Theorem 4.2.1. So we have to prove that the projection is a quasi-isomorphism for the second product, which follows from the Künneth formula and the fact that the algebra with zero multiplication has periodic cyclic homology equal to zero. □

Another corollary is the following.

COROLLARY 4.2.3. *Let (A, m) be an associative algebra, TA the tensor algebra over A and $J(A)$ the ideal in TA generated by*

$$\{a \otimes b - m(a, b) \mid a, b \in A\}.$$

Then

$$CC_\bullet^{per}(\overline{TA}^{J(A)}) \simeq CC_\bullet^{per}(A).$$

PROOF. The claim follows from the fact that, by the theorem **??**, for $n \geq m$ the quotient maps

$$T(A)/J(A)^n \to T(A)/J(A)^m$$

induce isomorphism on periodic cyclic homology. □

5. Excision in periodic cyclic homology

Recall the following notion of smoothness for non-commutative algebras, due to Cuntz and Quillen (see [**?**]).

DEFINITION 5.0.1. A unital algebra A is called quasi-free if $H^2(A, M) = 0$ for all A bimodules M.

REMARK 5.0.2. For future reference note that free algebras are quasifree.

PROPOSITION 5.0.3. *Suppose that A is a unital quasifree algebra.*

- $\Omega^1(A)$*, the kernel of the multiplication map* $A \otimes A \to A$*, is a projective* A*-bimodule;*
- A *is hereditary, i. e. any right (resp. left) submodule of a projective* A*-module is right (resp. left) projective.*

PROOF. To prove the first statement, look at the exact sequence of A-bimodules:

$$0 \longrightarrow \Omega^1(A) \longrightarrow A \otimes_k A \longrightarrow A \longrightarrow 0. \tag{5.1}$$

Since $A \otimes_k A$ is free A-bimodule, applying the functor $\mathbb{R}\mathrm{Hom}^{A^e}(\cdot, M)$ to it shows that

$$\mathrm{Ext}^i_{A^e}(\Omega^1(A), M) = \mathrm{Ext}^{i+1}_{A^e}(A, M).$$

Since A is quasifree, $\mathrm{Ext}^1_{A^e}(\Omega^1(A), M) = 0$ for every A-bimodule M and $\Omega^1(A)$ is projective as an A-bimodule.

For the second claim, suppose that M is a, say, left A module. Tensoring the split exact sequence 5.1 with M from the right, we get an exact sequence of left A-modules

$$0 \longrightarrow \Omega^1(A) \otimes_A M \longrightarrow A \otimes_k M \longrightarrow M \longrightarrow 0. \tag{5.2}$$

Since $\Omega^1(A)$ is projective as an A-bimodule, $\Omega^1(A) \otimes_A M$ is projective as a left A-module and the claim follows. □

Let us start by defining H-unitality in the context of pro-algebras.

DEFINITION 5.0.4. *Let A be an algebra over a field of characteristic zero and let $\cdots \subset A^n \subset A^{n-1} \subset \cdots \subset A^2 \subset A$ be the filtration of A by its increasing powers. A is* approximately H-unital *if the complex*

$$(h - \varprojlim_k C_\bullet(A^k), b')$$

is acyclic. In the following we set $C_\bullet(A^\infty) = h - \varprojlim_k C_\bullet(A^k))$.

The following lemma gives a useful criterion for approximate H-unitality.

LEMMA 5.0.5. *Let* A *be an algebra and let us denote by* $m : A \otimes A \to A$ *the multiplication map on* A*. Assume that there is a* $k \geq 1$ *and a left A-module map map* $\phi : A^k \to A \otimes A$ *which is a section of* m*, i. e.*

$$m \circ \phi(x) = x \text{ for all } x \in A^k.$$

Then A *is approximately* H*-unital.*

PROOF. For every q there is a $p \geq q$ and a A-linear splitting $\psi : A^p \to A^q \otimes A^q$ of the multiplication $m : A^q \otimes A^q \to A^{2q}$. which is obtained by iterating ϕ and then multiplying the last q variables together. Set

$$\Psi(a_0, \dots, a_n) = (a_0, \dots, \psi(a_n)).$$

Ψ is a contracting homotopy of the complex $(C_\bullet(A^\infty), b')$. □

COROLLARY 5.0.6. *A left ideal* J *in a unital quasi-free algebra* P *is approximately H-unital.*

PROOF. By the second clause in the proposition 5.0.3, J is a projective left R-module and hence there exists an R-linear lift $\phi : J \to R \otimes J$ for the multiplication map $m : R \otimes J \to J$. The restriction of ϕ to J^2 satisfies the conditions of the above lemma. □

The basic property of the approximate H-unitality is the following result.

THEOREM 5.0.7. *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be an exact sequence of algebras with A *unital and* J *approximately H-unital. Set*

$$K^n_\bullet = \mathit{Ker} \left\{ CC^{per}_\bullet(A) \overset{\pi_n}{\to} CC^{per}_\bullet(A/J^n) \right\},$$

where π_n *is induced by the quotient map* $A \to A/J^n$*. The morphism of complexes*

$$\mathbb{R} \varprojlim_n CC^{per}_\bullet(J^n) \to \mathbb{R} \varprojlim_n K^n_\bullet$$

induced by the inclusion $CC^{per}_\bullet(J^n) \to K^n_\bullet$ *is a quasi-isomorphism.*

PROOF. Set

$$\tilde{K}^n_\bullet = \text{Ker} \left\{ (C_\bullet(A), b) \overset{\pi_n}{\to} (C_\bullet(A/J^n), b) \right\},$$

Using the approximate H-unitality of it is straightforward to adapt the proof of the theorem 3.0.2 to the proof of the following statement.

LEMMA 5.0.8. *Given* k *and* n*, there exists an* $m > n$ *such that the following holds.*

$$\text{Im} \left\{ H_k(\tilde{K}^m, b) \to H_k(\tilde{K}^n, b) \right\} \subset \text{Im} \left\{ H_k((C_\bullet(J^n), b) \to H_k(\tilde{K}^n, b) \right\}$$

COROLLARY 5.0.9.

$$\mathbb{R} \varprojlim_n \text{Cone} \left\{ C_\bullet(J^n) \to (\tilde{K}^n_\bullet, b) \right\} = 0.$$

PROOF. Let C^* denote the cone of the morphism $\left\{ C_\bullet(J^n) \to (\tilde{K}^n_\bullet, b) \right\}$. The statement of the corollary is equivalent to the acyclicity of the following complex:

$$\begin{array}{ccccccccccc}
\dots & \longrightarrow & C^{*+1}_\bullet & \longrightarrow & C^*_\bullet & \longrightarrow \dots\dots \longrightarrow & C^3_\bullet & \longrightarrow & C^2_\bullet & \longrightarrow & C^1_\bullet \\
 & & \big\downarrow = & \searrow & \big\downarrow = & & \big\downarrow = & \searrow & \big\downarrow = & \searrow & \big\downarrow = \\
\dots & \longrightarrow & C^{*+1}_\bullet & \longrightarrow & C^*_\bullet & \longrightarrow \dots\dots \longrightarrow & C^3_\bullet & \longrightarrow & C^2_\bullet & \longrightarrow & C^1_\bullet .
\end{array}$$

But this follows from the above lemma by a straightforward diagram chasing. □

To complete the proof of the theorem, filter the mapping cone $\mathcal{C}$ of $\mathbb{R}\varprojlim_n CC^{per}_\bullet(J^n) \to \mathbb{R}\varprojlim_n K^n_\bullet$ by the powers of u. The associated spectral sequence has the E^1-term equal to zero, hence $\mathcal{C}$ is quasiisomorphic to zero. □

As the corollary, we get the following result.

THEOREM 5.0.10. *Let*

$$0 \longrightarrow J \longrightarrow A \longrightarrow A/J \longrightarrow 0$$

be an exact sequence of algebras with A *unital and* J *approximately H-unital. Then*

$$CC^{per}_\bullet(J) \to CC^{per}_\bullet(A) \to CC^{per}_\bullet(A/J)$$

is an exact triangle in the derived category of complexes of vector spaces.

PROOF. By the theorem 5.0.7, the following triangle is exact:

$$\mathbb{R}\varprojlim_n CC^{per}_\bullet(J^n) \to \varprojlim_n CC^{per}_\bullet(A) \to \varprojlim_n CC^{per}_\bullet(A/J^n).$$

Using Goodwilli theorem 4.1.1,

$$\varprojlim_n CC^{per}_\bullet(A/J^n) \simeq CC^{per}_\bullet(A/J) \text{ and } \varprojlim_n CC^{per}_\bullet(J^n) \simeq CC^{per}_\bullet(J).$$

The claimed result follows. □

THEOREM 5.0.11. *(Cuntz-Quillen) Suppose that*

$$0 \longrightarrow I \xrightarrow{\iota} A \xrightarrow{\pi} A/I \longrightarrow 0 .$$

is a short exact sequence of algebras over a field k *of characteristic cero. Then there is an induced exact triangle*

(5.3)
$$\begin{array}{ccc}
HC^{per}_\bullet(I) & \xrightarrow{\quad \iota_\bullet \quad} & HC^{per}_\bullet(A) \\
 {\scriptstyle [1]}\nwarrow & & \swarrow {\scriptstyle \pi_\bullet} \\
 & HC^{per}_\bullet(A/I) & .
\end{array}$$

PROOF. Let $T(A)$ denote the unital tensor algebra of A and let $I(A)$ denote the kernel of the natural quotient map $T(A) \to A$. and let $I(A/J))$ denote the kernel of the composition $\mathsf{T}(A) \to A \to A/J$. We have the following commuting diagram of short exact sequences.

$$
\begin{array}{ccccc}
\mathrm{I}(A) & \searrow & & & \\
\downarrow & & & & \\
\mathrm{I}(A/J) & \hookrightarrow & \mathsf{T}(A) & \twoheadrightarrow & A/J \\
\downarrow & & & \searrow & \\
J & & & & A
\end{array}
$$

Since both $\mathrm{I}(A)$ and $\mathrm{I}(A/J)$ are ideals in the free algebra $\mathsf{T}(A)$, the theorem 5.0.10 produces the following commuting diagram of morphisms of complexes with rows given by exact triangles.

$$
\begin{array}{ccccc}
\mathsf{CC}^{per}_\bullet(J) & \xrightarrow{[1]} & \mathsf{CC}^{per}_\bullet(\mathrm{I}(A)) & \longrightarrow & \mathsf{CC}^{per}_\bullet(\mathrm{I}(A/J)) \\
\downarrow & & \downarrow = & & \downarrow \\
\mathsf{CC}^{per}_\bullet(A) & \xrightarrow{[1]} & \mathsf{CC}^{per}_\bullet(\mathrm{I}(A)) & \longrightarrow & \mathsf{CC}^{per}_\bullet(\mathsf{T}(A)) \\
\downarrow & & \downarrow & & \downarrow = \\
\mathsf{CC}^{per}_\bullet(A/J) & \xrightarrow{[1]} & \mathsf{CC}^{per}_\bullet(\mathrm{I}(A/J)) & \longrightarrow & \mathsf{CC}^{per}_\bullet(\mathsf{T}(A))
\end{array}
$$

It follows immediately that the rightmost column is an exact triangle, and the claim of the theorem is the long exact homology sequence associated to it. □

6. Bibliographical notes

CHAPTER 8

Cyclic objects

1. Introduction

2. Some standard categorical constructions

2.1. The adjunction yoga. Let $\mathcal{B}$ and $\mathcal{C}$ be two small categories. Given a pair of functors

$$F : \mathcal{B} \to \mathcal{C} \text{ and } G : \mathcal{C} \to \mathcal{B},$$

F is left adjoint to G and G is right adjoint to F if, for any $X \in \mathsf{Obj}(\mathcal{B})$ and $Y \in \mathsf{Obj}(\mathcal{C})$, there exists a bijection

$$\mathsf{Hom}_{\mathcal{C}}(F(X), Y) \longrightarrow \mathsf{Hom}_{\mathcal{B}}(X, G(Y))$$

which is natural in X and Y. We will say that the pair of functors (F, G) is an adjunction. Given such an adjunction pair, the associated natural bijection

$$\mathsf{Hom}_{\mathcal{B}}(GF(X), GF(X)) \longrightarrow \mathsf{Hom}_{\mathcal{C}}(FGF(X), F(X))$$

applied to the identity functor $\mathsf{id}_{\mathcal{B}}$ gives a natural transformation

$$FGF \longrightarrow F.$$

Using this, the functor GF becomes a monoidal endofunctor on $\mathcal{B}$ with the composition given by

$$GF \circ GF = G(FGF) \longrightarrow GF.$$

2.2. Grothendieck six functor formalism. We will fix a commutative unital ring k and denote by $\mathsf{Mod}(k)$ the category of $k-\mathsf{modules}$.

DEFINITION 2.2.1. *Let $\mathcal{C}$ be a small category. A $\mathcal{C}$-module N is a functor*

$$N : \mathcal{C} \longrightarrow \mathrm{Mod}(k).$$

A morphism between two $\mathcal{C}$-modules is a natural transformation between the corresponding functors.

To spell it out, a $\mathcal{C}$-module N is a collection of k-modules N_c, $c \in \mathrm{Ob}(\mathcal{C})$ and k-linear morphisms $N(\gamma) : N_c \to N_d$ for any $\gamma \in \mathcal{C}(c, d)$ such that

$$N(\gamma_1\gamma_2)(m) = N(\gamma_1)(N(\gamma_2)(m)).$$

As a matter of notation, we will write γ instead of $N(\gamma)$ and, given a set X, we will use $k < X >$ to denote the free k-module with basis X. For future reference lat us state the following observation.

PROPOSITION 2.2.2. *Let $\mathcal{C}$ be a small category, k a commutative unital ring and let $\mathsf{Mod}_{\mathcal{C}}$ denote the category of $\mathcal{C}$-modules. The structure of abelian category on the category $\mathsf{Mod}(k)$ induces the structure of abelian category on $\mathsf{Mod}_{\mathcal{C}}$.*

PROOF. Let $\mathsf{N}, \mathsf{M} \in \mathsf{Mod}_{\mathcal{C}}$ and let $\Phi : \mathsf{N} \to \mathsf{M}$ be a natural transformation. The kernel (resp. cokernel) of Φ is the $\mathcal{C}$-module defined to be given by

$$(\mathsf{Ker}\ \Phi)_c = \mathsf{Ker}(\Phi_c) \text{ resp. } (\mathsf{Coker}(\Phi)_c = \mathsf{Coker}(\Phi_c).$$

It is straightforward to give $\mathsf{Ker}(\Phi)$ and $\mathsf{Coker}(\Phi)$ a structure of $\mathcal{C}$-modules and check that these have the properties in the definition of abelian category. □

DEFINITION 2.2.3. *Let $f : \mathcal{B} \to \mathcal{D}$ be a functor between small categories.*

(1) **The pullback functor** $f^* : \mathsf{Mod}_{\mathcal{D}} \to \mathsf{Mod}_{\mathcal{B}}$ *is given by the restriction*

$$f^*\mathsf{N} = \mathsf{N} \circ f : \mathcal{B} \to \mathrm{mod}(k).$$

(2) **the direct image functor** $f_! : \mathsf{Mod}_{\mathcal{B}} \to \mathsf{Mod}_{\mathcal{D}}$ **functor** *has the form*

$$(f_!\mathsf{M})_d = (\oplus_{b \in \mathrm{Ob}(B)} k < \mathcal{D}(f(b), d) > \otimes_k \mathsf{M}_d / \mathcal{K}$$

where $\mathcal{K}$ is the k-submodule generated by terms of the form

$$\{\gamma f(\beta) \otimes m - \gamma \otimes \beta m \mid \beta \in B(b_1, b_2),\ \gamma \in \Gamma(f(b_2), d),\ m \in \mathsf{M}_{b_2}\}$$

The $\mathcal{D}$-module structure is defined by $\gamma_0(\gamma \otimes m) = (\gamma_0\gamma) \otimes m$.

(3) **The pushforward functor** $f_* : \mathsf{Mod}_{\mathcal{B}} \to \mathsf{Mod}_{\mathcal{D}}$ *is defined as follows. Given $d \in \mathrm{Ob}(\mathcal{D})$,*

$$f_*(\mathsf{M})_d = \left\{ \{T_j\}_{j \in \mathrm{Ob}(\mathcal{B})} \mid T_j \in \mathrm{Hom}_k(k < \mathcal{D}(d, f(j)) >, \mathsf{M}_j) \right\}$$

such that, for any $\beta \in \mathcal{B}(j, k)$, the following diagram commutes.

$$\begin{array}{ccc} k < \mathcal{D}(d, f(j)) > & \xrightarrow{T_j} & \mathsf{M}_j \\ \Big\downarrow{\scriptstyle \circ f(\beta)} & & \Big\downarrow{\scriptstyle \beta} \\ k < \mathcal{D}(d, f(k)) > & \xrightarrow[T_k]{} & \mathsf{M}_k \end{array}$$

The $\mathcal{D}$-module structure is defined by $(\gamma_0 T)_j(\gamma) = T_j(\gamma\gamma_0)$.

REMARK 2.2.4. Let $*$ denote the point category with single object and single morphism and

$$p : \mathcal{C} \to *$$

the unique functor from a small category $\mathcal{C}$. Then, for a $\mathcal{C}$-module M, there exists a (functorial) isomorphism

$$p_!(\mathsf{M}) \xrightarrow{\sim} \mathrm{colim}_{\mathcal{C}} \mathsf{M}.$$

DEFINITION 2.2.5. *Let $\mathcal{C}$ be a small category, $\mathsf{M} \in \mathsf{Mod}_{\mathcal{C}}$ and $\mathsf{N} \in \mathsf{Mod}_{\mathcal{C}^{op}}$.*

(1) **The tensor product** $\mathsf{N} \otimes_{\mathcal{C}} \mathsf{M}$ *is the k-module of the form*

$$\left(\oplus_{b \in \mathrm{Ob}(C)} \mathsf{M}_b \otimes_k \mathsf{N}_b\right) / \mathcal{K}$$

where $\mathcal{K}$ is the k-submodule generated by all elements of the form

$$nf \otimes m - n \otimes fm$$

where $n \in \mathsf{N}_k$, $f \in C(j, k)$, and $m \in \mathsf{M}_j$.

REMARK 2.2.6. Let $k_\sharp$ be the constant $\mathcal{C}^{\mathrm{op}}$-module. Then

$$k_\sharp \otimes_{\mathcal{C}} \mathsf{M} \xrightarrow{\sim} \mathrm{colim}_C(\mathsf{M}) \qquad (2.1)$$

REMARK 2.2.7. In terms similar to construction of tensor product over category, one defines a fibered product over a category. Let M and N be respectively modules over $\mathcal{B}^{op}$ and $\mathcal{B}$. Then the fibered product

$$M \times_{\mathcal{B}} N$$

is the k-module of the form

$$M \times_{\mathcal{B}} N = \coprod_{x,y \in \mathrm{Obj}(\mathcal{B})} M_x \times N_y / \{(x, \delta y) \sim (x\delta, y) \mid \delta \in \mathrm{Hom}_{\mathcal{B}}(x, y)\}. \tag{2.2}$$

We will use the notation $\times_{\mathcal{B}}$, $\otimes_{\mathcal{B}}$, and $\otimes_{k\mathcal{B}}$ interchangeably.

LEMMA 2.2.8. *The following holds.*

1. *The functor* $f_!$ *is left adjoint to* f^**;*
2. *the functor* f_* *is right adjoint to* f^**;*
3. *the functor* $f_!$ *is right exact;*
4. *the bifunctor* $- \times_{\mathcal{B}} -$ *is right exact in both variables.*

PROOF. We will leave the proof as an exercise in linear algebra. □

DEFINITION 2.2.9. *Let* $\mathcal{C}$ *be a small category and* M *an* $\mathcal{C}$*-module. The homology of* $\mathbb{L}\mathrm{colim}_{\mathcal{C}}(M)$ *is denoted by* $H_\bullet(\mathcal{C}, M)$. *Here, as usual,* $\mathbb{L}$ *denotes the left derived functor.*

REMARK 2.2.10. By above remarks 2.2.4 and 2.2.6, the homology $H_*(\mathcal{C}, M)$ coincides with the homology of the two functors

$$M \longrightarrow \mathbb{L}p_!(M)$$

and

$$M \longrightarrow k_\# \otimes^{\mathbb{L}}_{\mathcal{C}} M.$$

.

REMARK 2.2.11. The general the "six functor formalism" of Grothendieck involves the proper pull back functor $f^!$ which is right adjoint to $f_!$ and which in the case of $\mathcal{C}$-modules coincides with f^*.

3. The simplicial and cyclic categories

3.1. Simplicial and polycyclic categories. Let X denote a monoid with a neutral element 1 and an automorphism α of order l. One can define the following operations on powers of X.

1. $d_j : X^{n+1} \to X^n$, $0 < n$, $0 \le j < n$
$$d_j(x_0, \dots, x_n) = (x_0, \dots, x_j x_{j+1}, \dots, x_n);$$
2. $d_n : X^{n+1} \to X^n$, $n \ge 0$
$$d_n(x_0, \dots, x_n) = (\alpha(x_n)x_0, x_1, \dots, x_{n-1});$$
3. $s_j : X^n \to X^{n+1}$, $n \ge 0$, $0 \le j \le n$
$$s_j(x_0, \dots, x_n) = (x_0, \dots, x_j, 1, \dots, x_n);$$
4. $\tau : X^{n+1} \to X^{n+1}, n \ge 0$
$$\tau(x_0, x_1, \dots, x_n) = (\alpha(x_n), x_0, \dots, x_{n-1}).$$

DEFINITION 3.1.1.

(1) Δ^{op} *is the category with objects* $[n]$, $n \in \mathbb{N} \cup \{0\}$, *whose set of morphisms* $\Delta^{\mathrm{op}}([n],[m])$ *is the set of natural (in* X*) operations* $X^{n+1} \to X^{m+1}$ *that can be obtained from* d_j *and* s_j *by composing them;*
(2) $\Lambda_\ell^{\mathrm{op}}$ *is the category with objects* $[n]$, $n \in \mathbb{N} \cup \{0\}$, *whose set of morphisms* $\Lambda_\ell([n],[m])$ *is the set of natural operations*

$$X^{n+1} \longrightarrow X^{m+1}$$

that can be obtained from d_j, s_j, *and* τ *by composing them;*
(3) $(\Delta')^{\mathrm{op}}$ *is the subcategory of* $\Lambda_\ell^{\mathrm{op}}$ *with the same objects and morphisms given by all morphisms in* $\Lambda_\ell^{\mathrm{op}}([n],[m])$ *for which the order of* $x_0, \ldots, x_n$ *is preserved.*

REMARK 3.1.2. The category (Δ^{op}) is opposite to the usual symplectic category Δ. Similarly, $\Lambda_\ell^{\mathrm{op}}$ is opposite to the polycyclic category Λ_ℓ. In the special case when $\ell = 1$, Λ_ℓ is denoted by Λ and called the cyclic category.

REMARK 3.1.3.

(1) Alternative description of Δ is as follows. The objects of Δ are sets $[n] = \{0, 1, \ldots, n\}$, $n \geq 0$ with their standard order, and morphisms are nondecreasing maps. A morphism in Δ can be written as a composition of the following:
   (a) Face maps are embeddings $\delta_i : [n] \to [n+1]$, $0 \leq i \leq n+1$ (i is not in the image of δ_i);
   (b) degeneracy maps $\sigma_i : [n+1] \to [n]$, $0 \leq i \leq n$ (i is the image of two succesive points under σ_i, all other $[i']$ have exactly one pre-image).
(2) $\Lambda^{\mathrm{op}}([n],[m])$ is the set of maps $X^{n+1} \to X^{m+1}$ of the form

$$(x_0, \ldots, x_m) \mapsto (x_{J_0}, \ldots, x_{J_m})$$

where each x_{J_q} is the product $x_j \ldots x_p$ for some j and p so that any x_i enters exactly one x_{J_q} and the cyclic order of the factors x_j is preserved. The product of zero factors x_j is by definition equal to 1.
(3) More generally, $\Lambda_\ell^{\mathrm{op}}([n],[m])$ is has the same description but now $x_{J_q} = \widetilde{x}_j \widetilde{x}_p$ where, for some integer r, $\widetilde{x}_j = \alpha^{r+1}(x_j)$ if $\widetilde{x}_j$ is strictly to the right from $\widetilde{x}_0$ and $\widetilde{x}_j = \alpha^r(x_j)$ otherwise. The subcategory Δ^{op} consists of those morphisms for which x_{J_0} contains x_0 as a factor.
(4) A morphism from $\Lambda^{\mathrm{op}}([n],[m])$ is in $(\Delta')^{\mathrm{op}}$ if, when we denote by r the smallest index for which $x_{J_r} \neq 1$, then $\widetilde{x}_0$ is the leftmost factor in x_{J_r}.

LEMMA 3.1.4.

(1) *The group* $\mathrm{Aut}_{\Lambda_\ell}([n])$ *is cyclic of order* $\ell(n+1)$ *and generated by* τ.
(2) *any morphism* $\lambda \in \Lambda_\ell([n],[m])$ *has a unique representation of the form*

$$\lambda = c\delta',$$

where $c \in \mathrm{Aut}_{\Lambda_\ell}([n])$ *and* $\delta' \in \Delta'([n],[m])$*;*
(3) *any morphism* $\lambda \in \Lambda_\ell([n],[m])$ *has a unique representation of the form*

$$\lambda = \delta c,$$

where $c \in \mathrm{Aut}_{\Lambda_\ell}([n])$ *and* $\delta \in \Delta'([n],[m])$.

PROOF. Follows immediately from the remark 3.1.3 above. ☐

We will have occasion to use the following extension of Δ.

DEFINITION 3.1.5. *The category* $\Delta_{\mathtt{big}}$ *is as follows.*

(1) *The objects of* $\Delta_{\mathtt{big}}$ *are linearly ordered finite sets:*
(2) *the morphisms are given by non-decreasing maps.*

3.2. Self-duality of Λ_ℓ.

PROPOSITION 3.2.1. $\Lambda_\ell^{\mathrm{op}}$*is isomorphic to* Λ_ℓ *for all* $\ell \in \mathbb{N}$.

PROOF. Consider all unital monoids X with an automorphism α such that $\alpha^\ell = \mathrm{id}$ and with an α-trace with values in a set K, i.e. with a map $\mathrm{tr} : X \to K$ such that $\mathrm{tr}(xy) = \mathrm{tr}(\alpha(y)x)$ for all x, y. For every $n \geq 0$ define a pairing

$$X^{n+1} \times X^{n+1} \to K;\ (\mathbf{x}, \mathbf{y}) \mapsto \langle \mathbf{x}, \mathbf{y} \rangle$$

by

$$\langle (x_0, \dots, x_n), (y_0, \dots, y_n) \rangle = \mathrm{tr}(x_0 y_0 \dots x_n y_n) \tag{3.1}$$

It is an elementary exercise to check that, for every $\lambda \in \Lambda_\ell^{\mathrm{op}}([n], [m])$ there is unique $\lambda^R \in \Lambda_\ell([n], [m])$ such that for all X, α, tr and for all λ, $\mathbf{x} \in X^{n+1}$, $\mathbf{y} \in X^{m+1}$

$$\langle \lambda \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{x}, \lambda^R \mathbf{y} \rangle \tag{3.2}$$

The map $\lambda \mapsto \lambda^R$ defines as isomorphism as in the statement of the proposition. $\square$

REMARK 3.2.2. The above isomorphism $\Lambda_\ell^{\mathrm{op}} \xrightarrow{\sim} \Lambda_\ell$ identifies $\Delta^{\mathrm{op}} \subset \Lambda_\ell^{\mathrm{op}}$ with $\Delta' \subset \Lambda_\ell$. Note also that, since the pairing $\langle\,,\,\rangle$ is not symmetric, the isomorphism $\lambda \mapsto \lambda^R$ is not an involution. For example, the isomorphism sends $\Delta' \subset \Lambda_\ell^{\mathrm{op}}$ to another subcategory of Λ isomorphic to $(\Delta^{\mathrm{op}})^{\mathrm{op}}$, the one for which

$$d_0(y_0, \dots, y_n) = (y_1, \dots, y_n \alpha^{-1}(y_0);$$
$$d_j(y_0, \dots, y_n) = (y_0, \dots, y_{j-1} y_j, \dots, y_n), j \geq 1;$$
$$s_j(y_0, \dots, y_n) = (y_0, \dots, 1, y_j, \dots, y_n)$$

The two embeddings of Δ^{op} into Λ_ℓ can be explained in terms of Hochschild complexes. Recall that $\mathcal{B}_\bullet$ denotes the standard bar resolution of A which is a simplicial bimodule (see below for the notion of symplicial object). The Hochschild complex is by definition $\mathcal{B}_\bullet \otimes_{A \otimes A^{\mathrm{op}}} A$. There are two ways to identify this with $A^{\otimes(n+1)} : (1 \otimes a_1 \otimes \dots \otimes a_n \otimes 1) \otimes a$ may go to $a \otimes a_1 \otimes \dots \otimes a_n$ or to $a_1 \otimes \dots \otimes a_n \otimes a$.

4. Simplicial and cyclic objects

4.1. Simplicial objects.

DEFINITION 4.1.1. *A simplicial object in a category* $\mathcal{C}$ *is a functor*

$$X : \Delta^{\mathrm{op}} \longrightarrow \mathcal{C}.$$

A simplicial object of $\mathcal{C}$ can be equivalently described as a collection of objects X_n of $\mathcal{C}$, $n \geq 0$, together with morphisms

$$d_i : X_{n+1} \to X_n,\ 0 \leq i \leq n+1 \tag{4.1}$$

$$s_i : X_n \to X_{n+1},\ 0 \leq i \leq n \tag{4.2}$$

subject to

$$d_i d_j = d_{j-1} d_i,\ i < j \tag{4.3}$$

$$s_i s_j = s_{j+1} s_i, \ i \le j \tag{4.4}$$

$$d_i s_j = s_{j-1} d_j, \ i < j; \ d_i s_i = d_{i+1} s_i = \mathrm{id}; \ d_i s_j = s_j d_{i-1}, \ i > j+1. \tag{4.5}$$

EXAMPLE 4.1.2. For a topological space X define $\mathrm{Sing}_n(X)$ to be the set of singular simplices of X. Then $\mathrm{Sing}(X)$ has a natural structure of a simplicial set.

EXAMPLE 4.1.3. Let A be a graded algebra. Put $A^\sharp_n = A^{\otimes(n+1)}$. Define

$$d_i(a_0 \otimes \ldots \otimes a_{n+1}) = (-1)^{\sum_{p\le i}|a_p|} a_0 \otimes \ldots a_i a_{i+1} \otimes \ldots \otimes a_{n+1} \tag{4.6}$$

for $i \le n$

$$d_{n+1}(a_0 \otimes \ldots \otimes a_{n+1}) = (-1)^{|a_{n+1}|\sum_{p\le n}|a_p|} a_{n+1} a_0 \otimes a_1 \otimes \ldots \otimes a_n \tag{4.7}$$

$$s_i(a_0 \otimes \ldots \otimes a_n) = (-1)^{\sum_{p\le i}|a_p|} a_0 \otimes \ldots a_i \otimes 1 \otimes \ldots \otimes a_n \tag{4.8}$$

for $i \le n$.

Let us record the following simple observation.

PROPOSITION 4.1.4. *The formulas* (4.6), (4.7) *and* (4.8) *give* $A^\sharp$ *the structure of a simplicial graded vector space.*

EXAMPLE 4.1.5. Suppose that $\mathcal{C}$ is a small category. Its nerve $N_\bullet\mathcal{C}$ is the simplicial set, where the set of 0-simplices coincides with the set of objects of $\mathcal{C}$ and, for $n > 0$, the set of n-simplices $N_n\mathcal{C}$ in $N\mathcal{C}$ coincides with the set of n-tuples of composable morphisms in $\mathcal{C}$. Face maps are given by composition (or omission, in the case of d_0 and d_n), $d_0, d_1 : N_1\mathcal{C} \to N_0\mathcal{C}$ are the target and source maps and degeneracy maps are given by inserting identity arrows.

EXAMPLE 4.1.6. For future reference, a particular case of the nerve of the category is $N_\bullet[n]$ which we will, whenever it does not lead to ambiguity, denote again by $[n]$.

4.2. Cyclic objects.

DEFINITION 4.2.1. *Let* $\ell \ge 1$. *An* ℓ*-cyclic object of a category* $\mathcal{C}$ *is a functor* $\Lambda_\ell^{op} \to \mathcal{C}$. *A* 1*-cyclic object is called cyclic.*

Explicitly, it is a simplicial object X in $\mathcal{C}$ together with morphisms $\tau : X_n \to X_n$ for all $n \ge 0$ such that the following identities hold.

$$d_1\tau = \tau d_0; \ d_2\tau = \tau d_1; \ \ldots; d_n\tau = \tau d_{n-1}; \ d_0\tau = d_n \tag{4.9}$$

$$s_1\tau = \tau s_0; \ s_2\tau = \tau s_1 \ \ldots; s_n\tau = \tau s_{n-1}; \ s_0\tau = \tau^2 s_n \tag{4.10}$$

$$\tau^{\ell(n+1)} = \mathrm{id}\,. \tag{4.11}$$

EXAMPLE 4.2.2. For a graded algebra A with an automorphism α such that $\alpha^\ell = \mathrm{id}$, put

$$\tau(a_0 \otimes a_1 \otimes \ldots \otimes a_n) = (-1)^{|a_n|\sum_{i<n} a_i}(a_n \otimes \alpha(a_0) \otimes \ldots \otimes a_{n-1}).$$

Set

$$({}_\alpha A^\sharp)_n = A^{\otimes(n+1)}.$$

The above formula for τ, together with (4.6), (4.7), (4.8), makes ${}_\alpha A^\sharp$ a cyclic graded k-module. For $\ell = 1$ and $\alpha = \mathrm{id}$ we write $A^\sharp$ instead of ${}_\alpha A^\sharp$.

EXAMPLE 4.2.3. The assignment

$$[n] \to \mathrm{Aut}_{\Lambda^{op}}[n]$$

extends to a cyclic object $\mathrm{Aut}(\Lambda^{op})$.

PROOF. Any element $\phi \in \mathrm{Hom}_{\Lambda^{op}}([n],[m])$ has unique representations

$$\phi = \alpha(\phi) \circ \beta(\phi) = \beta'(\phi) \circ \alpha'(\phi). \tag{4.12}$$

where $\alpha, \alpha' \in \mathrm{Hom}_{\Delta^{op}}([n],[m])$ and β and β' belong to the corresponding automorphism groups. Given $\phi \in \mathrm{Hom}_{\Lambda^{op}}([n],[m])$ define its action on $\mathrm{Aut}^{op}_{\Lambda}$,

$$\mathrm{Aut}^{op}_{\Lambda}(\phi) : \mathrm{Aut}_{\Lambda^{op}}([n]) \to \mathrm{Aut}_{(\Lambda^{op}}([m]),$$

by

$$\mathrm{Aut}^{op}_{\Lambda}(\phi)(\sigma) = \beta'(\phi)\beta'(\alpha'(\phi)\sigma).$$

It is easily seen that the uniqueness of the representation (4.12) implies that this defines an action of Λ^{op}. □

EXAMPLE 4.2.4. Let $\mathbb{A}^n$ denote the standard (geometric) n-simplex, i. e.

$$\mathbb{A}^n = \{0 \le t_1 \le \ldots \le t_n \le 1\} \subset [0,1]^n$$

Vertices of $\mathbb{A}^n$ are the extreme points in lexicographic ordering, so the jth vertex is the point

$$(0, \ldots, 0, \underbrace{1, \ldots, 1}_{j}).$$

Putting vertex j in correspondence to the morphism $[0] \to [n]$ in Δ that sends 0 to j, we identify vertices of $\mathbb{A}^n$ with $\Delta([0],[n])$. This implies that the collection

$$\{\text{vertices of } \mathbb{A}^n, n \in \mathbb{N}\}$$

forms a cosimplicial set. Since the simplex $\mathbb{A}^n$ can be identified with the formal convex hull of $\Delta([0],[n])$, the collection of n-simplices forms a cosimplicial topological space that we denote by $\mathbb{A}^\bullet$.

For future reference, let us record the following observation.

LEMMA 4.2.5. $\mathbb{A}^\bullet$ *is a cocyclic space.*

PROOF. This follows immediately from the fact that $\Lambda([0],[n]) = \Delta([0],[n])$. □

PROPOSITION 4.2.6. *Let* j *be the forgetful functor from cyclic to simplicial objects. It has the left adjoint functor* $j_!$ *given by the following construction. Let* X *be a simplial object. Then* $X \times_\Delta \Lambda$ *is the cyclic object given by*

$$j_!(X)_n = X \times_\Delta \Lambda([n],[-]). \tag{4.13}$$

with the obvious action of Λ^{op}. *We will also use the notation*

$$j_!(X) = X \times_\Delta \Lambda.$$

PROOF. The proof is a straightforward consequence of the definitions. □

4.3. Cyclic complexes. For an ℓ-cyclic k-module M define

$$b : M_n \to M_{n-1};\ b = \sum_{i=0}^{n} (-1)^i d_i;\ t = (-1)^{n+1}\tau : M_n \to M_n; \tag{4.14}$$

$$N = \sum_{i=0}^{\ell(n+1)-1} t^i;\ B : M_n \to M_{n+1};\ B = (1-t)sN \tag{4.15}$$

where

$$s = \tau s_n : M_n \to M_{n+1} \tag{4.16}$$

(s represents the morphism $(x_0, \dots, x_n) \mapsto (1, x_0, \dots, x_n)$). As in the case of algebras, the following holds.

PROPOSITION 4.3.1.

$$b^2 = bB + Bb = B^2 = 0$$

DEFINITION 4.3.2. *For an ℓ-cyclic k-module put*

$$C_\bullet(M) = (M_\bullet, b)$$
$$CC^-_\bullet(M) = (M_\bullet[[u]], b + uB)$$
$$CC_\bullet(M) = (M_\bullet[u^{-1}, u]]/uM_\bullet[[u]], b + uB)$$
$$CC^{\mathrm{per}}_\bullet(M) = (M_\bullet[u^{-1}, u]], b + uB)$$

where u is a formal parameter of degree -2.

4.4. Hochschild and cyclic homology as derived functors.

THEOREM 4.4.1. *One has, for an integer $\ell \geq 1$ and an ℓ-cyclic k-module M,*

$$HH_\bullet(M) = H_\bullet(\Delta^{\mathrm{op}}, M)$$
$$HC_\bullet(M) = H_\bullet(\Lambda_\ell, M)$$

PROOF. As above, let $k_\sharp$ be the constant Δ- or Λ_ℓ-module. By the remark 2.2.10 the right hand side is the homology of $k_\sharp \otimes^{\mathbb{L}}_{B} M$ where B is either Δ^{op} or Λ_ℓ. We will construct a projective resolution of the B^{op}-module $k_\sharp$ in both cases. By lemma 3.1.4, a free Λ_ℓ-module is also free as a Δ^{op}-module, hence it is sufficient to construct the resolution of $k_\#$ over Λ_ℓ.

Resolution of $k_\#$ over Λ_ℓ.

Define the complex of Λ^{op}-modules $\mathcal{P}_*$ by

$$\mathcal{P}_n([m]) =< \Lambda_\ell([m],[n]) >;\ b : \mathcal{P}_n \to \mathcal{P}_{n-1};\ b\lambda = \sum_{j=0}^{n} (-1)^j d_j \lambda. \tag{4.17}$$

Recall that d_0 as an operation in $\Lambda_\ell([m],[m-1])$ is given by

$$d_0 : (x_0, \dots, x_m) \mapsto (x_0 x_1, x_2, \dots, x_m) \tag{4.18}$$

Set

$$\kappa_m = \sum_{j=0}^{(m+1)\ell-1} d_0^{m-1} t^j \tag{4.19}$$

where t is the signed cyclic permutation, see (4.14).

Claim: For every $m \geq 0$

$$H_*(\mathcal{P}([m])) \tilde{\to} \begin{cases} k[d_0^m] & \text{for } * = 0 \\ k[\kappa_m] & \text{for } * = 1 \\ 0 & \text{for } * \geq 1 \end{cases}$$

PROOF OF THE CLAIM. Let $\mathcal{F}_m$ denote the free algebra with generators $\alpha^j(x_i)$ where $0 \leq i \leq m$ and $0 \leq j < \ell$. Let α be the automorphism of $\mathcal{F}_m$ defined on the generators by

$$\alpha^j(x_i) \mapsto \alpha^{j+1}(x_i), \alpha^\ell(x_i) = x_i$$

for $0 \leq i \leq m$ and $0 \leq j < \ell$. The complex $\mathcal{P}_*([m])$ is the direct summand of the Hochschild complex $C_\bullet(\mathcal{F}, {}_\alpha\mathcal{F})$, k-linearly generated by expressions of the form

$$\omega_0 \otimes \omega_1 \otimes \ldots \otimes \omega_k$$

where ω*s are words in the generators of $\mathcal{F}_m$ such that, for each $0 \leq i \leq m$, there is exactly one factor of the form $\alpha^j(x_i)$. Let V_m denote the k-linear span of the generators $\{\alpha^j(x_i)\}$ of $\mathcal{F}_m$. The Koszul resolution of the free algebra $\mathcal{F}_m$ has the form

$$\mathcal{K}: \qquad \mathcal{F}_m \otimes V_m \otimes \mathcal{F}_m \xrightarrow{b'} \mathcal{F}_m \otimes \mathcal{F}_m \xrightarrow{b'} \mathcal{F}_m \tag{4.20}$$

where, in the notation of (4.14), $b' = \sum_{j=0}^{n-1} d_j$. The complex

$$\mathcal{K} \otimes_{\mathcal{F}_m^e} {}_\alpha\mathcal{F}_m$$

is a subcomplex of the full Hochschild complex C_* of $\mathcal{F}_m$ with coefficients in ${}_\alpha\mathcal{F}_m$) and moreover, the inclusion

$$\mathcal{K} \otimes_{\mathcal{F}_m^e} {}_\alpha\mathcal{F}_m \hookrightarrow \mathcal{C}$$

is a quasiisomorphism. Hence the homology of $\mathcal{P}([m])$ can be computed as the homology of the intersection

$$\mathcal{P}([m]) \cap \mathcal{K} \otimes_{\mathcal{F}_m^e} {}_\alpha\mathcal{F}_m.$$

This subcomplex the following structure.

(1) The basis of zero-chains is of the form$\{d_0^m t^j | 0 \leq j < \ell(m+1)\}$;
(2) the basis of one-chains is of the form $\{d_0^{m-1} t^j | 0 \leq j < \ell(m+1)\}$;
(3) the differential b acts by

$$d_0^{m-1} t^j \mapsto d_0^m t^j (1-t).$$

Hence, as claimed, b has kernel and cokernel both of rank one, with free generators κ_m and d_0^m. □

Claim.

$$(\mathcal{P}_*([m])((u))/u\mathcal{P}_*([m])[[u]], b + uB), \; B : \lambda \mapsto \lambda B \tag{4.21}$$

is a resolution of $k_\sharp$.

PROOF OF THE CLAIM. Indeed, since $\mathcal{P}_*([m])$ is a cyclic module, $(b+uB)^2 = 0$ and it is straightforward to check that, on the level of homology, B sends the generator d_0^m to κ_m. The spectral sequence computing the homology of the double complex (4.21) degenerates at the E_2-level and, for all m, the homology of (4.21) is concentated in degree zero, where it is isomorphic to $k[d_0^m]$. □

Since $k_\sharp \otimes^{\mathbb{L}}_{\mathcal{B}} -$ coincides with the homology of $\mathcal{P}_* \otimes_{\mathcal{B}} -$ (for $\mathcal{B}$ equal to Δ^{op} or Λ_ℓ) applied to a cyclic object, we get the statement of the theorem - the explicit check that the resulting complexes indeed coincide with the corresponding complexes defining Hochschild and cyclic homology will be left as an exercise for the reader. □

Corollary 4.4.2. *One has, for an associative algebra* A*,*

$$HH_\bullet(A) \tilde{\to} H_\bullet(\Delta^{op}, A^\sharp)$$

$$HC_\bullet(A) = H_\bullet(\Lambda, A^\sharp)$$

4.4.1. *Other versions of the cyclic complex.* For an ℓ-cyclic object M, define

$$b' : M_n \to M_{n-1};\ b = \sum_{j=0}^{n-1} (-1)^j d_j \tag{4.22}$$

Note that the complex $(M_\bullet, b')$ is contractible, s from (4.16) being the homotopy). Define also

$$\tau = (-1)^n : M_n \to M_n;\ N = \sum_{j=0}^{\ell(n+1)-1} \tau^j \tag{4.23}$$

Lemma 4.4.3.

$$b(1-\tau) = (1-\tau)b';\ b'N = Nb; (1-\tau)N = N(1-\tau) = 0$$

Therefore the sequence of complexes

$$\dots \xrightarrow{N} (M_\bullet, -b') \xrightarrow{1-\tau} (M_\bullet, b) \xrightarrow{N} (M_\bullet, -b') \xrightarrow{1-\tau} (M_\bullet, b) \tag{4.24}$$

is a double complex. We will denote it by $\widetilde{CC}_\bullet(M)$; it is a generalization of $\widetilde{CC}_\bullet(A)$ from (**??**).

Lemma 4.4.4. *The homology of the total complex of* $\widetilde{CC}_\bullet(M)$ *computes* $HC_\bullet(M)$.

Proof. The same argument as in the proof of Theorem 4.4.1 shows that

$$[n] \mapsto (\dots \xrightarrow{N} (\mathcal{P}_*([n]), -b') \xrightarrow{1-\tau} (\mathcal{P}_*([n]), b))$$

is a projective resolution of $k_\sharp$. Alternatively, one can compare the two double complexes directly.

To do that, denote by $\widetilde{C}_\bullet(M)$ the total complex of the double complex

$$(C_\bullet(M), -b') \xrightarrow{1-\tau} (C_\bullet(M), b) \tag{4.25}$$

Define the map

$$p : \widetilde{C}_\bullet(M) \to C_\bullet(M) \tag{4.26}$$

as follows:

$$p(m) = m$$

if m is in the b column, and

$$p(m) = (1-\tau)sm$$

if m is in the $-b'$ column. Define also

$$\widetilde{B} : \widetilde{C}_\bullet(M) \to \widetilde{C}_{\bullet+1}(M) \tag{4.27}$$

as follows: if m is in the $-b'$ column then $\widetilde{B}m = 0$; if m is in the b column then $\widetilde{B}m = Nm$ which is located in the $-b'$ column. A straightforward check shows that p is a morphism of complexes and $\widetilde{B}$ anti-commutes with the differential. The complex $\widetilde{CC}_\bullet(M)$ is isomorphic to $\widetilde{C}_\bullet(M)[[u, u^{-1}]/u^{-1}\widetilde{C}_\bullet(M)[[u]]$, and p defines a morphism

$$\widetilde{CC}_\bullet(M) \to CC_\bullet(M) \tag{4.28}$$

This morphism is a quasi-isomorphism for columns, therefore a quasi-isomorphism of total complexes. □

DEFINITION 4.4.5. *For an integer $\ell \geq 1$ and for an ℓ-cyclic k-module M put*

$$C^\lambda_\bullet(M) = (M_\bullet/\mathrm{im}(1-\tau), b)$$

COROLLARY 4.4.6. *Assume that either $\mathbb{Q} \subset k$ or each M_n is free as a $\mathbb{Z}/\ell(n+1)\mathbb{Z}$-module. Then the projection to the rightmost column induces a quasi-isomorphism*

$$\widetilde{CC}_\bullet(M) \to C^\lambda_\bullet(M)$$

Indeed, it is a quasi-isomorphism of row complexes. (See also Proposition 2.0.1).

5. Functors between various cyclic and simplicial categories

5.1. The functors j_ℓ and j'_ℓ. For any ℓ, let

$$j_\ell : \Delta^{\mathrm{op}} \to \Lambda_\ell \tag{5.1}$$

and

$$j'_\ell : \Delta' \to \Lambda_\ell \tag{5.2}$$

be the embeddings of the subcategories from 3.

5.2. The functors π_ℓ. Let

$$\pi_\ell : \Lambda_\ell \to \Lambda \tag{5.3}$$

be the functor which is identical on objects, sends d_j to d_j, s_j to s_j, and τ to τ^ℓ (and therefore $\sigma = t^{n+1} \in \Lambda_\ell([n],[n])$ to the identity).

5.3. The functors i_ℓ. Observe first that for any monoid X and any ℓ there is a monoid X^ℓ with an automorphism

$$\alpha(x_1, \ldots, x_\ell) = (x_\ell, x_1, \ldots, x_{\ell-1}) \tag{5.4}$$

Identify $(X^l)^{n+1}$ with $X^{l(n+1)}$ via

$$((x_0^{(1)}, \ldots, x_0^{(\ell)}), \ldots, (x_n^{(1)}, \ldots, x_n^{(\ell)})) \mapsto (x_0^{(1)}, \ldots, x_n^{(1)}, \ldots, x_0^{(\ell)}, \ldots, x_n^{(\ell)}) \tag{5.5}$$

Under this identification, any morphism λ from $\Lambda_\ell([n],[m])$ defines a map $i_\ell(\lambda) : X^{l(n+1)} \to X^{\ell(m+1)}$. Let us observe that $i_\ell(\lambda)$ is defined by a unique morphism in $\Lambda(i_\ell[n], i_\ell[m])$ where

$$i_\ell[n] = \ell(n+1) - 1 \tag{5.6}$$

We have constructed a functor

$$i_\ell : \Lambda_\ell \to \Lambda \tag{5.7}$$

Example: the morphism $d_0 : [4] \to [3]$ in Λ_2 is mapped by i_2 to $d_0 d_5 : [9] \to [7]$ in Λ.

Indeed:

$$i_2(d_0) : (x_0^{(1)}, x_1^{(1)}, \ldots, x_4^{(1)}, x_0^{(2)}, \ldots, x_4^{(2)}) \mapsto ((x_0^{(1)}, x_0^{(2)}), \ldots, (x_4^{(1)}, x_4^{(2)})) \mapsto$$

$$((x_0^{(1)}x_1^{(1)}, x_0^{(2)}x_1^{(2)}), \ldots, (x_4^{(1)}, x_4^{(2)})) \mapsto ((x_0^{(1)}x_1^{(1)}, \ldots, x_4^{(1)}, x_0^{(2)}x_1^{(2)}, \ldots, x_4^{(2)}) = d_0 d_5((x_0^{(1)}, x_1^{(1)}, \ldots, x_4^{(1)}, x_0^{(2)}, \ldots, x_4^{(2)}))$$

More generally, the restriction of i_ℓ to Δ^{op} defines a functor

$$i_\ell|\Delta^{\mathrm{op}} = r_\ell : \Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \tag{5.8}$$

If we identify Δ^{op} with the opposite of the standard Δ, then r_ℓ is the subdivision functor: it sends the totally ordered set $[n] = \{0, \ldots, n\}$ to the set $[n] \times \{1, \ldots, \ell\}$ with the lexicographic order.

The functor i_ℓ also preserves the subcategory Δ'.

5.4. Cyclic homology of $(A^{\otimes \ell}, \alpha)$. For any algebra A, consider the algebra $A^{\otimes \ell}$ with the automorphism $\alpha(a_1 \otimes \ldots \otimes a_\ell) = a_\ell \otimes a_1 \otimes \ldots a_{l-1}$. By definition,

$$CC_\bullet(A^{\otimes \ell}, \alpha) \xrightarrow{\sim} CC_\bullet(i_\ell^* A^\sharp) \tag{5.9}$$

Proposition 5.4.1. *For any cyclic k-module M, there are natural quasi-isomorphisms of complexes*

$$C_\bullet(i_\ell^* M) \xrightarrow{\sim} C_\bullet(M); \; CC_\bullet(i_\ell^* M) \xrightarrow{\sim} CC_\bullet(M)$$

Proof. Let us start with the Hochschild complex $C_\bullet$ and the case $M = A^\sharp$ for an asociative unital algebra A. Let $\mathcal{B}_\bullet$ be the bar resolution of the bimodule A. We view it as a simplicial A-bimodule. Consider the tensor product of simplicial k-modules with the diagonal simplicial structure

$$\mathcal{B}_\bullet^{(\ell)} = \mathcal{B}_\bullet \boxtimes_k \ldots \boxtimes_k \mathcal{B}_\bullet$$

(ℓ times). (We used the symbol $\otimes$ in the proof above but are using $\boxtimes$ here to avoid confusion with the tensor product of complexes). Note that $\mathcal{B}_\bullet^{(\ell)}$ is the bar resolution of $A^{\otimes \ell}$. One has

$${}_\alpha(A^{\otimes \ell}) \otimes_{A^{\boxtimes \ell} \otimes (A^{\boxtimes \ell})^{\mathrm{op}}} \mathcal{B}_\bullet^{(\ell)} \xrightarrow{\sim} A \otimes_{A \otimes A^{\mathrm{op}}} (\mathcal{B}_\bullet \boxtimes_A \ldots \boxtimes_A \mathcal{B}_\bullet)$$

But $\mathcal{B}_\bullet \boxtimes_A \ldots \boxtimes_A \mathcal{B}_\bullet$ is a free A-bimodule resolution of A. Therefore the right hand side is quasi-isomorphic to the Hochschild complex $C_\bullet(A)$. To construct the quasi-isomorphism of the right hand side and $C_\bullet(A, A)$, all is needed is a bimodule morphism of complexes

$$F : \mathcal{B}_\bullet \boxtimes_A \ldots \boxtimes_A \mathcal{B}_\bullet \to \mathcal{B}_\bullet \tag{5.10}$$

(together with a morphism G in the opposite direction and homotopies for $\mathrm{id} - FG$ and $\mathrm{id} - GF$). We will get a homotopy equivalence

$$(i_\ell^* A_\bullet^\sharp, i_\ell^*(b)) \xrightarrow{\sim} (A_\bullet^\sharp, b) \tag{5.11}$$

Choose, for example, the composition of the Alexander-Whitney morphism

$$\mathrm{AW} : \mathcal{B}_\bullet \boxtimes_A \ldots \boxtimes_A \mathcal{B}_\bullet \to \mathcal{B}_\bullet \otimes_A \ldots \otimes_A \mathcal{B}_\bullet$$

with

$$\epsilon \otimes_A \ldots \epsilon \ldots \otimes_A \mathrm{id} : \mathcal{B}_\bullet \otimes_A \ldots \otimes_A \mathcal{B}_\bullet \to \mathcal{B}_\bullet \otimes_A A \ldots \otimes_A A = \mathcal{B}_\bullet$$

where $\epsilon : \mathcal{B}_\bullet \to A$ is the augmentation. We get

$$a_0^{(1)} \otimes \ldots \otimes a_n^{(1)} \otimes \ldots \otimes a_0^{(\ell)} \otimes \ldots \otimes a_n^{(\ell)} \mapsto a_0^{(1)} \ldots a_n^{(\ell-1)} a_0^{(\ell)} \otimes a_1^{(\ell)} \otimes \ldots \otimes a_n^{(\ell)}$$

This is clearly given by a morphism

$$F_0 \in \Delta^{\mathrm{op}}(i_\ell[n], [n]). \tag{5.12}$$

Furthermore, the morphism F, as well as the homotopies, are also given by ***MORE*** This implies the homotopy equivalence$(i_\ell^*(M)_\bullet, i_\ell(b)) \to (M_\bullet, b)$ for an arbitrary cyclic object M.

Now we need to pass from Hochschild to cyclic complexes. For this, consider complexes of Λ^{op}-modules (cf. also (4.21))

$$\mathcal{R}_\bullet = (\mathcal{P}_\bullet((u))/u\mathcal{P}_\bullet[[u]], b+uB);\ \mathcal{R}_\bullet^{(\ell)} = (\mathcal{P}_\bullet^{(\ell)}((u))/u\mathcal{P}_\bullet^{(\ell)}[[u]], i_\ell b + u i_\ell B) \tag{5.13}$$

Here $\mathcal{P}_\bullet$ and $\mathcal{P}_\bullet^{(\ell)}$ are cyclic objects in the category of complexes; they put in correspondance to an object $[m]$ of Λ the normalization (i.e. quotient by degenerate chains) of the complexes

$$(k\Lambda([m],[\bullet]), b);\ (k\Lambda([m], i_\ell[\bullet]), i_\ell(b)) \tag{5.14}$$

We have

$$M \otimes_\Lambda \mathcal{R}_\bullet \xrightarrow{\sim} CC_\bullet(M);\ M \otimes_\Lambda \mathcal{R}_\bullet^{(\ell)} \xrightarrow{\sim} CC_\bullet(i_\ell M)$$

It remains to construct a morphism of complexes of Λ^{op}-modules

$$F = F_0 + u^{-1}F_1 + \ldots : \mathcal{R}_\bullet^{(\ell)} \to \mathcal{R}_\bullet \tag{5.15}$$

where F_0 is the composition from the left with the morphism from (5.12).

There is precisely one obstruction for constructing F_1, F_2, etc. Indeed, we know that the homology of $\mathcal{P}_\bullet$ is nonzero only in degrees 0 and 1. So, as soon as we check that $BF_0 - F_0 i_\ell B : i_\ell[0] \to [1]$ is zero in $H_1(\mathcal{P}_\bullet)$, we will have no non-zero obstructions: we will know that $BF_0 - F_0 i_\ell(B)$ is homotopic to zero, F_1 will be a homotopy, etc. But

$$BF_0 : a_0^{(1)} \otimes \ldots \otimes a_0^{(\ell)} \mapsto B(a_0^{(1)} \ldots a_0^{(\ell)}) = 1 \otimes a_0^{(1)} \ldots a_0^{(\ell)},$$

whereas

$$F_0 i_\ell(B) : a_0^{(1)} \otimes \ldots \otimes a_0^{(\ell)} \mapsto F_0(\sum_{i=1}^{\ell} (1 \otimes \ldots \otimes 1) \otimes (a_0^{(i+1} \otimes \ldots \otimes a^{(i)})) =$$

$$\sum_{i=1}^{\ell} a_0^{(i+1)} \ldots a_0^{(i-1} \otimes a_0^{(i)}$$

and therefore

$$-BF_0 + F_0 i_\ell(B) : a_0^{(1)} \otimes \ldots \otimes a_0^{(\ell)} \mapsto b(\sum_{i=1}^{\ell-1} a_0^{(1)} \ldots a_0^{(i-1} \otimes a_0^{(i)} \otimes a_0^{(i+1)} \ldots a_0^{(\ell)})$$

□

6. The Kaledin resolution of a cyclic object

Here we will construct a resolution of any cyclic module M. It will not be projective but its value at any object $[n]$ will be free as a $C_n = \mathbb{Z}/(n+1)\mathbb{Z}$-module. As a result, we can compute $HC_\bullet(M)$ by applying to this resolution the generalized construction of the cyclic complex C^λ from 3. The result is the double complex $\widetilde{CC}_\bullet(M)$ from 4.4.1.

Let us start with the case $M = k^\sharp$. Note that the category Λ can be interpreted as follows. Objects are homotopy classes of triangulations of the circle S^1; the object $[n]$ corresponds to the triangulation by points $0, 1, \ldots, n$ located counterclockwise.

Morphisms are homotopy classes of maps $S^1 \to S^1$ that are of degree one and map triangulation to triangulation. Explicitly, the morphism

$$(x_0, \dots, x_n) \mapsto (x_{J_0}, \dots, x_{J_m}) \tag{6.1}$$

in $\Lambda([n],[m])$ corresponds to the counterclockwise-nondecreasing continuous map that sends the vertex i to the vertex k if x_i is a factor in x_{J_k}. Let

$$C_1(S^1,[n]) \xrightarrow{\partial} C_0(S^1,[n])$$

be the chain complex of the triangulation corresponding to $[n]$. It is a cyclic object in the category of chain complexes, as well as a length two extension of cyclic modules

$$0 \to k^\sharp \to C_1(S^1,[\_]) \xrightarrow{\partial} C_0(S^1,[\_]) \to k^\sharp \to 0 \tag{6.2}$$

which we also denote by

$$0 \to k^\sharp \to \mathbb{K}_1 \xrightarrow{\partial} \mathbb{K}_0 \to k^\sharp \to 0 \tag{6.3}$$

We also define the map $\mathbb{K}_0 \xrightarrow{N} \mathbb{K}_1$ to be the composition $\mathbb{K}_0 \to k^\sharp \to \mathbb{K}_1$. We get a resolution of $k^\sharp$:

$$\dots \xrightarrow{N} \mathbb{K}_1 \xrightarrow{\partial} \mathbb{K}_0 \xrightarrow{N} \mathbb{K}_1 \xrightarrow{\partial} \mathbb{K}_0 \to k^\sharp \to 0 \tag{6.4}$$

Now define for any cyclic module M

$$\mathbb{K}_i(M)_n = (\mathbb{K}_i)_n \otimes M_n$$

with the diagonal action of morphisms in Λ. We get the resolution of M

$$\dots \xrightarrow{N} \mathbb{K}_1(M) \xrightarrow{\partial} \mathbb{K}_0(M) \xrightarrow{N} \mathbb{K}_1(M) \xrightarrow{\partial} \mathbb{K}_0(M) \to M \to 0 \tag{6.5}$$

Lemma 6.0.1. *One has*

$$\mathbb{K}_0(M) \xrightarrow{\sim} j_! j^* M; \quad \mathbb{K}_1(M) \xrightarrow{\sim} j'_* j'^* M$$

Proof. Start with $M = k^\sharp$. Identify $j_! j^* k^\sharp$ with $C^0(S^1,[n])$ as follows: $t^j \otimes 1$ corresponds to the vertex j for all j. Now look at the unique decomposition of any morphism in Λ into $\lambda = t^i \delta$ where $\delta \in \Delta^{op}$, cf. Lemma 3.1.4. Observe that, when λ is identified with the corresponding triangulated map $S^1 \to S^1$, $j = \lambda(0)$. Consequently, if $\lambda t^k = t^j \delta$ for $\lambda \in \Lambda([n_1],[n])$ and $\delta \in \Delta^{op}([n_1],[n])$, then $j = \lambda(k)$. Therefore the action of λ on $t^k \otimes 1 \in (j_! j^* k^\sharp)_{n_1})$ agrees with the action of the corresponding map on the kth vertex of the triangulation $[n_1]$.

Now identify $C_1(S^1,[n])$ with $(j'_* j^* k^\sharp)_n$ as follows. By definition, the edge e_p from $p-1$ to p will correspond to the collection $\varphi^{(p)} = \{\varphi_j^{(p)}\}$ defined by

$$\varphi_j^{(p)} : (\delta' t^{-k}) \mapsto \delta_p^k$$

for any k and any $\delta' \in \Delta'([j],[n])$. (Here $n+1 = 0$, and δ_p^k is the Kronecker symbol). Let us look at the decomposition $\lambda = \delta' t^{-p}$ from Lemma 3.1.4. Let λ correspond to the map $(x_0, \dots, x_n) \mapsto (x_{J_0}, \dots, x_{J_m})$. Let r be the smallest index for which $x_{J_r} \neq 1$. Then the leftmost factor in x_{J_r} is x_p. More generally, assume

$$t^{-k}\lambda = \delta' t^{-p} \tag{6.6}$$

Let k' be the smallest index $k' \geq k$ (in the cyclic order) for which $x_{J_{k'}} \neq 1$. Then the leftmost factor in $x_{J_{k'}}$ is x_p.

Consequently, the action of λ on $j'_*j^*k^\sharp$ sends $\varphi^{(p)}$ to the sum of all $\varphi^{(k)}$ with given k' as above (this sum may be empty). In the language of triangulated maps, λ sends the edge e_p to the sum of all e_k that are contained in $\lambda(e_p)$.

For a general M, note that $j_!j^*M \xrightarrow{\sim} M \otimes j_!j^*k^\sharp$ and $M \otimes j'_*j'^*k^\sharp \xrightarrow{\sim} j'_*j'^*M$ The first isomorphism sends $\sigma\delta \otimes m = \sigma \otimes \delta m$ to $\sigma \otimes \sigma\delta m$ where δ is a morphism in Δ^0 and σ is a power of t. The second sends $m \otimes (\varphi_j)$ to $(\widetilde{\varphi}_j)$ defined by $\widetilde{\varphi}_j(\delta'\sigma) = \varphi(\sigma)\delta'\sigma^{-1}m$. Here δ' is a morphism in Δ' and σ is a power of t. **ELABORATE*** $\square$

LEMMA 6.0.2. *There are isomorphisms of double complexes*

$$(C^\lambda_\bullet(\mathbb{K}_1(M)) \xrightarrow{\partial} C^\lambda_\bullet(\mathbb{K}_0(M))) \xrightarrow{\sim} ((M_\bullet, b') \xrightarrow{1-t} (M_\bullet, b))$$

$$(C^\lambda_\bullet(\mathbb{K}_0(M)) \xrightarrow{N} C^\lambda_\bullet(\mathbb{K}_1(M))) \xrightarrow{\sim} ((M_\bullet, b) \xrightarrow{N} (M_\bullet, b'))$$

PROOF. Identify $C^\lambda_\bullet(\mathbb{K}_0(M))$, resp. $C^\lambda_\bullet(\mathbb{K}_1(M))$, with $M_\bullet$ by choosing $\mathbb{Z}/(n+1)\mathbb{Z}$-free generators of the nth component of $\mathbb{K}_0(M)$, resp. of $\mathbb{K}_1(M)$, to be $t^0 \otimes M_n$, resp. $(\varphi_j^{(0)}) \otimes M_n$ (cf. the proof above). As we saw in this proof, d_j act on t^0 by identity for all j; on $(\varphi^{(0)}$, d_j act by identity if $j < n$ and by zero for $j = n$. Indeed, $d_n : (x_0, \dots, x_n) \mapsto (x_n x_0, \dots, x_{n-1})$ and therefore x_0 is not a leftmost factor of any monomial x_{J_k}. Now, ∂ sends $(\varphi^{(0)}$ to $(t^0 - t^{-1}) \otimes m = t^0 \otimes (1-\tau)m$ in the quotient by the image of $1-\tau$ (here τ acts diagonally). As for N, the map $\mathbb{K}_0(k)_n \to k$ sends all $\tau^j \otimes 1$ to 1, and the map $k \to \mathbb{K}_1(k)_n$ sends 1 to the sum of all $\varphi^{(0)}$. After tensoring with M the composition of the two maps becomes the following: for every q,$t^q \otimes m \mapsto \sum_p t^p(\varphi^{(0)}) \otimes m = \varphi^{(0)} \otimes \sum_p \tau^p m$ modulo the image of $1-\tau$. $\square$

REMARK 6.0.3. It is straightforward that both factors in the composition $j_!j^*M \to M \to j'_*j'^*M$ are the standard adjunction maps.

6.1. The case of an ℓ-cyclic module. The above can be easily generalized to the following

PROPOSITION 6.1.1. *Let ℓ be an integer ≥ 1. For an ℓ-cyclic k-module M define*

$$\mathbb{K}_0(M) = j_{\ell!}j_\ell^*M;\ \mathbb{K}_1(M) = {j_\ell}'_*{j'_\ell}^*M \tag{6.7}$$

There are morphisms ∂, and N of ℓ-cyclic objects such that

$$\dots \xrightarrow{N} \mathbb{K}_1(M) \xrightarrow{\partial} \mathbb{K}_0(M) \xrightarrow{N} \mathbb{K}_1(M) \xrightarrow{\partial} \mathbb{K}_0(M) \to M \to 0 \tag{6.8}$$

is an acyclic complex of ℓ-modules. For every object $[n]$, both $\mathbb{K}_0(M)_n$ and $\mathbb{K}_1(M)_n$ are free $\mathbb{Z}/\ell(n+1)\mathbb{Z}$-modules. The sequence of complexes

$$\dots \xrightarrow{N} C^\lambda_\bullet(\mathbb{K}_1(M)) \xrightarrow{\partial} C^\lambda_\bullet(\mathbb{K}_0(M)) \xrightarrow{N} C^\lambda_\bullet(\mathbb{K}_1(M)) \xrightarrow{\partial} C^\lambda_\bullet(\mathbb{K}_0(M)) \tag{6.9}$$

is isomorphic to

$$\dots \xrightarrow{N} (M_\bullet, b') \xrightarrow{1-\tau} (M_\bullet, b) \xrightarrow{N} (M_\bullet, b') \xrightarrow{1-\tau} (M_\bullet, b)$$

PROOF. Let us start with interpreting Λ_ℓ in terms of triangulations. Let $[n]_\ell$ be the triangulation of S^1 with vertices $j^{(p)} = j + p(n+1)$, $0 \leq j \leq n$, $0 \leq p < \ell$ going counterclockwise. In other words, $[n]_\ell = i_\ell[n] = [\ell(n+1)-1]$. Then $\Lambda_\ell([n],[m])$ can be identified with homotopy classes of triangulated maps $(S^1, [n]_\ell) \to (S^1, [m]_\ell)$ which are:

(1) non-decreasing in counterclockwise order;
(2) of degree one;
(3) commuting with the shift $\sigma : [j]^{(p)} \mapsto [j]^{(p+1)}$ for all j and p (where $j^{(\ell)} = j^{(0)}$).

The identification is as follows. Start with a triangulated map $(S^1, [n]_\ell) \to (S^1, [m]_\ell)$ and construct a morphism in Λ_ℓ represented by $(x_0, \dots, x_n) \mapsto (x_{J_0}, \dots, x_{J_m})$ as in (2). First, for every $0 \le j \le n$, $\widetilde{x}_j$ will be a factor in x_{J_k} if the map sends $j^{(0)}$ to some $k^{(p)}$. Second, if the map sends $0^{(0)}$ to $i^{(r)}$, then $\widetilde{x}_0 = \alpha^r(x_0)$. As for other $\widetilde{x}_j$: we put $\widetilde{x}_j = \alpha^{r+1}(x_j)$ for all $j \le n$ such that $j^{(\ell-1)}$ is mapped to the same point $i^{(r)}$ as $0^{(0)}$, and $\widetilde{x}_j = \alpha^r(x_j)$ otherwise. This determines λ uniquely.

For example, there are four morphisms in $\Lambda_2([1], [0])$. They are represented my maps sending (x_0, x_1) to:

$$1)\,(x_0 x_1);\ 2)\,(\alpha(x_1), x_0);\ 3)\,(\alpha(x_0), \alpha(x_1));\ 4)\,(x_1, \alpha(x_0)).$$

The triangulations $[1]_2$ and $[0]_2$ are, respectively, the four points $0^{(0)}, 1^{(0)}, 0^{(1)}, 1^{(1)}$ and $0^{(0)}, 0^{(1)}$ located counterclockwise. The four morphisms above correspond to the four triangulated maps $(S^1, [1]_2) \to (S^1, [0]_2)$ as follows:

$$1)\,0^{(0)}, 1^{(0)} \mapsto 0^{(0)};\ 0^{(1)}, 1^{(1)} \mapsto 0^{(1)}$$

$$2)\,1^{(1)}, 0^{(0)} \mapsto 0^{(0)};\ 0^{(1)}, 1^{(0)} \mapsto 0^{(1)}$$

$$3)\,0^{(0)}, 1^{(0)} \mapsto 0^{(1)};\ 0^{(1)}, 1^{(1)} \mapsto 0^{(0)}$$

$$4)\,1^{(1)}, 0^{(0)} \mapsto 0^{(1)};\ 0^{(1)}, 1^{(0)} \mapsto 0^{(0)}$$

After these identifications, the above proof for Λ works for Λ_ℓ without any change. □

REMARK 6.1.2. We have identified $\Lambda_\ell([n], [m])$ with those morphisms in $\Lambda(i_\ell[n], i_\ell[m])$ that commute with $\sigma = \tau^{n+1}$. This identification is the functor i_ℓ from (5.7).

7. Cyclotomic objects

7.1. **Naive? Toy? cyclotomic modules.** As we saw in 4.2, the cyclic vector space $A^\sharp$ of an algebra over a perfect field of characteristic $p > 0$ carries an additional structure. Namely, the zeroth Tate cohomology of $C_p = \mathbb{Z}/p\mathbb{Z}$ with values in $i_p^* A^\sharp$ is F-linearly isomorphic to $A^\sharp$.

**Give a definition for iterations of i_p^*, and also over $\mathbb{Z}$ for all integer p?

To what extent can one replace the zeroth cohomology $\check{H}^0$ by the full complex $\check{C}^*$?

In many respects the answer turns out to be easier if one passes from algebras to *ring spectras*. In fact, C_p-equivariant spectra happen to admit both an analog of the full Cech complex (denoted by $X \mapsto X^{tC_p}$ together with a diagonal morphism $X \to (\wedge^p X)^{tC_p}$. As soon as basic properties if these two constructions are established, we get a full analog of what was defined in 7.1. We do this below in **??**. The category of algebras, esecially differential graded algebras, makes carrying out such a construction more complicated (for example, the Frobenius map $x \mapsto x^{\otimes p}$ does not commute with the differential). Nevertheless, a theory of cyclotomic modules exists; it is due to Kaledin. We outline in partially in **??**.

8. Appendix. Cyclic objects, topoi, and tropical projective geometry

9. Bibliographical notes

Connes, Loday, FT, Kaledin, Connes-Consani,

CHAPTER 9

Cyclic objects and the circle

1. Introduction

2. The action of $\mathbb{T}$

2.1. Realisation of cyclic spaces. Put

$$\mathbb{E}^n = \{\mathbb{Z} - \text{equivariant nondecreasing maps } \frac{1}{n+1}\mathbb{Z} \to \mathbb{R}\}/\mathbb{Z} \tag{2.1}$$

This is a cocyclic object in the category of topological spaces with an action of $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (the topology is ***). We have the functor

$$X \mapsto |X|_{\text{cyc}} = X \times_{\Lambda} \mathbb{E} \tag{2.2}$$

from cyclic spaces to spaces with an action of $\mathbb{T}$

Lemma 2.1.1. *For a cyclic topological space* X, $|X|_{\text{cyc}}$ *is homeomorphic to the geometric realisation of the underlying simplicial space* j^*X.

Proof. □

3. The action of $B\mathbb{Z}$

Let $E_n\mathbb{Z} = \mathbb{Z}^{n+1}$, $n \geq 0$. We write As usual, we write

$$d_j(m_0, m_1, \ldots, m_n) = (m_1, \ldots, m_j + m_{j+1}, \ldots, m_n),\ 1 \leq j \leq n-1; \tag{3.1}$$

for $0 \leq j < n$;

$$d_n(m_0, m_1, \ldots, m_n) = (m_1, \ldots, m_{n-1}) \tag{3.2}$$

$$s_j(m_0, \ldots, m_n) = (m_0, \ldots, m_j, 0, \ldots, m_n),\ 0 \leq j \leq n \tag{3.3}$$

We get a simplicial Abelian group that we denote by $E\mathbb{Z}$. The group $\mathbb{Z}$ acts on $E\mathbb{Z}$ freely by

$$(m_0, m_1 \ldots, m_n) \mapsto (m_0 + a, m_1, \ldots, m_n) \tag{3.4}$$

for $a \in \mathbb{Z}$. Define

$$B\mathbb{Z} = E\mathbb{Z}/\mathbb{Z} \tag{3.5}$$

We have $B_n\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}^n$, $n \geq 0$. As usual, we write

$$d_0(m_1, \ldots, m_n) = (m_2, \ldots, m_n); \tag{3.6}$$

$$d_j(m_1, \ldots, m_n) = (m_1, \ldots, m_j + m_{j+1}, \ldots, m_n),\ 1 \leq j \leq n-1; \tag{3.7}$$

$$d_n(m_1, \ldots, m_n) = (m_1, \ldots, m_{n-1}) \tag{3.8}$$

$$s_j(m_1, \ldots, m_n) = (m_1, \ldots, m_{j-1}, 0, m_j, \ldots, m_n),\ 0 \leq j \leq n \tag{3.9}$$

3.1. Realisation of cyclic objects. Define

$$\mathbb{E}^n_m = \{\mathbb{Z} - \text{equivariant maps } \frac{1}{n+1}\mathbb{Z} \to \mathsf{E}_m\mathbb{Z}\}/\mathbb{Z} \tag{3.10}$$

This is a cocyclic object in simplicial sets with an action of $B\mathbb{Z}$. Let $\mathcal{C}$ be a category with colimits. As above, we have a functor

$$X \mapsto \|X\|_{\mathrm{cyc}} = X \times_\Lambda \mathbb{E} \tag{3.11}$$

from cyclic objects to simplicial objects with an action of $B\mathbb{Z}$.

3.2. The action on $\mathrm{hocolim}_{\Lambda_\infty^{\mathrm{op}}}$. Now consider the category Λ_∞. Recall that $\mathbb{Z}$ is in the center of it. For any $j \geq 0$, denote by t^m the morphism in $\Lambda_\infty([j],[j])$ corresponding to an integer m. Let X be a cyclic object of $\mathcal{C}$, i.e. a functor $\Lambda_\infty^{\mathrm{op}} \to \mathcal{C}$ that sends all t^m to identity morphisms. Define

$$B\mathbb{Z} \times \mathrm{hocolim}_{\Lambda_\infty^{\mathrm{op}}} X \to \mathrm{hocolim}_{\Lambda_\infty^{\mathrm{op}}} X \tag{3.12}$$

$$(m_1, \ldots, m_n) \times (x, \lambda_1, \ldots, \lambda_n) \mapsto (x, t^{m_1}\lambda_1, \ldots, t^{m_n}\lambda_n) \tag{3.13}$$

Here, as usual,

$$\mathrm{hocolim}_{\Lambda_\infty^{\mathrm{op}}} X = X \times^h_{\Lambda_\infty} \mathrm{pt}$$

Note that

$$\mathrm{hocolim}_{\Delta^{\mathrm{op}}} X \xrightarrow{\sim} \mathrm{hocolim}_{\Lambda_\infty^{\mathrm{op}}} X \tag{3.14}$$

Explain

The above is a partial case of the construction used in the proof of Proposition B5 in [**?**] .

3.3. Comparison of 3.1. and 3.2

3.4. Comparison of 2.1 and 3.1.

4. Mixed complexes

A mixed complex is a DG module over the DG algebra $k[\epsilon]$ where $|\epsilon| = -1$ (if we use cohomological grading).

4.1. From cyclic modules to mixed complexes.

4.1.1. *The* (b, B) *mixed complex of a cyclic Abelian group.* For a cyclic object E in Abelian groups, we define a mixed complex $(E_\bullet, b)$ on which ϵ acts by B.

4.2. From simplicial modules over $B_\bullet\mathbb{Z}$ to mixed complexes. Given a simplicial module E over the simplicial group $k[B\mathbb{Z}]$, we pass to the DG module over the DG algebra $k[B\mathbb{Z}]$ by the Eilenberg-Zilber transformation (***ref). The generator γ of $H_1(k[B\mathbb{Z}])$ is represented by the cycle

$$\gamma = (1) \in \mathbb{Z} = B_1(\mathbb{Z}) \tag{4.1}$$

We claim that the square of this one-chain with respect to the EZ product is zero. Indeed, ***

Proposition 4.2.1. *For a cyclic object E in Abelian groups, there is a natural equivalence of mixed complexes (***improve)*

$$(\mathrm{hocolim}_{\Lambda_\infty^{\mathrm{op}}}(E), \gamma) \xrightarrow{\sim} (E_\bullet, b, B)$$

This is Theorem 2.3 from [**?**].

PROOF. □

5. Cyclic homologies and the circle action

5.1. Functors $(\_\_)_{h\mathbb{T}}$ and $(\_\_)^{h\mathbb{T}}$.

5.1.1. *For mixed complexes.* For a DG module M over $k[\epsilon]$, define

$$M_{h\mathbb{T}} = k \otimes^{\mathbb{L}}_{k[\epsilon]} M; \quad M^{h\mathbb{T}} = \mathbb{R}\mathrm{Hom}_{k]\epsilon]}(k, M) \tag{5.1}$$

5.1.2. *For simplicial objects with an action of* $B\mathbb{Z}$. *A more general construction in the context of* ∞ *categories*: $B\mathbb{T} = B(B\mathbb{Z})$ is an ∞ category; a $\mathbb{T}$-equivapiant object of an ∞ category $\mathcal{C}$ is a functor $\mathfrak{X} : B\mathbb{T} \to \mathcal{C}$; *when (co)limits exist*, define

$$\mathfrak{X}_{h\mathbb{T}} = \mathrm{colim}_{B\mathbb{T}}(\mathfrak{X}); \quad \mathfrak{X}^{h\mathbb{T}} = \lim_{B\mathbb{T}}(\mathfrak{X}) \tag{5.2}$$

More concrete versions: simplicial Abelian groups/spaces/spectra/othogonal spectra $\mathfrak{X}$ with an action of $\mathbb{T} = B\mathbb{Z}$: Some variation on the (co)simplicial objects

$$E\mathbb{T} \times_{\mathbb{T}} \mathfrak{X} \tag{5.3}$$

and

$$\mathrm{Maps}_{\mathbb{T}}(E\mathbb{T}, \mathfrak{X}) \tag{5.4}$$

(***What is a standard reference?)

THEOREM 5.1.1. *For a DG category* A*, let* $A^\sharp$ *be the corresponding cyclic object in the category of complexes and let* $\|A^\sharp\|_{\mathrm{cyc}}$ *be its realisation* (3.11)*. Then*

$$\mathrm{CC}_\bullet(A) \xrightarrow{\sim} (\|A^\sharp\|_{\mathrm{cyc}})_{hB\mathbb{Z}}$$

$$\mathrm{CC}^-_\bullet(A) \xrightarrow{\sim} (\|A^\sharp\|_{\mathrm{cyc}})^{hB\mathbb{Z}}$$

PROOF. □

5.2. The Tate construction.

5.2.1. *For mixed complexes.* For a DG module M over $k[\epsilon]$, there is a morphism

$$k \otimes_{k[\epsilon]} M \to \mathrm{Hom}_{k[\epsilon]}(k, M)[-1] \tag{5.5}$$

which is given by multiplication by ϵ. Now consider the composition

$$k \otimes^{\mathbb{L}}_{k[\epsilon]} M \to k \otimes_{k[\epsilon]} M \to \mathrm{Hom}_{k[\epsilon]}(k, M)[-1] \to \mathbb{R}\mathrm{Hom}_{k[\epsilon]}(k, M)[-1] \tag{5.6}$$

which becomes

$$\nu : M_{h\mathbb{T}} \to M^{h\mathbb{T}}[-1] \tag{5.7}$$

More

5.2.2. *For objects with an action of* $\mathbb{T}$, *etc.* *****The Tate construction is defined in Chapter I of [?].**

OR: one paragraph in [?] (before Defintion 1.3).***

Along these lines:

Start with a *finite* group G acting on an Abelian group E.

First, observe that for the diagonal functor

$$\delta : BG \to BG \times BG \tag{5.8}$$

there is a functorial isomorphism

$$\delta_! E \xrightarrow{\sim} \delta_* E \tag{5.9}$$

Indeed,

$$\delta_! E = (G \times G)_{\text{diag } G} E$$

and its elements have a unique presentation

$$\sum_{g \in G} (g, 1) \times x_g,\ x_g \in E \tag{5.10}$$

On the other hand

$$\delta_* E = \text{Maps}_{\text{diag } G}(G \times G, E)$$

and its elements have a unique presentation

$$(hg, h) \mapsto hy(g),\ y(g) \in E. \tag{5.11}$$

The map

$$(x_g) \mapsto (y(g) = x_{g^{-1}}) \tag{5.12}$$

intertwines the actions of $G \times G$ and just defines an isomorphism (5.9).

Now let $p_0, p_1 : BG \times BG \to BG$ be the two projections. Also, we write $f : BG \to \text{pt}$. If we write

$$E_{hG} = f_! E;\ \ E^{hG} = f_* E \tag{5.13}$$

then ***Finish following Construction I.1.7 of Nikolaus and Scholze***

Let us assume that $\delta_! \tilde{\to} \delta_*[-k]$ (so far $k = 0$). Then we get a morphism

$$p_0^* \to \delta_* \delta^* = \delta_* \tilde{\to} \delta_![k] = \delta_! \delta^* p_1^*[k] \to p_1^*[k]$$

which leads to

$$\text{id} \to p_{0*} p_0^* \to p_{0*} p_1^*[k] \tag{5.14}$$

Also, there is an isomorphism ***Explain, draw diagram***

$$f^* f_* \tilde{\to} p_{0*} p_1^* \tag{5.15}$$

Therefore we get

$$\text{id} \to f^* f_*[k] \tag{5.16}$$

or by adjunction

$$\text{Nm}_f : f_! \to f_*[k] \tag{5.17}$$

When we replace G with $\mathbb{T}$: more or less tautological (Σ is for suspension):

$$\delta_! E = \mathbb{T} \times E \tilde{\to} \Sigma \text{Maps}(\mathbb{T}, E) \tag{5.18}$$

with the actions of $\mathbb{T}$... Use/reconcile with: [?] and p. 29–33 of [?].

(generality: $\mathbb{T} = B\mathbb{Z}$; E...).

Next: we want to generalize this argument to the situation when we have a functor

$$\mathfrak{X} : BG \to \mathcal{C}. \tag{5.19}$$

What should the properties of $\mathcal{C}$ be? First, since $f_!$ is based on coproducts and f_* on products, any comparison between them requires the two being the same, as in is for Abelian groups. Also suspension plays a role. The appropriate context for it is *a stable* ∞ *category*.

THEOREM 5.2.1. *For a DG category* A*, let* $A^\sharp$ *be the corresponding cyclic object in the category of complexes and let* $\|A^\sharp\|_{\mathrm{cyc}}$ *be its realisation* (3.11). *Then*

$$CC^{\mathrm{per}}_\bullet(A) \xrightarrow{\sim} (\|A^\sharp\|_{\mathrm{cyc}})^{\mathrm{thB}\mathbb{Z}}$$

PROOF. □

6. Appendix. Cyclic homology and factorization homology

Cyclic homology of an associative algebra is a partial case of the following general construction. Let X be a framed manifold, i.e. an n-dimensional manifold with a trivialization of the tangent bundle. Let $\mathcal{A}$ be an algebra over the operad of little n-discs (see ***REF). Then one can define the complex $\int_X \mathcal{A}$. The homology of this complex is called factorization homology ***REF. Algebras over the operad of little intervals (or little 1-discs) are essentially the same as A_∞ algebras. When $X = \mathbb{T}^1$, we recover a complex computing the cyclic homology of an A_∞ algebra. We outline the construction and the comparison below.

Recall that an operad $\mathcal{O}$ in a symmetric monoidal category $(\mathcal{C}, \otimes)$ is:

(1) a collection of objects $\mathcal{O}(n)$, $n \geq 1$, with an action of the symmetric group Σ_n on $\mathcal{O}(n)$ on $\mathcal{O}(n)$ for all n;
(2) morphisms

$$\mathcal{O}(k) \otimes \mathcal{O}(n_1) \otimes \ldots \otimes \mathcal{O}(n_k) \to \mathcal{O}(n_1 + \ldots + n_k)$$

subject to the condition of associativity and compatibility with the symmetrc group actions;
(3) A morphism $e : \mathbf{k} \to \mathcal{O}(1)$ (where $\mathbf{k}$ is the terminal object of $\mathcal{C}$) subject to the unitality condition.

A morphism $\mathcal{O}_1 \to \mathcal{O}_2$ of operads is a collection of morphisms $\mathcal{O}_1(n) \to \mathcal{O}_2(n)$ which is compatible with the action of Σ_n, the composition, and the unit. For an object V, $\mathrm{End}_{\mathcal{C}}(V)$ is an operad in $\mathcal{S}ets$ if one puts $\mathrm{End}_{\mathcal{C}}(V)(n) = \mathrm{Mor}(V^{\otimes n}, V)$ and the image of e is id_V. For an object V, a structure of an $\mathcal{O}$-algebra on V is a morphism $\mathcal{O} \to \mathrm{End}_{\mathcal{C}}(V)$.

If $\mathcal{O}$ is an operad in $\mathcal{T}op$ then $C_{-\bullet}(\mathcal{O}, k)$ is an operad in $\mathrm{dgmod}(k)$ for any commutative unital ring k.

The operad **Ass** in $\mathcal{S}ets$ is defined as follows: $\mathbf{Ass}(n) = \Sigma_n$; we view this as the set of natural maps $A^{\times n} \to A$ for a monoid A. Namely: we interpret a permutation σ as an operation $(a_1, \ldots, a_n) \mapsto a_{\sigma 1} \ldots a_{\sigma n}$. This explains how to define compositions.

The operad of little discs Disc_k in $\mathcal{T}op$ is defined as follows. A point of the space $\mathrm{Disc}_k(n)$ is an *ordered* n-tuple of disjoint subdiscs $\{|x - x_j| \leq r_j\}$, $1 \leq j \leq n$, in the standard disc $\{|x| \leq 1\}$ in $\mathbb{R}^k$. In other words, Disc_k is the suboperad of the operad $\mathrm{End}_{\mathcal{T}op}(B^k(1))$ where $B^k(1) = \{|x| \leq 1\}$ in $\mathbb{R}^k$; points of the space $\mathrm{Disc}_k(n)$ are maps $\coprod_n B^k(1) \to B^k(1)$ whose restriction to any component $B^k(1)$ is dilation at the center followed by the standard embedding of a disc to a larger disc.

There is a morphism of operads in $\mathcal{T}op$ $\mathrm{Disc}_1 \to \mathbf{Ass}$. It acts as follows. Take a configuration of n intervals and number them as $S_1, \ldots, S_n$ where the interval S_i is to the left of S_j for $i < j$. Then, for some permutation σ, the segment S_j is labeled by the number σj for all j. The morphism sends such a configuration to σ.

Given any operad $\mathcal{O}$ in $\mathcal{C}$, define the category $\mathbf{P}_{\mathcal{O}}$ enriched in $\mathcal{C}$ as follows. Objects of $\mathbf{P}_{\mathcal{O}}$ are natural numbers $n \geq 1$. Define the morphisms as follows:

$$\mathbf{P}_{\mathcal{O}}(n, m) = \oplus_{\ell \geq 1} \oplus_{S_1, \ldots, S_\ell} \mathcal{O}(|S_1|) \otimes \ldots \otimes \mathcal{O}(|S_\ell|) \tag{6.1}$$

The sum is taken over all subdivisions

$$\{1, \ldots, n\} = \coprod_1^\ell S_j$$

where all S_j are nonempty.

In other words (when $\mathcal{C}$ is $\mathcal{S}ets$ or $\mathcal{T}ops$): $\mathbf{P}_{\mathcal{O}}(n, m)$ consists of natural operations $A^{\otimes n} \to A^{\otimes m}$ for any $\mathcal{O}$-algebra A. This explains the rule of composition of morphisms. Tautologically, for an algebra A over $\mathcal{O}$, $A^\sharp(n) = A^{\otimes n}$ defines a left module over $\mathbf{P}_{\mathcal{O}}$.

Now let M be an m-dimensional manifold with a framing. For simplicity, let us assume first that $M = \mathbb{T}^m$. In this case, define

$$M^\sharp(n) = \{j : \coprod_1^n B^m(1) \to M \mid j \text{ is standard}\} \tag{6.2}$$

Here j is called standard if its restriction to every component $B^m(1)$ is standard (i.e. is a composition of dilation at the center and a standard embedding of a disc into the flat torus). This defines a right action of $\mathbf{P}_{\mathrm{Disc}_m}$ on $M^\sharp$. Now for every Disc_m-algebra $\mathcal{A}$ define

$$\int_M \mathcal{A} = M^\sharp \times^h_{\mathbf{P}_{\mathrm{Disc}_m}} \mathcal{A}^\sharp \tag{6.3}$$

Let us look at the case $m = 1$ and $M = \mathbb{T}^1$. Then we have a functor

$$\mathbf{P}_{\mathrm{Disc}_1} \to \mathbf{P}_{\mathbf{Ass}}$$

As for modules, put

7. Bibliographical notes

Connes; Besser, Drinfeld; Loday; Nikolaus-Scholze; Kaledin; Ben-Zvi–Francis–Nadler;

CHAPTER 10

Examples

1. Introduction

2. Polynomial algebras

2.1. Hochschild homology of algebras of polynomials. Let $A = k[x_1, \dots, x_n]$. Let $\mathcal{B}_\bullet$ be the bar resolution of the A-bimodule A. One has

$$\mathcal{B}_p = A \otimes \overline{A}^{\otimes p} \otimes A \tag{2.1}$$

with the differential b' as in (1.6). Let $\mathcal{K}_\bullet$ be the Koszul resolution. By definition,

$$\mathcal{K}_p = A \otimes \wedge V \otimes A$$

where $V = \oplus_{j=1}^n k \cdot x_j$. The differential, that we also denote by b', acts as follows:

$$b'(a \otimes (v_1 \wedge \dots \wedge v_p) \otimes b) = \sum_{j-1}^{p} (-1)^{j-1} a v_j \otimes (v_1 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_p) \otimes b \tag{2.2}$$

$$- \sum_{j-1}^{p} (-1)^{j+p} a \otimes (v_1 \wedge \dots \wedge \widehat{v_j} \wedge \dots \wedge v_p) \otimes v_j b$$

This is a complex of A-bimodules (A acts on it by left and right multiplication). Moreover, it is a free bimodule resolution of A. One can see this, for example, by observing that the complex $\mathcal{K}_\bullet$ is the nth tensor power of $\mathcal{K}_\bullet(1)$, the latter being the Koszul resolution for $n = 1$.

There is an embedding $\mathcal{K}_\bullet \xrightarrow{i} \mathcal{B}_\bullet$ given by

$$i(a \otimes (v_1 \wedge \dots \wedge v_p) \otimes b) = \sum_{\sigma \in \Sigma_p} (-1)^{\mathrm{sgn}(\sigma)} a \otimes (v_{\sigma 1} \otimes \dots \otimes v_{\sigma p}) \otimes b \tag{2.3}$$

For any commutative k-algebra A, let $\Omega^1_{A/k}$ be the module of Kähler differentials of A. Denote

$$\Omega^p_{A/k} = \wedge^p_A \Omega^1_{A/k} \tag{2.4}$$

The exterior product makes $\Omega^\bullet_{A/k}$ a graded commutative algebra. There is unique graded derivation d of degree one that sends a to da and da to zero for every a in A.

Proposition 2.1.1. *The embedding* (2.3) *induces an isomorphism*

$$\Omega^p_{k[x_1,\dots,x_n]/k} \xrightarrow{\sim} \mathrm{HH}_p(k[x_1, \dots, x_n])$$

Proof. Follows from the isomorphism $\mathcal{K} \otimes_{A \otimes A^{\mathrm{op}}} A \xrightarrow{\sim} \Omega^\bullet_{A/k}$. □

2.2. The HKR map. For any commutative algebra A, define the map [?]

$$I_{\mathrm{HKR}} : C_\bullet(A) \to \Omega^\bullet_{A/k}$$

by

$$I_{\mathrm{HKR}} : a_0 \otimes \ldots \otimes a_p \mapsto \frac{1}{p!} a_0 da_1 \ldots \otimes da_p \tag{2.5}$$

LEMMA 2.2.1. *The above map is a morphism of complexes* $(C_\bullet(A), b) \to (\Omega^\bullet_{A/k}, 0)$. *One has* $I_{\mathrm{HKR}} \circ B = d \circ I_{\mathrm{HKR}}$.

This is verified by a direct computation. For $A = k[x_1, \ldots, x_n]$, the HKR map is a left inverse to the map induced by i as in (2.3). Therefore, when A is a polynomial algebra, the HKR map is a is a quasi-isomorphism of complexes. We will later specified by which morphism of free resolutions it is induced.

2.3. More details on the Hochschild homology of polynomial algebras. The standard procedure of homological algebra provides morphisms of resolutions $\mathcal{B}_\bullet \overset{\longrightarrow}{\longleftarrow} \mathcal{K}_\bullet$ over A which are homotopy inverse. In this subsection we will construct them explicitly, together with the homotopies and with the maps induced by them on the Hochschild complex and on Kähler differentials. This will be used later to establish analogues of Proposition (2.1.1).

Start with observing that $A = k[x_1, \ldots, x_n] \xrightarrow{\sim} A_1^{\otimes n}$ where $A_1 = k[x]$. So start with the case $n = 1$. Let $\mathcal{B}_\bullet(1)$ and $\mathcal{K}_\bullet(1)$ be the bar and Koszul resolutions for $n = 1$. We have the map $j : \mathcal{B}_\bullet(1) \to \mathcal{K}_\bullet(1)$ given by $j(a_0 \otimes a_1) = a_0 \otimes a_1$;

$$j(a_0 \otimes x^m \otimes a_2) = \sum_{k=0}^{m-1} a_0 x^k \otimes x \otimes x^{m-1-k} a_2; \tag{2.6}$$

$j = 0$ on $\mathcal{B}_p(1)$ for $p = 1$. In other words, if we identify $\mathcal{B}_1(1)$ with $k[x, y, z]$ and $\mathcal{K}_1(1)$ with $k[x, z] \otimes ky$, then

$$j : f(x, y, z) \mapsto \frac{f(x, x, z) - f(x, z, z)}{x - z} \otimes y.$$

We have $j \circ i = \mathrm{id}$, whereas $i \circ j = [b', s]$ where $s : \mathcal{B}_p(1) \to \mathcal{B}_{p+1}(1)$ can be chosen as follows. Let us use the notation

$$\frac{f(x) - f(y)}{x - y} = \sum f^{(1)}(x) f^{(2)}(y) \tag{2.7}$$

Define

$$s(a_0 \otimes \ldots \otimes a_{p+1}) = (-1)^p \sum a_0 \otimes \ldots \otimes a_{p-1} \otimes a_p^{(1)} \otimes x \otimes a_p^{(2)} a_{p+1}$$

for $p > 0$ and $s(a_0 \otimes a_1) = 0$. The fact that s is indeed a homotopy for $i \circ j - \mathrm{id}$ follows from the identities

$$\sum (a^{(1)} x \otimes a^{(2)} - a^{(1)} \otimes x a^{(2)}) = a \otimes 1 - 1 \otimes a;$$

$$\sum (a_1 a_2)^{(1)} \otimes (a_1 a_2)^{(2)} = \sum a_1 a_2^{(1)} \otimes a_2^{(2)} + \sum a_1^{(1)} \otimes a_1^{(2)} a_2$$

For general n, as we mentioned before, $\mathcal{K}_\bullet \xrightarrow{\sim} \mathcal{K}_\bullet(1)^{\otimes n}$. There are two standard morphisms of resolutions (0.4), (2.2)

$$\mathrm{EZ} : \mathcal{B}_\bullet(1)^{\otimes n} \longrightarrow \mathcal{B}_\bullet; \ \mathrm{AW} : \mathcal{B}_\bullet(1)^{\otimes n} \longleftarrow \mathcal{B}_\bullet.$$

Both morphisms EZ and AW are associative in the obvious sense. This allows us to define

$$\mathrm{EZ} : \otimes_{j=1}^{n} \mathcal{B}_\bullet(A_j) \longrightarrow \mathcal{B}_\bullet(\otimes_{j=1}^{n} A_j)$$

and

$$\mathrm{AW} : \otimes_{j=1}^{n} \mathcal{B}_\bullet(A_j) \longleftarrow \mathcal{B}_\bullet(\otimes_{j=1}^{n} A_j)$$

One has $\mathrm{AW} \circ \mathrm{EZ} = \mathrm{id}$; in Lemma 2.0.2 we constructed an explicit homotopy t for $\mathrm{id} - \mathrm{EZ} \circ \mathrm{AW}$ for $n = 2$. We can easily extend it to the case of any n. All that we need to know here is that the element

$$t[a_1 \otimes \ldots \otimes a_n | a_1' \otimes \ldots \otimes a_n' | \ldots]$$

is given by an algebraic expression involving taking elements a_j, a_j', *etc.* from one position to another and multiplying them with some other elements.

Note that while EZ is commutative in the obvious sense, AW is not. For $\sigma \in \Sigma_n$, let AW^σ be the map AW constructed for the product $A_{\sigma 1} \otimes \ldots \otimes A_{\sigma_n}$ Put

$$\mathrm{AW}^{\mathrm{sym}} = \frac{1}{n!} \sum_{\sigma} \mathrm{AW}^\sigma \tag{2.8}$$

The same argument as above allows to construct a homotopy t for $\mathrm{AW}^{\mathrm{sym}}$, of the same algebraic nature as discussed above. Now apply this to the case when $A_1 = \ldots = A_n = k[x]$. We have morphisms

$$\mathrm{EZ} : \mathcal{B}_\bullet(1)^{\otimes n} \longrightarrow \mathcal{B}_\bullet;\ \mathrm{AW}^{\mathrm{sym}} : \mathcal{B}_\bullet(1)^{\otimes n} \longleftarrow \mathcal{B}_\bullet,$$

as well as

$$i^{\otimes n} : \mathcal{K}_\bullet \longrightarrow \mathcal{B}_\bullet(1)^{\otimes n};\ j^{\otimes n} : \mathcal{K}_\bullet \longleftarrow \mathcal{B}_\bullet(1)^{\otimes n}.$$

The homotopy for $\mathrm{id} - i^{\otimes n} \circ j^{\otimes n}$ can be easily constructed from the one for $\mathrm{id} - ij$ for $n = 1$, for example one can take

$$s^{\otimes n} = \sum_{k=1}^{n} (\mathrm{id} - ij)^{\otimes (k-1)} \otimes s \otimes \mathrm{id}^{n-1-k}$$

Observe that

$$i = i^{\otimes n} \circ \mathrm{EZ};$$

define

$$j = \mathrm{AW}^{\mathrm{sym}} \circ j^{\otimes n};$$

we have

$$ij = i^{\otimes n} \mathrm{EZ} \circ \mathrm{AW}^{\mathrm{sym}} j^{\otimes n} = i^{\otimes n} j^{\otimes n} - i^{\otimes n} [b', t] j^{\otimes n} = -[b', s^{\otimes n}] - i^{\otimes n} [b', t] j^{\otimes n}$$

therefore we can chose the homotopy for $\mathrm{id} - ij$ to be

$$h = -s^{\otimes n} - i^{\otimes n} t j^{\otimes n}$$

Note also that the map $C_\bullet(A) \to \Omega^\bullet_{A/k}$ induced by j is I_{HKR}.

Definition 2.3.1. *Set*

$$C_p(n) = k[x_j^{(k)} | 1 \leq j \leq n; 0 \leq k \leq p]$$

A generalized differential operator is a linear map $C_p(n) \to C_q(n)$ *that is a linear combination of compositions of the following maps:*

1) $T_j(k, l)$ *that substitutes* $x_j^{(k)}$ *in place of* $x_j^{(l)}$

2) The map

$$D_j(k;l,m)f = \frac{T_j(k,l)f - T_j(k,m)f}{x_j^{(l)} - x_j^{(m)}};$$

3) partial derivatives.

More generally, if a generalized differential operator sends a subspace L *to a subspace* L'*, the induced operator on quotients will be also called a generalized differential operator.*

Let us identify $C_p(k[x_1, \ldots, x_n])$ with the quotient of $C_p(n)$. Recall the HKR map (2.5). Put

$$i(f dx_{j_1} \ldots dx_{j_p}) = (-1)^{\mathrm{sign}\,\sigma} \frac{1}{p!} \sum_{\sigma \in \Sigma_p} f \otimes x_{j_{\sigma 1}} \otimes \ldots \otimes x_{j_{\sigma p}} \tag{2.9}$$

PROPOSITION 2.3.2. *There is a generalized differential operator* $h : C_\bullet(k[x_1, \ldots, k_n]) \to C_{\bullet+1}(k[x_1, \ldots, k_n])$ *such that*

$$\mathrm{id} - i \circ I_{\mathrm{HKR}} = [b, h].$$

2.4. Completed Hochschild complexes of commutative algebras. Now consider an ideal I in any commutative algebra P. Consider the embeddings $i_k : P \to P^{\otimes(m+1)}$, $0 \le k \le m$, given by $a \mapsto 1 \otimes \ldots \otimes a \otimes \ldots \otimes 1$. For every $m \ge 0$, let I_Δ be the ideal in $P^{\otimes m+1}$ generated by $i_k(a) - i_l(a)$, $a \in P$, and by $i_k(a)$, $a \in I$, for all possible k and l. Denote by $\widehat{C}_m^{\mathrm{un}}(P)_{\Delta,I}$ the completion of $P^{\otimes m+1}$ with respect to I_Δ.

We write

$$d_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots a_i a_{i+1} \otimes \ldots \otimes a_{n+1} \tag{2.10}$$

for $i < n$;

$$d_n(a_0 \otimes \ldots \otimes a_{n+1}) = a_n a_0 \otimes a_1 \otimes \ldots \otimes a_{n-1}; \tag{2.11}$$

$$s_i(a_0 \otimes \ldots \otimes a_n) = a_0 \otimes \ldots a_i \otimes 1 \otimes \ldots \otimes a_n \tag{2.12}$$

for $i \le n$. One has

$$b = \sum_{j=0}^{n} (-1)^j d_j$$

Note that all d_j and s_j are algebra homomorphisms preserving I_Δ. Therefore they all extend to $\widehat{C}_m^{\mathrm{un}}(P)_{\Delta,I}$. We denote the quotient by the sum of images of all s_j by $\widehat{C}_m(P)_{\Delta,I}$.

PROPOSITION 2.4.1. *Let* $P = k[x_1, \ldots, x_n]$. *Then* I_{HKR} *induces a quasi-isomorphism*

$$(\widehat{C}_m(P)_{\Delta,I}, b) \to (\widehat{\Omega}^\bullet_{P/k}, 0)$$

where the right hand side stands for the I*-adic completion.*

PROOF. In fact, any map from 1), Definition 2.3.1, is a ring homomorphism preserving I_Δ. Any map D from 2) satisfies $D(fg) = S(f)D(g) + D(f)T(g)$ where S and T are as in 1). Therefore D sends I_Δ^{N+1} to I_Δ^N. We see that any generalized differential operator extends to the completed complex, and the statement follows from Proposition 2.3.2. □

For a commutative algebra A, denote by $\widehat{C}_\bullet(A)$ the complex $\widehat{C}_\bullet(A)_{\Delta,0}$ defined before Proposition 2.4.1 (i.e. the case $I = 0$).

PROPOSITION 2.4.2. *For a Noetherian commutative algebra* A, *the inclusion*

$$C_\bullet(A) \to \widehat{C}_\bullet(A)$$

is a quasi-isomorphism.

PROOF. For any algebra P, any P-module M, and any ideal I of P, denote by $\widehat{M}_I$ the I-adic completion of M.

LEMMA 2.4.3. *Let* P *a Noetherian algebra and let* $J_0 \subset J$ *be two ideals of* P. *Then the map*

$$\widehat{P}_J/J_0\widehat{P}_J \longrightarrow (\widehat{P/J_0})_{J/J_0}$$

is an isomorphism.

PROOF. To prove that the map is injective, note that the right hand side is the same as $(\widehat{P/J_0})_J$ and that completion is right exact (actually exact) on finitely generated modules. To prove injectivity, let $(p_N \in P/J^{N+1})|N \geq 0$ be an element of the kernel, such that $p_0 = 0$. Then $p_N \in J_0/J_0 \cap J^{N+1}$. Lift p_N to elements $\widetilde{p}_N$ of J_0. Then $\widetilde{p}_N - \widetilde{p}_{N+1} \in J_0 \cap J^N$. By Artin-Rees lemma [?], there exists $d \geq 0$ such that $J_0 \cap J^N = J^{N-d}(J_0 \cap J^d)$. Let $x_1, \ldots, x_m$ be generators of J_0. Then

$$\widetilde{p}_{N+1} - \widetilde{p}_N = \sum_{j=1}^{m} a_j^{(N)} x_j$$

where $a_j^{(N)}$ is in J^{N-d}. Put

$$a_j = \sum_{N=1}^{\infty} a_j^{(N)}.$$

Then

$$\sum_{N=1}^{\infty} (p_{N+1} - p_N) = \sum_{j=1}^{m} a_j x_j$$

□

Let $\widehat{\mathcal{B}}_m$ be the completion of $A^{\otimes(m+2)}$ by the ideal J_Δ generated by all $i_k(a) - i_l(a)$ for $0 \leq k, l \leq m+1$. Let J_δ be the ideal generated by all $i_0(a) - i_{m+1}(a)$. Apply the lemma to J_Δ instead of J_0 and J_Δ instead of J. We see that

$$\mathcal{B}_m \otimes_{A^e} A \xrightarrow{\sim} \widehat{C}^{un}(A).$$

We have ring morphisms

$$A \otimes A \longrightarrow A^{\otimes(m+2)} \longrightarrow \widehat{\mathcal{B}}_m$$

(the one to the left given by $i_0 \otimes 1^{\otimes m} \otimes i_m$). Each algebra to the right is flat over its neighbor on the left [?]. Therefore $\widehat{\mathcal{B}}_m$ is flat over $A \otimes A$. Endowed with the differential b', it is a flat resolution of A over A^e because the usual homotopy $a_0 \otimes \ldots \mapsto 1 \otimes a_0 \otimes \ldots$ extends to it. We conclude that $\widehat{C}^{un}_\bullet(A)$, and therefore $\widehat{C}_\bullet(A)$, computes the Hochschild homology of A. □

3. Periodic cyclic homology of finitely generated commutative algebras

For a finitely generated commutative algebra A, choose an algebra of polynomials $P = k[x_1, \dots, x_n]$ and an epimorphism $P \to A$ with the kernel I. Put

$$\widehat{CC}^{per}_{\bullet}(P)_{\Delta,I} = (\widehat{C}_{\bullet}(P)_{\Delta,I}((u)), b + uB)$$

$$\widehat{CC}_{\bullet}(A) = (\widehat{C}_{\bullet}(A)((u)), b + uB)$$

$$(\widehat{\Omega}^{\bullet}_{P/k})_I = \varprojlim \Omega^{\bullet}_{P/k}/I^{N+1}\Omega^{\bullet}_{P/k}$$

THEOREM 3.0.1. *Both morphisms*

$$CC^{per}(A) \longrightarrow \widehat{CC}_{\bullet}(A) \longleftarrow \widehat{CC}^{per}_{\bullet}(P)_{\Delta,I} \longrightarrow ((\widehat{\Omega}^{\bullet}_{P/k})_I((u)), ud)$$

are quasi-isomorphisms.

PROOF. The first map is a quasi-isomorphism by Proposition 2.4.2, the second by Theorem 4.2.2, and the third by Proposition 2.4.1. □

4. Smooth Noetherian algebras

By definition, a commutative Noetherian k-algebra A is smooth if, for any k-algebra C and any ideal I of C such that $I^2 = 0$, the map $\mathrm{Hom}(A, C) \to \mathrm{Hom}(A, C/I)$ is surjective. The class of smooth algebras includes the class of coordinate rings of nonsingular affine varieties over k.

THEOREM 4.0.1. *(see* [?], [?]*) The HKR map from 2.2 defines quasi-isomorphisms of complexes*

$$C_{\bullet}(A) \to (\Omega^{\bullet}_{A/k}, 0)$$

$$CC^{-}_{\bullet}(A) \to (\Omega^{\bullet}_{A/k}[[u]], ud)$$

$$CC_{\bullet}(A) \to (\Omega^{\bullet}_{A/k}[u^{-1}, u]]/u\Omega^{\bullet}_{A/k}[[u]], ud)$$

$$CC^{per}_{\bullet}(A) \to (\Omega^{\bullet}_{A/k}[u^{-1}, u]], ud)$$

PROOF. We will need a few standard results from commutative algebra.

(1) Let $\mathfrak{m}$ be a maximal ideal in A. Since A is smooth, its localization $A_{\mathfrak{m}}$ at $\mathfrak{m}$ is a regular local ring and a basis $x_1, \dots x_n$ for $\mathfrak{m}/\mathfrak{m}^2$ over k is a regular generating sequence for the ideal $\mathfrak{m}A_{\mathfrak{m}}$ in $A_{\mathfrak{m}}$.
(2) A morphism of two A-modules is an isomorphism if its localisations at all maximal ideals are isomorphisms.
(3) Suppose that $x_1, \dots x_n$ is a regular sequence generating an ideal $I \subset A$. The associated Koszul complex is a free A-resolution of A/I of the form

$$(\Lambda^*_A(A^n), d), \tag{4.1}$$

where

$$d = \sum_i \iota_i \otimes x_i,$$

(ι_i is the contraction with the i'th standard basis vector in k^n). In particular,

$$\mathrm{Tor}^A_*(A/I, A/I) \simeq \Lambda^*_{A/I}(I/I^2)$$

as algebras over A.

The proof procedes as follows. Let $\mu : A \otimes A \to A$ denote the multiplication in A and suppose that $\mathfrak{m}$ is a maximal ideal in A. Applying the Koszul complex computation to the ideal $\mu^{-1}(\mathfrak{m})(A \otimes A)_{\mu^{-1}(\mathfrak{m})} \subset (A \otimes A)_{\mu^{-1}(\mathfrak{m})}$, we get

$$\mathrm{Tor}_*^{(A\otimes A)_{\mu^{-1}(\mathfrak{m})}}(A_\mathfrak{m}, A_\mathfrak{m}) \simeq \Lambda^*_{A_\mathfrak{m}}(\mathfrak{m}/\mathfrak{m}^2).$$

Since the Hochshild complex computes the Tor-functor, the fact that the map

$$(C_\bullet(A), b) \to (\Omega^\bullet_{A/k}, 0)$$

given by (2.5) is a quasiisomorphism of complexes follows now from the isomorphisms

$$\mathrm{Tor}_*^{A\otimes A}(A, A)_\mathfrak{m} \leftarrow \mathrm{Tor}_*^{(A\otimes A)_{\mu^{-1}(\mathfrak{m})}}(A_\mathfrak{m}, A_\mathfrak{m}) \to \Omega^*_{A_\mathfrak{m}/k} \simeq (\Omega^*_{A/k})_\mathfrak{m}.$$

To deal with cyclic complexes one notices that the B-boundary map becomes the de Rham differential on $\Omega^\bullet_{A/k}$. As the result we get a morphism of double complexes, say for negative cyclic complex,

$$(\Omega^\bullet_{A/k}[[u]], ud) \to (CC^-_\bullet(A)[[u]], b + uB)$$

which, by above Hochschild homology case, is a quasiisomorphism on the rows and hence quasiisomorphism of double complexes. The claimed result follows. □

5. Finitely generated commutative algebras

For a finitely generated commutative algebra A over k, choose a surjective homomorphism $f : P \to A$ where P is a ring of polynomials. Let I be the kernel of f. Consider the complexes

$$0 \to P/I^{n+1} \xrightarrow{d} \Omega^1_{P/k}/I^n\Omega^1_{P/k} \xrightarrow{d} \dots \xrightarrow{d} \Omega^n_{P/k}/I\Omega^n_{P/k} \xrightarrow{d} 0 \tag{5.1}$$

for all $n \geq 0$. Denote their cohomologies by $H^*_{\text{cris}}(A/k; n)$ Denote by $H^\bullet_{\text{cris}}(A/k)$ the cohomology of the complex $(\widehat{\Omega}^\bullet_{P/k}, d)$, the I-adic completion of the De Rham complex of P. It is well known that the cohomologies above are independent of a choice of P and f.

THEOREM 5.0.1. *(cf. [**234**]). 1. There exists the canonical morphism*

$$\mu : HC_n(A) \to \bigoplus_{i\geq 0} H^{n-2i}_{\text{cris}}(A/k; n - i);$$

if A is smooth then μ is induced by the map from Theorem 4.0.1.

2. If A is a locally complete intersection then μ is an isomorphism.

3. There is the canonical isomorphism

$$\mu : HC^{\text{per}}_\bullet(A) \to H^\bullet_{\text{cris}}(A/k)$$

PROOF. □

5.1. Free commutative resolutions. For a commutative algebra morphism $A \to B$, a free commutative resolution of B over A is a differential graded A-algebra $R_\bullet$ with the differential $\partial : R_\bullet \to R_{\bullet-1}$ together with a morphism of DGAs $R_\bullet \xrightarrow{\epsilon} B$ such that:

(1) $R_\bullet$ is concentrated in nonnegative degrees;
(2) as a graded algebra, R_δ is free commutative over A;
(3) the morphism ϵ is a surjective quasi-isomorphism.

Here we view B as a DGA concentrated in degree zero. A morphism of two resolutions $(R_\bullet, \epsilon)$ and $(R'_\bullet, \epsilon')$ is a morphism f of DGA over A such that $\epsilon' f = \epsilon$. Two morphisms $f, g : (R_\bullet, \epsilon) \to (R'_\bullet, \epsilon')$ are homotopic if **** The following facts about resolutions are standard.

Proposition 5.1.1. (1) *A free resolution of B over A always exists;*
(2) *for every two free resolutions of B over A there is a quasi-isomorphism from one to another;*
(3) *every two morphisms between two resolutions are homotopic.*

Proof. □

5.2. Holomorphic functions. Let M be a complex manifold with the structure sheaf $\mathcal{O}_M$ and the sheaf of holomorphic forms $\Omega^\bullet_M$. If one uses one of the following definitions of the tensor product, then $C_\bullet(\mathcal{O}_M)$, etc. are complexes of sheaves:

$$\mathcal{O}_M^{\otimes n} = \mathrm{germs}_\Delta\, \mathcal{O}_{M^n}; \tag{5.2}$$

$$\mathcal{O}_M^{\otimes n} = \mathrm{jets}_\Delta\, \mathcal{O}_{M^n} \tag{5.3}$$

where Δ is the diagonal.

Theorem 5.2.1. *The map*

$$\mu : f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

defines a quasi-isomorphism of complexes of sheaves

$$C_\bullet(\mathcal{O}_M) \to (\Omega^\bullet_M, 0)$$

and a $\mathbb{C}[[u]]$-linear, (u)-adically quasi-isomorphism of complexes of sheaves

$$CC^-_\bullet(\mathcal{O}_M) \to (\Omega^\bullet_M[[u]], ud)$$

Similarly for the complexes $CC_\bullet$ and CC^{per}.

6. Group rings

Let G be a discrete group. Given $x \in G$ we will use $\mathcal{Z}_x$ to denote its centralizer

$$\mathcal{Z}_x = \{g \in G \mid gx = xg\}$$

$< x >$ to denote the cyclic subgroup of $\mathcal{Z}_x$ generated by x and $k(x)$ will denote denote the ring of "trigonometric series" $k[x, x^{-1}]$. Let $G/Ad(G)$ denote the conjugacy classes in G. We will use notation $\dot{x}$ to denote the conjugacy class of $x \in G$ and, in this section, we will fix the choice of representative x for each conjugacy class of G.

Theorem 6.0.1. [**89**] *Let G be a discrete group.*
(1) *Both Hochschild and cyclic homology splits into direct sum over conjugacy classes of G,*

$$HH_\bullet(k[G]) = \bigoplus_{\dot{x} \in G/Ad(G)} HH_\bullet(k[G])_{\dot{x}}$$
$$HC_\bullet(k[G]) = \bigoplus_{\dot{x} \in G/Ad(G)} HC_\bullet(k[G])_{\dot{x}}$$

and the splitting is compatible with the Connes-Gysin long exact sequence;

(2) *the components of the Hochschild homology of* G *are given by*

$$HH_\bullet(k[G])_{\dot{x}} \simeq \bigoplus_{\dot{x} \in G/Ad(G)} H_\bullet(\mathcal{Z}_x, k);$$

(3) *for an elliptic conjugacy class* $\dot{x}$*,* B *vanishes on* $HH_\bullet(k[G])_{\dot{x}}$ *and*

$$HC_\bullet(k[G])_{\dot{x}} = \bigoplus_i H_{n-2i}(\mathcal{Z}_x, k)$$

(4) *for a non-elliptic conjugacy class* $\dot{x}$*,*

$$HC_\bullet(k[G])_{\dot{x}} = \bigoplus_i H_{n-2i}(\mathcal{Z}_x/ < x >, k)$$

Proof. We will work with the non-normalised Hochschild chains $\widetilde{C}_\bullet(A)$. To begin with note that the both Hochschield and cyclic complexes of $k[G]$ split into a direct sum of complexes

$$\widetilde{C}_\bullet(k[G]) = \bigoplus_{\dot{x}} \widetilde{C}_\bullet(k[G])_{\dot{x}},$$

where $\widetilde{C}_n(k[G])_{\dot{x}}$ is the subspace of $\widetilde{C}_n(k[G])$ with k-linear basis given by:

$$g_0 \otimes g_1 \ldots \otimes g_n, \; g_0 g_1 \ldots g_n \in \dot{x}.$$

In particular, the first statement of the theorem is obvious. Let $B_n(G)$ denote the free $k[G]$-module with generators

$$\{[g_1|g_2|\ldots|g_n] \mid g_1, \ldots, g_n \in G\}.$$

We will use the notation $x^g = g^{-1}xg$, $x, g \in G$. Set

$$\begin{aligned} T : \widetilde{C}_n(k[G]) &\to B_n(G) \\ (g_0 \otimes g_1 \otimes \otimes \ldots \otimes g_n) &\mapsto g_1 g_2 \ldots g_n g_0 [g_1|g_2|\ldots|g_n]. \end{aligned} \tag{6.1}$$

Note the following identities.

$$\begin{aligned}
&Td_0T^{-1}(x^g[g_1|g_2|\ldots|g_n]) = x^{gg_1}[g_2|\ldots|g_n],\\
&Td_iT^{-1}(x^g[g_1|g_2|\ldots|g_n]) = x^g[g_1|g_2|\ldots|g_ig_{i+1}|\ldots|g_n];\ 0 < i < n,\\
&Td_nT^{-1}(x^g[g_1|g_2|\ldots|g_n]) = x^g[g_1|g_2|\ldots|g_{n-1}];\\
&Ts_iT^{-1}(x^g[g_1|g_2|\ldots|g_n]) = x^g[g_1|\ldots|e|\ldots|g_n];\ 0 < i < n,\\
&T\tau T^{-1}(x^g[g_1|g_2|\ldots|g_n]) = x^{gg_1\cdots g_n}[(g_1\ldots g_n)^{-1}x^g|\ldots|g_{n-1}].
\end{aligned} \tag{6.2}$$

We will denote by $B^{\dot{x}}_\bullet(G)$ the simplicial set $B_\bullet(G)$ endowed with the following structure.

$$\begin{aligned}
&d_0'(g[g_1|g_2|\ldots|g_n]) = gg_1[g_2|\ldots|g_n];\\
&d_i'(g[g_1|g_2|\ldots|g_n]) = g[g_1|g_2|\ldots|g_ig_{i+1}|\ldots|g_n]; 0 < i < n;\\
&d_n'(x^g[g_1|g_2|\ldots|g_n]) = g[g_1|g_2|\ldots|g_{n-1}];\\
&s'_i(g[g_1|g_2|\ldots|g_n]) = g[g_1|\ldots|e|\ldots|g_n];\\
&\tau'(g[g_1|g_2|\ldots|g_n]) = gg_1\ldots g_n[(g_1\ldots g_n)^{-1}x^g|\ldots|g_{n-1}].
\end{aligned} \tag{6.3}$$

Ad. 1 Hochschild homology

Note that, by the equations (6.2) and (6.3),

$$Ad(T)(\widetilde{C}_\bullet(k[G])_{\dot{x}}, b) = (k \otimes_{\mathcal{Z}_x} B^{\dot{x}}_\bullet(G), d)$$

where $d = \sum_i (-1)^i d_i$. Since $(B^{\dot{x}}_\bullet G, d)$ is the bar complex of G, the right hand side computes $H_\bullet(\mathcal{Z}_x, k)$, as claimed.

Ad. 2 Cyclic homology.

The elliptic case

Recall that $(k \otimes_{\mathcal{Z}_x} B_\bullet^{\dot{x}}, b)$ computes the $\dot{x}$- component of the Hochschild homology of $k[G]$. But this factorises via $(k \otimes_{k(x)} B_\bullet^{\dot{x}}, b)$, which computes homology of $< x >$. Since the cyclic subgroup $< x >$ of G is finite and k has characteristic zero,

$$H_q(< x >, k) = \begin{cases} k & \text{for } q = 0 \\ 0 & \text{for } q \neq 0. \end{cases}$$

As the result, $B = 0$. Hence

$$u^{-p} B_q^{\dot{x}}(G) \to \begin{cases} u^{-p} k & \text{for } q = 0 \\ 0 & \text{for } q \neq 0. \end{cases}$$

is a quasiisomorphism and

$$(k \otimes_{k(x)} B_\bullet^{\dot{x}}[u^{-1}, u]]/uk[u], b + uB)$$

is a free resolution of $k[u^{-1}, u]]/uk[u]$ over $\mathcal{Z}_x/(x)$. As $B = 0$, the Connes-Gysin exact sequence splits into short exact sequences of the form

$$0 \longrightarrow H_n(G, k) \longrightarrow HC_n(k[G])_{\dot{e}} \longrightarrow HC_{n-2}(k[G])_{\dot{e}} \longrightarrow 0,$$

and the $\dot{x}$-component of the cyclic homology is as claimed.

The non-elliptic case

In this case the Hochschild homology of $k(x)$ has one generator in dimension zero and one in dimension one. The generator in degree zero is 1, while the generator in degree one is the class of $(x) \in B_1^{\dot{x}}$. By (6.3), $B1 = (x)$, hence

$$B : HH_0(k(x)) \to HH_1(k(x))$$

is an isomorphism. But this means that the map

$$u^{-p} B_q^{\dot{x}}(G) \to \begin{cases} k & \text{for } p = 0 \text{ and } q = 0 \\ 0 & \text{otherwise} \end{cases}$$

defines a quasiisomorphism

$$(k \otimes_{k(x)} B_\bullet^{\dot{x}}[u^{-1}, u]]/uk[u], b + uB) \longrightarrow k$$

of $\mathcal{Z}_x/(x)$-modules, i. e. $(k \otimes_{k(x)} B_\bullet^{\dot{x}}[u^{-1}, u]]/uk[u], b + uB)$ is a free resolution of k over $\mathcal{Z}_x/ < x >$.

Since the $\dot{x}$-component of cyclic homology of $k[G]$ is computed by $(k \otimes_{\mathcal{Z}_x} B_\bullet^{\dot{x}}[u^{-1}, u]]/uk[u], b + uB)$, we get

$$\begin{aligned} HC(k[G])_{\dot{x}} &\simeq (k \otimes_{\mathcal{Z}_x} B_\bullet^{\dot{x}}[u^{-1}, u]]/uk[u], b + uB) \\ &\simeq k \otimes_{\mathcal{Z}_x/<x>} \left(k \otimes_{k(x)} B_\bullet^{\dot{x}}[u^{-1}, u]]/uk[u], b + uB)\right) \\ &\simeq k \otimes^{\mathbb{L}}_{\mathcal{Z}_x/<x>} k[u^{-1}, u]]/uk[u]. \end{aligned}$$

□

6.1. Cyclic homology of group rings and group homology, II. Here we give another proof that the group homology with trivial coefficients is a direct summand of the (negative, periodic) cyclic homology of the group algebra. In other words, we prove again part (3) of Theorem 6.0.1 for the case of the conjugacy class of the identity. We use the derived functor/Cuntz-Quillen approach from ***FT***.

Recall that for any algebra A $C_1^{sh}(A) = C_1(A)/bC^2(A)$; $C_0^{sh}(A) = C_0(A)$; $C_j^{sh}(A) = 0$ for $j > 1$;

$$CC_\bullet^{-,sh}(A) = C_\bullet^{sh}(A)[[u]], b + uB) \tag{6.4}$$

as before, we denote the image of $a_0 \otimes a_1$ in $C_1^{sh}(A)$ by $a_0 da_1$. ***Ref***

For any group Γ and for any $\gamma_0,\ \gamma_1 \in \Gamma$, define

$$\gamma_0 D\gamma_1 = \gamma_1^{-1}\gamma_0 d\gamma_1 \tag{6.5}$$

in $C_1^{sh}(k[\Gamma])$. Then the differentials in the short De Rham complex become

$$b(\gamma_0 D\gamma_1) = -\gamma_0 + \mathrm{Ad}_{\gamma_1}^{-1}(\gamma_0);\ \ B(\gamma_0) = \gamma_0^{-1}D\gamma_0 \tag{6.6}$$

For any group Γ, let $(C_\bullet(\Gamma, k[\Gamma]^{\mathrm{Ad}}), b)$ be the standard chain complex of Γ with coefficients in $k[\Gamma]$ on which Γ acts by adjoint representation. (In the notation of the proof of Theorem 6.0.1: it is the direct sum of all $k[B_\bullet^{\dot{x}}(\Gamma)]$ for all conjugacy classes $\dot{x}$). Put

$$C_1^{sh}(\Gamma, k[\Gamma]^{\mathrm{Ad}}) = C_1(\Gamma, k[\Gamma]^{\mathrm{Ad}})/bC_2(\Gamma, k[\Gamma]^{\mathrm{Ad}}); \tag{6.7}$$

$$C_0^{sh}(\Gamma, k[\Gamma]^{\mathrm{Ad}}) = C_0(\Gamma, k[\Gamma]^{\mathrm{Ad}});\ C_j^{sh}(\Gamma, k[\Gamma]^{\mathrm{Ad}}) = 0, j > 1. \tag{6.8}$$

We denote the class of $\gamma_0[\gamma_1]$ in the right hand side of (6.7) by $\gamma_0 D\gamma_1$ We have identified $CC_\bullet^{-,sh}(k[\Gamma])$ with

$$(C_\bullet^{sh}(\Gamma, k[\Gamma]^{\mathrm{Ad}})[[u]], b + uB) \tag{6.9}$$

with b and B given by (6.6).

When Γ has a normal subgroup Γ', the double complex (6.9) has a direct summand $(C_\bullet^{sh}(\Gamma, k[\Gamma']^{\mathrm{Ad}})[[u]], b + uB)$. Finally, when Γ is free,

$$(C_\bullet^{sh}(\Gamma, k)[[u]], b) \to (C_\bullet^{sh}(\Gamma, k[\Gamma']^{\mathrm{Ad}})[[u]], b + uB) \tag{6.10}$$

is a quasi-isomorphism.

Let π be a discrete group. Consider a free simplicial resolution $\Gamma_\bullet \xrightarrow{\epsilon} \pi$. Let $\Gamma'_\bullet = \mathrm{Ker}(\epsilon)$. Let

$$\partial = \sum_{j=0}^{n} (-1)^j d_j$$

be the simplicial differential on $k[\Gamma_\bullet]$.

We have quasi-isomorphisms

$$CC_\bullet^-(k[\pi]) \longleftarrow CC_\bullet^-(k[\Gamma_\bullet]) \longrightarrow CC_\bullet^{-,sh}(k[\Gamma_\bullet]) \xrightarrow{\sim} (C^{sh}(\Gamma_\bullet, k[\Gamma_\bullet]^{\mathrm{Ad}})[[u]], \partial + b + uB), \tag{6.11}$$

the projection

$$(C_\bullet^{sh}(\Gamma_\bullet, k[\Gamma_\bullet]^{\mathrm{Ad}}[[u]], \partial + b + uB) \to (C_\bullet^{sh}(\Gamma_\bullet, k[\Gamma'_\bullet]^{\mathrm{Ad}})[[u]], \partial + b + uB) \tag{6.12}$$

and the quasi-isomorphisms

$$(C_\bullet(\Gamma_\bullet, k)[[u]], \partial + b) \to (C_\bullet^{sh}(\Gamma_\bullet, k))[[u]], \partial + b) \to (C^{sh}(\Gamma_\bullet, k[\Gamma'_\bullet]^{\mathrm{Ad}})[[u]], \partial + b + uB) \tag{6.13}$$

as well as

$$(C_\bullet(\pi, k)[[u]], b) \longleftarrow (C_\bullet(\Gamma_\bullet, k)[[u]], \partial + b) \tag{6.14}$$

Therefore, up to quasi-isomorphism, $(C_\bullet(\pi, k)[[u]], b)$ is a direct summand of $CC^-_\bullet(k[\pi])$.

REMARK 6.1.1. It may be worth mentioning that there also is a double complex

$$C^{sh}_\bullet(\Gamma, k[\Gamma'^{ab}])[[u]], b + uB \tag{6.15}$$

Here $\Gamma'^{ab} = \Gamma'/[\Gamma', \Gamma']$ is the Abelianization. If Γ is free then Γ' is a free group, and Γ'^{ab} is a free Abelian group. One can choose a set of free generators as follows. Choose a system of free generators of Γ. Then Γ' is the group of based loops on the Cayley graph of $\pi = \Gamma/\Gamma'$. Choose a spanning tree on this graph; there is a system of free generators of Γ' indexed by edges not on the tree. Denote them by y_e where e is such an edge. Also denote the free generators of Γ by x_j.

Note that

$$C^{sh}_1(\Gamma, k[\Gamma'^{ab}]) \xrightarrow{\sim} \oplus_j k[\Gamma'^{ab}] D x_j$$

The operator B is well defined but its Dx_j components are not differential operators on $k[y_e^{\pm 1}]$. Rather, they are differential-difference (or integro-differential) operatos. In fact, they are of the form

$$\sum_e \sum_{k=1}^{\ell(e)} \pm \mathrm{Ad}_{\gamma_{e,k}} y_e \frac{\partial}{\partial y_e} \tag{6.16}$$

Here $\gamma_{e,k}$ are elements of Γ representing the paths on the spanning tree that are parts of the loop y_e.

This is worth bearing in mind when one tries to replace $k[\Gamma]$, $k[\Gamma']$, and $k[\Gamma'^{ab}]$ by topological completions. In case of the latter, one would need a certain ring (or space) of smooth (generalized) functions on the *infinite dimensional* torus, and one would need to take extra care to be sure that the differential B of the form (6.16) extends to this completion.

6.2. Periodic cyclic homology of the group algebra of a free group. Here we compute the periodic cyclic homology of $\mathbb{C}[\Gamma]$ when Γ is a free group. We use a general argument due to J. Cuntz. This argument extends to other functors; in particular, it might allow to compute periodic cyclic homology of certain C^∞ completions of the group algebra.

PROPOSITION 6.2.1. *Let Γ be a free group with free generators x_j. Let $\mathbb{C}[X_\Gamma]$ be the algebra generated by invertible x_j subject to relations*

$$(x_j - 1)(x_k - 1) = 0$$

for all $j \neq k$. Then the projection

$$CC^{per}_\bullet(\mathbb{C}[\Gamma]) \to CC^{per}_\bullet(\mathbb{C}[X_\Gamma])$$

is a quasi-isomorphism.

PROOF. We may assume Γ to be finitely generated. Let d be the number of free generators x_j. In addition to the obvious projection $p : \mathbb{C}[\Gamma] \to \mathbb{C}[X_\Gamma]$, consider the morphism

$$q : \mathbb{C}[X_\Gamma] \to M_d(\mathbb{C}[\Gamma])$$

that sends x_j to the diagonal matrix $\sum_{k \neq j} E_{kk} + x_j E_{jj}$. Extend pq and qp to

$$pq : M(\mathbb{C}[X_\Gamma]) \to M(\mathbb{C}[X_\Gamma])$$

and

$$qp : M(\mathbb{C}[\Gamma]) \to M(\mathbb{C}[\Gamma])$$

Both are homotopic to identity with a polynomial homotopy. ***Ref*** Therefore they induce chain homotopic morphisms of periodic cyclic complexes. The statement now follows from Morita invariance. □

COROLLARY 6.2.2. *Let* Γ *be a free group. Then*

$$(C_\bullet^{\mathrm{sh}}(\Gamma, \mathbb{C})((u)), b + uB) \to CC^{\mathrm{per,sh}}(\mathbb{C}[\Gamma])$$

is a quasi-isomorphism.

7. Rings of differential operators

For a C^∞ manifold X of dimension n let $D(X)$ be the ring of differential operators on X. We use the tensor products defined analogously to (11.2), (11.3).

THEOREM 7.0.1. *(*[**86**], [**?**]*). There is a quasi-isomorphism*

$$C_\bullet(D(X)) \to (\Omega^{2n-\bullet}(T^*X), d)$$

which extends to a $\mathbb{C}[[u]]$*-linear,* (u)*-adically continuous quasi-isomorphism*

$$CC_\bullet^-(D(X)) \to (\Omega^{2n-\bullet}(T^*X)[[u]], d)$$

As in 11.3, one also has analogous statements for the cyclic and periodic cyclic complexes.

7.1. Holomorphic differential operators. Let X be a complex manifold of complex dimension n. For the sheaf D_X of holomorphic differential operators, define the Hochschild, cyclic, etc. complexes of sheaves using tensor products analogous to those in 5.2. Let $\pi : T^*X \to X$ be the projection.

THEOREM 7.1.1. [**80**] *There exists an isomorphism*

$$\pi^{-1}C_\bullet(D_X) \to (\Omega^\bullet_{T^*X}[2n], d)$$

in the derived category of the category of sheaves on T^*X*, which extends to a* $\mathbb{C}[[u]]$*-linear,* (u)*-adically continuous isomorphism in the derived category*

$$\pi^{-1}CC_\bullet^-(D_X) \to (\Omega^\bullet_{T^*X}[2n][[u]], d)$$

As in 5.2, similar isomorphisms exist for the cyclic and periodic cyclic complexes.

8. Rings of complete symbols

For a compact smooth manifold X, let $CL(X)$ be the algebra of classical pseudodifferential operators. By $L_\infty(X)$ denote the algebra of smoothing operators (i.e. integral operators with smooth kernel), and put

$$CS(X) = CL(X)/L_\infty(X).$$

We use the projective tensor products.

For any manifold M, denote by $\widehat{\Omega}^*(M \times S^1)$ the space of power series

$$\sum_{\epsilon=0,\ 1;\ i=-\infty}^{N} \alpha_{i,\ \epsilon} z^i dz^\epsilon$$

where α_i are forms on M. Denote by S^*X the cosphere bundle of X.

THEOREM 8.0.1. [?] *There exists a quasi-isomorphism*

$$C_\bullet(CS(X)) \to (\widehat{\Omega}^{2n-\bullet}(S^*X \times S^1), d)$$

which extends to a $\mathbb{C}[[u]]$*-linear,* (u)*-adically continuous quasi-isomorphism*

$$CC^-_\bullet(CS(X)) \to (\widehat{\Omega}^{2n-\bullet}(S^*X \times S^1)[[u]], d)$$

Similarly for $CC_\bullet$, $CC^{\mathrm{per}}_\bullet$. In particular:

COROLLARY 8.0.2.

$$HH_p(CS(X)) = H^{2n-p}(S^*X \times S^1)$$

$$HC_p(CS(X)) = \bigoplus_{i \geq 0} H^{2n-p+2i}(S^*X \times S^1)$$

Combined with the pairing with the fundamental class of $S^*X \times S^1$, the first of the above isomorphisms gives an isomorphism

$$HH_0(CS(X)) = CS(X)/[CS(X), CS(X)] \to \mathbb{C} \tag{8.1}$$

This isomorphism is given by the Wodzicki-Guillemin residue [?]. The above theorem has also a holomorphic version where the ring of complete symbols is replaced by the sheaf of microdifferential operators [**67**].

9. Rings of pseudodifferential operators

THEOREM 9.0.1.

$$HH_0(L_\infty(X)) \simeq \mathbb{C}; \;\; HH_p(L_\infty(X)) = 0, \; p > 0;$$

$$HC_{2p}(L_\infty(X)) \simeq \mathbb{C}; \quad HC_{2p+1}(L_\infty(X)) = 0$$

THEOREM 9.0.2. [?]

$$HH_p(CL(X)) \simeq HH_p(CS(X)) \text{ for } p \neq 1;$$

there is an exact sequence

$$0 \to HH_1(CL(X)) \to HH_1(CS(X)) \to \mathbb{C} \to 0$$

THEOREM 9.0.3. *For all* $p \geq 0$

$$HC_{2p}(CL(X)) \simeq HC_{2p}(CS(X))$$

and there is an exact sequence

$$0 \to HC_{2p+1}(CL(X)) \to HC_{2p+1}(CS(X)) \to HC_{2p}(\mathbb{C}) \to 0$$

Theorems 9.0.2 and 9.0.3 follow from Theorem 9.0.1 and from Wodzicki excision theorem 3.0.2.

The results of the two previous subsections were extended to more general rings of symbols and of pseudodifferential operators by Melrose-Nistor and Benameur-Nistor ([?], [?]). A survey of these results can be found in [?].

10. Noncommutative tori

Let $\{\exp 2\pi i\theta_{ij}\}_{i,j}$ be a $n \times n$ matrix representing a class ω in $H^2(\mathbb{Z}^n, \mathbb{T})$. In particular, $\theta_{ij} = -\theta_{ji} \in \mathbb{R}$.

DEFINITION 10.0.1. $Q(\mathbb{T}^n_\theta)$ denote the $*$-algebra over $\mathbb{C}$ with unitary generators $u_1, \dots, u_n$ subject to relations relation

$$u_i u_j = \exp 2\pi i\theta_{ij} u_j u_i \tag{10.1}$$

We will call $Q(\mathbb{T}^n_\theta)$ the algebra of *rational functions* on the non-commutative n-torus $(\mathbb{T}^n_\theta)$ and will let $\mathcal{Z}$ denote its center.

THEOREM 10.0.2. [**115, ?**] *Let* $(f_1, \dots, f_n)$ *be the standard orthonormal basis of* $\mathbb{C}^n$*. The formula*

$$u_i \otimes \omega \to u_i \otimes f_i \wedge \omega$$

extends uniquely to a derivation $d : Q(\mathbb{T}^n_\theta) \otimes \Lambda^\bullet \mathbb{C}^n \to Q(\mathbb{T}^n_\theta) \otimes \Lambda^{\bullet+1}\mathbb{C}^n$*. The following holds.*

(1) $H_\bullet(Q(\mathbb{T}^n_\theta)) \simeq \mathcal{Z} \otimes \Lambda^\bullet \mathbb{C}^n$
(2) $HC_k(Q(\mathbb{T}^n_\theta)) \simeq \begin{cases} \Lambda^k\mathbb{C}^n, & \textit{for } k < n \\ \mathcal{Z} \otimes \Lambda^n \mathbb{C}^n, & \textit{for } k = n \end{cases}$
(3) $HC^-_\bullet(Q(\mathbb{T}^n_\theta)) \simeq (\mathcal{Z} \otimes \Lambda^\bullet \mathbb{C}^n[[u]], ud)$.
(4) $HC^{per}_\bullet(Q(\mathbb{T}^n_\theta)) \simeq (\Lambda^\bullet \mathbb{C}^n[u^{-1}, u]], ud)$.

PROOF. For $\alpha = \{\alpha_1, \dots, \alpha_n\} \in \mathbb{Z}^n$, we set

$$u^\alpha = u_1^{\alpha_1} \dots u_n^{\alpha_n} \text{ and } u_i u^\alpha u_i^{-1} = < i \mid \alpha > u^\alpha.$$

For simplicity, we will denote the algebra $Q(\mathbb{T}^n_\theta)$ by $\mathcal{A}$. It is easy to check that a free resolution of $\mathcal{A}$ as an $\mathcal{A}$-bimodule has a form

$$(\mathcal{A}^e \otimes \Lambda^\bullet \mathbb{C}^n, t) \xrightarrow{\epsilon} \mathcal{A}, \tag{10.2}$$

where $\epsilon(a\overset{\circ}{b}) = ab$ and

$$t(1 \otimes \omega) = \sum_{i=1}^n (1 - u_i(\overset{\circ}{u}_i)^{-1}) \otimes \iota(f_i)\omega.$$

Here ι_v denotes contraction with the vector $v \in \mathbb{C}^n$. Set $v_\alpha = \sum_i (1- < i \mid \alpha >) f_i$. Tensoring the resolution (10.2) with $\mathcal{A}$ over $\mathcal{A}^e$ produces a direct sum of complexes parametrised by $\alpha \in \mathbb{Z}^n$:

$$\bigoplus_{\alpha \in \mathbb{Z}^n} (\mathbb{C}u^\alpha \otimes \Lambda^\bullet \mathbb{C}^n, 1 \otimes \iota_{v_\alpha}). \tag{10.3}$$

But, since

$$\iota_v(v \wedge \omega) + v \wedge (\iota_v \omega) = ||v||^2 \omega$$

for all $\omega \in \Lambda\mathbb{C}^n$, the complex (10.3) is contractible precisely when $v_\alpha \neq 0$ or, equivalently, when $u^\alpha \notin \mathcal{Z}$. The first statement, about Hochschild homology of $\mathcal{A}$, follows.

The quasi-isomorphism

$$\Phi : \bigoplus_{u^\alpha} (\mathbb{C}u^\alpha \otimes \Lambda^\bullet \mathbb{C}^n, 1 \otimes \iota_{v_\alpha}) \to (C_\bullet(\mathcal{A}), b) \tag{10.4}$$

is given by

$$u^{\alpha}\otimes f_{i_1}\wedge\ldots\wedge f_{i_k}\mapsto u^{\alpha}\sum_{\sigma\in\mathfrak{S}_k}(-1)^{sgn(\sigma)}(u_{i_{\sigma(1)}}\ldots u_{i_{\sigma(k)}})^{-1}\otimes u_{i_{\sigma(1)}}\otimes\ldots\otimes u_{i_{\sigma(k)}}.$$

Restricted to the components with $u^{\alpha}\in\mathcal{Z}$, the Connes-Gysin spectral sequence degenerates at the E_2-term, under the quasi-isomorphism (10.4) d_2 becomes identified with d and the rest of the proof is the same as the proof of the theorem 11.3.1. □

11. Topological algebras

Let us start with some introductory remarks. Suppose that A is a topological algebra with locally convex topology. By fiat, the modules over A will be rquired to have locally convex topology such that the action $A\times M\to M$ of A on M is jointly continuos, i.e. extends to a continuous map from the projective tensor product

$$A\otimes_{\pi}M\to M.$$

A morphism of A-modules is a continuous map $\phi:M\to N$ of A-modules admitting a continuous linear section $\sigma:N\to M$.

All the standard methods of homological algebra work in this case and, in particular, the Hochshild homology coincides with the Tor-functor and can be computed using any projective resolution of A as A-bimodule. An example is the bar resolution

$$\big((A\otimes_{\pi}A^{op})\otimes_{\pi}(A/k)^{\otimes_{\pi}\star},d^{\star}\big)\to A$$

with the standard boundary maps. The reason for the choice of the projective tensor product is to ensure that the boundary maps are continuous. The easiest case is that of nuclear algebras, when projective and injective tensor products coincide. Below some examples.

11.1. H-unitality.

A consequence of the choice above is that, in the topological situation, the homology of a complex (C_*,d) is defined by the quotients of the form

$$\mathrm{Ker}(d)/\overline{\mathrm{Im}(d)},$$

where the closure is taken in the toplogy on the space of chains. In this context, a Frechet algebra is H-unital, if the complex $(C_*(A),b^{'})$ is contractible, i. e. the closure of the range of $b^{'}$ coincides with its kernel. Given this definition, it is not difficult to check that the proof of the excision theorem for Hochschild and cyclic homology extends to this case. As a useful source of examples, let us give the following result.

PROPOSITION 11.1.1. *Suppose that $\mathcal{A}$ is a Frechet algebra admitting a two-sided approximate unit, i. e. a sequence of $u_n\in\mathcal{A}$ such that, for all $a\in\mathcal{A}$*

$$\lim_{n\to\infty}u_n a=\lim_{n\to\infty}au_n=a\ .$$

Then $\mathcal{A}$ is H-unital

PROOF. In fact, it is easy to check that

$$B_0^n(a_1\otimes\ldots a_n)=u_n\otimes a_1\otimes\ldots a_n$$

satisfies

$$\lim_{n\to\infty}[B_0^n,b^{'}]=id$$

□

11.2. Smooth non-commutative torus. Let $\mathcal{S}(\mathbb{Z}^n)$ denote the Schwartz space of functions $a : \mathbb{Z}^n \to \mathbb{C}$ satisfying

$$p_N(a) = \sum_{\alpha \in \mathbb{Z}^n} |\alpha|^N |a_\alpha| < \infty \text{ for all } N \in \mathbb{N}$$

where, as usual, $|\alpha| = \sum_k |\alpha_k|$. $\mathcal{S}(\mathbb{Z}^n)$ with the locally convex topology given by the collection of seminorms $p_N, N \in \mathbb{N}$ is a nuclear Frechet algebra.

Definition 11.2.1. Let $u_1, \dots u_n$ satisfy the relations in the definition 10.0.1. The smooth non-commutative torus is the $C^\infty(\mathbb{T}^n_\theta)$ is the completion of $Q(\mathbb{T}^n_\theta)$ in the topology induced by the seminorms p_N above.

A convenient way of representing the elements of $C^\infty(\mathbb{T}^n_\theta)$ is as the sums of the form

$$\sum_{\alpha \in \mathbb{Z}^n} a_\alpha u^\alpha, \ \{a_\alpha\} \in \mathcal{S}(\mathbb{Z}^n).$$

Theorem 11.2.2. *In the notation of the theorem 10.0.2*

(1) *Hochschild homology of* $C^\infty(\mathbb{T}^n_\theta)$ *is equal to the direct sum of the spaces*

$\mathcal{Z} \otimes \Lambda^\bullet \mathbb{C}^n$ *and*
$(\{c_\alpha\} \in \mathcal{S}(\mathbb{Z}^n) \mid u^\alpha \notin \mathcal{Z}) \,/\, \big(\{c_\alpha\} \mid \{\|\nu_\alpha\|^{-2} c_\alpha\} \notin \mathcal{S}(\mathbb{Z}^n)\big) \otimes \Lambda \mathbb{C}^n.$

(2) *In the case when* $\{\|\nu_\alpha\|^{-1}\}$ *is a multiplier of* $\mathcal{S}(\mathbb{Z}^n)$*, Hochschild, cyclic and periodic cyclic homologies of* $C^\infty(\mathbb{T}^n_\theta)$ *coincide and are equal to* $H^*_{DR}(\mathbb{T}^n)$*. This holds in particular when* θ *is generic, i. e. if*

$$\mathrm{dist}(e^{2\pi i \theta_{ij}}, \{\lambda \mid \lambda^k = 1\}) = O(\frac{1}{k^2}), \ i, j = 1 \dots, n.$$

(3) *For general* θ *the cyclic periodic homology of* $C^\infty(\mathbb{T}^n_\theta)$ *is equal to* $H^*_{DR}(\mathbb{T}^n)$.

Proof. The first two parts of the theorem follow immediately from the fact that the projective resolution of the algebraic torus $Q\mathbb{T}^n_\theta$ constructed in the proof of the theorem 10.0.2 lifts to a projective resolution of the smooth torus.

The rest of the computation requires an explicit choice of the homotopy inverse H of the the quasiisomorphism Φ in 10.4. Instead of the general explicit formulas, let us describe the algorithm giving H. Let us write the elements of the $\mathcal{A} = C^\infty(\mathbb{T}^n_\theta)$-bimodules $\mathcal{A}^e \otimes \overline{\mathcal{A}}^{\otimes l}$ in the form

$$\mathcal{A} \otimes \overline{\mathcal{A}}^{\otimes l} \otimes \mathcal{A}$$

The construction the $\mathcal{A}$ bimodule map is on the induction in k of the terms of the form

$$1 \otimes \omega \otimes u^{\alpha_k} \otimes u^{\alpha_{k+1}} \otimes \dots \otimes u^{\alpha_l} \otimes 1, \text{ where } \omega \in \Lambda^k \mathbb{C}^n$$

It proceeds as follows. Let $\alpha_k = \{t_1, \dots, t_n\}$.

(1) Replace u^{α_k} by

$$\sum_{p=1}^{n} \sum_{q=0}^{t_p} u_1^{t_1} \dots u_{p-1}^{t_{p-1}} u_p^q \otimes f_p \otimes u_p^{t_p - q} \otimes u_{p+1}^{t_{p+1}} \dots u_n^{t_n}$$

for $t_q > 0$ and by

$$-\sum_{p=1}^{n}\sum_{q=t_p-1}^{-1} u_1^{t_1}\dots u_{p-1}^{t_{p-1}}u_p^q\otimes f_p\otimes u_p^{t_p-q}\otimes u_{p+1}^{t_{p+1}}\dots u_n^{t_n}$$

for $t_p < 0$.

(2) Replace an expression of the form

$$1\otimes f_{i_1}\wedge\dots\wedge f_{i_k}(A\otimes f_p\otimes B)\otimes u^{\alpha_{k+1}}\otimes\dots\otimes u^{\alpha_l}\otimes 1$$

by

$$A\,(1\otimes f_{i_1}\wedge\dots\wedge f_{i_k}\wedge f_p\otimes u^{\alpha_{k+1}}\otimes\dots\otimes u^{\alpha_l}\otimes 1)\,(u^{\alpha_{k+1}}\dots u^{\alpha_l})^{-1}Bu^{\alpha_{k+1}}\dots u^{\alpha_l}$$

A direct computation shows that this procedure produces the homotopy inverse

$$H:(\mathcal{A}\otimes\overline{\mathcal{A}}^{\otimes\star}\otimes\mathcal{A},b)\to(\mathcal{A}\otimes\Lambda^\star\mathbb{C}^n\otimes\mathcal{A},t).$$

to Φ. In particular, in the spectral (b,B) sequrnce computing cyclic periodic homology, the second differential $d_2 = HB$ has the form

$$d_2(u^\alpha\otimes\omega)=u^\alpha\otimes\left(\sum_i\sum_{k=0}^{\alpha_i}u^{-\alpha}u_i^{\alpha_i-k}\prod_{l>i}u_l^{\alpha_l}\prod_{l<i}u_l^{\alpha_l}u_i^kf_i\right)\wedge\omega,$$

hence is given by exterior product with the sequence of vectors

$$w=\{w_\alpha\}=\left\{\sum_i\sum_{k=0}^{\alpha_i}u^{-\alpha}u_i^{\alpha_i-k}\prod_{l>i}u_l^{\alpha_l}\prod_{l<i}u_l^{\alpha_l}u_i^kf_i\right\}_{\alpha\in\mathbb{Z}^n}.$$

Note that, since $bB + Bb = 0$, $v_\alpha \perp w_\alpha$. We will need the following two results.

LEMMA 11.2.3.

(1) *For* $\alpha\neq(0,\dots,0)$, $\|v_\alpha\|+\|w_\alpha\|\geq\frac{\pi^2}{8n}\left(\sum_{k=1}^n|\alpha_k|^2\right)^{-1}$;

(2) *Suppose that* $x, y \in \mathcal{S}(\mathbb{Z}^n,\Lambda\mathbb{C}^n)$ *satisfy*

$$w\wedge x=\iota_v y \text{ and } \iota_v x=0.$$

Then there exist x_1 *and* x_2 *in* $\mathcal{S}(\mathbb{Z}^n,\Lambda\mathbb{C}^n)$ *such that*

$$x=w\wedge x_1+\iota_v x_2.$$

Given lemma, let us complete the proof of the theorem. It will be enough to show that the component of the Hochschild homology (see the theorem 11.2.2) given by

$$(\{c_\alpha\}\in\mathcal{S}(\mathbb{Z}^n)\mid u^\alpha\notin\mathcal{Z})\,/\,\big(\{c_\alpha\}\mid\{\|v_\alpha\|^{-2}c_\alpha\}\notin\mathcal{S}(\mathbb{Z}^n)\big)\otimes\Lambda\mathbb{C}^n$$

does not contribute to the (b,B)- spectral sequence computing periodic cyclic homology. But it follows immediately from the second part of the lemma (it gets killed at the third page of the spectral sequence). □

PROOF OF THE LEMMA 11.2.3. This is essentially due to Alain Connes [**111**]. Set $\lambda_k=\prod_j\lambda_{kj}^{\alpha_j}$. Then

$$\|v_\alpha\|^2=\sum_k|1-\lambda_k|^2 \text{ and } \|w_\alpha\|^2=\sum_k\left|\frac{1-\lambda_k^{\alpha_k}}{1-\lambda_k}\right|^2.$$

If we write $\lambda_k = e^{i\theta_k}$ then either

$$\left|\frac{1-\lambda_k^{\alpha_k}}{1-\lambda_k}\right|>1$$

or

$$\alpha_k\theta_k \notin [-\frac{\pi}{2}, \frac{\pi}{2}].$$

Since in the second case $\theta_k > \frac{\pi}{2|\alpha_k|}$ and hence $|1-\lambda_k| > \frac{\pi^2}{8|\alpha_k|^2}$, the inequality in the first part of the lemma follows. We will leave the second part as an exercise for the reader. □

11.3. Smooth functions. For a compact smooth manifold M one can compute the Hochschild and cyclic homology of the algebra $C^\infty(M)$ where the tensor product in the definition of the Hochschild complex is one of the following three:

$$C^\infty(M)^{\otimes n} = C^\infty(M^n); \tag{11.1}$$

$$C^\infty(M)^{\otimes n} = \text{germs}_\Delta\, C^\infty(M^n); \tag{11.2}$$

$$C^\infty(M)^{\otimes n} = \text{jets}_\Delta\, C^\infty(M^n) \tag{11.3}$$

where Δ is the diagonal.

THEOREM 11.3.1. *The map*

$$\mu : f_0 \otimes f_1 \otimes \ldots \otimes f_n \mapsto \frac{1}{n!} f_0 df_1 \ldots df_n$$

defines a quasi-isomorphism of complexes

$$C_\bullet(C^\infty(M)) \to (\Omega^\bullet(M), 0)$$

and a $\mathbb{C}[[u]]$*-linear,* (u)*-adically continuous quasi-isomorphism*

$$CC^-_\bullet(C^\infty(M)) \to (\Omega^\bullet(M)[[u]], ud)$$

Localizing with respect to u*, we also get quasi-isomorphisms*

$$CC_\bullet(C^\infty(M)) \to (\Omega^\bullet(M)[u^{-1}, u]]/u\Omega^\bullet(M)[[u]], ud)$$

$$CC^{\text{per}}_\bullet(C^\infty(M)) \to (\Omega^\bullet(M)[u^{-1}, u]], ud)$$

PROOF. The statement for the Hochschild complex for tensor products (11.2, 11.3), follows from Proposition 2.3.2. Indeed, this proposition implies that the homotopy h extends to these tensor products. For the first tensor product, the following construction, due to Alain Connes (see [?]), provides a resolution of $C^\infty(M)$ which can be used to prove that μ is a quasi-isomorphism. Suppose first that $\chi(M)$ is zero and hence there exists an everywhere non-zero vector field V on M. Fix a metric on M and define a vector field W in a geodesic neighbourhood U of the diagonal $\Delta \subset M \times M$ by

$$(\exp_x(tV), y) \to \exp_x(tV)_*(tV) \oplus 0 \in T_{(\exp_x(tV),y)}(M \times M).$$

Let W_1 be a vector field on $M \times M$ which vanishes on a neighbourhood U_1 of the diagonal and such that $\|W_1\| \geq 1$ on $M \times M \setminus U$, hence, in particular, $\overline{U}_1 \subset U$. Let

$$\pi : M \times M \to M$$

be the projection onto the first factor. The complex

$$(\Gamma(M \times M, \pi^*(\Lambda^\bullet T^*M \otimes \mathbb{C}), \iota_{W+iW_1})$$

is quasiisomorphic to the complex of $(C^\infty(M \times M \times M^{\times\bullet}), b)$ of $C^\infty(M \times M)$-modules and one easily concludes that μ is a quasi-isomorphism. In the case when $\chi(M) \neq 0$, one replaces M by $M \times \mathbb{T}^1$ and uses Künneth formula.

The claim of the theorem for the cyclic complexes follows from the Hochschild-to cyclic spectral sequence. In fact, the HKR map is a quasi-isomorphism at the level of E_1 and therefore is a quasi-isomorphism. □

COROLLARY 11.3.2. *Let $\mathcal{S}(\mathbb{R}^n)$ be the algebra of functions on $\mathbb{R}^k$ satisfying*

$$p_n(f) = \sup(1+|x|^2)^{\frac{n}{2}} \sum_{|\alpha|\leq n} |\partial^\alpha f| < \infty,\ n \in \mathbb{N}.$$

Then

$$\mathsf{HC}^{per}(\mathcal{S}(\mathbb{R}^n)) \simeq \mathbb{C}[n].$$

PROOF. This follows immediately from the short exact sequence

$$0 \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathcal{S}(\mathbb{R}^n) \longrightarrow \mathbb{C}[[x_1,\dots,x_n]] \longrightarrow 0,$$

since $\mathcal{S}(\mathbb{R}^n)$ is unital, $\mathsf{HC}^{per}(\mathcal{S}(\mathbb{R}^n)) \simeq \mathsf{H}_{DR}(S^n)$ and the formal power series algebra $\mathbb{C}[[x_1,\dots,x_n]]$ is homotopic to $\mathbb{C}$. □

For completeness, let us add the following, topological version of Morita invariance of periodic cyclic homology.

THEOREM 11.3.3. *Let $\mathcal{K}^\infty$ denote the Frechet algebra of smoothing operators on $\mathbb{R}^n$, i. e. integral operators with kernels in $\mathcal{S}(\mathbb{R}^{2n})$. Then, for any Frechet algebra $\mathcal{A}$,*

$$\mathsf{HC}^{per}_n(A\otimes\mathcal{K}^\infty) \simeq \mathsf{HC}^{per}_n(A).$$

PROOF. Note first that $\mathcal{K}^\infty$ is nuclear, hence no distinction between projective and injective tensor products. Let $e \in \mathcal{K}^\infty$ be a rank one idempotent in $\mathcal{K}^\infty$ and let

$$\mathsf{Tr} : \mathcal{K}^\infty \to \mathbb{C}$$

denote the standard trace on $\mathcal{K}^\infty$. Since the flip of the two factors in the tensor product $\mathcal{K}^\infty(\mathbb{R}^n)\otimes\mathcal{K}^\infty(\mathbb{R}^n) \simeq \mathcal{K}^\infty(\mathbb{R}^{2n})$ is homotopic to the identity, the two homomorphisms

$$A\otimes\mathcal{K}^\infty \ni A\otimes T \xrightarrow{i} A\otimes T\otimes e \in A\otimes\mathcal{K}^\infty\otimes\mathcal{K}^\infty$$

and

$$A\otimes\mathcal{K}^\infty \ni A\otimes T \xrightarrow{j} A\otimes e\otimes T \in A\otimes\mathcal{K}^\infty\otimes\mathcal{K}^\infty$$

are also homotopic. Since homotopic homomorphisms define the same map on periodic cyclic homology,

$$\mathrm{id}\#\mathsf{Tr}\circ i_* = \mathrm{id}\#\mathsf{Tr}\circ j_*.$$

Since the first map is the identity on $A\otimes\mathcal{K}^\infty$ and the range of the second map is equal to the subspace $\mathsf{HC}^{per}(A\otimes\mathbb{C}e) = \mathsf{HC}^{per}(A)$, the claimed result follows. □

12. Algebroid stacks

12.1. Introduction.

12.2. Definition and basic properties. Let M be a smooth manifold (C^∞ or complex). In by a descent datum for an algebroid stack on M we will mean the following data:

1) an open cover $M = \cup U_i$;

2) a sheaf of rings $\mathcal{A}_i^\bullet$ on every U_i;

3) an isomorphism of sheaves of rings $G_{ij} : \mathcal{A}_j|(U_i \cap U_j) \xrightarrow{\sim} \mathcal{A}_i|(U_i \cap U_j)$ for every i, j;

4) an invertible element $c_{ijk} \in \mathcal{A}_i(U_i \cap U_j \cap U_k)$ for every i, j, k satisfying

$$G_{ij} G_{jk} = \mathrm{Ad}(c_{ijk}) G_{ik} \tag{12.1}$$

such that, for every i, j, k, l,

$$c_{ijk} c_{ikl} = G_{ij}(c_{jkl}) c_{ijl} \tag{12.2}$$

If two such descent data $(U_i', \mathcal{A}_i', G_{ij}', c_{ijk}')$ and $(U_i'', \mathcal{A}_i'', G_{ij}'', c_{ijk}'')$ are given on M, an isomorphism between them is an open cover $M = \cup U_i$ refining both $\{U_i'\}$ and $\{U_i''\}$ together with isomorphisms $H_i : \mathcal{A}_i' \xrightarrow{\sim} \mathcal{A}_i''$ on U_i and invertible elements b_{ij} of $\mathcal{A}_i'(U_i \cap U_j)$ such that

$$G_{ij}'' = H_i \mathrm{Ad}(b_{ij}) G_{ij}' H_j^{-1} \tag{12.3}$$

and

$$H_i^{-1}(c_{ijk}'') = b_{ij} G_{ij}'(b_{jk}) c_{ijk}' b_{ik}^{-1} \tag{12.4}$$

A descent datum for a gerbe is a descent datum for an algebroid stack for which $\mathcal{A}_i = \mathcal{O}_{U_i}$ and $G_{ij} = \mathrm{id}$. In this case c_{ijk} form a two-cocycle in $Z^2(M, \mathcal{O}_M^*)$.

12.3. Categorical interpretation. A datum defined as above gives rise to the following categorical data:

(1) A sheaf of categories $\mathcal{C}_i$ on U_i for every i;

(2) an invertible functor $G_{ij} : \mathcal{C}_j|(U_i \cap U_j) \xrightarrow{\sim} \mathcal{C}_i|(U_i \cap U_j)$ for every i, j;

(3) an invertible natural transformation

$$c_{ijk} : G_{ij} G_{jk}|(U_i \cap U_j \cap U_k) \xrightarrow{\sim} G_{ik}|(U_i \cap U_j \cap U_k)$$

such that, for any i, j, k, l, the two natural transformations from $G_{ij} G_{jk} G_{kl}$ to G_{il} that one can obtain from the c_{ijk}'s are the same on $U_i \cap U_j \cap U_k \cap U_l$.

If two such categorical data $(U_i', \mathcal{C}_i', G_{ij}', c_{ijk}')$ and $(U_i'', \mathcal{C}_i'', G_{ij}'', c_{ijk}'')$ are given on M, an isomorphism between them is an open cover $M = \cup U_i$ refining both $\{U_i'\}$ and $\{U_i''\}$, together with invertible functors $H_i : \mathcal{C}_i' \xrightarrow{\sim} \mathcal{C}_i''$ on U_i and invertible natural transformations $b_{ij} : H_i G_{ij}'|(U_i \cap U_j) \xrightarrow{\sim} G_{ij}'' H_j|(U_i \cap U_j)$ such that, on any $U_i \cap U_j \cap U_k$, the two natural transformations $H_i G_{ij}' G_{jk}' \xrightarrow{\sim} G_{ij}'' G_{jk}'' H_k$ that can be obtained using H_i's, b_{ij}'s, and c_{ijk}'s are the same. More precisely:

$$((c_{ijk}'')^{-1} H_k)(b_{ik})(H_i c_{ijk}') = (G_{ij}'' b_{jk})(b_{ij} G_{jk}') \tag{12.5}$$

The above categorical data are defined from $(\mathcal{A}_i^\bullet, G_{ij}, c_{ijk})$ as follows:

1) $\mathcal{C}_i$ is the sheaf of categories of $\mathcal{A}_i^\bullet$-modules;

2) given an $\mathcal{A}_i^\bullet$-module $\mathcal{M}$, the $\mathcal{A}_j^\bullet$-module $G_{ij}(\mathcal{M})$ is the sheaf $\mathcal{M}$ on which $a \in \mathcal{A}_i^\bullet$ acts via $G_{ij}^{-1}(a)$;

3) the natural transformation c_{ijk} between $G_{ij} G_{jk}(\mathcal{M})$ and $G_{jk}(\mathcal{M})$ is given by multiplication by $G_{ik}^{-1}(c_{ijk}^{-1})$.

From the categorical data defined above, one defines a sheaf of categories on M as follows. For an open V in M, an object of $\mathcal{C}(V)$ is a collection of objects X_i of $\mathcal{C}_i(U_i \cap V)$, together with isomorphisms $g_{ij} : G_{ij}(X_j) \tilde{\to} X_i$ on every $U_i \cap U_j \cap V$, such that

$$g_{ij} G_{ij}(g_{jk}) = g_{ik} c_{ijk}$$

on every $U_i \cap U_j \cap U_k \cap V$. A morphism between objects (X'_i, g'_{ij}) and (X''_i, g''_{ij}) is a collection of morphisms $f_i : X'_i \to X''_i$ (defined for some common refinement of the covers), such that $f_i g'_{ij} = g''_{ij} G_{ij}(f_j)$.

12.4. Algebras associated to a stack, the smooth case. The basic example of the categorical interpretation is a gerbe, where the categories $\mathcal{C}_i$ coincide with the category of *-representations of the algebra $\mathcal{K}^\infty$ of compact operators on a separable Hilbert space H. Recall that all irreducible representations of $\mathcal{K}$ are unitarily equivalent. The definition above reduces in this case to a bundle of compact operators on M.

$$\begin{array}{ccc} \mathcal{K} & \longrightarrow & \mathcal{E} \\ & & \downarrow \\ & & M. \end{array}$$

The associated descent data has the following form.

(1) A finite open cover $\{U_i\}_{i \in I}$ of M;
(2) A family of continuous maps

$$U_i \cap U_j \to G_{ij} \in U(H),$$

where $U(H)$ is the unitary group of H,
(3) the cocycle condition - on $U_i \cap U_j \cap U_k$

$$Ad(U_{ij}) Ad(U_{jk}) Ad(U_{ki}) = id.$$

(4) Since the center of $U(H)$ coincides with the unit circle, the collection

$$\{U_{ij} U_{jk} U_{ki}\}_{i,j,k}$$

defines a cocycle c_{ijk} in $Z^2(M, \mathcal{O}^*_M)$.

Definition 12.4.1. In principle, the corresponding cocycle has values in $C(M)^*$, but it is not difficult to check that it is homotopic to one with values in $C^\infty(M)^*$ and that the corresponding cohomology class is independent of the choices made. The image of c_{ijk} under the boundary map

$$\delta : H^2(M, \mathcal{O}^*_M) \to H^3(M, \mathbb{Z})$$

is the Dixmier-Douady class of the gerbe.

The associative algebra $\Gamma(M, \mathcal{E})$of smooth sections of $\mathcal{E}$ has a Morita equivalent representation of the following form. Set

$$Mat_{tw}(\mathcal{A}) = \{m \in M_{|I|}(C^\infty(M)) \mid supp(m_{ij} \subset U_i \cap U_j)\}$$

with the matrix product twisted by the cocycle c, i.e. of the form

$$m_{ij} \cdot m_{jk} = c_{ijk} m_{ij} m_{jk}.$$

The support condition in this definition restricts usefullness of this construction to the smooth case, hence the following version.

12.5. Matroid algebras. Suppose that $(U_i,\ \mathcal{A}_i,\ G_{ij},\ c_{ijk})$ is a descent datum of an algebroid stack. Denote by $\mathfrak{N}$ the nerve of the covering $\{U_i\}$ of M and, for any simplex $\sigma = (i_0, \dots, i_p)$ of $\mathfrak{N}$ set

$$I_\sigma = \{i_0, \dots, i_p\} \text{ and } U_\sigma = \bigcap_{i \in I_\sigma} U_i.$$

Definition 12.5.1. For $\sigma \in \mathfrak{N}$, $\mathsf{Mat}^\sigma_{tw}(\mathcal{A})$ is the slgebra of finite sums

$$\sum_{i,j \in I_\sigma} a_{ij} e_{ij}$$

where $a_{ij} \in \mathcal{A}_i(U_\sigma)$, $\{e_{i,j}\}$, $i, j \in I_\sigma$ are the matrix units and the product is defined by

$$a_{ij} e_{ij} \cdot b_{kl} e_{kl} = \delta_{jk} a_{ij} G_{ij}(b_{jl}) c_{ijk} e_{ik}.$$

An inclusion of simplexes $\iota : \sigma \to \tau$ induces a (non unital) homomorphism

$$\iota_* : \mathsf{Mat}^\sigma_{tw}(\mathcal{A}) \to \mathsf{Mat}^\tau_{tw}(\mathcal{A})$$

given by

$$\{\iota_*(a)\}_{i,j} = \delta_{\iota(k)i} \delta_{\iota(l)j} a_{kl}.$$

The collection $\mathsf{Mat}^\sigma_{tw}(\mathcal{A})$, $\sigma \in \mathfrak{N}$ is easily seen to admit pullback over refinements, hence the following definition makes sense.

Definition 12.5.2. Given an algebroid stack $\mathcal{A}$, set

$$\mathrm{CC}^-_*(\mathcal{A}) = \left(\lim_{\overrightarrow{\mathfrak{U}}} \prod_{\sigma_0 \subset \sigma_1 \subset \dots \subset \sigma_p} \mathsf{CC}^-_{*-p}\left(\mathrm{Matr}^{\sigma_p}_{\mathrm{tw}}(\mathcal{A})\right), b + uB + \check{\partial} \right)$$

where σ_i run through simplices of $N(\mathfrak{U})$ and

$$\check{\partial} s_{\sigma_0 \dots \sigma_p} = \sum_{k=0}^{p-1} (-1)^k s_{\sigma_0 \dots \widehat{\sigma_k} \dots \sigma_p} + (-1)^p \, i_{\sigma_{p-1}\sigma_p}\left(s_{\sigma_0 \dots \sigma_{p-1}}\right).$$

Note that $\mathsf{CC}^-_*(\mathcal{A})$ is a complex of sheaves on M.

The definition of the Hochschild, cyclic, and periodic cyclic complexes are similar.

Theorem 12.5.3. *If $\mathcal{A}$ is a gerbe on a smooth manifold M, the cyclic periodic cohomology of $\mathcal{A}$ is isomorphic to the twisted de Rham cohomology*

$$(\Omega^*(M), d + H\wedge),$$

where H is a representative of the image of the Dixmier Douady class under the map

$$H^3(M, \mathbb{Z}) \to H^3(M, \mathbb{R}) = H^3_{DR}(M).$$

Sketch of the proof. Suppose that the cover $\{U_i\}_{i \in I}$ of M is good (all intersections of U_i's are contractible). Then the cocycle $c_{ijk}|_{U_\sigma}$ is a coboundary, i.e. there exists a $\mathbb{T}$- valued two cochain d_{ij} such that

$$c_{ijk}|_{U_\sigma} = d_{ij} d_{jk} d_{ik}^{-1}.$$

The map

$$\mathrm{Matr}^\sigma_{\mathrm{tw}} \ni \{T_{ij}\} \mapsto \{T_{ij} d_{ij}\} \in M_{|I|}(C^\infty(U_\sigma))$$

is an isomorphism of algebras and hence

$$(C_* \left(\mathrm{Matr}_{\mathrm{tw}}^{\sigma_p}(\mathcal{A})\right)[u^{-1}, u]], b+uB) \simeq (C_*(M_{|I|}(C^\infty(U_\sigma)))[u^{-1}, u]], b+uB) \simeq (\Omega^*(U_\sigma)[u^{-1}, u]], b+uB).$$

We will leave it as an exercise for the reader to work out the rest of the $(b+uB, \check{\partial})$ spectral sequence. □

13. Bibliographical notes

CHAPTER 11

Characteristic classes

1. Introduction

2. Chern character on K_0

Let A be an associative algebra, as usual over a commutative unital ring k. Recall that the abelian group $K_0(A)$ is defined as the universal abelian group generated by the stable isomorphism classes of idempotents in $M_\infty(A)$ under the addition given by direct sum.

DEFINITION 2.0.1. *Let p be an idempotent in $M_n(A)$. The chern character $ch(p)$ of p is the image of the class of 1 in $CC_0^-(k)$ under the composition*

$$CC_0^-(k) \simeq CC_0^-(M_n(k)) \to CC_0^-(M_n(A)) \simeq CC_0^-(A)$$

where the middle map is induced by the homomorphism

$$\phi_p : k \ni \lambda \mapsto \lambda p \in M_n(A). \tag{2.1}$$

It is easy to see that it extends to a homomorphism

$$ch : K_0(A) \to CC_0^-(A).$$

An easy computation gives the following formula

PROPOSITION 2.0.2. *Let $p \in M_n(A)$ be an idempotent. Then*

$$ch(p) = \left(p + \sum_{n>0} (-1)^n \frac{(2n)!}{(n!)^2} u^n (p - \frac{1}{2}) \otimes p^{\otimes 2n} \right)$$

PROOF. It is easy to check directly that the above formula does indeed define a class in $CC_0^-(A)$. To see that it is indeed the image of the class of $1 \in CC_0^-(k)$, it is enough to check that our formula is true in the case when $A = kp \oplus k(1-p)$. This is easily seen using the splitting exact sequence of negative cyclic homology associated to the split exact sequence

$$0 \longrightarrow k \xrightarrow{\phi_p} A \longrightarrow k \longrightarrow 0.$$

□

3. Chern character on higher algebraic K-theory of algebras

The starting point is a simple observation.

LEMMA 3.0.1. *The map*

$$G^n \ni (g_1, \dots, g_n) \to (g_1 g_2 \dots g_n)^{-1} \otimes g_1 \dots \otimes g_n \in C_n(k[G])$$

extends to a morphism of complexes

$$C_\bullet(BG, \mathbb{Z}) \to C_\bullet(k[G])_{<e>},$$

where the left hand side stands for the singular chain complex of the standard simplicial model of BG *and the right hand side stands for the part of the Hochschild complex of the group ring* $k[G]$ *localised at the cojugacy class of the unit* $e \in G$.

Let us apply it to the case of discrete group $G = GL_n(A)$, where A is an associative algebra. Let

$$\tau : k[GL_n(A)] \to M_n(A) \tag{3.1}$$

be the homomorphism of rings induced by the inclusion $GL_n(A) \subset M_n(A)$.

PROPOSITION 3.0.2. *The composition*

$$C_n(BGL_k(A), \mathbb{Z}) \longrightarrow C_n(k[GL_k(A)])_{<e>} \xrightarrow{\tau_*} C_n(M_k(A)) \xrightarrow{\#} C_n(A), \tag{3.2}$$

where # is the trace map (3.7), extends to a morphism of complexes

$$C_\bullet(BGL_k(A), \mathbb{Z}) \to C^-_\bullet(A).$$

PROOF. Recall that, by the remark ??, B vanishes on the image of $H_n(k[GL_k(A)])_{<e>}$. Another way of formulating this is thet the map

$$C_\bullet(k[GL_k(A)])_{<e>} \to CC_\bullet(k[GL_k(A)])_{<e>}$$

is injective on homology. So suppose that x_n is a cycle representing a class in $H_n(k[GL_k(A)])_{<e>}$. Then uBx_n is zero cycle in cyclic homology, hence it is zero in Hochschild homology, i. e. $uBx_n = buy_{n+1}$ for some uy_{n+1}. Set $x_{n+2} = By_{n+1}$. Then $-u^2x_{n+2}$ is a cycle in Hochschild homology which vanishes in cyclic homology (in fact it is the boundary of $-uy_{n+1} + x_n$), hence $x_{n+2} = by_{n+3}$ for some y_{n+3}. By induction we get a sequence $(x_{n+2k},\ k \geq 0)$ and

$$\tilde{x}_n = \sum_{k \geq 0} u^k x_{n+2k}$$

is a class in negative cyclic homology extending x_n. It is easy to see that the class of $\tilde{x}_n$ in $CC^-_n(k[GL_k(A)])$ is independent of the choices made in this construction. □

Before formulating the definition below, recall that algebraic K-theory of a ring R is defined as follows. One constructs the space $BGl_\infty(A)^+$ by adding a few cells to $BGL_\infty(A)$ - essentially by killing the commutator subgroup of $GL_\infty(A)$ - so that

(1) $H_\bullet(BGL_\infty(A), \mathbb{Z}) \to H_\bullet(BGL_\infty(A)^+, \mathbb{Z})$ is an isomorphism;
(2) $K^{alg}_i(A) = \pi_i(BGL_\infty(A)^+)$.

DEFINITION 3.0.3. *The Chern character*

$$ch_n : K^{alg}_n(A) \to CC^-_n(A)$$

for $n \geq 1$ *is given by the composition*

$$\begin{aligned} K^{alg}_n(A) = \pi_n(BGL_\infty(A)_+) \to H_n(BGL_\infty(A)^+) \simeq \\ \simeq H_n(BGL_\infty(A)) \to CC^-_n(A). \end{aligned}$$

Here the first arrow is the Hurewicz homomorphism and the second arrow is constructed in the proposition 3.0.2 above.

The particular case of K_1 deserves a separate formulation.

PROPOSITION 3.0.4. *Let* $U \in GL_\infty(A)$. *Then*

$$ch_1(U) = \#\left(\sum_{k=0}^{\infty}(-1)^k\frac{(k+1)!}{(2k+1)!}(U^{-1}-1)\otimes U\otimes(U^{-1}\otimes U)^{\otimes k}u^k\right).$$

?

PROOF. Given an invertible element U of $GL_\infty(A)$, the associated Hochschild cycle is of the form $U^{-1}\otimes U$ and it has an extension (unique) to a negative cyclic homology class, easily seen to be of the claimed form.

REMARK 3.0.5. An alternative construction of the chern character follows from the following simple observation.

PROPOSITION 3.0.6. *The map*

$$t:(g_0,\dots,g_n)\mapsto((g_0\dots g_n)^{-1},g_1\dots g_n,g_1\dots,g_{n-1})$$

gives the simplicial model of $BGL(A)$ *above the structure of a cyclic module and the Dennis map from the lemma 3.0.1 is a morphism of cyclic objects.*

□

4. The Chern character in topological and relative K-theory

In this section we will work with a topological algebra A over a field k of characteristic zero. The (mild) assumptions that we make will be listed below.

4.1. The topological and the relative K-theory. Recall that the topological K-theory $K^{top}(A)$ of A is given by the homotopy groups of $BGL_\infty(A)^{top}$, where $GL_\infty(A)^{top}$ is considered as a topological group. We assume that the commutator subgroup of $GL_\infty(A)^{top}$ is in the connected component of identity. Then there is a homotopy fibration

$$BGL_\infty(A)^+\to BGL_\infty(A)^{top}. \tag{4.1}$$

DEFINITION 4.1.1. *The relative K-theory* $K^{rel}_\bullet(A)$ *of* A *is given by the homotopy groups of the fiber of the fibration* (4.1). *In particular there exists a boundary map*

$$K^{top}_{k+1}(A)\xrightarrow{\delta}K^{rel}_k(A)\ ,\ k\geq 1$$

and the relative K-theory fits into a long exact sequence:

$$\cdots\longrightarrow K^{rel}_k(A)\longrightarrow K^{alg}_k(A)\longrightarrow K^{top}_k(A)\dashrightarrow K^{rel}_1(A)\longrightarrow K^{alg}_1(A)\longrightarrow K^{top}_1(A)\longrightarrow 0.$$

In what follows, we will define the completed (negative, periodic) cyclic homology $\widehat{HC}_\bullet(A)$ and the Chern character on the topological K-theory is the map

$$ch: K^{top}_\bullet(A)\to\widehat{HC}^{per}_\bullet(A)$$

We will prove the following

THEOREM 4.1.2. *The restriction of the Chern character from (3.0.3) to the image of the relative K-theory in the algebraic K-theory factors through a map* $K^{rel}_\bullet(A) \to HC_{\bullet-1}(A)$ *and the following diagram is commutative*

$$\begin{array}{ccccccccccc} \dots & \longrightarrow & K^{rel}_k(A) & \longrightarrow & K^{alg}_k(A) & \longrightarrow & K^{top}_k(A) & \longrightarrow & K^{rel}_{k-1}(A) & \longrightarrow & \dots. \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \dots & \longrightarrow & \widehat{HC}_{k-1}(A) & \longrightarrow & \widehat{HC}^-_k(A) & \longrightarrow & \widehat{HC}^{per}_k(A) & \longrightarrow & \widehat{HC}_{k-2}(A) & \longrightarrow & \dots \end{array}$$

The bottom exact sequence here is induced by (the completed version of) the short exact sequence of complexes (1.13).

The rest of this section is devoted to constructing the Chern characters from K^{top} and from K^{top} and to proving the theorem.

4.2. The Dold-Kan correspondence. One can pass from complexes of Abelian groups to simplicial Abelian groups as follows. Let $\mathbf{D}$ be the category whose objects are $[m]$, $m \in \mathbb{Z}$, and the only morphisms are multiples iof the units and of $d : [m] \to [m-1]$ such that $d^2 = 0$. In other words, $\mathbf{D}$-modules are complexes of Abelian groups. Consider the $(\mathbf{D}, \mathbb{Z}\Delta)$-bimodule $C_\bullet(\Delta^*)$ or, in other words, the cosimplicial object in complexes whose value at $[n]$ is the normalized chain complex of the n-simplex.

This is precisely the complex of cosimplicial Abelian groups

$$\dots \xrightarrow{b} \Delta^{op}([\bullet],[1]) \xrightarrow{b} \Delta^{op}([\bullet],[0]) \tag{4.2}$$

with $b = \sum_{j=0}^{n}(-1)^j d_j$ that we used in **??**. One has

$$C_\bullet(\Delta^*) \otimes_{\mathbb{Z}\Delta} A_* = (A_\bullet, b),$$

the standard chain complex of a simplicial Abelian group A_*. Now we would like to use the same bimodule to pass from complexes to simplicial Abelian groups:

$$|C_\bullet|_{DK} = \mathrm{Hom}_{\mathbf{D}}(C_\bullet(\Delta^*), C_\bullet) \tag{4.3}$$

We have

$$\mathrm{Hom}_{\mathrm{Complexes}}((A_*, b), C_\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{Z}\Delta^{op}}(A_\bullet, |C_\bullet|_{DK}) \tag{4.4}$$

4.3. Chern character in topological K theory. For a topological algebra A, let $A(\Delta^n)$ denote ***the space of *appropriate*** functions $\Delta^n \to A$. We get a simplicial algebra $A(\Delta^n)$. The Chern character provides a morphism of bisimplicial Abelian groups

$$\mathbb{Z}BGL_\infty(A(\Delta^*)) \to |CC^-_\bullet(A(\Delta^*))|_{DK} \tag{4.5}$$

and therefore

$$\mathbb{Z}BGL_\infty(A(\Delta^*)) \to |CC^{per}_\bullet(A(\Delta^*))|_{DK} \tag{4.6}$$

4.3.1. *The morphism* $\mathrm{CC}^-_\bullet(\widehat{A}(\Delta^*)) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A)$. We assume that there are:

1) completions $A^{\otimes(n+1)} \to A^{\widehat{\otimes}(n+1)}$;

2) sheaves $C^{\mathrm{sm}}_{\mathbb{R}^m}(A^{\widehat{\otimes}n+1})$ of $A^{\widehat{\otimes}n+1}$-valued functions on $\mathbb{R}^m$ such that:

3) $C^{\mathrm{sm}}_{\mathbb{R}^m}$ contains all polynomial $A^{\widehat{\otimes}n+1}$-valued functions.

We assume that all the maps between $A^{\otimes(n+1)}[t_1, \ldots, t_m]$ that are induced by a)-f) below extend to C^{sm} :

a) partial derivatives $\frac{\partial}{\partial t_j}$;

b) affine maps $L : \mathbb{R}^m \to \mathbb{R}^{m'}$;

c) maps $A^{\otimes n+1} \to A^{\otimes n'+1}$ that are compositions of :

d) permutations of tensor factors;

e) $a_0 \otimes a_1 \otimes \ldots \mapsto a_0 a_1 \otimes \ldots$; $a_0 \otimes a_1 \otimes \ldots \mapsto 1 \otimes a_0 \otimes a_1 \otimes \ldots$;

f) for a bounded polytope K in $\mathbb{R}^k$, the map

$$f \mapsto \int_K f dt_{n+1} \ldots dt_{n+k} : k[t_1, \ldots, t_{n+k}] \to k[t_1, \ldots, t_n]$$

We denote by $C^{\mathrm{sm}}(K)$ the space of restrictions to k of functions from of C^{sm}; by $\Omega^\bullet_{\mathrm{sm}}(K)$, $C^{\mathrm{sm}}(K) \otimes \wedge^\bullet(dt^1, \ldots, dt_m)$; we denote $d = \sum \frac{\partial}{\partial t_j} dt_j$.

We assume all the usual relations to be satisfied, namely: the map L^* as in b) commutes with d; $d^2 = 0$; the maps c), d), e) commute with d and with f); the Stokes formula is true for f) and d; the usual relationship between L^* and f) holds; $d(fc) = df \cdot c + f dc$ for a polynomial f.

We will write

$$\widehat{C}_\bullet(A) = (A^{\widehat{\otimes}(\bullet+1)}, b);\ \widehat{\mathrm{CC}}^-_\bullet(A) = (\widehat{C}_\bullet(A)[[u]], b + uB, \tag{4.7}$$

etc.

For any algebra A and a commutative algebra B, consider the composition of the comultiplication **??** with the HKR map

$$\mathrm{CC}^-_\bullet(A \otimes B) \to \mathrm{CC}^-_\bullet(A) \otimes_{k[[u]]} \mathrm{CC}^-_\bullet(B) \to \mathrm{CC}^-(A) \otimes_{k[[u]]} (\Omega^\bullet_{B/k}, ud)$$

Under the above assumptions,when $B = k[t_0, \ldots, t_n]/(t_0 + \ldots + t_n - 1)$, this extends to a morphism

$$\mathrm{CC}^-_\bullet(\widehat{A}(\Delta^*)) \to (\Omega^\bullet_{\mathrm{sm}}(\Delta^*, \widehat{\mathrm{CC}}^-_\bullet(A)), b + uB + ud) \tag{4.8}$$

where

$$\widehat{A}(\Delta^n) = C^{\mathrm{sm}}(\Delta^n, \widehat{A}) \tag{4.9}$$

Now consider the map

$$\int : \Omega^\bullet_{\mathrm{sm}}(\Delta^*, \widehat{\mathrm{CC}}^-_\bullet(A)) \to \widehat{\mathrm{CC}}^-_\bullet(A))$$

sending a form ω on Δ^n to $u^{-n} \int_{\Delta^n} \omega$. This is a map preserving the degree (because an n-form on Δ^n contributes homological degree $2n$ and u is of degree $-2n$. It also preserves the differential because of the Stokes formula. Composing (4.9) with this map, we get a morphism of complexes

$$\mathrm{CC}^-_\bullet(\widehat{A}(\Delta^*)) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A) \tag{4.10}$$

Now denote

$$\mathbb{K}^{\mathrm{top}}(A) = \mathrm{BGL}_\infty(\widehat{A}(\Delta^*)) \tag{4.11}$$

$$\mathbb{K}^{\mathrm{alg}}(A) = \mathrm{BGL}_\infty(A)^+ \tag{4.12}$$

We get the Chern cahracter in topological K theory given by

$$\mathbb{Z}\mathrm{BGL}_\infty(\widehat{A}(\Delta^*)) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A) \tag{4.13}$$

or

$$\mathrm{ch}\colon \mathbb{K}^{\mathrm{top}}(A) \to |\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A)|_{\mathrm{DK}} \tag{4.14}$$

4.4. The Karoubi regulator. ***When $\mathbb{K}^{\mathrm{top}}(A)$ is an H-space,*** by universality of the + construction, there is the natural morphism

$$\mathbb{K}^{\mathrm{alg}}(A) = \mathrm{BGL}_\infty(A)^+ \to \mathrm{BGL}_\infty(\widehat{A}(\Delta^*)) = \mathbb{K}^{\mathrm{top}}(A)$$

and the natural commutative diagram

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{alg}}(A) & \longrightarrow & \mathbb{K}^{\mathrm{top}}(A) \\ \downarrow & & \downarrow \\ |\mathrm{CC}^-_\bullet(A)|_{\mathrm{DK}} & \longrightarrow & |\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A)|_{\mathrm{DK}} \end{array} \tag{4.15}$$

or

$$\mathbb{K}^{\mathrm{alg}}(A) \to |\mathrm{CC}^-_\bullet(A)|_{\mathrm{DK}} \times_{|\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(A)|_{\mathrm{DK}}} \mathbb{K}^{\mathrm{top}}(A) \tag{4.16}$$

as well as

$$\mathbb{K}^{\mathrm{rel}}(A) \to |\widehat{\mathrm{CC}}_\bullet(A)[1]|_{\mathrm{DK}} \tag{4.17}$$

(indeed, if we replace CC^- by $\widehat{\mathrm{CC}}^-$ in the bottom left corner of (4.15) then the fiber of the bottom line becomes the right hand side of (**??**)

5. Karoubi-Villamayor K theory

Recall the definition of the Karoubi-Villamayor K theory of a ring A. For $n \ge 0$, define

$$A[\Delta^n] = A[t_0, \ldots, t_n]/(t_0 + \ldots + t_n - 1) \tag{5.1}$$

These rings form a simplicial ring in the usual way: informally, the action of morphisms from Δ^{op} is induced from the action Δ on simplices Δ^n. More precisely, all d_j and s_j act by identity on A; on the generators t_k they act by

$$d_j : t_k \mapsto t_k,\ k < j;\ t_j \mapsto 0;\ t_k \mapsto t_{k-1},\ k > j; \tag{5.2}$$

$$s_j : t_k \mapsto t_k,\ k < j;\ t_j \mapsto t_j + t_{j+1};\ t_k \mapsto t_{k+1},\ k > j. \tag{5.3}$$

Define

$$\mathbb{K}^{\mathrm{KV}}(A) = \mathrm{BGL}(A[\Delta^*]) \tag{5.4}$$

Since the above is an H-space, there is no need for the plus construction. By the universal property of the plus construction, there is a natural morphism

$$\mathbb{K}^{\mathrm{alg}}(A) \to \mathbb{K}^{\mathrm{KV}}(A) \tag{5.5}$$

We get the following version of (4.17)

$$\mathrm{fiber}(\mathbb{K}^{\mathrm{alg}}(A) \to \mathbb{K}^{\mathrm{KV}}(A)) \to |\mathrm{CC}_\bullet(A)[1]|_{\mathrm{DK}} \tag{5.6}$$

5.1. Karoubi-Villamayor K theory of a filtered ring. Given a filtered ring A with an decreasing filtration Fkj, $k \geq 0$, one can define a refinement of the above definitions as follows. Choose $0 \leq j \leq n$. Put

$$A[\Delta^n]_F = \{\sum_\alpha t^\alpha F^{|\alpha|} A\} \subset A[\Delta^n]$$

where $\alpha = (k_0, \ldots, \widehat{k_j}, \ldots, k_n)$, $t^\alpha = (t_0^{k_0}, \ldots \widehat{t_j^{k_j}}, \ldots, t_n^{k_n})$, and $|\alpha| = \sum_{l \neq j} k_l$. It is easy to see that $A[\Delta^n]_F$ is a subring of $A[\Delta^n]$ that does not depend on j. We define

$$\mathbb{K}^{\mathrm{KV},F}(A) = \mathrm{BGL}(A[\Delta^*]_F)^+ \tag{5.7}$$

5.2. Relative Karoubi-Villamayor K theory of an ideal. Let R be a ring with an ideal J, and let $F^m = J^m$ be the filtration by powers of J. Let

$$\mathbb{K}^{\mathrm{KV},J}(R) = \mathbb{K}^{\mathrm{KV},F}(R) \tag{5.8}$$

with respect to this filtration. Define

$$\mathbb{K}^{\mathrm{KV}}(R, J) = \mathrm{fiber}(\mathbb{K}^{\mathrm{KV},J}(R) \to \mathbb{K}^{\mathrm{alg}}(R/J)) \tag{5.9}$$

***Seems true: this is the same as

$$\mathbb{K}^{\mathrm{KV}}(R, J) = \mathrm{fiber}(\mathrm{BGL}(R[\Delta^*]_F) \to \mathrm{BGL}(R/J)) \tag{5.10}$$

where, as above, F is the filtration by powers of J.

6. Relative K theory and relative cyclic homology of a nilpotent ideal

Let J be a *nilpotent* ideal in a ring R.

LEMMA 6.0.1. *The simplicial set* $\mathbb{K}^{\mathrm{KV}}(R, J)$ *is contractible.*

PROOF. Define simplicial subsets

$$X_m = \mathrm{Matr}(\{\sum_{|\alpha| \leq m} J^{|\alpha|} t^\alpha\}) \cap \mathrm{Ker}(\mathrm{GL}(R[\Delta^*]_F) \to \mathrm{GL}(R/J))$$

We have

$$X_N = \mathrm{Ker}(\mathrm{GL}(R[\Delta^*]_F) \to \mathrm{GL}(R/J))$$

if $J^N = 0$. Also, there are fibrations ***EXPAND*** for $m > 0$

$$X_{m-1} \to X_m \to Y_m$$

where $Y_m = \mathrm{Matr}(\{\sum_{|\alpha|=m} J^m t^\alpha\})$ with the simplicial structure that we will describe next.

Observe that formulas (5.2) and (5.3) define a simplicial structure on $W_n = \mathbb{Z}t_0 + \ldots + \mathbb{Z}t_n$. Let

$$c_n = t_0 + \ldots + t_n$$

All the morphisms in Δ^{op} sent c_* to c_*. Let

$$V_n = W_n / c_n.$$

The W_n, resp. V_n, $n \geq 0$, form a simplicial $\mathbb{Z}$-module W_*, resp.V_*.

Identify $\{\sum_{|\alpha|=m} J^m t^\alpha\}$ with $\mathrm{Sym}^m(W_*) \otimes_{\mathbb{Z}} J^m$. This gives rise to the simplicial structure on Y_m.

It remains to show that all Y_m with $m \geq 2$ are contractible, as is X_1. In fact, W_* is contractible, and V_* has one nonzero homotopy group $\pi_1 \xrightarrow{\sim} \mathbb{Z}$. Therefore by Künneth formula

$$\pi_k(\mathrm{Sym}^m(V_*)) \xrightarrow{\sim} \mathrm{Sym}^m \pi_1(V_*) \xrightarrow{\sim} \mathrm{Sym}^m \mathbb{Z}[1] = 0$$

for $m > 1$. ***Improve*** □

Define

$$\mathbb{K}^{\mathrm{alg}}(R, J) = \mathrm{fiber}(\mathbb{K}^{\mathrm{alg}}(R) \to \mathbb{K}^{\mathrm{alg}}(R/J)) \tag{6.1}$$

$$\mathrm{CC}_\bullet(R, J) = \mathrm{fiber}(\mathrm{CC}_\bullet(R) \to \mathrm{CC}_\bullet(R/J)) \tag{6.2}$$

and similarly for other types of cyclic complexes.

We get a commutative diagram

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{alg}}(R, J) & \longrightarrow & \mathbb{K}^{\mathrm{KV}}(R, J) \\ \downarrow & & \downarrow \\ |\mathrm{CC}^-_\bullet(R, J)|_{\mathrm{DK}} & \longrightarrow & |\mathrm{CC}^-_\bullet(R[\Delta]_F)|_{\mathrm{DK}} \end{array}$$

(Here, again, F is the filtration by powers of J).

7. The characteristic classes of Goodwillie and Beilinson

****Double check* Now, for any ring R with an ideal J, we have the analog of (4.10)

$$\mathrm{CC}^-_\bullet(R[\Delta]_F) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(R, J)_{\mathbb{Q}} \tag{7.1}$$

where the completion on the right is the J-adic completion.

Definition 7.0.1. *Let r be a positive integer. An ideal J of a ring R is a r-pd ideal if*

1) the J-adic completion $\widehat{R}^{\otimes n}$ has no $\mathbb{Z}$-torsion, and

2) for any n there is m such that the J-adic completion $\widehat{J}^{\otimes n}$ is inside $n!^r \widehat{J}^{\otimes m}$, and $m \to \infty$ as $n \to \infty$.

Lemma 7.0.2. *Let J be a 2-pd ideal. The map* (7.1) *factors through*

$$\mathrm{CC}^-_\bullet(R[\Delta]_F) \to \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(R, J) \tag{7.2}$$

Proof. The map (7.1) is the sum of maps of the following form. Fix n and recall that $t = (t_0, \ldots, \widehat{t_j}, \ldots, t_n)$ for some j. Start with a Hochschild chain $a_0 t^{\alpha_0} \otimes \ldots \otimes a_N t^{\alpha_N}$ where $a_k \in J^{|\alpha_k|}$ and at least one a_k is in J. Subdivide the monomials t^{α_k} into $n+1$ groups and then multiply the members of each group. We get monomials $t^{\beta_0}, \ldots, t^{\beta_n}$ where $\sum_{k=0}^N |\alpha_k| = \sum_{l=0}^n |\beta_l|$. Then compute

$$\int_{\Delta^n} t^{\beta_0} dt^{\beta_1} \ldots dt^{\beta_n}. \tag{7.3}$$

Then subdivide the a_k into segments in cyclic order; multiply elements of each segment. We get a new Hochschild chain $b_0 \otimes \ldots \otimes b_m$ with $b_j \in J^{r_j}$ and $\sum r_j \geq \sum |\alpha_k|$. Multiply this chain by (7.3). We obtain the general form of a component of (7.1).

Let us assume that $t = (t_0, \ldots, t_{n-1})$. The integral (7.3) is of the form

$$\int_{\Delta^n} t_0^{m_0} \ldots t_{n-1}^{m_{n-1}} dt_0 \ldots dt_{n-1} = \frac{1}{n!} \frac{m_0! \ldots m_{n-1}!}{(m_0 + \ldots + m_{n-1} + n)!} \tag{7.4}$$

where $\sum_{j=0}^{n-1}(m_j + 1) = \sum_{k=0}^{N} |\alpha_j|$ and $m_j \geq 0$ for all j. Therefore it is an integer times $\frac{1}{(\sum |\alpha_k|)!^2}$. □

As a consequence, we have

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{alg}}(R, J) & \longrightarrow & \varprojlim_N \mathbb{K}^{\mathrm{KV}}(R/J^N, J/J^N) \\ \downarrow & & \downarrow \\ |\mathrm{CC}^-_\bullet(R, J)|_{\mathrm{DK}} & \longrightarrow & |\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(R, J)_{\mathbb{Q}}|_{\mathrm{DK}} \end{array}$$

for any R and J, and

$$\begin{array}{ccc} \mathbb{K}^{\mathrm{alg}}(R, J) & \longrightarrow & \varprojlim_N \mathbb{K}^{\mathrm{KV}}(R/J^N, J/J^N) \\ \downarrow & & \downarrow \\ |\mathrm{CC}^-_\bullet(R, J)|_{\mathrm{DK}} & \longrightarrow & |\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(R, J)|_{\mathrm{DK}} \end{array}$$

when J is a 2-pd ideal. Note that the top right corners are contractible ***a few more words***. After completing the bottom left corners, we get the characteristic classes of Goodwillie

$$\mathbb{K}^{\mathrm{alg}}(R, J) \to |\widehat{\mathrm{CC}}_{\bullet-1}(R, J)_{\mathbb{Q}}|_{\mathrm{DK}} \tag{7.5}$$

for any R and J, and of Beilinson

$$\mathbb{K}^{\mathrm{alg}}(R, J) \to |\widehat{\mathrm{CC}}_{\bullet-1}(R, J)|_{\mathrm{DK}} \tag{7.6}$$

for any R and any 2-pd ideal J.

8. K-homology cycles and pairing to topological K-theory

This section is mainly for the notation and we refere the reader to the textbooks on the subject for analytic details. All the way through this section we will work with will unital, $\mathbb{Z}72\mathbb{Z}$-graded C*-algebra.

8.1. K-homology.

Definition 8.1.1. Let A be unital, $\mathbb{Z}/2\mathbb{Z}$-graded C*-algebra. An even (resp. odd) K-homology cycle of A is the following data

(1) A $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space and an even $*$-homomorphism $\rho : A \to \mathcal{L}(H)$, where $\mathcal{L}(H)$ stands for the algebra of bounded operators on H with grading induced by the grading on H;
(2) An odd (resp. even) bounded operator $F \in \mathcal{L}(H)$ such that $F^2 - 1$, $F - F^*$ and $[F, \rho(a)]$ are compact operators for all $a \in A$

The following theorem of Kasparov allowes a free passage between even and odd K-homology classes.

THEOREM 8.1.2 (Formal Bott periodicity, Kasparov [**?**]). *Let $\mathcal{C}_1$ denote the complexified Clifford algebra of one dimensional real Euclidean space,π its two dimensional spin representation on $\mathbb{C}^{(1|1)}$ and γ the grading operator on $\mathsf{C}^{(1|1)}$. Then the map*

$$K^*(A) \ni [(H, \rho, F)] \to [(H \otimes \mathbb{C}^{(1|1)}, \rho \otimes \pi, F \otimes \gamma)] \in K^{*+1}(A)$$

is an isomorphism.

We will need the unbounded version of K-homology cycles.

DEFINITION 8.1.3. Let A be unital, $\mathbb{Z}/2\mathbb{Z}$-graded C*-algebra. An unbounded even (resp. odd) K-homology cycle of A is the following data

(1) A $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space and an even $*$-homomorphism $\rho : A \to \mathcal{L}(H)$, where $\mathcal{L}(H)$ stands for the algebra of bounded operators on H with grading induced by the grading on H;
(2) An odd (resp. even) self-adjoint operator D with compact resolvent and such that $[D, \rho(a)]$ are bounded for a dense subalgebra $\mathcal{A} \in A$.
(3) D can be always chosen to be invertible and the corresponding unbounded K-homology cycle defines a bounded K-homology cycle by setting $F = D|D|^{-1}$.

A cycle (H, ρ, F) over A is trivial if moreover $F = F^*$ and all the commutators $[\rho(a), F]$ vanish. Twe cycles are equivalent if, up to an addition of trivial cycles, they are homotopic. The corresponding equivalence classes form the K-homology groups $K^*(A)$ and there exists a pairing

$$K_*(A) \times K^*(A) \to \mathbb{Z}$$

we will call the Chern character

$$Ch : K^*(A) \to Hom(K_*(A), \mathbb{Z}).$$

Thre following observation is fairly convenient.

LEMMA 8.1.4. *Suppose that (ρ, H, F) is a bounded K-homology cycle. A compact perturbation$G$ of F satisfies $G^2 = 1$*

PROOF. $P = \frac{1+F}{2}$ is an idempotent in $\mathcal{L}(H)/\mathcal{K}(H)$. Cince $K_1(\mathcal{K}) = 0$, the six term exact sequence in K theory associated to the exact sequence

$$0 \longrightarrow \mathcal{K}(H) \longrightarrow \mathcal{L}(H \longrightarrow \mathcal{L}(H)/\mathcal{K}(H) \longrightarrow 0$$

gives a lift $P_1 \in \mathcal{L}(H)$ such that $P_1^2 = P_1$. $G = 2P_1 - 1$ provides the required perturbation of F. □

REMARK 8.1.5. For future reference, let us describe the pairing of K-homology cycles to K-theory.

Let (π, H, F) be an even cycle over A.

The grading γ on H corresponds to splitting the $H = H_+ \oplus H_-$ and, with respect to this splitting,

$$F = \begin{pmatrix} 0 & F_- \\ F_+ & 0 \end{pmatrix} \text{ and, for an idempotent } e \in A,\ \pi(e) = \begin{pmatrix} \pi_+(e) & 0 \\ 0 & \pi_-(e) \end{pmatrix}.$$

F_+ is a Fredholm operator from the range of $e_+ = \pi_+(e)$ to the range of $e_- = \pi_-(e)$ and

$$< [F], [e] >= index(e_- F_+ e_+)$$

All of this stabilises, i. e. we can replace A by $A \otimes \mathcal{K}$ and $[e]$ by a difference classes $[e]-[f]$. In the unbounded picture of K-homology, the pairing with K_0 has the same form:

$$< [D], [e] >= \mathrm{index}(e_- D_+ e_+ : e_+ H \to e_- H)$$

The odd case.

Let P denote a projection which is a compact perturbation of $\frac{1+F}{2}$ - such one always exists - and let u be a unitary in A representing a class in $K_1(A)$. The operator

$$P\rho(u)P : PH \to PH$$

is Fredholm and the pairing of K-homology to K-theory has the form

$$< [u], [F]) >= \mathrm{Index}(P\rho(u)P) \in \mathbb{Z}.$$

Suppose that (A, π, H, D) is an unbounded cycle. Then the pairing of the associated bounded cycle is given by the spectral flow of the path

$$[0, 1] \ni t \to D_t = (1 - t)D + t\rho(u)^* D\rho(u)$$

i. e. the number (with multiplicity) of eigenvalues of D_t that cross any fixed $\lambda \notin \sigma(D_0) \cup \sigma(D_1)$ from below to above as t runs from zero to one.

8.2. Infinite cochains and the Chern character on K-homology. Suppose that A is a unital C^*-algebra and (ρ, H, F) is a K-homology cycle on A. Under some regularity conditions, the pairing of the K-homology class $[(\rho, H, F)]$ with $K_*(A)$ (after tensoring with $\mathbb{Q}$) factorises through the chern character on K-theory. There are two caveats to this statement.

(1) In general, the periodic cyclic homology of a C*-algebra tends to reduce to traces, hence one neede to choose a (non-canonical) Banach algebra $\mathcal{A}$ and a continuous homomorphism

$$\mathcal{A} \to A$$

which induces an isomorphism on the topological K-theory. The particular choice of $\mathcal{A}$ might depend on the class of the cycle $[(H, \rho, F)] \in K_*(A)$.

(2) The dual of cyclic periodic homology consists of collections of cochains $\phi_k \in C^k(\mathcal{A}) = \mathrm{Hom}(C_k(\mathcal{A}), \mathbb{C}),\ k \in \mathbb{N}$, which satisfy the identity

$$b\phi_k + B\phi_{k+2} = 0$$

and are non-zero for finite number of indices k. To detect most of the classes in the periodic cyclic cohomology, one needs in general infinite nonvanishing collections of such cochains. For those to define linear functionals on periodic cyclic homology, the periodic cyclic chains in question have to satisfy some growth conditions. One of those is given by the following definition, due to Alain Connes.

Definition 8.2.1. Suppose that A is a Banach algebra. The entire cyclic periodic complex of A is given by

$$(CC^e_*(A), b + uB),$$

where the entire chains are the complection of $\oplus_n A \otimes \overline{A}^{\otimes n}$ with respect to the seminorms

$$\|\sum_n \omega_n\|_k = \sum_n \frac{k^n \|\omega_n\|_\pi}{(n!)^{\frac{1}{2}}},\ \omega_n \in A \otimes \overline{A}^{\otimes n},\ k \in \mathbb{N}.$$

REMARK 8.2.2. The name comes from the fact that, for an idempotent $e \in A$, the infinite chain $\mathsf{ch}(ze)$ (see the proposition 2.0.2) is an entire function of $z \in \mathbb{C}$.

DEFINITION 8.2.3. Let A be a C*-algebra, $\mathcal{A} \to A$ a continuous map inducing an isomorphism on K-theory and $x \in K_*(A)$. Suppose that there exists an entire cyclic cocycle $\mathsf{Ch}(x)$ making commutative the following diagram.

$$
\begin{array}{ccccc}
K_*(A) & \xleftarrow[\simeq]{} & K_*(\mathcal{A}) & \xrightarrow{<x,\cdot>} & \mathbb{Z} \\
 & & \Big\downarrow{\scriptstyle \mathsf{ch}} & & \Big\downarrow \\
 & & CC^{e}_{*}(\mathcal{A}) & \xrightarrow[\mathsf{Ch}(x)]{} & \mathbb{C}.
\end{array}
$$

$\mathsf{Ch}(x)$ is called the Chern character of the homology class x. We will often write $\mathsf{Ch}(F)$ or $\mathsf{Ch}(D)$ when the corresponding class x is represented by a concrete bounded or unbounded K-cycle.

9. Chern character of finitely summable Fredholm modules

DEFINITION 9.0.1. A Fredholm module over A is a bounded K-homology cycle (ρ, H, F) where $F^2 = 1$. It is p-summable, if

$$[F, \rho(a)] \in \mathcal{L}^p \text{ for all}$$

for all a in a dense subalgebra $\mathcal{A}$ of A which is closed under holomorphic functional calculus. Here $\mathcal{L}^p$ denotes the Schatten ideal of bounded operators T on H such that $|T|^p$ is trace class. Below we will typically supress ρ from the notation and use γ for the grading operator on H.

Given a $\mathbb{Z}/2\mathbb{Z}$-graded Hilbert space H, Tr_s denotes the graded trace on $\mathcal{L}(H)$, i. e.

$$\mathsf{Tr}_s(x) = \mathsf{Tr}(\gamma x) \text{ for } x \in \mathcal{L}^1.$$

Then

LEMMA 9.0.2.

$$\tau_0(x) = \frac{1}{2}\mathrm{Tr}_s(F[F, x]) \tag{9.1}$$

is a graded trace on the subalgebra $\mathcal{L}^{1,F}(\mathcal{H})$ *of operators* x *on* $\mathcal{H}$ *such that* $[F, x] \in \mathcal{L}^1$.

PROOF. Suppose for example that x and y in $\mathcal{L}^1$ are even. Then

$$2\tau_0(xy) = \mathsf{Tr}_s(F([F, x]y + x[F, y])) = \mathsf{Tr}_s(-[F, x]Fy + Fx[F, y]) =$$
$$\mathsf{Tr}_s(Fy[F, x] - x[F, y]F) = \mathsf{Tr}_s(Fy[F, x] + xF[F, y]) = \mathsf{Tr}_s(Fy[F, x] + F[F, y]x) = 2\tau_0(yx),$$

We used the fact that F is odd and the identity

$$F[F, x] + [F, x]F = [F^2, x] = 0.$$

The case when one or both of x and y are odd follow the same pattern. □

PROPOSITION 9.0.3. *Suppose that* (ρ, H, F) *is a* $p+1$*-summable Fredholm module over* A*. The cochain*

$$\tau_p(a_0, \ldots, a_p) = \frac{(-1)^p}{2^{p+1}}\mathrm{Tr}_s(\gamma F[F, a_0] \ldots [F, a_p]). \tag{9.2}$$

is a cocycle of the cyclic complex $C^\bullet_\lambda(\mathcal{A})$*. Here, in the odd (ungraded) case,* H *has trivial grading and* $\mathsf{Tr}_s = \mathsf{Tr}$.

PROOF. *Even case*

Let (ρ, H, F) be an even K-homology cycle. Since $F^2 = 1$, $ad(F)^2 = 0$, hence τ_p is a composition

$$\mathcal{A}^{\otimes(p+1)} \to \Omega^p_{\mathcal{A}} \to \mathcal{L}^{1,F}(\mathcal{H}) \xrightarrow{\tau_0} \mathbb{C}$$

of

$$(a_0, \dots, a_p) \mapsto a_0 da_1 \mapsto \dots da_p \mapsto a_0[F, a_1] \dots [F, a_p]$$

with τ_0. By the above lemma 9.0.2, τ_0 is a closed graded trace on $\Omega(\mathcal{A})$ and hence τ_p is a cyclic cocycle. Note that, for $\mathcal{A}$ even, $\tau_p = 0$ for p odd.

Odd case

Suppose that (ρ, H, F) is an K-homology class, so F is even. By Kasparov formal Bott periodicity, the theorem 8.1.2,

$$(\rho \otimes \pi, H \otimes \mathbb{C}^{(1|1)}, F \otimes \epsilon)$$

is an even K-homology class over $A \otimes \mathcal{C}^1$ Let $\tilde{\tau}_p$ be its Chern character. Then

(1) $\tilde{\tau}_p = \tau_p \# tr_s$, where tr_s is the standard graded trace on the Clifford algebra $\mathcal{C}^1$.
(2) If $\mathcal{A}$ is even, τ_p vanishes for even p.

□

REMARK 9.0.4. An alternative proof that τ_p is a cocycle for F odd (even K-homology class) and A even follows from the fact that, since $ad(F)^2 = 0$, $\wedge^n ad(F)$ is a reduced cyclic homology class of the algebra $\mathcal{A}$ generated by F and $\mathcal{A}$ and hence

$$\tau_n = \chi_{\tau_0}(\wedge^n ad(F))$$

is a cyclic cocycle. Moreover, the image of $\wedge^n ad(F)$ under the boundary map

$$\overline{C}_\lambda(\mathcal{A}) \to C^{per}_{*+1}(\mathbb{C})$$

coincides with $ch(1)$. In other words, $\tau_{2n} = \tau_0$ as cyclic periodic cocycles on $\mathcal{L}^{1,F}(\mathcal{H})$.

THEOREM 9.0.5 ([**111**]). *Suppose that* (ρ, H, F) *is a* $p+1$*-summable Fredholm module over* A. *Then its Chern character*

$$Ch[F] : K_*(A) \to \mathbb{Z}$$

has the following form.

(1) *For* p *and* (ρ, H, F) *even and* $e \in M_n(\mathcal{A})$, $e^2 = e$,
$$< [F], [e] > = < \tau_p, ch(e) > .$$
(2) *For* p *and* (ρ, H, F) *odd and* $u \in M_n(\mathcal{A})$, u *invertible,*
$$< [F], [u] > = < \tau_p, ch(u) >$$

PROOF. Let us sketch the proof for the case $p = 0$. We will use the notation from the remark 8.1.5. As usual, given Fredholm operator $G : H_1 \to H_2$ and its inverse R modulo trace class operators, the index of G can be computed using the formula

$$Tr(1_{H_2} - GR) - Tr(1_{H_1} - RG).$$

Hence the index of $e_-F_+e_+$ can be computed as

$$Tr(e_+ - e_+F_-e_-F_+e_+) - Tr(e_- - e_-F_+e_+F_-e_-) = Tr_s(e - eFeFe) = \tau_0(e).$$

The general case of even p is similar and uses the fact that, if R is an inverse of G modulo $\mathcal{L}^p$, then

$$\mathrm{Index}(G) = \mathrm{Tr}(1_{H_2} - GR)^p) - \mathrm{Tr}((1_{H_1} - RG)^p).$$

Similarly the odd case uses the fact that the pairing of the K-homology cycle (ρ, H, F) is given by index of the operator of the form PuP, where $P = \frac{1+F}{2}$ and $u \in A$ represents a class in $K_1(A)$. □

PROPOSITION 9.0.6.

$$\tau_{p+2} = S\tau_p$$

in $\mathrm{HC}^{p+2}(\mathcal{A})$ *(for both for even and odd p+1 summable Fredholm modules).*

PROOF. This follows from realizing the cochain τ_p as the image of the characteristic map

$$\tau_p = \mathrm{const}_p \chi_{\tau_0}(F^{\wedge p}).$$

□

PROPOSITION 9.0.7. *Let* $(\rho(t), H, F(t))_{t\in[0,1]}$ *be a family of p+1 summable Fredholm modules over* A *such that, for all* $a \in A$, $t \to \rho(t(a))$ *and* $t \to F(t)aF(t)$ *are piecewise strongly* C^1*s. Moreover assume that, for all* $a \in \mathcal{A}$,

$$t \to F(t)[F(t), \rho(t)(a)] \in \mathcal{L}^p$$

is C^1. *Then*

$$t \to SCh(F(t)) \in HC^{p+2}(\mathcal{A})$$

is constant.

PROOF. Conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & F_+(t) \end{pmatrix}$ replaces $\rho(t)_-$ by $F_-(t)\rho(t)_-F_+(t)$ and $F(t)$ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since $Ch(F)$ is invariant under similarities, we can just as well assume that $F(t) = F$ is constant independent of t. Set

$$\psi = \int_0^1 dt(\iota_{\frac{d}{dt}}\tau).$$

ψ is a Hochschild cocycle and $B\psi = \tau(1) - \tau(1)$. □

10. Theta summable Fredholm modules

Recall the construction in 2.1.1 in Chapter 7. Let A be a $\mathbb{Z}/2\mathbb{Z}$-graded algebra and D an odd element of A. Then $\nabla_D = -D^2\epsilon + D$ is an element of the Chevalley-Eilenberg complex of $\mathfrak{g}_A = C^*(A, A)[1]$. It satsfies the Maurer-Cartan equation

$$(\delta + \partial_{Lie})\nabla^D + \frac{1}{2}[\nabla^D, \nabla^D] = 0.$$

Suppose that, moreover, τ is a $\mathfrak{g}$-invariant, graded trace on A. Then the associated map χ from the corollary 3.0.6 produces an infinite cyclic periodic cochain $\chi(\exp(\nabla^D)) = \phi^D$. For future reference we will replace D with $D_t = \sqrt{t}D$.

PROPOSITION 10.0.1. *The components of* ϕ^{D_t} *have the form*

$$\phi_n^{D_t}(a_0, \ldots, a_n) = t^{\frac{n}{2}} \int_{\Delta_t} \tau(a_0 e^{-t_0 D^2}[D, a_1] e^{-t_1 D^2} \ldots [D, a_n] e^{-t_n D^2}) dt_1 \ldots dt_n. \tag{10.1}$$

formally satisfying the cocycle condition:

$$b\phi_n + B\phi_{n+2} = 0 \tag{10.2}$$

Δ_t *denotes the simplex* $\{t_0 \ldots t_n \geq 0,\ t_0 + \ldots t_n = t\}$.

PROOF. The content of the claim is just the Duhamel formula for the exponential

$$\exp(\iota_{t D^2} + t^{\frac{1}{2}} \mathrm{ad}\, D).$$

□

DEFINITION 10.0.2. The infinite cochain $\phi_n^{D_t}$, $t > 0$, is called the Jaffe Lesniewski Osterwalder (JLO) cocycle.

To begin with, let us state the following general result.

THEOREM 10.0.3 (A. Connes, see [**114**]). *Let* (ρ, H, D) *be an unbounded Fredholm module over* A *and set* $\mathcal{A}$ *to be the* C^∞*-domain of* $\mathrm{ad}\, D$*. Suppose moreover that* D *satisfies the estimate*

$$\exp(-t D^2) \in \mathcal{L}^1(\tau),\ \ \theta < t,\ \textit{for some}\ 0 < \theta < 1.$$

Then the JLO cocycle associated to D *defines a continuous linear functional on the entire cyclic periodic homology of* $\mathcal{A}$*. The class of* ϕ^D *(see* (10.1)*) is invariant under perturbations of the form* $D \rightsquigarrow D + V$ *with* $V \in \mathcal{A}$ *and coincides with* $\mathrm{Ch}(D)$.

A HINT TOWARDS THE PROOF. We refere the reader to the original papers, let us just remark that the algebra involved is the same as before, and the convergence of various series involved in the proofs is based on the following Hölder estimate (see Lemma 2.1, [**276**]).

LEMMA 10.0.4. *Let* $A_i = a_i D + b_i,\ i = 1, \ldots n$*, where* $a_i,\ b_i \in \mathcal{A}$*,* $a_i = 0$ *for* $i > k$*. Then, both in the even and in the odd case,*

$$\left| \int_\Delta \mathrm{Tr}(a_0 e^{-t_0 D^2}[D, a_1] e^{-t_1 D^2} \ldots [D, a_n] e^{-t_n D^2}) dt_1 \ldots dt_n \right| \leq \frac{(1-\theta)^{-\frac{k}{2}} \tau(e^{-\theta D^2})}{(n-k)!)} \prod_{i=0}^{n} (\|a_i\| + \|b_i\|).$$

The proof of the fact that $< \phi^D, \mathrm{ch}[e] >=< [(\rho, H, D)], [e] >$ for an idempotent $e \in \mathcal{A}$ follows the standard method.

$$\tilde{D} = D - (e[D, e] - [D, e]e)$$

is a bounded perturbation of D, hence $\phi^D = \phi^{\tilde{D}}$ in entire cyclic cohomology of $\mathcal{A}$ and, moreover, $[\tilde{D}, e] = 0$. As the result,

$$< \phi^D, \mathrm{ch}[e] >=< \phi^{\tilde{D}}, \mathrm{ch}[e] >= \mathrm{Tr}_s(e \exp(-D^2)) =< [(\rho, H, D)], [e] >,$$

where the last equality is the index formula of McKean and Singer. □

11. The residue cochain

The Duhamel type expansion of $\chi(\exp(t\nabla^D))$ produces an infinite cochain

(11.1)
$$\chi(\exp(t\nabla^D))_n(a_0,\dots,a_n) = t^n\int_{\Delta_t}\tau(a_0e^{-t_0D^2}[D,a_1]e^{-t_1D^2}\dots[D,a_n]e^{-t_nD^2})dt_1\dots dt_n.$$

This does not satisfy the cocycle identity 10.2 since $t\nabla$ is not a Maurer-Cartan element but, comparing to the expansion of the JLO cocycle (10.1) , we get the componentwise identity

(11.2)
$$\phi_n^{D_t} = t^{-\frac{n}{2}}\chi(\exp(t\nabla^D))_n$$

To continue, assume that $R = D^2$ is invertible in A. A formal version of the Cauchy formula gives the identity:

(11.3)
$$\exp(t\nabla^D) = \frac{1}{2\pi i}\int_\gamma e^{-st}\frac{1}{s-\nabla^D}ds,$$

where the contour of integration is given by $i\mathbb{R}+\epsilon$, where $\mathbb{R}$ is negatively oriented and ϵ is positive and "separates the spectrum of R" from zero.

Similarly to the Duhamel formula which gave the expansion $\phi_D = \sum_n u^n\phi_{2n}$ with the components ϕ_{2n}, the Dyson resolvent expansion of $\psi = \chi(\frac{1}{s-R+D})$ gives the infinite cochain with components

(11.4)
$$\chi(\exp(t\nabla^D))(a_0,\dots,a_n) = \frac{1}{2\pi i}\int_\gamma e^{-st}\tau(a_0(s-R)^{-1}[D,a_1](s-R)^{-1}\dots[D,a_n](s-R)^{-1})ds$$

Hence the following corollary.

Corollary 11.0.1.
$$(a_0,\dots,a_n)\mapsto t^{-\frac{n}{2}}\frac{1}{2\pi i}\int_\gamma e^{-st}\tau(a_0(s-R)^{-1}[D,a_1](s-R)^{-1}\dots[D,a_n](s-R)^{-1})ds,\ n\in\mathbb{N}$$

is an infinite cyclic periodic cocycle which, term by term, coincides with the JLO cocycle ϕ^{D_t} *from 10.1 .*

Applying the Mellin transform to the formula for the JLO cocycle in the corollary 11.0.1, we get the following identity.

(11.5)
$$M(\phi^{D_t})(z)(a_0,\dots,a_n) = \int_{\mathbb{R}_+} t^z\phi^{D_t}(a_0,\dots,a_n)\frac{dt}{t} = \frac{1}{2\pi i}\Gamma(z)\int_\gamma s^{\frac{n}{2}-z}\tau(a_0(s-R)^{-1}[D,a_1](s-R)^{-1}\dots[D,a_n](s-R)^{-1})ds.$$

To continue we need the following.

Assumptions

- $\mathcal{A}$ *is a filtered algebra*
$$\mathcal{A} = \bigcup \mathcal{A}_n,$$
with $\mathcal{A}_n\mathcal{A}_m \subset \mathcal{A}_{n+m}$ *and* $[\mathcal{A}_n,\mathcal{A}_m]\subset\mathcal{A}_{n+m-1}$.

- *The filtration is complete in the sense that the sums of the form*
$$\sum_{n=-\infty}^{N} a_n,\ N<\infty,\ a_n\in\mathcal{A}_n,$$
converge.
- $D\in\mathcal{A}_1$. *The expressions of the form*
$$\tau(XR^{-z}),\ X\in\mathcal{A}$$
are well defined and analytic for $\Re z\gg 0$ *and admit meromorphic extension to the half plane* $\Re z>-k$ *for some positive* k

Applying repeatedly the identity
$$[X,(s-R)^{-1}]=-(s-R)^{-1}[X,R](s-R)^{-1},$$
the term
$$a_0(s-R)^{-1}[D,a_1](s-R)^{-1}\dots[D,a_n](s-R)^{-1}$$
can be rewritten as a formal infinite sum of the form
$$\sum_{k>n}X_{n,k}(s-R)^{-k}$$
where $X_{n,k}=X_{n,k}(a_0,\dots,a_n;D)$ are linear in the variables $(a_0,\dots,a_n)$ with values in $\mathcal{A}_n$. As the result, the equation 11.5 has the form
$$M(\phi^{D_t})_n(z)=\frac{1}{2\pi i}\Gamma(z)\int_\gamma s^{\frac{n}{2}-z}\tau(\sum_{k>n}X_{n,k}(s-R)^{-k})ds.$$
Again using the Cauchy formula in the form
$$\binom{\frac{n}{2}-z}{k}R^{\frac{n}{2}-z-k}=\int_\gamma s^{\frac{n}{2}-z}(s-R)^{-k}ds$$
we get the identity
$$M(\phi^{D_t})_n(z)=\frac{1}{2\pi i}\Gamma(z)\sum_{k>n}\binom{\frac{n}{2}-z}{k}\tau(X_{n,k}R^{\frac{n}{2}-z-k}).$$

THEOREM 11.0.2. *(see*[**132**] *and* [**320**]*) Under the above assumptions the following holds.*

(1) *The collection of the maps*
$$(a_0,\dots,a_n)\to\frac{1}{2\pi i}\Gamma(z)\sum_{k>n}\binom{\frac{n}{2}-z}{k}\tau(X_{n,k}(a_0,\dots,a_n;D)R^{\frac{n}{2}-z-k})$$
defines infinite cochain on the subalgebra $\mathcal{A}$ *of* A_0 *consisting of the elements* $a\in\mathcal{A}_0$ *such that* $\tau(e^{-zR}a)$ *are meromorphic functions on a hyperplane* $H_k=\{\Re z>-k\}$ *with values in meromorphic functions on* H_k *and satisfying the cocycle condition*
$$bM(\phi^{D_t})_n+BM(\phi^{D_t})_{n+2}=0.$$
(2) *The residue* res *at* $z=0$ *of* $M(\phi^{D_t})(z)$ *is an infinite cyclic periodic cocycle.*

(3) *The JLO cocycle admits an assymptotic expansion as* $t\to 0^+$ *and, in the case when* $\tau(aR^{-z})$ *has simple poles for* $a\in\mathcal{A}$*, the residue cocycle is the constant term of this expansion.*

(4) *Denote by* $X^{(k)}$ *the k-fold commutator of* R *with* X*, i.e.*

$$X^{(k)} = (adR)^k(X).$$

Then the residue cocycle has the form

$$res(a_0, \ldots, a_n) = \sum_k c_{n,k} Res_{z=0} \tau(a_0[D, a_1]^{(k_1)} \ldots [D, a_n]^{(k_n)} R^{\frac{n}{2}-z-|k|}),$$

where the sum is over n-tuples $k = (k_1, \ldots, k_n))$ *of non-negative integers and the coefficients* $c_{n,k}$ *have the form*

$$c_{n,k} = \frac{(-1)^{|k|}}{k!} \frac{\Gamma(|k| + \frac{n}{2})}{(k_1+1)(k_1+k_2+2)\ldots(k_1+\ldots+k_n+n)}.$$

Comments about the proof. The analysis involved is somewhat outside the subject of this book but, in principle, it is a consequence of the standard set of tools developed in the analysis of complex powers and heat kernel expansions for elliptic pseudodifferential operators. Our assumptions replace the corresponding results about zeta-functions of elliptic pseudodifferential operators in the classical case. For explicit formulas for the terms $X_{n,k}$ we refere to the original papers of Connes-Moscovici and the excellent survey article of Nigel Higson. An alternative is of course for the reader to work out the combinatorics as an exercise. The relation between the JLO and residue cocycles is essentially the corollary of the properties of the inverse Mellin transform. □

11.1. The local index formula. The residue cocycle can be interpreted as being "local" in the following sense. Suppose that M is a compact, Riemannian manifold with a $Spin^c$-structure and D the associated Dirac operator. Then, given a function $f \in C^\infty(M)$, the residue is simple and can be interpreted as the measure om M, in fact

$$res_{z=0} Tr(f|D|^{-z}) = \int_M f d\mu$$

where $d\mu$ is the measure associated to the Riemannian metric on M. The principal symbols of the commutators of functions on M with D have the form

$$\sigma_p([D, f]) = \gamma(df)$$

where γ is the Clifford multiplication with $df \in \Gamma(M, T^*M)$. Hence the residue cocycle becomes in this case a sum of terms of the form

$$res(f_0, f_1, \ldots, f_k) = \int_M c_k f_0 df_1 \ldots df_k$$

where $\{\int_M c_2 k\}_{k=0,\ldots n} \in H_{ev}(M)$ can be identified with the $\hat{A}^{\frac{1}{2}}$-genus of the complexified cotangent bundle of M. Hence in ths case the fact that

$$< res, [e] >=< JLO, [e] >= index\{e_- D_+ e_+ : \mathfrak{Rg}(e_+) \to \mathfrak{Rg}(e_-)\}$$

becomes the local index formula of Atiyah and Singer.

12. Chern character of perfect complexes

The following is contained in [**58**].

12.1. Perfect complexes and twisting cochains. Consider a sheaf of algebras $\mathcal{A}$ on a topological space X. Fix an open cover $\mathfrak{U} = \{U_j | j \in J\}$. We denote

$$\mathcal{A}_{j_0 \ldots j_p} = \mathcal{A}|(U_{j_0} \cap \ldots \cap U_{j_p}) \tag{12.1}$$

Following Toledo and Tong, we define *a twisting cochain* as:

(1) a collection of strictly perfect complexes of $\mathcal{A}_{U_j}$-modules $\mathcal{F}_j$;
(2) a collection of morphisms of degree $1 - p$, $p \geq 0$, of $\mathcal{A}_{j_0 \ldots j_p}$-modules on $U_{j_0} \cap \ldots \cap U_{j_p}$

$$\rho_{j_0 \ldots j_p} : \mathcal{F}_{j_0} \longleftarrow \mathcal{F}_{j_p}$$

such that
(3)

$$\check{\partial}\rho + \rho \smile \rho = 0$$

Here, for two collections of homogenous $\rho_{j_0 \ldots j_p}$ and $\varphi_{j_0 \ldots j_p}$ of any degree, we put

$$(\rho \smile \varphi)_{j_0 \ldots j_m} = \sum_{p=0}^{m} (-1)^{|\rho_{j_0 \ldots j_p}|} \rho_{j_0 \ldots j_p} \varphi_{j_p \ldots j_m} \tag{12.2}$$

$$(\check{\partial}\rho)_{j_0 \ldots j_{m+1}} = \sum_{p=1}^{m} (-1)^p \rho_{j_0 \ldots \widehat{j_p} \ldots j_m} \tag{12.3}$$

Note that $(\mathcal{F}_j, \rho_j)$ is a complex of $\mathcal{A}_j$-modules.

Definition 12.1.1. *Let* $\mathrm{Tw}(\mathfrak{U}, \mathcal{A})$ *be the following DG category.*

(1) *Objects are twisting cochains* ρ*.*
(2) *A morphism of degree* n *between* ρ *and* rho' *is a collection of morphisms*

$$\varphi_{j_0 \ldots j_p} : \mathcal{F}'_{j_p} \to \mathcal{F}_{j_0}$$

of degree $n - p$.
(3) *The differential acts by*

$$(d\varphi) = \check{\partial}\varphi + \rho \smile \varphi - (-1)^{|\varphi|} \varphi\rho.$$

(4) *The composition of* φ *and* ψ *is* $\varphi \smile \psi$*.*

Let $I_{\mathfrak{U}}$ be the category whose set of objects is J and such that there is unique morphism between any two objects. A twisting cochain ρ satisfies the same relations as an A_∞ functor

$$\rho \colon kI_{\mathfrak{U}} \to \mathrm{sPerf}(\mathcal{A}) \tag{12.4}$$

that sends j to $(\mathcal{F}_j, \rho_j)$. Similarly, $\mathrm{Tw}(\mathfrak{U}, \mathcal{A})$ is defined almost exactly in the same way as the DG category $\mathbf{C}(kI_{\mathfrak{U}}, \mathrm{sPerf}(\mathcal{A}))$ as in **??**. The only difference is that the targets of the components vary, namely, objects $\mathcal{F}_j$ lie in different categories, and so do morphisms $\rho_{j_0 \ldots j_p}$. We will first recall the construction that we would like to carry out, and then outline a minor variation on it that suits our purpose.

12.2. A character map from the category of A_∞ functors. For a category Γ and a DG category $\mathcal{P}$, we have constructed ***REF*** a DG functor

$$CC^-_\bullet(\mathbf{C}(k\Gamma,\mathcal{P}))\otimes CC^-_\bullet(k\Gamma)\to CC^-_\bullet(\mathcal{P}) \tag{12.5}$$

If Γ is a groupoid then we also have

$$C_\bullet(\Gamma,k)\to CC^-_\bullet(k\Gamma) \tag{12.6}$$

??. Combiming, we get

$$CC^-_\bullet(\mathbf{C}(k\Gamma,\mathcal{P}))\to \underline{\mathrm{Hom}}(C_\bullet(\Gamma,k),CC^-_\bullet(\mathcal{P})) \tag{12.7}$$

12.3. Chern character of a twisting cochail. We need a modification of the above. For simplicity, let Γ be the groupoid $I_{\mathfrak{U}}$ where $\mathfrak{U}$ is a set. We assume that there is a presheaf $\mathcal{P}$ of categories on the cyclic nerve of $I_{\mathfrak{U}}$ is given, that is, a category $\mathcal{P}_J$ for any finite subset $J=\{j_0,\dots,j_p\}$ of $\mathfrak{U}$ together with functors $r_{JK}:\mathcal{P}_K\to\mathcal{P}_J$ for $J\subset K$ so that $r_{JJ}=\mathrm{id}$ and $r_{IJ}r_{JK}=r_{IK}$ for any $I\subset J\subset K$.

For example, when $\mathfrak{U}$ is an open cover of X and $\mathcal{A}$ is a sheaf of rings on X then one can put

$$\mathcal{P}_{\{j_0,\dots,j_p\}}=\mathrm{Perf}(\mathcal{A}|U_{j_0}\cap\dots\cap\mathcal{U}_{j_p})$$

Having Γ and $\mathcal{P}$ as above, we can modify the definition of the Hochschild chain complex and of an A_∞ functor as follows. For an object F of $\mathcal{P}_j$ and a subset J containing j, we will write

$$F|J=r_{J\{j\}}F \tag{12.8}$$

Fix two collections $\{F_j,\ G_j\in\mathrm{Ob}(\mathcal{P}_j)|j\in\mathrm{Ob}(\Gamma)\}$. Define the *local Hochschild cochain complex*

$$C^\bullet_{\mathrm{loc}}(k\Gamma,_F\mathcal{P}_G)=\prod_J \underline{\mathrm{Hom}}(k\Gamma(j_0,j_1)[1]\otimes\dots\otimes k\Gamma(j_{n-1},j_n)[1],\mathcal{P}_J(F_{j_0}|J,G_{j_n}|J))$$

where the product is over all $J=j_0,\dots,j_n\in\mathrm{Ob}(\Gamma)$. (The category $\mathcal{P}_J$ depends only on the underlying set. In fact everything we do only requires it to be the same for the underlying cyclically ordered set).

The differential δ on $C^\bullet_{\mathrm{loc}}(k\Gamma,_F\mathcal{P}_G)$ and the product

$$\smile: C^\bullet_{\mathrm{loc}}(k\Gamma,_F\mathcal{P}_G)\otimes C^\bullet_{\mathrm{loc}}(k\Gamma,_G\mathcal{P}_H)\to C^\bullet_{\mathrm{loc}}(k\Gamma,_F\mathcal{P}_H)$$

are defined exactly as in the case of two DGA categories. A local $A\infty$ functor $k\Gamma\to\mathcal{P}$ is a collection $F=\{F_j\}$ and a cochain of degree one in $C^\bullet_{\mathrm{loc}}(k\Gamma,_F\mathcal{P}_F)$ satisfying $\delta\rho+\rho\smile\rho=0$. We denote the pair of F and ρ simply by F. As in **??**, local A_∞ functors form a DG category that we denote by $\mathbf{C}_{\mathrm{loc}}(k\Gamma,\mathcal{P})$. Exactly as in (12.7) we get

$$CC^-_\bullet(\mathbf{C}_{\mathrm{loc}}(k\Gamma,\mathcal{P}))\to \underline{\mathrm{Hom}}_{\mathrm{loc}}(C_\bullet(\Gamma,k),CC^-_\bullet(\mathcal{P})) \tag{12.9}$$

Here $\underline{\mathrm{Hom}}_{\mathrm{loc}}$ in the right hand side stands for

$$\prod_{j_0,\dots,j_n}\underline{\mathrm{Hom}}(k\Gamma(j_0,j_1)[1]\otimes\dots\otimes k\Gamma(j_{n-1},j_n)[1],CC^-_\bullet(\mathcal{P}_{\{j_0,\dots,j_n\}}))$$

Composing with REF?***

$$CC^-_\bullet(\mathrm{Perf}(\mathcal{A}))\to CC^-_\bullet(\mathbf{C}_{\mathrm{loc}}(k\Gamma,\mathcal{P})) \tag{12.10}$$

and observing that

$$\underline{\mathrm{Hom}}_{\mathrm{loc}}(C_\bullet(\Gamma,k),CC^-_\bullet(\mathcal{P}))\xrightarrow{\sim}\check{C}^*(\mathfrak{U},CC^-_\bullet(\mathcal{A})), \tag{12.11}$$

and passing to the direct limit in $\mathfrak{U}$, we get

$$\mathrm{ch}: \mathrm{CC}^-_\bullet(\mathrm{Perf}(\mathcal{A})) \to \check{C}^*(X, \mathrm{CC}^-_\bullet(\mathcal{A})) \tag{12.12}$$

REMARK 12.3.1. For a twisting cochain ρ, the value of Čech-negative cyclic cochain $\mathrm{ch}(\rho)$ on $U_{j_0} \cap \ldots \cap U_{j_p}$ is the sum of terms as follows. Let

$$(\ell_0, \ldots, \ell_N, \ell_0) = (j_k, j_{k+1}, \ldots, j_k, \ldots, j_k, j_{k+1}, \ldots, j_k) \tag{12.13}$$

which is $(j_0, \ldots, j_p)$ shifted cyclically by k and then repeated m times for some m (so $N + 1 = m(p + 1)$). Let $(i_0, i_1, ..., i_M)$ be an ordered subset of the ordered set $(\ell_0, \ldots, \ell_N)$, and choose an ordered subset $(i_k, q_1, \ldots, q_{r_k}, i_{k+1})$ in the segment $i_k \leq p \leq i_{k+1}$ in (12.13), $0 \leq k \leq M$ (we put $i_{M+1} = i_0$). Define

$$\rho_(i_k, i_{k+1}) = \rho_{i_k \ldots q_1} \ldots \rho_{q_{r_k} \ldots i_{k+1}} \tag{12.14}$$

$$\mathrm{ch}(\rho)_{j_0 \ldots j_p} = \sum_{m \geq 0} \sum_{i_0, \ldots, i_{p+2m}} c(i_0, \ldots, i_{p+2m}) u^m \rho(i_0, i_1) \otimes \ldots \otimes \rho(i_{p+2m}, i_0)$$

Only terms $\rho_{j_q \ldots j_{q+r}}$ with $r > 0$ participate.

For example, if the twisting cochain consists of the transition isomorphisms $\rho_j k$ (i.e. when our sheaf of modules $\mathcal{M}$ is locally free, then

$$\mathrm{ch}(\rho)_{j_0 \ldots j_p} =$$

*****TO BE CORRECTED****

13. Bibliographical notes

Connes; Connes-Karoubi; Karoubi; Goodwillie; Beilinson; JLO, Getzler-Szenes; Wodzicki; Connes-Moscovici; Higson; BGNT

CHAPTER 12

Examples II. Algebraic index theorem and deformation quantization

1. Jets

Let M be a smooth manifold. We will denote by $\mathcal{O}_M$ the ring of smooth functions on M and by $\mathcal{D}_M$ the ring of differential operators on M. Both $\mathcal{O}_M$ and $\mathcal{D}_M$ are left modules over $\mathcal{O}_M$ and are global sections of sheaves over M. $\mathcal{D}_M$ has the degree filtration $\mathcal{D}_M = \cup_n \mathcal{D}_M^n$, where $\mathcal{D}_M^n$ denotes the differential operators of degree n. Set

$$J_M^n = \mathsf{Hom}_{\mathcal{O}_M}(\mathcal{D}_M, \mathcal{O}_M).$$

and $J_M^\infty = \varprojlim J_M^n$ with respect to the obvious restriction maps.

A jet of a smooth function f is, by definition, an element of J of the form

$$j^\infty(f)(D) = Df.$$

Lemma 1.0.1. *J is a sheaf of algebras.*

Proof. In fact, $\mathcal{D}_M$ has a coassociative coproduct

$$\Delta : \mathcal{D}_M \to \mathcal{D}_M \otimes_{\mathcal{O}_M} \mathcal{D}_M$$

defined by setting, for $f \in \mathcal{O}_M$ and $X \in \mathsf{Vect}(M)$ by

$$\Delta f = f \otimes 1 \text{ and } \Delta(X) = X \otimes 1 + 1 \otimes X$$

and extending it to all of differential operators by requiring thar Δ is a ring homomorphism. The convolution product

$$l_1 * l_2(D) = l_1 \otimes l_2(\Delta D)$$

makes J into a bundle of algebras. □

Lemma 1.0.2. *For n=1,2,...,∞,*

$$U \to J_U^n,\ U \text{ open subset of } M$$

are locally free sheaves of algebras on M, hence sections of algebra bundles of n-jets on M.

Proof. We will deal with the case $n = \infty$, the case of finite jets being an immediate corollary. We will need some notation. Set

$$\mathbb{O} = \mathbb{C},$$

the ring of formal power series (in n variables, where n will stand for the dimension of M). Similarly, $\mathbb{D}$ will denote the rings of formal differential operators in n variables, generated over $\mathbb{O}$ by $\partial_{\hat{x}_1}, \ldots, \partial_{\hat{x}_n}$. Let $(U, x_1, \ldots x_n)$ be a local coordinate system on M. Then a jet $l \in J_U$ is uniquely determined by its "Taylor coefficients"

$$l_\alpha = l(\partial_x^\alpha) \in \mathcal{O}_U$$

and we can write it in the form

$$l = \sum_\alpha l_\alpha \frac{\widehat{x}^\alpha}{\alpha!} \in \mathcal{O}_U[[\widehat{x}_1, \ldots, \widehat{x}_n]]$$

with the convention

$$l(\partial_x^\alpha) = \partial_{\widehat{x}}^\alpha l|_{\widehat{x}=0}.$$

In particular, J_M has local trivializations $J_U = \mathcal{O}_U \times \mathbb{O}$. One checks that this trivialisation is consistent with the product structure on jets, hence J_M is the space of sections of an algebra bundle with fiber $\mathbb{O}$. □

In the future we will write J whenever M is understood.

1.1. Kazhdan connection. Just to recall some basic notions, let $\mathcal{E} \to M$ be a vector bundle over M. A connection in E is a linear map

$$\nabla : \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E}) \otimes_{\mathcal{O}_M} \Omega(M)$$

which satisfies the Leibnitz identity

$$\nabla(f\sigma) = \sigma \otimes df + f\nabla\sigma.$$

We will often write

$$\nabla_X =< \nabla, X >: \Gamma(M, \mathcal{E}) \to \Gamma(M, \mathcal{E}),$$

where $<,>$ is the standard $\mathcal{O}$-valued pairing of vector fields with one-forms.

In particular locally, in a local trivialisation

$$\mathcal{E}|_U \simeq U \times \mathbb{C}^k,$$

it has the form

$$\nabla|_U = d|_U + A_U,\ A_U \in \Omega^1(U, \mathrm{End}(\mathcal{E})).$$

Set, for $\omega \in \Omega^*(M)$,

$$\nabla(\sigma \otimes \omega) = \nabla(\sigma)\omega + \sigma \otimes d\omega.$$

This makes

$$\nabla : \Omega^*(M, \mathcal{E}) \to \Omega^{*+1}(M, \mathcal{E})$$

into an odd, graded derivation. One checks that $\nabla^2(f\sigma) = f\nabla^2(\sigma)$, hence

$$\nabla^2(\sigma) = R\sigma$$

for some $R \in \Omega^2(M, \mathrm{End}(\mathcal{E}))$. R is the curvature of ∇ and the connection is flat if $\nabla^2 = 0$. In local trivialisation of $\mathcal{E}$ as above,

$$R|_U = dA_U + \frac{1}{2}[A_U, A_U].$$

Note that all our commutators are graded.

LEMMA 1.1.1. *Let* $l \in J_M$. *then*

$$\nabla_X(l)(D) = X\nabla(D) - \nabla(XD)$$

is a flat connection on J. *Moreover,*

$$\nabla(l) = 0 \Longleftrightarrow \{\textit{there exists } f \in \mathcal{O}_M \textit{ such that } l(D) = j^\infty(f)\}.$$

WE LEAVE THE PROOF AS AN EXERCISE FOR THE READER. As a hint for the second statement, $f = l(1)$. □

1.2. Local computations. Recall that, given local coordinate system $(U, x_1, \dots x_n)$, we constructed a local trivialisation of J of the form $J|_U = \mathcal{O} \otimes \mathbb{O}$ such that

$$l(\partial_x^\alpha) = \partial_{\widehat{x}}^\alpha l|_{\widehat{x}=0}.$$

LEMMA 1.2.1. *In local coordinates, the Kazhdan connection has the form*

$$\nabla = \sum_i dx_i(\partial_{x_i} - \partial_{\widehat{x}_i}).$$

PROOF. Recall that, in local coordinates,

$$l = \sum_\alpha l_\alpha \frac{\widehat{x}^\alpha}{\alpha!}$$

where $l_\alpha = l(\partial_x^\alpha)$. In these terms,

$$\left(\nabla_{\partial_{x_i}} l\right)_\alpha = \partial_{x_i}(l_\alpha) - l(\partial_{x_i}\partial_x^\alpha) = \partial_{x_i}(l_\alpha) - l_{\alpha+\delta_i}$$

which means that

$$\nabla_{\partial_{x_i}} l = (\partial_{x_i} - \partial_{\widehat{x}_i})l$$

as claimed. □

Since $\nabla^2 = 0$, $(\Omega^*(M, J), \nabla)$ is a complex.

PROPOSITION 1.2.2. *The complex*

$$(\Omega^*(M, J), \nabla)$$

is contractible in positive dimensions and its 0-th cohomology is isomorphic to $\mathcal{O}_M$.

PROOF. We have already seen that $\ker(\nabla) = \mathcal{O}_M$. Set

$$\rho = \sum_i \widehat{x}_i \iota_{\partial_{x_i}}.$$

Then

$$[\nabla, \rho]|_{\Omega^{>0}} = \mathrm{id}$$

and hence our complex is contractible in positive dimensions. □

REMARK 1.2.3. For completeness, let us give an explicit computation of the kernel of $\nabla|_J$. In local coordinates, a jet l is a function in two sets of coordinates, $(x_1, \dots, x_n, \widehat{x}_1, \dots, \widehat{x}_n)$, smooth in x's and formal power series in $\widehat{x}$'s. Hence $\nabla l = 0$ translates into

$$(\partial_{x_i} - \partial_{\widehat{x}_i})l = 0, \ i = 1, \dots, n.$$

Hence

$$l(x, \widehat{x}) = \phi(x + \widehat{x}), \ \phi \in C^\infty(\mathbb{R}^n).$$

The meaning of the formal expression on the right hand side is: expand ϕ into full Taylor series at x with increments $\widehat{x}$.

2. Formal geometry

The general idea is as follows. Let M be a smooth connected manifold. Then M is a homogeneous space of the form $M \simeq \mathsf{Diff}(M)/\mathsf{Diff}(M)_m$.

While Diff(M) is not a Lie group, it has a pro-Lie group model which is small enough to use Lie group methods and big enough to recover M as a homogeneous space. In particular, Kazhdan connection turns out to be an infinite dimensional analogue of Cartan connection and, more to the point, the general notion of geometric object corresponds to the Cartan model of group cohomology. More about it later.

2.1. Jets of coordinate systems. Let M be a smooth manifold and $m \in M$. The infinitesimal neighbourhood of m is the ringed space

$$(m, \mathbb{O}_m(M))$$

where $\mathbb{O}_m(M)$ is the completion of the ring $\mathcal{O}_M$ at the ideal $I_m = \{f \in \mathcal{O}_M \mid f(m) = 0\}$, i. e. the complete local ring

$$\mathbb{O}_m(M) = \varprojlim_n \mathcal{O}_M/I_m^n.$$

In local coordinates $(x_1, \ldots, x_n)$ centered at m, this is our old "friend", the ring of formal power series $\mathbb{O}$, the identification given by sending $f \in C^\infty(M)$ to its full Taylor series at the point m expressed in the coordinates $(x_1, \ldots, x_n)$.

All our maps between complete local rings are required to be continuous.

Below we will need the real forms of both $\mathbb{O}_m$ and $\mathbb{O}$ given by jets of real valued functions at $m \in M$ and $0 \in \mathbb{R}^n$, which we will denote by $\mathbb{O}_m(\mathbb{R})$ and $\mathbb{O}(\mathbb{R})$.

Set

$$G = \mathsf{Aut}(\mathbb{O}(\mathbb{R})) \text{ and } \mathbb{W} = \mathsf{Der}(\mathbb{O}(\mathbb{R})).$$

In particular, $\mathbb{W}$ is the Lie algebra of formal vector fields. Note that $\mathbb{O}(\mathbb{R})$ is filtered by degree - it is not graded, since it is a direct product and not a direct sum of monomials in $\widehat{x}$'s. We give $\mathbb{W}$ the associated grading with $\mathbb{W}_{-1}$ being the Lie algebra of vector field with constant coefficients and the Lie subalgebra $\mathbb{W}_{\geq 0}$ of formal vector fields vanishing at $\widehat{x} = 0$ is the Lie algebra of G.

Before stating the main definition note that, given a local coordinate system $(x_1, \ldots, x_n)$,

$$\mathbb{O}_m(\mathbb{R}) = \mathbb{R}[[\widehat{x}_1, \ldots, \widehat{x}_n]]$$

where $\widehat{x}_i = dx_i$, hence $J^1(M) = T^*M$. A continuous homomorphism $\phi : \mathbb{O}_m(\mathbb{R}) \xrightarrow{\sim} \mathbb{O}(\mathbb{R})$ is uniquelly determined by

$$\phi(\widehat{x}_i) = \sum_\alpha c_i^\alpha \widehat{x}^\alpha, \; i = 1, \ldots, n.$$

The first jet of ϕ is, in these terms, the linear part of ϕ,

$$j^1(\phi)(\widehat{x}_i) = \sum_j c_i^j \widehat{x}_j, \; i = 1, \ldots, n, \; \{c_i^j\} \in M_n(\mathbb{R}).$$

Definition 2.1.1. Let M be a smooth manifold of dimension n. The manifold $\tilde{M}$ of jets of coordinate systems on M is the following.

$$\widetilde{M} = \{(m, \phi) \mid m \in M, \phi : \mathbb{O}_m(\mathbb{R}) \xrightarrow{\sim} \mathbb{O}(\mathbb{R})\}.$$

Theorem 2.1.2.

(1) $\widetilde{M}$ *is a principal bundle over* M *with the structure group* G*;*
(2) *the jet bundle has the form*

$$J = \widetilde{M} \times_G \mathbb{O};$$

(3) $\widetilde{M}$ *has trivial tangent bundle, in fact*

$$T\widetilde{M} \simeq \widetilde{M} \times \mathbb{W};$$

(4) *The Kazhdan connection on* $J(M)$ *is induced by the trivialisation of the tangent bundle of* $\widetilde{M}$ *above.*

PROOF. The first two claims are straightforward, with G acting from the right by

$$(m, \phi)g = (m, g^{-1} \circ \phi).$$

To prove the third claim, note first that, since G acts freely on $\widetilde{M}$, its Lie algebra $\mathbb{W}_{\geq 0}$ maps to vertical vector fields on $\widetilde{M}$. let $(m, \phi) \in \widetilde{M}$. By the classical Borel theorem, ϕ is a jet of a local diffeomorphism

$$M \supset U \xrightarrow{\Phi} V \subset \mathbb{R}^n$$

which maps the point m to the center $0 \in \mathbb{R}^n$. Given element $w \in \mathbb{W}_{-1} = T_0(\mathbb{R}^n)$,

$$[-\epsilon, \epsilon] \ni t \to (\Phi^{-1}(0 + tw), j^{\infty}_{\Phi^{-1}(0+tw)}\Phi)$$

is a smooth path in $\widetilde{M}$ and hence defines an element in $T_{(m,\phi)}\widetilde{M}$. It is easy to check that the two maps, $\mathbb{W}_{\geq 0}$ to vertical vector fields and $\mathbb{W}_{-1}$ to $\mathsf{Vect}(\widetilde{M})$, together define a Lie algebra homomorphism and a global trivialisation of the tangent bundle of $\widetilde{M}$.

Since both the trivialisation of the tangent bundle of $\widetilde{M}$ and Kazhdan connection are functorial w. r. to. smooth imbeddings and diffeomorphisms, the last claim can be checked in a local coordinate system $(U, x_1, \ldots, x_n)$. So,

$$\widetilde{U} = U \times G, \ \widetilde{U} = TU \times TG = U \times G \times (\mathbb{W}_{-1} \oplus \mathbb{W}_{\geq 0}) = \widetilde{U} \times \mathbb{W}$$

and the induced connection on $J(U) = \widetilde{U} \times_G \mathbb{O} = U \times \mathbb{O}$ is given by the map

$$TU = U \times \mathbb{R}^n \to U \times \mathbb{W}_{-1} \subset U \times \mathbb{W}$$

which is just the map

$$\partial_{x_i} \mapsto \partial_{\widehat{x}_i}, \ i + 1, \ldots, n$$

which is precisely the local expression of the Kazhdan connection. □

3. Gelfand-Fuks construction

Suppose that $(\mathfrak{g}, H)$ is a Gelfand pair. This means that $\mathfrak{g}$ is a Lie algebra, H is a Lie group acting reductively on $\mathfrak{g}$, the Lie algebra $\mathfrak{h}$ of H is a subalgebra of $\mathfrak{g}$ and the action of H on $\mathfrak{g}$ restricts to the adjoint action on $\mathfrak{h}$.

Suppose moreover that is a principal H-bundle and $A \in \Omega^1(\mathcal{P}, \mathfrak{g})$ is a basic 1-form, i.e. it is H-invariant and vanishes on vertical vectors in $T\mathcal{P}$ and

$$(d + A)^2 = 0.$$

Under these conditions $d + A$ descends to a $\mathfrak{g}$-valued, flat connection ∇ on M.

DEFINITION 3.0.1. Let $(\mathbb{L}, \delta)$ be a $(\mathfrak{g}, H)$-complex. Then

$$\mathcal{L} = \mathcal{P} \times_H \mathbb{L}$$

is a sheaf on M with fiber $\mathbb{L}$,

$$\Omega^*(M, \mathcal{L}) = (\Omega(\mathcal{P}) \otimes \mathbb{L})^H_{basic}$$

is the sheaf of $\mathcal{L}$-valued differential forms and

$$(\Omega^*(M, \mathcal{L}), \nabla + \delta)$$

is a complex. Here $\omega \in \Omega(\mathcal{P}) \otimes \mathbb{L}$ is basic means that it is H-invariant and, for $h \in \mathfrak{h}$,

$$\iota_h \otimes 1 + 1 \otimes h \cdot \omega = 0.$$

THEOREM 3.0.2 (Gelfand - Fuks construction). *Suppose that, as above, ∇ is a flat $\mathfrak{g}$-valued connection of the form $d + A$, $A \in \Omega^1(\mathcal{P}, \mathfrak{g})^H_{basic}$. The following defines a map of complexes*

$$GF : (C_{Lie}(\mathfrak{g}, H; \mathbb{L}), \partial_{Lie} + \delta) \to (\Omega^*(M, \mathcal{L}), \nabla + \delta)$$

let $l \in (C^k_{Lie}(\mathfrak{g}, H; \mathbb{L})$, $(X_1, \dots, X_k)$ be vector field on M and $(\tilde{X}_1, \dots, \tilde{X}_k)$ their lifts to $\mathcal{P}$. Then

$$GF(l)(X_1, \dots, X_k) = l(A(\tilde{X}_1), \dots, A(\tilde{X}_k)).$$

THE PROOF IS STRAIGHTFORWARD. □

EXAMPLE 3.0.3. Let, as before, $\widetilde{M}$ be the manifold of jets of coordinate systems on M. As a bundle over M, it has the structure group $G = Aut(\mathbb{O})$. The restriction to the first jet

$$\phi \to j^1\phi \in GL(n, \mathbb{R})$$

has contractible fiber and, since O(n) is the maximal compact subgroup of $GL(n, \mathbb{R})$, we can reduce the structure group of $\widetilde{M} \to M$ to O(n). In other words, there exists a sequence of fibrations

$$\begin{array}{ccc} \widetilde{M} & \longleftarrow & G \\ \downarrow & & \\ \mathcal{P} & \longleftarrow & O(n) \\ \downarrow & & \\ M & & \end{array}$$

$\mathcal{P}$ is a bundle of orthonormal frames with respect to some Riemannian structure on M and there exists a $O(n)$-equivariant section $\sigma : \mathcal{P} \to \widetilde{M}$. Let A be a $\mathbb{W}$-valued 1-form on $\widetilde{M}$ given by the projection

$$T\widetilde{M} \simeq \widetilde{M} \times \mathbb{W} \to \mathbb{W}$$

and A_0 its pullback along σ. Then $(\mathbb{W}, O(n))$ is a Gelfand pair and $\nabla = d + A_0$ is a flat $\mathbb{W}$-valued connection on M.

(1) Let $\mathbb{L} = \mathbb{C}$. Then $\mathcal{L} = C^\infty(M)$, the complex $(\Omega^*(M,\mathcal{L}),\nabla)$ coincides with the de Rham complex of M and

$$GF : (C^*_{Lie}(\mathbb{W}; H, \mathbb{C}), \partial_{Lie}) \to (\Omega^*(M), d)$$

is the Chern Weyl map giving characteristic classes of the tangent bundle of M.

(2) Let $\mathbb{L} = \mathbb{O}$. Then $\mathcal{L}$ is the jet bundle J of M, the complex $(\Omega^*(M,J),\nabla)$ is the complex of jet-valued differential forms with Kazhdan connection and, as we have seen before,

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathcal{O}_M & \longrightarrow & 0 & & \\ & & \downarrow & & & & \\ 0 & \longrightarrow & \Omega^0(M,J) & \xrightarrow{\nabla} & \Omega^1(M,J) & \xrightarrow{\nabla} & \dots \end{array}$$

is a quasiisomorphism of complexes. The GF map is concentrated in degree zero and represents the unit of $\mathbb{O}$.

(3) Similar arguments to the ones given in the above example show that all the chern classes of complex bundles are in the image of an appropriate GF transform. To be more precise, let E be a complex N-dimensional vector bundle over n-dimensional smooth manifold M. We replace $\tilde{M}$ by the profinite manifold of jets of joint coordinates for both M and the fibers of E:

$$\begin{array}{ccc} U \times \mathbb{C}^N & \overset{\phi}{\to} & E \\ \downarrow & & \downarrow \\ U & \overset{\phi_0}{\to} & M, \end{array}$$

where ϕ_0 a local coordinate system at m and ϕ a bundle map which is a linear isomorphism on the fibers. The construction of the Gelfand-Fuks transform goes through and produces a map of complexes:

$$(C^*_{Lie}(\mathbb{W}_n \ltimes \mathfrak{gl}(N,\mathbb{C})[[\widehat{x}_1,\dots,\widehat{x}_n]]), O(n) \times U(N); \mathbb{C}), \partial_{Lie}) \to (\Omega^*(M), d).$$

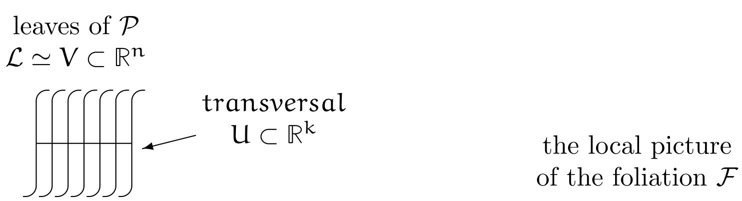
It is not difficult to see that the cohomology classes produced this way coincide with the Chern classes constructed with the help of a connection on the vector bundle E.

(4) Let $\mathbb{L} = CC^{per}_*(\mathbb{O}), b + uB)$, the cyclic periodic complex of $\mathbb{O}$. Then

$$\mathcal{L} = (J^\infty_\Delta(CC^{per}_*(\mathcal{O}(M)), b + uB) \simeq HC^{per}_*(\mathcal{O}(M)) \simeq H^*_{DR}(M)[u^{-1}u]].$$

4. Characteristic classes of foliations

In this subsection we will apply the method of the previous subsection for the construction of the secondary classes of a foliated manifold $(M,\mathcal{F})$.



Locally, M is diffeomorphic to the product

$$V \times U \subset \mathbb{R}^n \times \mathbb{R}^k$$

where k denotes the codimansion of the foliation and n the dimension of the leaves of $\mathcal{F}$. We will use $(x_1, \ldots, x_n)$ to denote the local leafwise coordinates on V and $(\lambda_1, \ldots, \lambda_k)$ for the local transversal coordinates on U. In paricular we have

$$\mathcal{F}|_{V \times U} \simeq T(V). \tag{4.1}$$

The construction below is essentially identical with the one described in the previous subsection, and we will just provide a "dictionary" for the changes required.

The infinitesimal model
$\mathbb{O} \rightsquigarrow \mathbb{O} = \mathbb{R}[[\hat{x}_1, \ldots, \hat{x}_n, \hat{\lambda}_1, \ldots, \hat{\lambda}_k]]$.
Set $\mathbb{O}^{\mathcal{F}}_m$ to be the subring of $\mathbb{O}_m$ consisting of jets at $m \in M$ of leafwise constant smooth functions on M.

The manifold of non linear frames

$$\widetilde{M}^{\mathcal{F}} = \left\{ (m, \phi) \mid \ m \in M,\ \phi : \mathbb{O}_m(M) \tilde{\to} \mathbb{O},\ \phi \text{ continuous},\ \phi(\mathbb{O}^{\mathcal{F}}_m) = \mathbb{R}[[\hat{\lambda}_1, \ldots, \hat{\lambda}_k]] \ \right\}$$

The structure group (of the jets of point-preserving diffeomorphisms $G^{\mathcal{F}}_0$

$$\left\{ J^\infty(g) \in \widehat{\mathrm{Diff}}(\mathbb{R}^{n+k})_{id} \mid g(0) = 0 \text{ and } g(x, \lambda) = \left(\tilde{x}(x, \lambda), \tilde{\lambda}(\lambda) \right) \right\}.$$

The Lie algebra of infinitesimal diffeomorphisms

$$\mathbb{W}_k \ltimes \mathbb{W}_n[[\hat{\lambda}_1, \ldots, \hat{\lambda}_k]].$$

The bundle $\mathcal{P}^F$ *of* $\mathcal{F}$*-adapted linear frames*

$$\{(X_1, \ldots, X_n, Y_1, \ldots, Y_k) \in \mathcal{P} \mid (X_1, \ldots, X_n) \text{ is a linear frame of} \mathcal{F}_m\}.$$

The maximal compact subgroup H *of the structure group of* $\mathcal{P}^{\mathcal{F}}$

$$O(n) \times O(k).$$

The arguments of the previous section apply verbatim to produce an H-equivariant section

$$\sigma : \mathcal{P}^{\mathcal{F}} \to \widetilde{M}^{\mathcal{F}}$$

and the Kazdan connection:

$$\omega_{\mathcal{F}} : T_{(m,\phi)}(\tilde{M}^{\mathcal{F}}) \to \mathbb{W}_k \ltimes \mathbb{W}_n[[\hat{\lambda}_1, \ldots, \hat{\lambda}_k]] \tag{4.2}$$

The GF construction gives us now the map of complexes:

$$(4.3)\qquad \begin{array}{c}(C^*_{Lie}(\mathbb{W}_k \ltimes \mathbb{W}_n[[\hat{\lambda}_1, \ldots, \hat{\lambda}_k]], O(k) \times O(n); \mathbb{C}), \partial_{Lie}) \\ \downarrow \\ (\Omega^*(M), d).\end{array}$$

In particular, since $\mathbb{W}_k$ is a quotient of $\mathbb{W}_k \ltimes \mathbb{W}_n[[\hat{\lambda}_1, \ldots, \hat{\lambda}_k]]$, we get:

$$\begin{array}{ccc} H^*_{Lie}(\mathbb{W}_k, O(k); \mathbb{C}) & \longrightarrow & H^*_{Lie}(\mathbb{W}_k \ltimes \mathbb{W}_n, O(k) \times O(n); \mathbb{C}) \\ & \searrow & \downarrow \\ & & H^*_{DR}(M) \end{array}$$

The dotted arrow in the diagram above produces *Gelfand-Fuks classes of the foliation* $\mathcal{F}$. We will end this subsection with an alternative description, due to Fuks. Let $\mathcal{N}$ denote the normal bundle of the foliation, i.e.

$$\mathcal{N} = TM/\mathcal{F} \text{ i. e. the cokernel of } \mathcal{F} \hookrightarrow T(M).$$

We choose some metric on $\mathcal{N}$, and let

$$p : \mathcal{P}_0 \to M$$

be the bundle of orthonormal frames in $\mathcal{N}$. For each point $\underline{m} = (m, (X_1(m), \ldots, X_k(m))$ of $\mathcal{P}_0$ we can choose a local immersion:

$$\iota_{\underline{m}} : \mathbb{R}^k \to \mathcal{P}_0, \ 0 \mapsto m$$

which is transversal to the foliation $p^*(\mathcal{F})$ of $\mathcal{P}_0$ and such that, for $\underline{m}'$ in a neighbourhood of $\underline{m}$,

$$(d\iota_{\underline{m}})_{\iota^{-1}_{\underline{m}}(\underline{m}')}(\partial_{x_i}) = X_i(\underline{m}') \mod p^*\mathcal{F}.$$

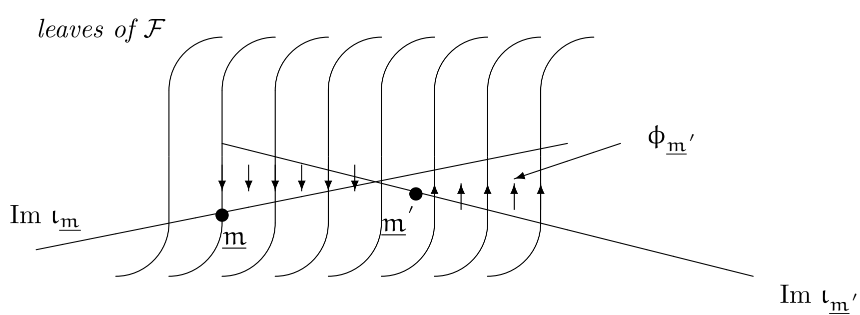
Now, given a point $\underline{m}$ in $\mathcal{P}_0$, each $\underline{m}'$ sufficiently close to $\underline{m}$ defines a local diffeomorphism:

$$\phi_{\underline{m}'} : \mathbb{R}^k \to (\mathbb{R}^k, 0)$$

given by the composition:

$$(4.4)\qquad \mathbb{R}^k \overset{\iota_{\underline{m}'}}{\to} \text{Im } \iota_{\underline{m}'} \overset{\mathcal{F}}{\to} \text{Im}\iota_{\underline{m}} \overset{\iota_{\underline{m}}^{-1}}{\to} \mathbb{R}^k,$$

where the $\mathcal{F}$-arrow referes to the flow along the leaves of the foliation:



The differential of the map:

$$\underline{m}' \to \phi_{\underline{m}'}$$

at $\underline{m}' = \underline{m}$ give an $O(k)$-equivariant one-form:

$$A : T_{\underline{m}}(\mathcal{P}_0) \to \mathbb{W}_k.$$

It is not difficult to see that $dA + \frac{1}{2}[A, A] = 0$ and that the Gelfand-Fuks classes as constructed above are given by the Gelfand-Fuks transform with respect to this A.

5. The Weyl algebra

Start with the algebra of h-differential operators on $\mathbb{C}^n$ with polynomial coefficients, i.e. the algebra over $\mathbb{C}[h]$ generated by x_j, ξ_j, $1 \le j \le n$, subject to relations

$$[x_j, x_k] = [\xi_j, \xi_k] = 0;\ [\xi_j, x_k] = \delta_{jk}. \tag{5.1}$$

We identify this algebra with $\mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n, h]$ as follows. For every monomial $x_1^{p_1} \ldots \xi_n^{q_n}$ write the products of p_1 copies of x_1, …, q_n copies of ξ_n in all possible orders, and then take the average of all these products. We get an associative multiplication on $\mathbb{C}[x_1, \ldots, x_n, \xi_1, \ldots, \xi_n, h]$. It is well known, and easy to see, that this multiplication is given by

$$f \star_W g(x, \xi) = \exp(\frac{h}{2}(\partial_\xi \partial_y - \partial_\eta \partial_x)(f(x, \xi) g(y, \eta)|_{x=y, \xi=\eta} \tag{5.2}$$

Here we write

$$x = (x_1, \ldots, x_n);\ \xi = (\xi_1, \ldots, \xi_n);\ \partial_y \partial_\xi = \sum_{j=1}^{n} \frac{\partial}{\partial y_j} \frac{\partial}{\partial \xi_j} \tag{5.3}$$

etc. We call this product the Moyal-Weyl product. Note that it extends to $C^\infty(\mathbb{R}^n)[[h]]$ and $\mathbb{C}[[x_1, \ldots, \xi_n, h]]$. In the latter case we change how we denote the generators (to stress that they are now formal variables) and write

$$\widehat{\mathbb{A}} = (\mathbb{C}[[\widehat{x}, \widehat{\xi}, h]], \star_W) = (\mathbb{C}[[\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{\xi}_1, \ldots, \widehat{\xi}_n, h]], \star_W) \tag{5.4}$$

Note that the symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ acts on $\widehat{\mathbb{A}}$ by automorphisms. In fact,

Lemma 5.0.1. *The Lie algebra*

$$\mathfrak{g}_0 = \{\frac{1}{h} q(\widehat{x}, \widehat{\xi})\}$$

where q are all quadratic polynomials acts on $\widehat{\mathbb{A}}$ by commutators, and this action coincides with the infinitesimal action of the Lie algebra $\mathfrak{sp}_{2n}$.

Proof. □

Since the product is $\mathfrak{sp}_{2n}$-invariant, it is better to pass to a more invariant notation and write

$$\widehat{y} = (\widehat{y}_1, \ldots, \widehat{y}_{2n}) = (\widehat{x}_1, \ldots, \widehat{x}_n, \widehat{\xi}_1, \ldots, \widehat{\xi}_n); \tag{5.5}$$

$$\omega_{j,j+n} = -\omega_{j+n,j} = 1, 1 \le j \le n;\ \omega_{kl} = 0 \tag{5.6}$$

in all other cases.

5.0.1. *Hochschild and cyclic homology of the Weyl algebra.* We will view $\widehat{\mathbb{A}}[h^{-1}]$ as an algebra over

$$K_h = \mathbb{C}((h)). \tag{5.7}$$

All our Hochschild and cyclic complexes will be over $\mathbb{C}[[h]]$ *or* K_h, *and all tensor products involved will be automatically completed.* In other words, by definition

$$\widehat{\mathbb{A}}^{\otimes(m+1)} = \mathbb{C}[[\widehat{y}^{(0)}, \ldots, \widehat{y}^{(m)}, h]] \tag{5.8}$$

Define the formal forms

$$\widehat{\Omega}^{\bullet} = \mathbb{C}[[\widehat{y}_1, \ldots, \widehat{y}_{2n}]]\{d\widehat{y}_1, \ldots, d\widehat{y}_{2n}\} \tag{5.9}$$

PROPOSITION 5.0.2. *There is a quasi-isomorphism*

$$(C_{\bullet}(\widehat{\mathbb{A}}), b) \xrightarrow{\sim} (\widehat{\Omega}^{2n-\bullet}[[h]], hd)$$

PROOF. Use the Koszul resolution

$$\widehat{\mathbb{A}} \otimes \wedge^{\bullet}(\mathbb{C}^{2n}) \otimes \widehat{\mathbb{A}}$$

(compare to ***ref***). □

Inverting h and applying the Poincaré lemma, we get

PROPOSITION 5.0.3. *a)*

$$\mathrm{HH}_{2n}(\widehat{\mathbb{A}}[h^{-1}]) \xrightarrow{\sim} K_h;$$
$$\mathrm{HH}_j(\widehat{\mathbb{A}}[h^{-1}]) = 0, \ j \neq 2n$$

The space HH_{2n} *is generated by the homology class of the cycle*

$$U_n = \frac{1}{h^n} 1 \otimes \mathrm{Alt}_{S_{2n}}(\widehat{y}_1 \otimes \ldots \otimes \widehat{y}_{2n})$$

b)

$$\mathrm{HC}^-_{2n-2k}(\widehat{\mathbb{A}}[h^{-1}]) \xrightarrow{\sim} K_h, \ k \geq 0;$$
$$\mathrm{HC}^-_j(\widehat{\mathbb{A}}[h^{-1}]) = 0, \ j \neq 2n$$

for all other j. *The space* HC^-_{2n-2k} *is generated by the class of* $u^k U_n$.

c)

$$\mathrm{HC}_{2n+2k}(\widehat{\mathbb{A}}[h^{-1}]) \xrightarrow{\sim} K_h, \ k \geq 0;$$
$$\mathrm{HC}_j(\widehat{\mathbb{A}}[h^{-1}]) = 0$$

for all other j. *The space* HC_{2n+2k} *is generated over* K_h *by the class of* $u^{-k}U_n$.

COROLLARY 5.0.4.

$$\overline{\mathrm{HC}}_{2n-1-2k}(\widehat{\mathbb{A}}[h^{-1}]) \xrightarrow{\sim} K_h, \ k \geq 0; \ \mathrm{HC}_j(\widehat{\mathbb{A}}[h^{-1}]) = 0$$

for all other j.

The class of

$$\overline{U}_n = \frac{1}{2nh^n} \mathrm{Alt}_{S_{2n}}(\widehat{y}_1 \otimes \ldots \otimes \widehat{y}_{2n}) \in \overline{C}^{\lambda}_{2n-1}(\widehat{\mathbb{A}}[h^{-1}])$$

generates $\overline{\mathrm{HC}}_{2n-1}(\widehat{\mathbb{A}}[h^{-1}])$ *over* K_h.

DEFINITION 5.0.5. *Let*

$$\mathrm{tr} : \mathrm{HC}^-_{\bullet}(\widehat{\mathbb{A}}[h^{-1}]) \to K_h[2n][[u]]$$

be the $K_h[[u]]$*-linear isomorphism sending* U_n *to* 1.

Let us go back to Proposition 5.0.2. The right hand side has the following invariant meaning. Let

$$\pi = \sum_{j,k} \omega_{jk} \frac{\partial}{\partial \widehat{y}_j} \frac{\partial}{\partial \widehat{y}_k} \tag{5.10}$$

be the Poisson bivector of the symplectic form ω. Put

$$L_\pi = [d, \iota_\pi] : \widehat{\Omega}^\bullet \to \widehat{\Omega}^{\bullet-1} \tag{5.11}$$

Note that $[d, L_\pi] = 0$ and that there is an isomorphism

$$(\widehat{\Omega}^\bullet, L_\pi) \tilde{\to} (\widehat{\Omega}^{2n-\bullet}, d) \tag{5.12}$$

Define the Brylinski double complex

$$(\widehat{\Omega}^\bullet[[h]][[u]], hL_\pi + ud) \tag{5.13}$$

One might expect a quasi-isomorphism between the above and the negative cyclic complex of $\widehat{\mathbb{A}}$. This is certainly the case after we invert h (Lemma 5.0.3 can be immediately upgraded to give this). We will show below that this is indeed the case even without inverting h.

5.1. The trace density map. We will now describe an explicit representative of the trace map from Definition 5.0.5.

For $0 \le a, b \le m$, define

$$\pi_{ab} : \widehat{\mathbb{A}}^{\otimes(m+1)} \to \widehat{\mathbb{A}}^{\otimes(m+1)};\ \pi_{ab} = \sum_{j,k} \omega_{jk} \frac{\partial}{\partial \widehat{y}_j^{(a)}} \frac{\partial}{\partial \widehat{y}_k^{(b)}} \tag{5.14}$$

In this notation, for example,

$$f \star_W g = \mu \circ \exp(\frac{h}{2}\pi_{01})(f(\widehat{y}^{(1)})g(\widehat{y}^{(2)})) \tag{5.15}$$

Here

$$\mu : F(\widehat{y}^{(1)}, \widehat{y}^{(2)})) \mapsto F(\widehat{y}, \widehat{y}). \tag{5.16}$$

Now define

$$\mathrm{TR} : \widehat{\mathbb{A}}^{\otimes m+1} \to \widehat{\Omega}^m[[h]] \tag{5.17}$$

$$\mathrm{TR} = \mathrm{HKR} \circ \int_{\Delta^m} \exp(\frac{h}{2} \sum_{0 \le a < b \le m} (2t_b - 2t_a - 1)\pi_{ab}) dt_1 \dots dt_m \tag{5.18}$$

Here

$$\Delta^m = \{(t_0, \dots, t_m) | 0 = t_0 \le \dots \le t_m \le 1\} \tag{5.19}$$

is the standard simplex and

$$\mathrm{HKR} : \mathbb{C}[[\widehat{y}_1, \dots, \widehat{y}_{2n}]]^{\otimes m+1} \to \widehat{\Omega}^m$$

is the usual HKR map (we extend the scalars to $\mathbb{C}[[h]]$).

Proposition 5.1.1. *The maps* (5.17) *define an* $\mathfrak{sp}_{2n}$*-equivariant quasi-morphism of complexes*

$$\mathrm{TR} : CC_\bullet^-(\widehat{\mathbb{A}}) \to (\widehat{\Omega}^\bullet[[h, u]], hL_\pi + ud) \tag{5.20}$$

Proof. □

5.2. Lie algebra homology and Hochschild and cyclic complexes of the Weyl algebra.

5.2.1. *Derivations of the Weyl algebra.* The Lie algebra

$$\widetilde{\mathfrak{g}} = \frac{1}{h}\widehat{\mathbb{A}} \tag{5.21}$$

with the bracket $[a, b] = a \star_W b - b \star_W a$ acts on $\widehat{\mathbb{A}}$ by derivations. Define also the quotient

$$\mathfrak{g} = \frac{1}{h}\widehat{\mathbb{A}}/\frac{1}{h}\mathbb{C}[[h]] \tag{5.22}$$

It can be easily shown that the Lie algebra of all continuous derivations of $\widehat{\mathbb{A}}$ coincides with $\mathfrak{g}$.

We have the complex

$$C^\bullet(\mathfrak{g}, \mathrm{sp}_{2n}; \mathrm{CC}^-(\widehat{\mathbb{A}}[h^{-1}])) \tag{5.23}$$

and its counterparts when $\mathrm{CC}^-_\bullet$ is replaced by $C_\bullet$, $\mathrm{CC}_\bullet$, etc.

5.3. The distinguished (co)homology classes.

LEMMA 5.3.1. *There are cochains that are unique up to homology:*

a)

$$\mathbb{U} = \sum_{j,k\geq 0} u^j U^{(k)}_{2(n+j)} \in C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; \mathrm{CC}^-_\bullet(\widehat{\mathbb{A}}[h^{-1}]))$$

where

$$U^{(k)}_{n,j} \in C^k(\mathfrak{g}, \mathfrak{sp}_{2n}; C_{2(n+j)+k}(\widehat{\mathbb{A}}[h^{-1]}]))$$

and $U^{(0)}_{n,0} = U_n$ *as in Proposition 5.0.2. The class* $\mathbb{U}$ *freely generates*

$$\mathbb{H}^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; \mathrm{CC}^-_\bullet(\widehat{\mathbb{A}}[h^{-1}]))$$

over $H^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h[[u]])$.

b)

$$\overline{\mathbb{U}} = \sum \overline{U}^{(k)}_n \in C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; \overline{\mathrm{CC}}_\bullet(\widehat{\mathbb{A}}[h^{-1}]))$$

where

$$\overline{U}^{(k)}_n \in C^k(\mathfrak{g}, \mathfrak{sp}_{2n}; \overline{\mathrm{CC}}_{2n-1+k}(\widehat{\mathbb{A}}[h^{-1}]))$$

and $\overline{U}^{(0)}_n = \overline{U}_n$ *as in Corollary 5.0.4. The class* $\overline{\mathbb{U}}$ *freely generates*

$$\mathbb{H}^{1-2n}(\mathfrak{g}, \mathfrak{sp}_{2n}; \overline{\mathrm{CC}}_\bullet(\widehat{\mathbb{A}}[h^{-1}]))$$

over $H^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h)$.

PROOF. Follows from Proposition 5.0.2 and Corollary 5.0.4. ◻

Recall the boundary map

$$B : \overline{\mathrm{CC}}_{\bullet-1}(A) \to \mathrm{CC}^{\mathrm{per}}_\bullet(A)$$

for any algebra A.

LEMMA 5.3.2.

$$B\overline{\mathbb{U}} = \mathbb{U}$$

PROOF. ◻

Definition 5.3.3. *Let*

$$\mathbf{tr} : \mathbb{H}^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; CC^-_\bullet(\widehat{\mathbb{A}}[h^{-1}])) \to \mathbb{H}^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h[2n][[u]])$$

be the $H^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h[[u]])$*-linear isomorphism sending* $\mathbb{U}$ *to* 1.

6. The algebraic index theorem

We now have two free generators of

$$\mathbb{H}^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; CC^{per}_\bullet(\widehat{\mathbb{A}}[h^{-1}]))$$

over $H^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h((u)))$. They are both of degree zero. One is equal to 1, the image of the basis element of $CC^{per}(K_h)$. The other is $u^{-n}\mathbb{U}$. One version of the algebraic index theorem compares them to one another. Another, equivalent form of the theorem is the statement about $\mathbf{tr}(1)$ where $\mathbf{tr}$ is as in Defintion 5.3.3.

Let us first introduce the elements we need in $H^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h)$.

6.1. Characteristic classes in Lie algebra cohomology. Given a Lie algebra $\mathfrak{g}$ over k and a Lie subalgebra $\mathfrak{h}$ that acts on $\mathfrak{g}$ reductively, define the morphism

$$c: S^\bullet(\mathfrak{h}^*)^{\mathfrak{h}} \otimes_k K \to H^{2\bullet}(\mathfrak{g}, \mathfrak{h}; K) \tag{6.1}$$

(where $k \subset K$) as follows. Choose an $\mathfrak{h}$-equivariant projection $A : \mathfrak{g} \to \mathfrak{h}$ and define

$$R(X, Y) = [A(X), A(Y)] - A([X, Y]) \in C^2(\mathfrak{g}, \mathfrak{h}; \mathfrak{h}) \tag{6.2}$$

Now for any $P \in S^m\mathfrak{h})^{\mathfrak{h}}$ define

$$c_P = P(R) \in C^{2m}(\mathfrak{g}, \mathfrak{h}; K) \tag{6.3}$$

Lemma 6.1.1. *The cochain* c_P *is a cocycle whose class does not depend on a choice of* A.

Proof. Note that

$$R = \partial^{\mathrm{Lie}} A + \frac{1}{2}[A, A];$$

therefore

$$\partial^{\mathrm{Lie}} R + [A, R] = 0;$$
$$\partial^{\mathrm{Lie}} P(R) = \pm(\mathrm{ad}_A P)(R) = 0.$$

If we have another choice for A (that we denote by A'), then

$$A' = A + B,\ B \in C^1(\mathfrak{g}, \mathfrak{h}; \mathfrak{h});$$

$$R' = \partial^{\mathrm{Lie}} A' + \frac{1}{2}[A', A'] = R + \partial^{\mathrm{Lie}} B + [A, B];$$
$$P(R') =$$

FINISH □

Define also $\frac{1}{h}\theta \in H^2(\mathfrak{g}, \mathfrak{h}; K_h)$ as the class of the extension

$$0 \to \frac{1}{h}\mathbb{C}[[h]] \to \widetilde{\mathfrak{g}} \to \mathfrak{g} \to 0 \tag{6.4}$$

(cf. (5.22)). In other words,

$$\frac{1}{h}\theta = c_P \in H^2(\widetilde{\mathfrak{g}}, \mathfrak{sp}_{2n} \oplus \frac{1}{h}\mathbb{C}[[h]]; K_h) = H^2(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h)$$

where P is the invariant polynomial of degree one that consists of the projection along $\mathfrak{sp}_{2n}$ followed by the embedding into K_h.

Define the invariant power series on $\mathfrak{gl}_n$ by its restriction to the subalgebra of diagonal matrices:

$$\widehat{A}(x_1, \ldots, x_n) = \prod_{j=1}^{n} \frac{x_j}{e^{x_j/2} - e^{-x_j/2}}$$

Consider the restrictions

$$S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n} \xrightarrow{j} S(\mathfrak{sp}_{2n}^*)^{\mathfrak{sp}_{2n}} \xrightarrow{i} S(\mathfrak{gl}_n^*)^{\mathfrak{gl}_n}$$

Note that j is onto and i is an embedding and $\widehat{A}$ is in the image of i.. ***Ref?*** We use the identification

$$\sqrt{\widehat{A}} = j(\sqrt{\widehat{A}}) \in S(\mathfrak{sp}_{2n}^*)^{\mathfrak{sp}_{2n}} \tag{6.5}$$

as well as

$$\sqrt{\widehat{A}} = c_{\sqrt{\widehat{A}}} \in H^{\mathrm{ev}}(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h) \tag{6.6}$$

Remark 6.1.2. Note that $ij(\sqrt{\widehat{A}}) = \widehat{A}$. In our earlier works [?] and [**69**], the main motivation was the case $M = T^*X$ (see below) whose cohomology we identified with that of X. That is why we denoted by $\widehat{A}$ what we are now denoting by $\sqrt{\widehat{A}}$.

6.2. The main theorem.

Theorem 6.2.1.

$$\mathrm{tr}(1) = \sum_{m=0}^{\infty} u^{-m}(\sqrt{\widehat{A}} e^{\theta/h})_{2m}$$

in $\mathbb{H}^0(\mathfrak{g}, \mathfrak{sp}_{2n}; K_h((u)))$.

Or equivalently:

Theorem 6.2.2.

$$u^{-n}\mathbb{U} = \sum_{m=0}^{\infty} u^{-m}((\sqrt{\widehat{A}} e^{\theta/h})^{-1})_{2m}$$

in $\mathbb{H}^0(\mathfrak{g}, \mathfrak{sp}_{2n}; CC_\bullet^{\mathrm{per}}(\widehat{\mathbb{A}}[h^{-1}]))$.

We will present two proofs: one from [?] and [?], using the computation of $\partial\overline{\mathbb{U}}$; the other, a mix of an argument from ***FT RR*** with some sort of an algebraic version of the heat kernel proof of the index theorem. We also will introduce a refined version for the trace map TR.

6.3. Generalized characteristic classes. Under the assumptions of 6.1, define $\mathfrak{g}[\epsilon]$ to be the dg Lie algebra with the differential $\frac{\partial}{\partial \epsilon}$, where $\epsilon^2 = 0$ and $|\epsilon| = -1$.

Consider a dg $\mathfrak{g}[\epsilon]$-module V. We assume that V is completely reducible over $\mathfrak{h}$. Define the generalized characteristic map

$$c : C^\bullet(\mathfrak{h}[\epsilon], \mathfrak{h}; V) \to C^\bullet(\mathfrak{g}, \mathfrak{h}; V) \tag{6.7}$$

as follows. Define

$$\iota \in C^1(\mathfrak{g}, \mathfrak{h}; \mathrm{End}(V)) \tag{6.8}$$

by

$$\iota(X)(v) = (X - A(X))v \tag{6.9}$$

for $X \in \mathfrak{g}$ and $v \in V$. Now, for $\varphi \in C^\bullet(\mathfrak{h}[\epsilon], \mathfrak{h}; V)$ define

$$c(\varphi) = \sum_{k,l \geq 0} \iota^k \varphi(\epsilon R, \dots, \epsilon R) \tag{6.10}$$

where there are l arguments ϵR in φ.

Lemma 6.3.1. *Formula* (6.10) *defines a morphism of complexes.*

Proof. □

6.4. First proof: the boundary map. Consider the image of $\mathbb{U} = B\overline{\mathbb{U}}$ in $\mathbb{H}^{-2n}(\mathfrak{g}, \mathfrak{sp}_{2n}; CC_\bullet^{per}(\widehat{\mathbb{A}}[h^{-1}]))$. Observe that this is equal to the following: take $\partial\overline{\mathbb{U}}$ in $\mathbb{H}^{2-2n}(\mathfrak{g}, \mathfrak{sp}_{2n}; CC_\bullet(K_h))$; use the embedding $CC_\bullet(K_h) \to CC_{\bullet+2}^{per}(K_h)$; and then embed $CC_\bullet^{per}(K_h)$ into $CC_\bullet^{per}(\widehat{\mathbb{A}}[h^{-1}])$. To see that the two classes are indeed the same, represent $\overline{\mathbb{U}}$ by a cocycle in $C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; \overline{CC}_\bullet(\widehat{\mathbb{A}}[h^{-1}]))$. Lift this cocycle to a cochain in $C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; CC_\bullet(\widehat{\mathbb{A}}[h^{-1}]))$. View this cochain as an element of $C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; CC_\bullet^{per}(\widehat{\mathbb{A}}[h^{-1}]))$. (More precisely: multiply this cochain by u^{-1} and then define all the coefficients at u^j, $j \geq 0$, to be zero). The differential of this element is the difference of cochains representing the two classes defined above.

Now compute $\partial\overline{\mathbb{U}}$. For that, note that $\overline{C}_\bullet^\lambda(\widehat{\mathbb{A}}[h^{-1}])$ is a dg module over $\mathfrak{g}[\epsilon]$. ***Ref*** Now interprete $\partial\overline{\mathbb{U}}$ as the value of the characteristic map c (cf. (6.10)) on an analogue of $\partial\overline{\mathbb{U}}$ in $\mathbb{H}^{1-2n}(\mathfrak{h}[\epsilon], \mathfrak{h}; \overline{C}_\bullet^\lambda(\widehat{\mathbb{A}}[h^{-1}]))$. The latter can be defined explicitly as *** Now apply to the above the Brodzki cocycle ***ref ***FINISH

6.5. Algebraic index theorem for TR. Now we will state the algebraic index theorem for the trace map TR instead of tr. First extend TR to the Lie algebra cohomology setting.

Proposition 6.5.1. *The trace density morphism* TR (5.20) *extends to a cocycle*

$$\mathrm{TR} \in C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; \mathrm{Hom}^\bullet_{\mathbb{C}[[h,u]]}(CC^-(\widehat{\mathbb{A}}), \widehat{\Omega}^\bullet((h))[[u]])) \tag{6.11}$$

Proof. First observe that both sides in TR are $\mathfrak{g}$-modules. Indeed, the action of $\mathfrak{g}$ on the right hand side factors through the projection

$$\frac{1}{h} f \mapsto f \bmod h \tag{6.12}$$

This projection intertwines $[\,,]$ with the Poisson bracket $\{\,,\}$. The Lie algebra $(\mathbb{C}[[\widehat{y}]]/\mathbb{C}, \{\,,\})$ acts on $\widehat{\Omega}^\bullet[[h,u]]$ preserving the differential $hL_\pi + ud$.

These modules are not dg modules over $\mathfrak{g}[\epsilon]$, and neither is

$$\mathrm{Hom}^\bullet_{\mathbb{C}[[h,u]]}(CC^-(\widehat{\mathbb{A}}), \widehat{\Omega}^\bullet[[h,u]]) \tag{6.13}$$

All three become ones after inverting h. Observe that TR is a cocycle in (6.13) on which $\epsilon\mathfrak{sp}_{2n}$ acts by zero. Indeed, ***Finish*** Apply the characteristic map (6.7) to TR. Note that

$$c(\mathrm{TR}) = \sum_{k \geq 0} \iota^k \mathrm{TR}$$

where ι is defined in (6.9). (This is because substitution of ϵR into TR gives zero). For a power series $F \in \mathbb{C}[[\widehat{y}, h]]$, we write

$$F = F(0) + hF(1) + \dots,\ F(j) \in \mathbb{C}[[\widehat{y}]];$$

we also write

$$\overline{F} = F - F^{(2)}(0)$$

where $F^{(2)}(0)$ is the quadratic part of $F(0)$. Then

$$\iota(\frac{1}{h}F)(\varphi) = \frac{1}{h}(\varphi L_{\overline{F}} - (d\overline{F}(0) \wedge\ )\varphi)$$

We would like to claim that ι^kTR do not contain negative powers of h. However, this does not seem to be the case for $k > 1$.

FINISH □

DEFINITION 6.5.2. *Define*

$$\mathrm{TR}_\pi = \exp(\frac{h\iota_\pi}{u}) \circ \mathrm{TR} \in C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; \mathrm{Hom}^\bullet_{\mathbb{C}[[h]]((u))}(\mathrm{CC}^{\mathrm{per}}_\bullet(\widehat{\mathbb{A}}), \widehat{\Omega}^\bullet((h))((u)))$$

LEMMA 6.5.3. *If $\widehat{\Omega}^\bullet[[h]]((u))$ is defined as a complex with the differential ud then TR_π is a cocycle.*

PROOF. Follows from the Cartan relation $[d, \iota_\pi] = L_\pi$. □

DEFINITION 6.5.4. *Let* I *be the $\mathfrak{g}$-invariant isomorphism*

$$\mathrm{CC}^{\mathrm{per}}_\bullet(\widehat{\mathbb{A}}) \xrightarrow{\sim} \mathrm{CC}^{\mathrm{per}}_\bullet(\mathbb{C}[[\widehat{y}, h]])$$

****REF*** (Goodwillie, ...) viewed as a cocycle in*

$$C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; \mathrm{Hom}^\bullet(\mathrm{CC}^{\mathrm{per}}(\widehat{\mathbb{A}}), \mathrm{CC}^{\mathrm{per}}(\mathbb{C}[[\widehat{y}, h]]))$$

Define

$$\sigma = \mathrm{HKR} \circ \mathrm{I} \in C^\bullet(\mathfrak{g}, \mathfrak{sp}_{2n}; \mathrm{Hom}^\bullet(\mathrm{CC}^{\mathrm{per}}(\widehat{\mathbb{A}}), \widehat{\Omega}^\bullet[[h]]((u)))$$

THEOREM 6.5.5.

$$\mathrm{TR}_\pi = \sigma \wedge \sum_k u^{-k}\widehat{A}_{2k}$$

7. Deformation quantization

Let (M, π) be a symplectic manifold. A deformation quantization of (M, π) is a formal series

$$f \star g = fg + \sum_{k=1}^{\infty} h^k P_k(f, g) \tag{7.1}$$

such that $\star$ is associative, $1 \star f = f \star 1 = f$, P_k are bidifferential operators, and

$$P_1(f, g) - P_1(g, f) = \{f, g\}_\pi \tag{7.2}$$

7.1. The Fedosov construction. We have defined the algebra $\widehat{\mathbb{A}}$ and the Lie algebras $\mathfrak{g}$ and $\widetilde{\mathfrak{g}}$ acting on it. The symplectic group SP_{2n} acts on all three and preserves the action. Therefore, given a symplectic manifold (M, ω), we get a bundle of algebras $\widehat{\mathbb{A}}_M$ and two bundles of Lie algebras $\widetilde{\mathfrak{g}}_M \to \mathfrak{g}_M$.

Denote by $\widehat{\mathbb{A}}^\bullet_M$, $\widetilde{\mathfrak{g}}^\bullet_M$, and $\mathfrak{g}^\bullet_M$ the sheaves of differential forms with values in $\widehat{\mathbb{A}}_M$, $\widetilde{\mathfrak{g}}_M$, and $\mathfrak{g}_M$, respectively.

8. Algebraic index theorem for deformation quantization

THEOREM 8.0.1.

$$\begin{array}{ccc} CC^{per}(\widehat{\mathbb{A}}_M) & \xrightarrow{TR_\pi} & \Omega^\bullet_M[[h]]((u)) \\ \downarrow I & & \downarrow \sqrt{\widehat{A}_u(TM)} \\ CC^{per}_\bullet(\mathcal{O}_M[[h]]) & \xrightarrow{HKR} & \Omega^\bullet_M[[h]]((u)) \end{array} \tag{8.1}$$

where:
I *is the Goodwillie isomorphism ***Ref****

$$TR_\pi = \exp(\frac{u\iota_\pi}{h}) \circ TR \tag{8.2}$$

9. Appendix: the general case

Here we discuss, with only a brief sketch of a proof, the algebraic index theorem for general deformation quantizations.

THEOREM 9.0.1. *There exists an* L_∞ *quasi-isomorphism*

$$K : \wedge^{\bullet+1}T_M \to C^{\bullet+1}(\mathcal{O}_M, \mathcal{O}_M) \tag{9.1}$$

and a compatible quasi-isomorphism of L_∞ *modules over* $\wedge^{\bullet+1}(T_M)$

$$S : CC^-_\bullet(\mathcal{O}_M, \mathcal{O}_M) \to (\Omega^\bullet_M[[u]], ud) \tag{9.2}$$

The initial terms of K *and are the* HKR *morphisms.*

Such pairs (K, S) are constructed from a piece of data called *a Drinfeld associator* and denoted by Φ. We will denote the pair constructed from Φ by (K_Φ, S_Φ).

DEFINITION 9.0.2. *A formal Poisson structure on* M *is a formal series*

$$\pi = \pi_0 + h\pi_1 + \ldots \in \wedge^2 T_M[[h]]$$

satisfying $\{\pi, \pi\} = 0$.

For a formal Poisson structure π put

$$\Pi_{\pi,\Phi} = \sum_{k=1}^{\infty} K_{\Phi,k}(h\pi, \ldots, h\pi) \tag{9.3}$$

(k arguments $h\pi$).

LEMMA 9.0.3. *Let*

$$f \star_{\pi,\Phi} g = fg + \Pi_{\pi,\Phi}(f, g)$$

Then $\star_{\pi,\Phi}$ *is a deformation quantization of* π_0.

The 2-cochain $\Pi_{\pi,\Phi}$ is an MC element of $C^2(\mathcal{O}_M, \mathcal{O}_M)[[h]]$, and we write

$$\mathcal{O}_{\pi,\Phi,M} = (\mathcal{O}_M[[h]], \star_{\pi,\Phi}) \tag{9.4}$$

LEMMA 9.0.4.

$$C^\bullet(\mathcal{O}_{\pi,\Phi,M}, \mathcal{O}_{\pi,\Phi,M}) \xrightarrow{\sim} (C^\bullet(\mathcal{O}_M, \mathcal{O}_M)[[h]], \delta + [\Pi_{\pi,\Phi}, \,]$$

$$CC^-_\bullet(\mathcal{O}_{\pi,\Phi,M}) \xrightarrow{\sim} (CC^-_\bullet(\mathcal{O}_M)[[h]], b + L_{\Pi_{\pi,\Phi}} + uB)$$

THEOREM 9.0.5.

$$\begin{array}{ccc} CC_\bullet^{per}(\mathcal{O}_{\pi,\Phi,M}) & \xrightarrow{TR_{\pi,\Phi}} & \Omega_M^\bullet[[h]]((u)) \\ \downarrow I & & \downarrow \sqrt{\widehat{A}_{u,\Phi}(TM)} \\ CC_\bullet^{per}(\mathcal{O}_M[[h]]) & \xrightarrow{HKR} & \Omega_M^\bullet[[h]]((u)) \end{array} \tag{9.5}$$

is homotopy commutative. Here I *is the Goodwillie rigidity isomorphism,*

$$\widehat{A}_{u,\Phi}(T_M) = \sum_{k\geq 0} u^{-k}\widehat{A}_{\Phi,2k}(T_M)$$

and

$$\widehat{A}_\Phi(T_M) = \sum_{k\geq 0} \widehat{A}_{\Phi,2k}(T_M)$$

is the characteristic class of the tangent bundle that is defined by an invariant power series $\widehat{A}_\Phi$ *whose restriction from* $\mathfrak{g}_{2n}$ *to* $\mathfrak{sp}_{2n}$ *is* $\widehat{A}$. ****Explain more****

9.1. Sketch of the proofs. We start with the formal situation. For any $n > 0$, put

$$\widehat{\mathcal{O}} = \widehat{O}_n = \mathbb{C}[[x_1, \ldots, x_n]];\ \ \widehat{\Omega} = \widehat{\mathcal{O}}\{dx_1, \ldots, dx_n\}; \tag{9.6}$$

$$\wedge^\bullet \widehat{T} = \widehat{\mathcal{O}}\{\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}\};\ \ \widehat{T} = \wedge^1(\widehat{T}); \tag{9.7}$$

The Schouten bracket makes $\wedge^{\bullet+1}(\widehat{T})$ a graded Lie algebra of which $\widehat{T}$ is a Lie subalgebra. In $\widehat{T}$, there is a Lie subalgebra $\mathfrak{gl}_n$ consisting of formal vector fields $\sum a_{jk}x_j\frac{\partial}{\partial x_k}$.

We start with L_∞ quasi-isomorphism

$$\widehat{K}_\Phi : \wedge^{\bullet+1}(\widehat{T}) \to C^{\bullet+1}(\widehat{\mathcal{O}}, \widehat{\mathcal{O}}) \tag{9.8}$$

and an L_∞ module quasi-isomorphism over $\wedge^{\bullet+1}(\widehat{T})$

$$\widehat{S}_\Phi : CC_\bullet^-(\widehat{\mathcal{O}}) \to (\widehat{\Omega}^\bullet[[u]], ud) \tag{9.9}$$

with the following additional properties:

(1) $\widehat{K}_\Phi$ and $\widehat{S}_\Phi$ are $\mathfrak{gl}_n$-equivariant;
(2) $\widehat{K}_{\Phi,1}(h) = h$ for $h \in \mathfrak{gl}_n$;
(3) For $h \in \mathfrak{gl}_n$, $\widehat{K}_{\Phi,k}(h, \ldots) = 0$ for $k > 1$ and $\widehat{S}_{\Phi,1}(h, \ldots) = 0$ for $k \geq 1$.

(By $\ldots$ we mean *any* arguments, not just those in $\mathfrak{gl}_n$).

From (9.8) and (9.9) we produce MC elements in

$$C^\bullet(\widehat{T}, \mathfrak{gl}_n; \mathrm{Hom}^\bullet(\wedge^{\bullet+1}(\widehat{T}), C^{\bullet+1}(\widehat{\mathcal{O}}, \widehat{\mathcal{O}})) \tag{9.10}$$

and

$$C^\bullet(\widehat{T}, \mathfrak{gl}_n; \mathrm{Hom}^\bullet(CC_\bullet^-(\widehat{\mathcal{O}}), \widehat{\Omega}^\bullet[[u]])) \tag{9.11}$$

(by splitting the arguments into two groups, one in $\widehat{T}$ and the second anywhere). The images of those under the Gelfand-Fuks map GF provide the morphisms K_Φ and S_Φ.

DEFINITION 9.1.1. *We denote by*

$$\sqrt{\widehat{A}_{u,\Phi}} \in \mathbb{H}^0(\widehat{T}, \mathfrak{gl}_n; \widehat{\Omega}^\bullet((u)))$$

the evaluation of (9.11) *at the periodic cyclic cycle* 1. *We also define*

$$\sqrt{\widehat{A}}_{\Phi,u}(T_M) = \mathrm{GF}(\widehat{A}_{u,\Phi}) \in \mathbb{H}^0(M, \Omega_M^\bullet((u)))$$

Recall that for any algebra A the negative cyclic complex $\mathrm{CC}_\bullet^-(A)$ is an L_∞ module over the DGLA

$$(\mathfrak{g}_A[\epsilon][[u]], \delta + u\frac{\partial}{\partial\epsilon}) \tag{9.12}$$

where $\mathfrak{g}_A = C^{\bullet+1}(A, A)$. Note that the DGLA $(\mathfrak{g}_M[\epsilon][[u]], u\frac{\partial}{\partial\epsilon})$ acts on $(\Omega^\bullet[[u]], ud)$ where $\mathfrak{g}_M = \wedge^{\bullet+1}(T_M)$. Our first claim is that S_Φ extends to an L_∞ module morphism over $(\mathfrak{g}_A[\epsilon][[u]], \delta + u\frac{\partial}{\partial\epsilon})$.

This follows from ***Homotopy calculus operad and the fact that (K_Φ, S_Φ) extend to a morphism of homotopy calculi-EXPLAIN MORE?***

From this we deduce that the diagram (8.1) commutes up to homotopy. Indeed, I is obtained by a universal formula in terms of the L_∞ action of $\mathfrak{g}_A[\epsilon][[u]]$. The only term not involving higher terms in the L_∞ module action is $\exp(\frac{h\iota_p}{u})$. ***Explain a bit more***

It remains to show that the restriction of $\sqrt{\widehat{A}_{\Phi,u}}$ to $\mathfrak{sp}_{2n}$ is $\sqrt{\widehat{A}}_u$.This follows from carrying out our construction in the symplectic case and from the fact that in that case TR_π is unique up to homotopy. ***Explain more***

9.2. Further remarks and questions.

9.2.1. *Relation to the symplectic case.* The general setting of 9 suggests that some version of the trace density morphism of Proposition 6.5.1. **The following is probably true**.

Let us go back to the symplectic case. Let

$$\mathfrak{g}_{\mathrm{Ham}} = \mathbb{C}[[\widehat{y}_1, \dots, \widehat{y}_{2n}]]/\mathbb{C} \tag{9.13}$$

be the Lie algebra of formal Hamiltonian vector fields. As we observed before (6.12), $\mathfrak{g}$ acts on formal forms via

$$\mathfrak{g} \to \mathfrak{g}_{\mathrm{Ham}};\ \frac{1}{h}f \mapsto (f \bmod h) \tag{9.14}$$

To the contrary, $\mathfrak{g}_{\mathrm{Ham}}$ does not act on the negative cyclic complex of the deformed algebra. However, *it acts up to inner derivations*, namely:

LEMMA 9.2.1. *For* $f \in \mathfrak{g}_{\mathrm{Ham}}$, *define* $D(f) : \widehat{\mathbb{A}} \to \widehat{\mathbb{A}}$; $D(f)(a) = \frac{1}{h}[f, a]$. *Then there are elements* $c(f, g) \in \widehat{\mathbb{A}}$, *bilinear and skew-symmetric in* f *and* g, *such that*

(1)
$$[D(f), D(g)] = D(\{f, g\}) + \mathrm{ad}(c(f, g));$$

(2)
$$D(f)c(g, h) + D(g)c(h, f) + D(h)c(f, g) - c(\{f, g\}, h) - c(\{g, h\}, f) - c(\{h, f\}, g) = 0$$

In addition,

$$c(q, f) = 0$$

for a quadratic function q.

For such an action of a Lie algebra on an algebra,on cal still define the Lie algebra cochain complex with coefficients in the Hochschild complex

$$\mathcal{L} \in C^{\bullet}(\mathfrak{g}_{\mathrm{Ham}}, \mathfrak{sp}_{2n}; C_{\bullet}(\widehat{\mathbb{A}})) \tag{9.15}$$

of the algebra. One just has to add a new term to the differential, namely: multiplication by a cochain

$$\mathcal{L} \in C^{2}(\mathfrak{g}_{\mathrm{Ham}}, \mathfrak{sp}_{2n}; \mathrm{End}^{-1} C_{\bullet}(\widehat{\mathbb{A}})) \tag{9.16}$$

$$\mathcal{L}(f, g) = L_{c(f,g)}$$

Similarly for negative cyclic, etc. complexes.

It is probably true that TR extends to a cocycle

$$\mathrm{TR} \in C^{\bullet}(\mathfrak{g}_{\mathrm{Ham}}, \mathfrak{sp}_{2n}; \mathrm{Hom}^{\bullet}(\mathrm{CC}^{-}_{\bullet}(\widehat{\mathbb{A}}), \widehat{\Omega}^{2n-\bullet}[[h, u]])) \tag{9.17}$$

(no need for inverting h).

9.2.2. *On the class* $\widehat{A}_{\Phi}$. So far we have established that $\widehat{A}_{\Phi}$ is a characteristic class defined by an invariant power series on $\mathfrak{gl}$ whose restriction to $\mathfrak{sp}$ is $\widehat{A}$. Next, we claim that

$$\widehat{A}_{\Phi}(x_1, \ldots, x_n) = \widehat{A}_{\Phi}(x_1) \ldots \widehat{A}_{\Phi}(x_n)$$

where $\widehat{A}_{\Phi}(x)$ is a power series in one variable whose even part is $\widehat{A}(x)$.

This should be based on the following. Recall that a homotopy Gerstenhaber structure on $C^{\bullet}(A, A)$ is an MC element in a DGLA

$$\mathrm{Def}^{\bullet}(C^{\bullet}) \tag{9.18}$$

of cochains of $C^{\bullet}$ with values in $C^{\bullet}$. A cochain is a collections of multi-linear maps

$$\varphi_{m_1,\ldots,m_k} : (C^{\bullet})^{\otimes m_1} \otimes \ldots \otimes (C^{\bullet})^{\otimes m_k} \to C^{\bullet} \tag{9.19}$$

with a certain symmetry property with respect to $S_k \times (S_{m_1} \times \ldots \times S_{m_k})$.

Recall that we write $\widehat{\mathcal{O}}_n = \mathbb{C}[[\widehat{x}_1, \ldots, \widehat{x}_n]]$. Note that the action of $\mathfrak{gl}_n$ on this cochain complex for $C^{\bullet}(\widehat{\mathcal{O}}_n, \widehat{\mathcal{O}}_n)$ extends to an action to $\mathfrak{gl}_n[\epsilon]$. Indeed, for $h \in \mathfrak{gl}_n$, $h\epsilon$ acts by

$$(\iota_h \varphi)(c_1^{(1)}, \ldots, c_{m_1}^{(1)}; \ldots; c_1^{(k)}, \ldots, c_{m_k}^{(k)})$$
$$= \varphi_{1,m_1,\ldots,m_k}(h; c_1^{(1)}, \ldots, c_{m_1}^{(1)}; \ldots; c_1^{(k)}, \ldots, c_{m_k}^{(k)})$$

Next we **should be able to** upgrade the Kontsevich formality to a MC element of the DGLA

$$C^{\bullet}(\mathfrak{gl}_n[\epsilon], \mathfrak{gl}_n;\ \mathrm{Def}^{\bullet}(C^{\bullet}(\widehat{\mathcal{O}}_n, \widehat{\mathcal{O}}_n))) \tag{9.20}$$

Similarly with the DGLAs governing a homotopy calculus structure and a morphism of homotopy calculi. After that we should use multiplicativity

$$\widehat{\mathcal{O}}_1^{\otimes n} \xrightarrow{\sim} \widehat{\mathcal{O}}_n$$

*****Parts of that may be contained in the works of Willwacher or others**

Remark 9.2.2. Here is the motivation for the DGLA (9.18). Let V be a graded vector space with a basis $\{v^j\}$. Let D be a derivation of the Gerstenhaber algebra $\mathrm{Sym}(\mathrm{Lie}(V))[-1]$ defined on generators by

$$Dv^i = \sum f^i_{jk} v^j v^k \tag{9.21}$$

We assume that $|D| = 1$ and $D^2 = 0$. In particular:

(1) V^* is a graded Lie algebra (which we denote by $\mathcal{G}$).

(2) For $\lambda \in V^*$, let ∂_λ be the derivation sending $v \in V$ to $\lambda(v) \cdot 1$ and all Lie brackets to zero. Then $\lambda \mapsto \mathcal{L}_\lambda = [\partial_\lambda, D]$ is an action of this Lie algebra by derivations.
(3) Moreover, $\lambda + \epsilon\mu \mapsto \mathcal{L}_\lambda + \partial_\mu$ defines an action of $\mathcal{G}[\epsilon]$ by derivations.
(4) Denote by $\mathrm{Sym}(\mathrm{Lie}(V)[-1])^+$ the ideal generated by Lie brackets. Then the action of $\mathcal{G}[\epsilon]$ preserves this ideal. Now define DGLA

$$\mathrm{Der}^+(\mathrm{Sym}(\mathrm{Lie}(V)[-1])) \tag{9.22}$$

to be the algebra of derivations whose image is inside $\mathrm{Sym}(\mathrm{Lie}(V)[-1])^+$, with the differential being $[D, \ ]$. The DGLA $\mathcal{G}[\epsilon]$ acts on the DGLA (9.22) by derivations.

Now consider any DGLA $C^\bullet[1]$. We apply the above formally to $V = C^\bullet[1]^*$. If we ignore the issue of duality and infinite dimensionality, then the resulting DGLA is $\mathrm{Def}(C^\bullet)$. Its MC elements are homotopy Gerstenhaber structures on $C^\bullet$ whose underlying L_∞ structure is the original DGLA structure on $C^\bullet[1]$.

10. Appendix II. An algebraic harmonic oscillator proof of the A.I.T.

10.1. Distinguished cyclic cohomology classes. Consider any algebra over k such that

$$\mathrm{HH}_{2n}(A) = k;\ \mathrm{HH}_j(A) = 0,\ j \neq 2n. \tag{10.1}$$

(In particular, A can be the Weyl algebra). Consider the DG algebra$(A[\eta], \frac{\partial}{\partial \eta})$ where $\eta^2 = 0$ and $|\eta| = 1$ (we are using the homological grading).

Since $A[\eta]$ is contractible, its cyclic homology is zero. Nevertheless, the cyclic complex carries important information if we consider its filtration by powers of η.

We start with computing the cyclic homology of $A[\eta]$ as a graded algebra. We write

$$\mathrm{CC}_\bullet(A[\eta]) = \bigoplus_{l \geq 0} \mathrm{CC}_\bullet^{(l)}(A[\eta]);\ \mathrm{CC}^\bullet(A[\eta]) = \prod_{\ell \geq 0} \mathrm{CC}^\bullet_{(\ell)}(A[\eta]) \tag{10.2}$$

Lemma 10.1.1. *For any* $\ell > 0$

$$\mathrm{HC}^{(\ell)}_{2(n+\ell)-1}(A[\eta], d = 0) \xrightarrow{\sim} k;$$

it is generated over k *by the image of the Hochschild homology class of*

$$U \cup \eta^{(\ell)} \tag{10.3}$$

where U *is any given generator of* $\mathrm{HH}_{2n}(A)$ *and*

$$\eta^{(\ell)} = \eta \cup (B\eta)^{\cup(\ell-1)} = (\ell-1)!\eta^{\otimes \ell} \tag{10.4}$$

$$\mathrm{HC}^{(\ell)}_j(A[\eta], d = 0) = 0,\ j \neq 2(n+\ell) - 1$$

Proof. By the Künneth formula,

$$\mathrm{HH}_\bullet(A[\eta]) \xrightarrow{\sim} \mathrm{HH}_\bullet(A) \otimes \mathrm{HH}_\bullet(k[\eta])$$

But $\mathrm{HH}_\bullet(k[\eta])$ has a basis consisting of $1 \otimes \eta^{\otimes \ell}$ and $\eta \otimes \eta^\ell$, $\ell \geq 0$. The differential B is

$$\eta^{\otimes \ell} \mapsto \ell \cdot 1 \otimes \eta^{\otimes \ell}.$$

The result now follows from the Hochschild-to-cyclic spectral sequence. □

COROLLARY 10.1.2. *Choose a generator* U *of* $HH_{2n}(A)$ *and the generator* τ *of* $HC^{2n}(A)$ *such that* $\tau(U) = 1$. *For every* $\ell > 0$ *there exists a cocycle* τ_ℓ *in* $CC^{2(n+\ell)-1}(A[\eta], \frac{\partial}{\partial\eta})$ *such that:*

1. *the value of the component of* τ_ℓ *in* $CC^{2(n+\ell)-1}_{(\ell)}(A[\eta])$ *on the cyclic cycle* (10.3) *is one;*
2. *All the components of* τ_ℓ *in* $CC^{2(n+\ell)-1}_{(m)}(A[\eta])$ *are zero for* $m < \ell$.

Such a cocycle is unique up to a coboundary of a cochain in $CC^{2(n+\ell)-2}_{(\geq\ell)}(A[\eta])$.

PROOF. □

10.2. Distinguished relative Lie algebra cohomology classes. Let A be as above, and let $\mathfrak{h}$ be a Lie subalgebra of A that acts on A reductively.

The construction of 10.1 immediately extends to a cocycle

$$\tau_\ell \in C^{2(n+\ell)-1}_{(\geq\ell)}(A[\eta], \mathfrak{h}; CC^\bullet(A[\eta])) \tag{10.5}$$

unique up to a coboundary of a cochain in $C^{2n-2+\ell}_{(\geq\ell)}$.

The right hand side is the relative Chevalley-Eilenberg complex of the Lie algebra A acting on the cyclic complex of the associative algebra A. The decomposition

$$CC^\bullet = \prod_{\ell\geq 0} C^\bullet_{(\ell)}$$

is with respect to the grading for which (cohomological) degree of η is -1.

Also, similarly to what we did in 10.1, we can define the cocycles

$$\tau_\ell^{\mathrm{Lie}} \in C^{2(n+\ell)}_{(\geq\ell)}(\mathfrak{gl}(A[\eta]), \mathfrak{h}) \tag{10.6}$$

This cocycle is unique up to a coboundary of a cochain in $C^{2(n+\ell)-1}_{(\geq\ell)}$.

EXPLAIN the issue of $\mathfrak{h}$ not being a Lie subalgebra of $\mathfrak{gl}$ This follows from

LEMMA 10.2.1.

$$H^j_{(\ell)}(\mathfrak{gl}(A[\eta])) = 0,\ j < 2(n+\ell);$$
$$H^{2(n+\ell)}_{(\ell)}(\mathfrak{gl}(A[\eta]), \mathfrak{h}) \xrightarrow{\sim} k$$

PROOF. The first statement follows from Lemma 10.1.1 and the expression of Lie algebra cohomology of $\mathfrak{gl}$ in terms of cyclic homology, the second from the spectral sequence

$$H^i(\mathfrak{h}) \otimes H^j(\mathfrak{g}, \mathfrak{h}) \Rightarrow H^{i+j}(\mathfrak{g}) \tag{10.7}$$

MORE? □

The explicit link between the two versions of τ_ℓ is as follows:

$$C^\bullet(\mathfrak{gl}(A[\eta]), \mathfrak{h}) \xrightarrow{\alpha} C^\bullet(\mathfrak{gl}(A[\eta]), \mathfrak{h}; C^\bullet(\mathfrak{gl}(A[\eta]))) \xrightarrow{\beta} C^\bullet(A[\eta], \mathfrak{h}; CC^{\bullet-1}(A[\eta]))$$

This also gives a normalisation of τ_ℓ^{Lie}. The first map is as follows: for a Lie algebra cochain φ and for $X_1, \ldots, X_k \in \mathfrak{gl}(A[\eta])$,

$$\alpha\varphi(X_1, \ldots, X_k) = \iota_{X_1} \ldots \iota_{X_k} \varphi;$$

the second is the restriction from $\mathfrak{gl}(A[\eta])$ to the "Lie subalgebra" $A[\eta]$ (***explain***), together with the projection from Lie to cyclic complex.

We also need to compare τ_ℓ for different ℓ. We have

$$\tau_{\ell+1} \in CC^{2(n+\ell)+1}_{(\geq \ell+1)}(A[\eta]);\ \tau_\ell \in CC^{2(n+\ell)-1}_{(\geq \ell)}(A[\eta]) \tag{10.8}$$

Both right hand sides map to $CC^{2(n+\ell)+1}_{(\geq \ell)}(A[\eta])$: one tautologically, the other by the periodicity operator S (dual to multiplication by u). Same for Chevalley-Eilenberg complexes $C^\bullet(A[\eta], \mathfrak{h}; CC^\bullet)$.

Lemma 10.2.2.

$$S\tau_\ell = \tau_{\ell+1}$$

in the cohomology of degree $2(n+\ell)+1$ *of* $C^\bullet_{(\geq \ell)}(A[\eta], \mathfrak{h}; CC^\bullet(A[\eta]))$.

Proof. Follows from

$$\eta \cup (B\eta)^{\cup \ell} \overset{\partial_\eta}{\mapsto} (B\eta)^{\cup \ell};\ \ \eta \cup (B\eta)^{\cup(\ell-1)} \overset{B}{\mapsto} (B\eta)^{\cup \ell}$$

***FINISH □

Another link between the $\tau_{(\ell)}$ for different ℓ. Namely, all τ^{Lie}_ℓ can be obtained from τ^{Lie}_1 as follows. We assume that, as is the case for the Weyl algebra, there is an $\mathfrak{h}$-invariant Hochschild $2n$-cocycle TR which is zero on the images of B and L_h, $h \in \mathfrak{h}$. We recall that

$$L_h(a_0 \otimes \ldots \otimes a_m) = \sum_{j=0}^{m} (-1)^j a_0 \otimes \ldots a_j \otimes h \otimes \ldots \otimes a_m \tag{10.9}$$

We normalise TR_0 so that $\mathrm{TR}(U) = 1$ on or chosen generator U of $HH_{2n}(A)$. For $a_1, \ldots, a_{2n}$ and $b_1, \ldots, b_\ell$ in A put

$$\tau^{\mathrm{Lie}}_\ell(\eta b_1, \ldots, \eta b_\ell, a_1, \ldots, a_{2n}) = \sum_{\sigma \in S_{2n}} \mathrm{sgn}(\sigma)\mathrm{TR}_0(b_0 \otimes a_{\sigma 1} \otimes \ldots \otimes a_{\sigma_{2n}}) \tag{10.10}$$

and

$$b_0 = \frac{1}{\ell!} \sum_{\tau \in S_\ell} b_{\tau 1} \ldots b_{\tau \ell} \tag{10.11}$$

This generalises immediately to $a_1, \ldots, a_{2n}$ and $b_1, \ldots, b_\ell$ in $\mathfrak{gl}(A)$. Namely, we precede (10.10) by the trace map

$$\wedge^\bullet \mathrm{Matr}(A[\eta]) \to \wedge^\bullet(A[\eta]) \tag{10.12}$$

We extend τ^{Lie}_ℓ unquely up to a coboundary to a cocycle that vanishes when there are more thatn $2n$ inputs from A (or more generally from $\mathfrak{gl}(A)$).

***FINISH REDUCING TO:

10.3. Riemann-Roch for Lie algebra cohomology. Now let $A = \widehat{\mathbb{A}}[h^{-1}]$. Let $\mathfrak{gl}_n$ be the Lie subalgebra of A with the basis $\widehat{x}_j \star_W \frac{\widehat{\xi}_k}{h}$. We view it as a "subalgebra" of $\mathfrak{gl}(A)$ ***. Also, $\mathfrak{gl}(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

Let $\mathfrak{h} = \mathfrak{gl}_n \oplus \mathfrak{gl}$. For an invariant power series P, we denote the characteristic class c_P corresponding to the subalgebra $\mathfrak{gl}_n$, resp. $\mathfrak{gl}$, by $P(T)$ and the characteristic class c_P, resp. $P(E)$.

Define TR_0 as the composition

$$C_{2n}(\widehat{\mathbb{A}})[h^{-1}] \overset{\mathrm{TR}}{\longrightarrow} \widehat{\Omega}^{2n}((h)) \overset{\iota_{\omega^n}}{\longrightarrow} \widehat{\Omega}^0((h)) \overset{\mathrm{ev}_{\widehat{y}=0}}{\longrightarrow} \mathbb{C}((h)) \tag{10.13}$$

Here TR is as in (5.17).

THEOREM 10.3.1. *For any* $\ell > 0$

$$\tau_\ell^{\mathrm{Lie}} = (\mathrm{Td}(T)\mathrm{ch}(E))^{2(n+\ell)}$$

in $H^{2(n+\ell)}(\mathfrak{gl}(\widehat{\mathbb{A}}), \mathfrak{gl}_n \oplus \mathfrak{gl})$.

10.4. Reduction to the algebraic harmonic oscillator calculation.

PROPOSITION 10.4.1. *Let* $n = 1$. ***

PROOF. We want to change $\frac{1}{\ell!}(\eta\frac{\widehat{x}\widehat{\xi}}{h})^{\wedge\ell}$ by a boundary to make its components in $\wedge^j(A) \wedge \wedge^\bullet(\eta A)$ equal to zero when $j < 2$. Denote

$$L_m = \widehat{x}^{m+1}\frac{\widehat{\xi}}{h} \tag{10.14}$$

One has

$$[L_m, L_n] = (n - m)L_{m+n} \tag{10.15}$$

For a formal parameter t we write

$$\exp(t\eta L_0) = 1 + \sum_{\ell=1}^{\infty}\frac{t^\ell}{\ell!}(\eta L_0)^{\wedge\ell} \tag{10.16}$$

□

We look for functions φ_m such that

$$\exp(t\eta L_0) - 1 = \partial_{\mathrm{Lie}}(\sum_{m=1}^{\infty}\varphi_m(\eta L_0)\frac{(\eta L_{-1})^{\wedge m}}{m!} \wedge L_m) \tag{10.17}$$

We see that

$$\varphi_1(q) = \frac{e^{qt} - 1}{2q} \tag{10.18}$$

and

$$(m + 2)\varphi_{m+1}(q) + m\varphi_m'(q) = 0 \tag{10.19}$$

which we deduce from

$$\sum_{m=1}^{\infty} m\varphi_m'(\eta L_0)\eta L_m\frac{(\eta L_{-1})^m}{m!} + \sum_{m=2}^{\infty}(m + 1)\varphi_m(\eta L_0)\eta L_{m-1}\frac{(\eta L_{-1})^{m-1}}{(m - 1)!} = 0$$

We get

$$\varphi_{m+1}(q) = (-1)^m\frac{1}{(m + 1)(m + 2)}(2\varphi_1)^{(m)}(q) \tag{10.20}$$

So we have

$$\exp(t\eta L_0) - 1 = \partial_{\mathrm{Lie}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m + 1)(m + 2)}2\varphi_1^{(m)}(\eta L_0)\frac{(\eta L_{-1})^{m+1}}{(m + 1)!} \wedge L_{m+1} \tag{10.21}$$

And if we apply $\frac{\partial}{\partial\eta}$ to the argument of ∂_{Lie} in the right hand side, we get

$$\partial_{\mathrm{Lie}}\sum_{m=0}^{\infty}\frac{(-1)^m}{(m + 1)(m + 2)}2\varphi_1^{(m)}(\eta L_0)\frac{(\eta L_{-1})^m}{m!} \wedge L_{-1} \wedge L_{m+1} \tag{10.22}$$

(recall that we are working with relative chains, so $\ldots \wedge L_0 \wedge \ldots = 0$).

Note also that $\eta L_{-1} = \frac{\partial}{\partial \widehat{x}}(\eta L_0)$. So

$$\exp(t\eta L_0) - 1 \sim \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)} \frac{1}{m!} (\frac{\partial}{\partial \widehat{x}})^m 2\varphi_1(\eta L_0) \wedge L_{m+1} \wedge L_{-1} \tag{10.23}$$

So we have to compute:

$$\mathrm{TR}_0 (\sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)} \frac{1}{m!} (\frac{\partial}{\partial \widehat{x}})^m (2\varphi_1)(L_0) \otimes (L_{m+1} \otimes L_{-1} - L_{-1} \otimes L_{m+1})) \tag{10.24}$$

where φ_1 is as in (10.18) and TR_0 as in (10.13).

Note also that TR commutes with $\iota_{L_{-1}}$. Therefore or calculation reduces to: compute

$$\mathrm{TR}_1 \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)(m+2)} \frac{1}{m!} (\frac{\partial}{\partial \widehat{x}})^m (2\varphi_1)(L_0) \otimes L_{m+1} \tag{10.25}$$

where TR_1 is the composition

$$C_1(\widehat{\mathbb{A}}) \xrightarrow{\mathrm{TR}} \widehat{\Omega}^1((h)) \xrightarrow{\iota_{\partial_{\widehat{x}}}} \widehat{\Omega}^1((h)) \xrightarrow{\mathrm{ev}_0} \mathbb{C}((h)) \tag{10.26}$$

Here ev_0 is the evaluation at $\widehat{x} = \widehat{\xi} = 0$ and TR is as in (5.17).

For any a_0 we have

$$\mathrm{TR}_1(a_0 \otimes L_{m+1}) = \sum_{k=0}^{\infty} (1 \otimes \partial_{\widehat{x}}) \frac{1}{(2k+1)!} (\frac{h}{2}(\partial_{\widehat{\xi}} \otimes \partial_{\widehat{x}} - \partial_{\widehat{x}} \otimes \partial_{\widehat{\xi}}))^{2k} (a_0 \otimes \frac{\widehat{x}^{m+1}\widehat{\xi}}{h})|_{\widehat{x}=\widehat{\xi}=0}$$

The only non-zero term in this sum is the component $(\partial_{\widehat{\xi}}^{m+1}\partial_{\widehat{x}} \otimes \partial_{\widehat{\xi}}\partial_{\widehat{x}}^{m+1})$ of

$$(\partial_{\widehat{\xi}} \otimes \partial_{\widehat{x}} - \partial_{\widehat{x}} \otimes \partial_{\widehat{\xi}})^{m+2}.$$

It enters with the coefficient $-(m+2)$. We get

$$-\frac{(m+2)}{(m+3)!} (\frac{h}{2})^{m+2} \frac{1}{h} (m+2)! \partial_{\widehat{\xi}}^{m+1} \partial_{\widehat{x}} a_0 |_{\widehat{x}=\widehat{\xi}=0} \tag{10.27}$$

if m is even.

Therefore

$$\sum_{\ell \geq 0} \frac{t^{\ell+1}}{(\ell+1)!} \tau_\ell (\eta L_0)^{\ell+1} = -\frac{1}{2(m+3)} \sum_{m \text{ even}} (\frac{h}{2})^{m+1} \frac{(\partial_{\widehat{\xi}} \partial_{\widehat{x}})^{m+1}}{(m+1)!} (\frac{e^{tL_0 - 1}}{L_0})|_{\widehat{x}=\widehat{\xi}=0}$$

Also note: $\frac{\partial}{\partial t}$ of the above is

$$-\frac{1}{2(m+3)} \sum_{m \text{ even}} (\frac{h}{2})^{m+1} \frac{(\partial_{\widehat{x}} \partial_{\widehat{\xi}})^{m+1}}{(m+1)!} \exp_\star(tL_0)|_{\widehat{x}=\widehat{\xi}=0} \tag{10.28}$$

where $\exp_\star$ is the exponential in the Weyl algebra.

10.5. The algebraic harmonic oscillator calculation.

10.5.1. *Summary of the calculation.* Define

$$\Phi(q,t) = \exp(q\partial_{\widehat{x}}\partial_{\widehat{\xi}}) \exp_\star(t\widehat{x}\widehat{\xi})|_{\widehat{x}=\widehat{\xi}=0} \tag{10.29}$$

and

$$\Psi(q,t) = -\frac{1}{4}q^{-2}\int\int q(\Phi(q,t) - \Phi(-q,t))dqdt \tag{10.30}$$

(note that (10.28) is equal to $\Psi(\frac{\hbar}{2}, \frac{t}{\hbar})$).

PROPOSITION 10.5.1. *Consider the algebra* $\mathbb{C}[[Z, Z^+, \hbar]]$ *with the Moyal-Weyl product* $\star_W$ *such that* $[Z^+, Z] = \hbar$. *Put*

$$\Phi(s,t) = -\frac{1}{2}\exp(s\partial_{Z^+}\partial_Z)\exp_\star(tZZ^+)|_{Z=Z^+=0}$$

(where the second factor is the exponential of $ZZ^+ = \frac{1}{2}(Z\star_W Z^+ + Z^+\star_W Z)$*). Then*

$$\frac{1}{2\hbar}s(\Phi(s,t) - \Phi(-s,t)) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\widehat{\mathcal{A}}_\hbar(s,t)$$

where

$$\widehat{\mathcal{A}}_\hbar(\frac{\hbar}{2},t) = \frac{\hbar t/2}{\mathrm{sh}(\hbar t/2)}$$

More precisely, we will show that ***doble check***

$$\frac{1}{2\hbar}s(\Phi(s,t) - \Phi(-s,t) = \frac{\partial}{\partial s}\frac{\partial}{\partial t}\widehat{\mathcal{A}}_\hbar(s,t) \tag{10.31}$$

where

$$\frac{\partial}{\partial t}\widehat{\mathcal{A}}_\hbar(s,t) = -\frac{1}{\mathrm{sh}(\frac{\hbar t}{2})}(s - \frac{\hbar}{2}\mathrm{arcth}(\frac{2s}{\hbar}\mathrm{th}(\frac{\hbar t}{2}))) \tag{10.32}$$

10.5.2. *The calculation.*

LEMMA 10.5.2.

$$\Phi(s,t) = -\frac{1}{2(\mathrm{ch}(\frac{\hbar t}{2}) - \frac{2s}{\hbar}\mathrm{sh}(\frac{\hbar t}{2}))}$$

PROOF. Let us look for an expression

$$\exp_\star(tZZ^+) = a(t)\exp(T(t)Z^+Z) \tag{10.33}$$

(the usual exponential, as opposed to the star exponential).

We have

$$\frac{d}{dt}\exp_\star(tZZ^+) = Z^+Z\star_W\exp_\star(tZ^+Z) \tag{10.34}$$

and

$$\frac{d}{dt}(a(t)\exp_\star(T(t)ZZ^+)) = (\frac{da}{dt}/a + \frac{dT}{dt}Z^+Z)\exp(T(t)Z^+Z) \tag{10.35}$$

Since

$$Z^+Z\star_W\exp(T(t)Z^+Z) = (Z^+Z - \frac{\hbar^2}{4}\partial_{Z^+}\partial_Z)\exp(T(t)Z^+Z) =$$

$$(Z^+Z - \frac{\hbar^2}{4}(T + T^2Z^+Z))\exp(T(t)Z^+Z),$$

we get the equations

$$\frac{da}{dt}/a = -\frac{\hbar^2}{4}T \tag{10.36}$$

and

$$\frac{dT}{dt} = 1 - \frac{h^2}{4}T^2 \tag{10.37}$$

which means

$$a(t) = \frac{1}{\mathrm{ch}(\frac{ht}{2})} \tag{10.38}$$

and

$$T(t) = \frac{2}{h}\mathrm{th}(\frac{ht}{2}); \tag{10.39}$$

Therefore

$$\exp_\star(tZ^+Z) = \frac{1}{\mathrm{ch}(\frac{ht}{2})}\exp(\frac{2}{h}\mathrm{th}(\frac{ht}{2})Z^+Z) \tag{10.40}$$

LEMMA 10.5.3.

$$\exp(s\partial_Z\partial_{Z^+})\exp(TZZ^+) = \frac{1}{1-sT}\exp(\frac{-T}{1-sT})$$

PROOF. Consider the $\mathfrak{sl}_2$ triple

$$E_+ = Z^+Z;\ E_- = -\partial_Z\partial_{Z^+};\ E_0 = Z\partial_Z + Z^+\partial_{Z^+} + 1 \tag{10.41}$$

The statement follows from

$$\begin{pmatrix}1 & 0\\ -s & 1\end{pmatrix}\begin{pmatrix}1 & T\\ 0 & 1\end{pmatrix} = \begin{pmatrix}1 & -T(1-sT)^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}(1-sT)^{-1} & 0\\ 0 & 1-Ts\end{pmatrix}\begin{pmatrix}1 & 0\\ -s(1-Ts)^{-1} & 1\end{pmatrix}$$

in $\exp(\mathfrak{sl}_2)$. □

REMARK 10.5.4. The above identity holds in 2×2 matrices over any ring R, commutative or not, as long as $1-sT$ is invertible. This provides an algebraic path between $1-sT$ and $1-Ts$ in GL(R) (in a sense that can be made precise). When s and T commute, we get an element of algebraic K_2 of the ring which is the Loday symbol $\langle s, T\rangle$. If s is invertible, this is the Milnor symbol $\{s, 1-sT\}$. Algebraic index theorem does have K-theoretical meaning (it computes the pairing of the canonical trace on an idempotent); however, we are not aware of any algebraic K-theoretical interpretation of the computation we are presenting here.

We get

$$\Phi(s,t) = -\frac{1}{2}(1 - \frac{2s}{h}\mathrm{th}(\frac{ht}{2}))^{-1}\frac{1}{\mathrm{ch}(\frac{ht}{2})} \tag{10.42}$$

Now we have

$$s(\Phi(s,t) - \Phi(-s,t)) = -\frac{2h}{\mathrm{sh}(\frac{ht}{2})}(-1 + \frac{1}{1-(\frac{2s}{h})^2\mathrm{th}^2(\frac{ht}{2})})$$

The anti-derivative of the right hand side with respect to s is

$$-\frac{2h}{\mathrm{sh}(\frac{ht}{2})}\frac{h}{\mathrm{th}(\frac{ht}{2})}\mathrm{arcth}(\frac{ht}{2})$$

***FINISH □

10.5.3. *Appendix to 10 : Generalized Bernoulli polynomials.* 0. Recall that if we put

$$\psi_0(x,t) = \frac{te^{tx}}{e^t - 1} \tag{10.43}$$

then the Bernoulli polynomials are defined by

$$\psi_0(x,t) = 1 + \sum_{n-1}^{\infty} B_n(x)\frac{t^n}{n!} \tag{10.44}$$

***Ref

There are two other generating functions for polynomials whose values at zero lead to Bernoulli numbers. They appear in the algebraic harmonic oscillator proof in 10.

2. Put

(10.45)

$$\psi_1(y,t) = \int (\frac{1}{(1-e^{-t})^2(y-1)}(y+(1-y)e^{-t})\log(y+(1-y)e^{-t}) - \frac{1}{1-e^{-t}})dt$$

normalised so that $\psi_1(y,0) = 1$. Then

$$\psi_1(0,t) = \frac{t}{1-e^{-t}}; \tag{10.46}$$

$$\psi_1(y,t) = 1 + \sum_{n-1}^{\infty} \frac{(-1)^n}{n^2(n+1)} t^{2n} P_n(y) \tag{10.47}$$

where $P_n(y)$ is a monic polynomial of degree n.

2. Now put

$$\psi_2(y,t) = \int \frac{1}{\mathrm{sh}(\frac{t}{2})}(1 - \frac{\mathrm{th}^{-1}((y+1)\mathrm{th}(\frac{t}{2}))}{(y+1)\mathrm{th}(\frac{t}{2})})dt \tag{10.48}$$

normalised so that $\psi_2(y,0) = 1$. Then

$$\psi_2(0,t) = \frac{t/2}{\mathrm{sh}(t/2)}; \tag{10.49}$$

$$\psi_2(y,t) = 1 + \sum_{n=1}^{\infty} \frac{t^{2n}}{2^{2n+1}n(2n+1)} Q_{2n}(y) \tag{10.50}$$

where Q_{2n} is a monic polynomial of degree $2n$, even with respect to $y+1$.

11. Appendix. Index theorem for elliptic pairs

For a complex analytic manifold A, let $\mathcal{M}$ be a $\mathcal{D}_X$-module and $\mathcal{F}$ an $\mathbb{R}$-constructible sheaf on X. We say that $(\mathcal{M},\mathcal{F})$ is an elliptic pair if $\mathrm{SS}(\mathcal{M}) \cap \mathrm{SS}(\mathcal{F})$ is compact in T^*X (it is then automatically inside the zero section of X). We refer the reader to [**503**], [**504**] for definitions.

The finiteness theorem of Schapira and Schneiders maintains that

$$\mathbb{R}\Gamma(X, \mathrm{DR}^\bullet(\mathcal{M}) \otimes_{\mathbb{C}_X} \mathcal{F})$$

has finite dimensional total cohomology (in other words, it is a perfect complex over $\mathbb{C}$).The Schapira-Schneiders index theorem for elliptic pairs states that

$$\sum_j \dim \mathbb{H}^j(X, \mathrm{DR}^\bullet(\mathcal{M}) \otimes_{\mathbb{C}_X} \mathcal{F}) = \int_{T^*X} \mu\mathrm{eu}(\mathcal{M}) \smile \mu\mathrm{eu}(\mathcal{F}) \tag{11.1}$$

Here

$$\mu\mathrm{eu}(\mathcal{M}) \in H^{2n}_{SS(\mathcal{M})}(X, \mathbb{C}); \quad \mu\mathrm{eu}(\mathcal{F}) \in H^{2n}_{SS(\mathcal{F})}(X, \mathbb{C}) \tag{11.2}$$

are the microlocal Euler classes and $n = \dim_{\mathbb{C}}(X)$. Schapira and Schneiders conjectured a formula for $\mu\mathrm{eu}(\mathcal{M})$ in terms of the principal symbol of $\mathcal{M}$ and the Todd class of X. This conjecture was proven in [**69**], [**67**] using the algebraic index theorem.

EXAMPLE 11.0.1. Let $\mathcal{M} = \mathcal{O}_X$ and $\mathcal{F} = \mathbb{C}_X$. We get the Kashiwara-Dubson formula for $\sum(-1)^j \dim H^j(X, \mathcal{F})$.

EXAMPLE 11.0.2. Let $\mathcal{F} = \mathbb{C}_X$. We get a formula for $\sum(-1)^j \dim H^j(X, \mathrm{DR}(\mathcal{M}))$.

EXAMPLE 11.0.3. In particular, let $\mathcal{E}$ be a holomorphic vector bundle and let $\mathcal{M} = \mathcal{D}(\mathcal{O}_x, \mathcal{E})$ be the sheaf of differential maps from $\mathcal{O}_X$ to $\mathcal{E}$. Let, as above, $\mathcal{F} = \mathbb{C}_X$. We get the Riemann-Roch formula for $\sum(-1)^j \dim H^j(X, \mathcal{E})$.

EXAMPLE 11.0.4. Let X be a real analytic manifold. Let $D : \mathcal{E}_+ \to \mathcal{E}_-$ be a real analytic elliptic differential operator between two real analytic vector bundles. Let $X_{\mathbb{C}}$, $\mathcal{E}_{+,\mathbb{C}}$, $\mathcal{E}_{-,\mathbb{C}}$, $D_{\mathbb{C}}$ be a complexification of X, resp. $\mathcal{E}_+$, $\mathcal{E}_-$, D. Let $\mathcal{M}$ be the two-term complex

$$\mathcal{D}(\mathcal{O}_{X_{\mathbb{C}}}, \mathcal{E}_{+,\mathbb{C}}) \xrightarrow{D\circ} \mathcal{D}(\mathcal{O}_{X_{\mathbb{C}}}, \mathcal{E}_{-,\mathbb{C}})$$

and $\mathcal{F} = \mathbb{C}_X$. We get the Atiyah-Singer index theorem for D.

It would be instructive to bring the proof of (11.1) closer into line with the methods of 5.1 and 5.2. This could be probably carried out using the approach of [**?**] and [**219**]. There, the following is proven. Let $\mathcal{E}$ be a holomorphic vector bundle on a compact complex manifold X. Let $D : \mathcal{E} \to \mathcal{E}$ be a holomorphic differential operator. Then

$$\sum_j (-1)^j \mathrm{tr}(D | H^j(X, \mathcal{E})) = \int_X [D] \tag{11.3}$$

for a cohomology class explicitly constructed from D. Note that this result generalizes the partial case from Example 11.0.3 by considering any differential operator D instead of the identity. Perhaps the Schapira-Schneiders index theorem can be generalized to involve an endomorphism of the D_X-module $\mathcal{M}$.

The method used in the proof of (11.3), topological quantum mechanics, seems to fit well with what was discussed in 5.2.

12. Bibliographical notes

Feigin-Tsygan; Nest-Tsygan; Feigin-Felder-Shoikhet; Feigin-Losev-Shoikhet; Schmitt; BFFLS;

Index theorem for deformation quantization: Fedosov, Nest-Tsygan, Bressler-Nest-Tsygan;

Formality theorem for chains: Shoikhet, Dolgushev-Tamarkin-Tsygan, Willwacher, ... Kontsevich-Soibelman...

Riemann-Roch for Lie algebra cohomology: FT, Bressler-Kapranov-Tsygan-Vasserot

Calaque-Rossi-van den Bergh...

CHAPTER 13

Operations on Hochschild and cyclic complexes, III

1. Introduction

Let A be an algebra and D be its derivation. When A is commutative, D acts on $\Omega^{\bullet}_{A/k}$ by Lie derivatives, as it does on any natural tensor construction applied to A. We denote this action either by L_D or simply by D.

We also define the contraction by D as the only graded derivation of degree -1 of $\Omega^{\bullet}_{A/k}$ such that

$$\iota_D(a) = 0;\ \iota(da) = a,\ a \in A \tag{1.1}$$

One has

$$[D, \iota_D] = 0;\ [d, \iota_D] = 0;\ \iota_D^2 = 0 \tag{1.2}$$

One can combine the last two identities into

$$(ud + \iota_D)^2 = 0 \tag{1.3}$$

Now let A be an algebra and D a derivation of A. We can ask whether something similar to (1.3) exists on Hochschild chains. The answer is yes, but with modifications. First, we relax (1.3) and ask for an operator $\mathcal{J}(D)$ which is no more linear in D but rather a formal combination

$$\mathcal{J}(D) = \sum_{n=1}^{\infty} \frac{J_{D^n}}{n!} \tag{1.4}$$

We are looking for $\mathcal{J}(D)$ satisfying

$$(b + uB + \mathcal{J}(D))^2 = uD \tag{1.5}$$

What we find instead is a series $\mathcal{I}(D)$ satisfying

$$(b + uB + \mathcal{I}(D))^2 = u(e^D - 1) \tag{1.6}$$

All the numerators I_{D^n} of the homogenous components of $\mathcal{I}(D)$ ar edefined over $\mathbb{Z}$.

In characteristic zero, one can indeed pass from one algebraic structure to another, but the procedure is somewhat awkward. One way of saying this is that the operators $\mathcal{I}(D)$ on Hochschild chains of a commutative algebra are A-linear modulo u, and the "classical" operators $\mathcal{J}(D)$ that we get from them are not.

Next we compare these operators $\mathcal{J}(D)$ to the standard ones (cf. (1.3)) via the HKR map. We first construct an extended HKR map that intertwines this $\mathcal{J}(D)$ with a nonstandard $\mathcal{J}(D)$ on forms (neither of the two is A-linear modulo u). Then we see that any such $\mathcal{J}(D)$ is determined by a power series $A(D)$ such that $A(0) = 1$ and is equivalent to the standard one.

In our case,

$$A(D) = \frac{\log(1+D)}{D} = \frac{X}{1-e^{-X}} \tag{1.7}$$

where $X = -\log(1+D)$. We are not sure what is the significance of this. Note that the Todd class does appear as a discrepancy in other constructions where we compare noncommutative and classical calculus, such as formality theorems and the algebraic index theorem 9.

Note also some resemblance with the contents of Chapter 25. Formula (1.5) is of course analogous to $\mathcal{D}^2 = W$ where W is a central element (not a zero divisor) of a ring $\mathcal{A}$. Moreover, such a $\mathcal{D}$ appears precisely from the action of a free resolution of $\mathcal{A}/W\mathcal{A}$ which is almost identical what we have here. In the notation of 2.3 below, this resolution is $\mathcal{A}_0(W)$ where the ring of scalars is $\mathcal{A}$. In other words: an action of $\mathcal{J}(D)$ describes a (derived) trivial action of a derivation, and an action of $\mathcal{I}(D)$ describes an action of a derivation whose exponential automorphism acts trivially. This is clearly related to the content of Chapter 19, in particular to 81 .

Note also that everything we do is valid not just in characteristic zero but if we multiply D and d by a formal parameter t such that t^n is (uniquely) divisible by $n!$.

Next we extend the noncommutative Cartan calculus to the entire Lie algebra of derivations of A, and then to the DG algebra of Hochschild cochains. Recall that the latter are a noncommutative analog of multivector fields. We first restrict ourselves to multivector fields of degree ≤ 1, namely vector fields and functions. In the classical case,

$$\iota_f(\omega) = f\omega;;L_f(\omega) = df \wedge \omega \tag{1.8}$$

for a function f and a form ω. Noncommutative Cartan calculus extends straightforwardly to the DG Lie algebra $A[1] \oplus \mathrm{Der}(A)$ with the differential

$$(a, D) \mapsto (0, \mathrm{ad}(a)).$$

We use this in Chapter 25.

Next we extend noncommutative Cartan calculus to the full DGA of Hochschild cochains (section 3). Note that comparing the two versions of Cartan calculus via HKR (when our algebra is commutative) becomes much more difficult; when A is regular, a positive answer is given by the formality theorem ***More

2. Noncommutative Cartan calculus

2.1. Noncommutative Cartan calculus of derivations. Now let A be any algebra. Consider Reinhart's pairing (5.4) from Chapter 4. It satisfies

$$[D, I_D] = 0;\ [b + uB, I_D] = uD;$$

as for $I_D^2 = 0$, this is true only up to homotopy. To see what structure I_D is part of at the chain level, introduce

$$I_{D^n} = \iota_{D^n} + uS_{D^n} \tag{2.1}$$

$$\iota_{D^n}(a_0 \otimes \ldots \otimes a_n) = a_0 D^n(a_1) \otimes a_2 \otimes \ldots \otimes a_n; \tag{2.2}$$

$$S_{D^n}(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} 1 \otimes a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes D^n(a_1 \otimes \ldots \otimes a_{j-1}) \tag{2.3}$$

Here D^n stands for the nth power of the action of D on the tensor $a_1 \otimes \ldots \otimes a_{j-1}$.

PROPOSITION 2.1.1.

$$[b + uB, I_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} I_{D^k} I_{D^{n-k}} = uD^n$$

for all $n > 0$.

PROOF. Direct computation. □

Compare this with the operations on $\Omega^{\bullet}_{A/k}$ for a commutative A. If we put

$$J_{D^n} = \iota_D,\ n = 1;\ J_{D^n} = 0,\ n > 1 \tag{2.4}$$

then

$$[ud, J_D] = uD;\ [ud, J_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} J_{D^k} J_{D^{n-k}} = 0,\ n > 1 \tag{2.5}$$

When $\mathbb{Q} \subset k$ and A is commutative, the HKR map ***Ref intertwines $b + uB$ and ud. But so far we have two different structures on the two sides.

REMARK 2.1.2. As Anton Alekseev pointed out to us, the systems of operators satisfying the same relations as the J_{D^n} have appeared in [**14**] and [**50**]. Systems satisfying the same relations as the I_{D^n} have appeared in [**?**].

To compare those two structures, note that (2.5) is equivalent to

$$(ud + \mathcal{J}(D))^2 = uD \tag{2.6}$$

where

$$\mathcal{J}(D) = \sum_{n=1}^{\infty} \frac{1}{n!} J_{D^n} \tag{2.7}$$

On the other hand, relations from Proposition 2.1.1 are equivalent to

$$(b + uB + \mathcal{I}(D))^2 = u(e^D - 1) \tag{2.8}$$

where

$$\mathcal{I}(D) = \sum_{n=1}^{\infty} \frac{1}{n!} I_{D^n} \tag{2.9}$$

Whenever $n!$ divide D^n (for example when $\mathbb{Q} \subset k$) and the infinite sums converge (for example if we replace D by tD where t is a formal parameter), the above sums are well defined. But under those assumptions, the following is true.

LEMMA 2.1.3. *Let* I_{D^n}, $n \geq 1$, *satisfy the relations* (2.8), (2.9). *Define*

$$\mathcal{J}(D) = \sum_{k,l \geq 0} c_{k,l} D^k I_{D^l} \tag{2.10}$$

where

$$\sum_{k,l \geq 0} c_{k,l} D^k I_{D^l} x^k y^l = \sum_{n=1}^{\infty} \frac{1}{n!} y(y - x) \ldots (y - (n-1)x) \tag{2.11}$$

Then

$$(b + uB + \mathcal{J}(D))^2 = uD$$

PROOF. We are solving

$$(\partial_{\mathrm{Cobar}} + uB)\mathcal{J}(D) + \mathcal{J}(D)^2 = uD \tag{2.12}$$

where

$$\partial_{\mathrm{Cobar}} J_{D^m} = \sum_{k=1}^{m-1} \binom{m}{k} J_{D^k} J_{D^{m-k}}$$

We are looking for a solution of the form

$$\mathcal{J}_F(D) = \sum_{n=1}^{\infty} x_n(D) I_{D^n} \text{ where } F(D, y) = \sum_{n=1}^{\infty} x_n(D) y^n \tag{2.13}$$

Equation (2.12) translates into the following two: first,

$$F(y_1 + y_2) - F(y_1) - F(y_2) + F(y_1)F(y_2) = 0$$

which implies

$$x_n(D) = \frac{1}{n!} f(D)^n$$

for some f, and second,

$$\sum_{n=1}^{\infty} \frac{1}{n!} f(D)^n D^n = D.$$

This implies

$$f(D)D = \log(1+D);\ 1 - F(D, y) = \exp(\frac{y}{D}\log(1+D)) = (1+D)^{\frac{y}{D}};$$

we conclude that

$$F(D, y) = -\sum_{n=1}^{\infty} \frac{1}{n!} y(y-D)\dots(y-(n-1)D) \tag{2.14}$$

□

A crucial observation for us is that the homogeneous part of $\mathcal{J}_F(R)$ ((2.13)) of total degree n in D has the denominator $n!$.

COROLLARY 2.1.4. *Let the* I_{D^n} *satisfy the relations as in Proposition 2.1.1. Let* J_{D^n} *denote the homogenous component of degree* n *in* D *of* $\mathcal{J}(D)$ *as in* (2.10). *Then the* J_{D^n} *satisfy the relations* (2.5).

2.2. Compatibility of Cartan calculus with HKR. Now let D be a derivation of a commutative algebra A over k containing $\mathbb{Q}$. By Lemma 2.11, operators D and J_{D^n}, $n \geq 1$, subject to relations (2.5) act on the left hand side of the HKR map. They also act on the right hand side by

$$J_D^0 = \iota_D;\ J_{D^n}^0 = 0,\ n > 1 \tag{2.15}$$

Recall that we write $\mathcal{J}(D) = \sum \frac{1}{n!} J_{D^n}$, etc.

THEOREM 2.2.1. *There is a natural* $k[[u]]$*-linear continuous morphism*

$$\mathrm{HKR}(D) : CC_\bullet^-(A)[[u]] \to \Omega^\bullet_{A/k}[[u]]$$

such that

$$\mathrm{HKR}(D) = \mathrm{HKR} + \sum_{n=1}^{\infty} \mathrm{HKR}_{D^n},$$

HKR_{D^n} *are homogeneous of degree* n *in* D*, and*

$$(ud + \mathcal{J}^0(D))\mathrm{HKR}(D) = \mathrm{HKR}(D)(b + uB + \mathcal{J}(D))$$

PROOF. Put

$$I_{D^n} = \iota_{D^n} + uS_{D^n} \tag{2.16}$$

Here ι_{D^n} on $C_\bullet(A)[[u]]$ is given by (2.2); on $\Omega^\bullet[[u]]$, $\iota_{D^n} = \iota_D$ for $n = 1$ and zero for $n > 1$. Also, S_{D^n} on $C_\bullet(A)[[u]]$ is given by (2.3), and we put

$$\iota_{D^n} = 0,\ n > 1;\ \ S_{D^n} = \frac{1}{n+1} dD^n \tag{2.17}$$

on $\Omega^\bullet[[u]]$. Also, on $\Omega^\bullet[[u]]$ we denote $b = 0$ and $B = d$. Observe that the I_{D^n} on both sides satisfy the relations as in Proposition 2.1.1. We will first construct

$$\Phi(D) = \mathrm{HKR} + \sum_{n=1}^{\infty} \frac{1}{n!} \Phi_{D^n} \tag{2.18}$$

satisfying

$$(ud + \mathcal{I}(D))\Phi(D) = \Phi(D)(b + uB + \mathcal{I}(D)) \tag{2.19}$$

Here, as above, $\mathcal{I}(D) = \sum \frac{1}{n!} I_{D^n}$.

We are looking for $k[[u]]$-linear continuous operators of degree zero

$$\Phi_{D^n} : C_\bullet(A)[[u]] \to \Omega^\bullet_{A/k}[[u]] \tag{2.20}$$

such that

$$\Phi_{D^0} = \mathrm{HKR} \tag{2.21}$$

and

$$[b + uB, \Phi_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} [I_{D^k}, \Phi_{D^{n-k}}] = 0 \tag{2.22}$$

If

$$\Phi_{D^n} = \phi_{D^n} + u\psi_{D^n} + \dots \tag{2.23}$$

(we will see later that all coefficients at u^j are zero for $j > 0$), then (2.22) becomes

$$[b, \phi_{D^n}] + \sum_{k=1}^{n-1} \binom{n}{k} [\iota_{D^k}, \phi_{D^{n-k}}] = 0 \tag{2.24}$$

$$[B, \phi_{D^n}] + [b, \psi_{D_n}] + \sum_{k=1}^{n-1} \binom{n}{k} [\iota_{D^k}, \psi_{D^{n-k}}] + [S_{D^k}, \phi_{D^{n-k}}] = 0 \tag{2.25}$$

etc.

We start by defining

(2.26)

$$\phi_D(a_0 \otimes \dots \otimes a_n) = \frac{1}{(n+1)!} \sum_{k=1}^{n} (n-k) a_0 da_1 \dots da_{k-1} dDa_k da_{k+1} \dots da_n$$

and $\psi_D = 0$. We observe that those ψ_D and ϕ_D satisfy (2.24) and (2.25). More precisely,

$$[b, \phi_D] + [\iota_D, \mathrm{HKR}] = 0; \tag{2.27}$$

$$d \circ \phi_D + S_D \circ \mathrm{HKR} = 0 \tag{2.28}$$

and

$$-\phi_D \circ B - \mathrm{HKR} \circ S_D = 0 \tag{2.29}$$

(this is checked straightforwardly but we will also prove a more general statement).

Below we will analyze the complex of operations from Hochschild chains to forms. Obstructions to construct higher ϕ_{D^m}, ψ_{D^m}, etc. are cohomology classes of this complex. We will easily see that the obstructions to constructing ϕ_{D^m} all vanish. Unfortunately, the obstructions to constructing the ψ_{D^m} do not vanish automatically, or at least we do not know an *a priori* reason for them to do. So we have to perform a calculation. We will start with looking at the operations in more detail.

2.2.1. *The complex of higher* HKR *operations.* For any $n \geq 0$ consider the operations $C_n(A) \to \Omega^\bullet_{A/k}$ which are linear combinations of

$$a_0 \otimes \ldots \otimes a_n \mapsto d^{\epsilon_0} D^{m_0}(a_0) \ldots d^{\epsilon_n} D^{m_n}(a_n) \tag{2.30}$$

for $\epsilon_j = 0$ or 1 and for $m_j \geq 0$. □

There are three differentials: precomposition with b and with B, and postcomposition with d. If we identify the operation (2.30) with the monomial

$$\eta^{\epsilon_0} x^{m_0} \otimes \ldots \otimes \eta^{\epsilon_n} x^{m_n} \tag{2.31}$$

then the k-module of operations is identified with the Hochschild cochain complex of the coalgebra $k[x, \eta]$; the three differentials become the coalgebra version of b and B, and the multiplication by η (which is a coderivation).

Operations with $m_0 = \epsilon_0 = 1$ form a subcomplex which is the cobar construction of $k[x, \eta]$. Its graded components of degree $m \geq 2$ in x are all acyclic. Note that if we define ϕ_D as in (2.26) then the obstruction to finding each ϕ_{D^m} for $m \geq 2$ lies in this subcomplex and therefore vanishes.

The complex of operations with the Hochschid differential is denoted by $\mathcal{C}^\bullet$. The subcomplex $m_0 = \epsilon_0 = 1$ is denoted by $\mathcal{C}^\bullet_0$.

For a monomial in (2.31), put

$$m = \sum_{j=0}^{n} m_j; \quad p = \sum_{j=0}^{n} \epsilon_j \tag{2.32}$$

Both b and B preserve these two numbers. This gives a decomposition of the complex of operations

$$\mathcal{C}^\bullet = \bigoplus_{m,p} \mathcal{C}^\bullet(m,p); \quad \mathcal{C}^\bullet_0 = \bigoplus_{m,p} \mathcal{C}^\bullet_0(m,p) \tag{2.33}$$

For any monomial α as in (2.31) with $m > 0$, we say that its principal part is α if all but one m_j are zero. Otherwise, we say that its principal part is zero. Extend the principal part by linearity.

LEMMA 2.2.2. (1) *The subcomplex of elements of* $\mathcal{C}^\bullet_0(m,p)$ *with zero principal part is acyclic for any* $m \geq 1$ *and* $p \geq 2$.

(2) *Let* α *be a linear combination of monomials* (2.31) *with all* $\eta_j = 1$. *Assume that the principal part of* α *is zero. If* α *is a Hochschild cocycle then* $\alpha = 0$.

PROOF. 1) For $m \geq 1$, the quotient of $\mathcal{C}_0^\bullet(m,*)$ by elements of principal part zero is the cobar construction of $k[\epsilon,\eta]$ where $\epsilon^2=0$. This quotient is therefore acyclic for $*\geq 2$, as is $\mathcal{C}_0^\bullet(m,*)$. Therefore the subcomplex is acyclic.

2) The cohomology of $\mathcal{C}^\bullet$ is the Hochschild cohomology of $k[x,\eta]$ which is isomorphic to $\mathrm{HH}^\bullet(k[x])\otimes \mathrm{HH}^\bullet(k[\eta])$. Cochains with all $\epsilon_j=1$ lie in $\oplus_p \mathcal{C}^p(m,p+1)$. There are no coboundaries there, and the only cocycles are multiples of products (cf. (4.2)) of $x^m \in \mathrm{HH}^0(k[x])$ with $\eta^{\otimes(p+1)} \in \mathrm{HH}^p(k[\eta])$. Explicitly, these cocycles are equal to multiples of

$$\sum \frac{m!}{m_0!\ldots m_p!} x^{m_0}\eta\otimes\ldots\otimes x^{m_p}\eta$$

Therefore any such cocycle with principal part zero is zero. □

2.2.2. *Proof of* (2.19). We already know that ϕ_{D^m} exist. We will show that the obstructions to constructing ψ_{D^m} have the principal part zero, and therefore are not only trivial but identically zero by Lemma 2.2.2 above. Therefore we will be able to choose $\Phi_{D^m}=\phi_{D^m}$ (no dependence on u). As a first step, we construct the principal parts of ϕ_{D^m} explicitly. We write $\alpha_0\sim\alpha_1$ if α_0 and α_1 have the same principal part. Denote

$$\phi_k^m(a_0\otimes\ldots\otimes a_n)=a_0da_1\ldots da_{k-1}dD^m(a_k)da_{k+1}\ldots da_n \tag{2.34}$$

$$\rho_k^m(a_0\otimes\ldots\otimes a_n)=a_0da_1\ldots da_{k-1}D^m(a_k)da_{k+1}\ldots da_n \tag{2.35}$$

for $1\leq k\leq n$

We will see that there exists a solution $\Phi_{D^m}=\phi_{D^m}$ of (2.21) satisfying

$$\phi_{D^m}\sim\sum_{k=1}^{n}\frac{(n+m-k)!}{(n+m)!(n-k)!}\phi_k^m \tag{2.36}$$

and $\psi_{D^m}\sim 0$. First we have to show that (2.36) implies

$$-\phi_{D^m}\circ b+\sum_{a=1}^{m-1}\binom{m}{a}[\iota_{D^{m-a}},\phi_{D^a}]+[\iota_{D^m},\mathrm{HKR}]\sim 0 \tag{2.37}$$

Observe that for $1\leq k\leq n-1$

$$\phi_k^m\circ b\sim(-1)^k(\rho_k^m+\rho_{k+1}^m) \tag{2.38}$$

Therefore

$$-\phi_{D^m}\circ b\sim-\sum_{k=1}^{n-1}\frac{(n-1+m-k)!}{(n-1+m)!(n-1-k)!}(-1)^k(\rho_k^m+\rho_{k+1}^m)\sim$$

$$-\sum_{k=1}^{n-1}\frac{(n-1+m-k)!}{(n-1+m)!(n-1-k)!}(-1)^k\rho_k^m+\sum_{k=2}^{n}\frac{(n+m-k)!}{(n-1+m)!(n-k)!}(-1)^k\rho_k^m\sim$$

$$-\frac{1}{(n+m-1)!}\sum_{k=2}^{n-1}\Big(\frac{(n-1+m-k)!}{(n-1-k)!}-\frac{(n+m-k)!}{(n-k)!}\Big)(-1)^k\rho_k^m+$$

$$\frac{1}{(n+m-1)!}\Big(\frac{(n+m-2)!}{(n-2)!}\rho_1^m+(-1)^n m!\rho_n^m\Big)$$

which gives

$$-\phi_{D^m}\circ b\sim\frac{1}{(n+m-1)!}\sum_{k=2}^{n-1}\frac{(n-1+m-k)!}{(n-k)!}m(-1)^k\rho_k^m+ \frac{1}{(n+m-1)!}(\frac{(n+m-2)!}{(n-2)!}\rho_1^m+(-1)^n m!\rho_n^m) \tag{2.39}$$

We also have

(2.40)

$$\sum_{a=1}^{m-1}\binom{m}{a}\iota_{D^{m-a}}\phi_{D^a}=m\iota_D\phi_{D^{m-1}}=\sum_{k=1}^{n}m\frac{(n+m-1-k)!}{(n+m-1)!(n-k)!}(-1)^{k-1}\rho_k^m$$

as well as

$$-\sum_{a=1}^{m-1}\binom{m}{a}\phi_{D^a}\iota_{D^{m-a}}\sim 0 \tag{2.41}$$

$$\iota_{D^m}\circ \mathrm{HKR}\sim 0 \tag{2.42}$$

$(m>1)$;

$$\iota_D\circ\mathrm{HKR}\sim\frac{1}{m!}\sum_{k=1}^{n}(-1)^{k-1}\rho_k^1 \tag{2.43}$$

$$-\mathrm{HKR}\circ\iota_{D^m}\sim-\frac{1}{(n-1)!}\rho_1^m \tag{2.44}$$

The sum of right hand sides from (2.39) through (2.44) is zero. Indeed, for $m>1$ the terms with ρ_k^m for $2\le k\le n$ are contained only in (2.39) and (2.40), and those two cancel each other out. The terms with ρ_1^m appear in (2.39), (2.39), and (2.44) and sum up to zero. For $m=1$, (2.40) is of course zero, and the remaining three terms sum up to zero.

Solving (2.19) at each m, we see that the obstruction to constructing ϕ_{D^m} is a Hochschild cocycle in $\mathcal{C}_0^\bullet$ whose principal part is zero. By Lemma 2.2.2, we can modify ϕ_{D^m} without changing its principal part so that the mth equation will be satisfied. This proves (2.24).

Now put

$$\Theta_k^m(a_0\otimes\ldots\otimes a_n)=da_0\ldots da_{k-1}dD^m(a_k)da_{k+1}\ldots da_n \tag{2.45}$$

for $0\le k$. We need to show that

$$d\phi_{D^m}-\phi_{D^m}\circ B+\sum_{a=1}^{m-1}\binom{m}{a}[S_{D^{m-a}},\phi_{D^a}]+S_{D^m}\circ\mathrm{HKR}-\mathrm{HKR}\circ S_{D^m}\sim 0 \tag{2.46}$$

In fact

$$d\phi_{D^m}\sim\sum_{k=1}^{n}\frac{(n+m-k)!}{(n+m)!(n-k)!}\Theta_k^m \tag{2.47}$$

$$\sum_{a=1}^{m-1}\binom{m}{a}S_{D^{m-a}}\phi_{D^a}\sim\sum_{k=1}^{n}\sum_{a=1}^{m-1}\frac{\binom{m}{a}}{m-a+1}\frac{(n+a-k)!}{(n+a)!(n-k)!}\Theta_k^m \tag{2.48}$$

$$-\sum_{a=1}^{m-1}\binom{m}{a}\phi_{D^a}S_{D^m}\sim-\sum_{k=1}^{n}\sum_{a=1}^{m-1}\binom{m}{a}\sum_{l=k}^{n}\frac{(n+a-l)!}{(n+a+1)!(n-l)!}\Theta_k^m \tag{2.49}$$

(indeed: for every k, $D^a(a_k)$ appears on each position from number k to number n in $S_{D^a}(a_0\otimes\ldots\otimes a_n)$).

$$-\operatorname{HKR}\circ S_{D^m}\sim-\sum_{k=1}^{n}\sum_{l=k}^{n}\frac{1}{(n+1)!}\Theta_k^m \tag{2.50}$$

$$-\phi_{D^m}\circ B\sim-\sum_{k=0}^{n}\sum_{l=1}^{n+1}\frac{(n+m+1-l)!}{(n+m+1)!(n+1-l)!}\Theta_k^m \tag{2.51}$$

(indeed: each a_k appears on each position from number one to number $n+1$ in $B(a_0\otimes\ldots\otimes a_n)$).

$$S_{D^m}\circ\operatorname{HKR}\sim\frac{1}{m+1}\frac{1}{n!}\sum_{k=0}^{n}\Theta_k^m \tag{2.52}$$

Let us compute (2.51). The opposite of the coefficient at Θ_m^k in the right hand side is

$$\sum_{l=1}^{n+1}\frac{(n+m+1-l)!}{(n+m+1)!(n+1-l)!}=\frac{1}{(n+m+1)!}\sum_{\lambda=0}^{n}\frac{(\lambda+m)!}{m!}=$$
$$\frac{m!}{(n+m+1)!}\sum_{\lambda=0}^{n}\binom{\lambda+m}{m}=\frac{m!}{(n+m+1)!}\binom{n+m+1}{m+1}=\frac{1}{(m+1)n!}$$

Therefore the sum of (2.51) and (2.52) is zero.

Now look at (2.50). Note that

$$\sum_{l=k}^{n}\frac{(n+a-l)!}{(n-l)!}=a!\sum_{l=k}^{n}\binom{n+a-l}{a}=a!\sum_{\lambda=0}^{n-k}\binom{\lambda+a}{a}=a!\binom{n-k+a+1}{a+1}$$

Note also that the sum of (2.50) and (2.49) is the same as (2.49) but with the sum over a is taken from 0 to $m-1$. Because of the above, the opposite of the coefficient at Θ_k^m in this sum is

$$\sum_{a=0}^{m-1}\binom{m}{a}\frac{1}{(n+a+1)!}a!\binom{n-k+a+1}{a+1}$$

Now, the sum of (2.47) and (2.48) is just (2.48) but with the sum over a taken from 1 to m. The coefficient of Θ_k^m in this sum is

$$\sum_{a=1}^{m}\frac{\binom{m}{a}}{m-a+1}\frac{(n+a-k)!}{(n+a)!(n-k)!}=\sum_{a=0}^{m-1}\frac{\binom{m}{a+1}}{m-a}\frac{(n+a+1-k)!}{(n+a+1)!(n-k)!}$$

Therefore the coefficient at Θ_k^m in the sum of (2.47) and (2.48) is opposite to the coefficient in the sum of (2.50) and (2.49), namely

$$\sum_{a=0}^{m-1}\frac{m!(n-k+a-1)!}{(a+1)!(m-a)!(n-k)!}$$

We conclude that (2.47), (2.48), (2.49), and (2.50) sum up to zero, and so do (2.51) and (2.52). This proves (2.46). Note that all terms of $\Phi(D)$ of degree >1 in u

vanish automatically. Indeed, there are no operations $C_\bullet(A) \to \Omega^\bullet_{A/k}$ of the type we allowthat have degree greater than 1. This completes the proof of (2.19).

By Lemma 2.1.3, we can pass from $\mathcal{I}(D)$ to $\mathcal{J}(D)$ satisfying (2.5). By modifying $\Phi(D)$, we can replace $\mathcal{I}(D)$ by $\mathcal{J}(D)$ in (2.19). (The easiest if to take $\mathcal{I}(D)+\xi\Phi(D)$ instead of $\mathcal{I}(D)$, apply to it the formula in Lemma 2.1.3, and take the coefficient at ξ). This does prove the theorem, except that we have a different $\mathcal{J}(D)$ on the right. In fact, if we take the I_{D^n} as in (2.17) and apply to them the construction from Lemma 2.1.3, we get

$$I_{D^n} = (-1)^{n-1}(n-1)!D^{n-1}\iota_D + uc_n dD^n \tag{2.53}$$

where $c_n = \int_0^1 y(y-1)\dots(y-n+1)dy$.

2.2.3. *Comparison to the standard Cartan calculus on forms.* We have established that a syslem of operators J_{D^n} acts naturally on te Hochschild complex, and that the HKR map extends to a morphism to forms on which their own J_{D^n} act. But the latter action, given by (2.53), is not standard.

It remains to show that all natural systems J_{D^n} on $\Omega^\bullet_{A/k}[[u]]$ are equivalent. Any natural system of J_{D^n} is of the form

$$J_{D^n} = a_n D^{n-1}\iota_D + ub_n dD^n \tag{2.54}$$

satisfying

$$a_1 = 1;\ a_n + \sum_{k=1}^{n-1}\binom{n}{k}a_k b_{n-k} = 0,\ n > 1 \tag{2.55}$$

In other words: *any* natural operation of degree -1 on $\Omega^\bullet_{A/k}$ is of the form

$$\mathcal{J}(D) = A(D)\iota_D + uB(D)d;\ B(D) = \frac{1}{A(D)} - 1 \tag{2.56}$$

where A and B are power series. It satisfies $(ud + \mathcal{J}(D))^2 = uD$ if and only if $A(0) = 1$ and $B(D) = \frac{1}{A(D)} - 1$, or

$$ud + \mathcal{J}(D) = A(D)\iota_D + \frac{u}{A(D)}d \tag{2.57}$$

Let $\mathcal{J}^0_{D^n}$ be as above when all $b_n = 0$ (and therefore all $a_n = 0$ for $n > 1$). In other words, $\mathcal{J}(D)$ is as in (2.56) with $A(D) = 1$. For a power series $P(D)$, put

$$\mathcal{F}(D) = 1 + P(D)\iota_D d \tag{2.58}$$

$$(ud + \mathcal{J}^0(D))\mathcal{F}(D) = \mathcal{F}(D)(ud + \mathcal{J}(D)) \tag{2.59}$$

is equivalent to

$$DP(D) = \frac{1}{A(D)} - 1 \tag{2.60}$$

In our case,

$$A(D) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(n-1)!}{n!}D^{n-1} = \frac{\log(1+D)}{D} \tag{2.61}$$

Let $\mathcal{F}(D)$ be as in (2.58) for this choice of $A(D)$. Composing $\mathcal{F}(D)$ with the above $\Phi(D)$, we get our HKR(D) as in Theorem 2.2. That completes the proof of this theorem.

2.3. Cartan calculus in terms of Lie algebras.

2.3.1. *The algebras* $\mathcal{A}_0$ *and* $\mathcal{A}_1$. For a variable D, denote by $\mathcal{A}_0(D)$ the algebra over $k[D][u]$ freely generated by J_{D^n}, $n \geq 1$, with the differential given by

$$\partial J_D = uD;\ \partial J_{D^n} + \sum_{k=1}^{n-1} \binom{n}{k} J_{D^k} J_{D^{n-k}} = 0,\ n > 1 \tag{2.62}$$

Define the Abelian DG Lie algebra over $k[u]$ as follows.

$$\mathfrak{a}_0(D) = k[u]D \oplus k[u]\epsilon D;\ \partial(\epsilon D) = D;\ \partial D = 0 \tag{2.63}$$

Then there is a quasi-isomorphism of DG algebras

$$\mathcal{A}_0(D) \to U(\mathfrak{a}_0(D)) \tag{2.64}$$

that sends D to D, J_D to ϵD, and J_{D^n} to zero for $n > 1$. (The universal enveloping algebra is defined over the ring of scalars $k[u]$).

Denote by $\mathcal{A}_1(D)$ the algebra over $k[D][u]$ freely generated by J_{D^n}, $n \geq 1$, with the differential given by

$$\partial I_{D^n} + \sum_{k=1}^{n-1} \binom{n}{k} I_{D^k} I_{D^{n-k}} = uD^n \tag{2.65}$$

Then Lemma 2.1.3 establishes an isomorphism

$$\mathcal{A}_0(D) \xrightarrow{\sim} \mathcal{A}_1(D) \tag{2.66}$$

when $\mathbb{Q} \subset k$.

REMARK 2.3.1. A module over $\mathcal{A}_0(D)$ is the same as an L_∞ module over $\mathfrak{a}_0(D)$ which is defined over $k(D)$. That is, the L_∞ module structure is defined by multilinear maps from $\mathfrak{a}_0(D)$ which vanish if one of the arguments is in D.

Because of (2.66), both sides of the HKR map are L_∞ modules over $\mathfrak{a}_0(D)$. Theorem 2.2.1 says that HKR extends to natural L_∞ morphism of DG modules

$$\mathrm{HKR}(D) : CC^-_\bullet(A) \to \Omega^\bullet_{A/k}[[u]]$$

This an L_∞ morphism over kD, meaning that

$$\mathrm{HKR}(D)(D, X_1, \ldots, X_n) = 0 \tag{2.67}$$

for all $j > 0$ and all X_j in $\mathfrak{a}_0(D)$.

2.3.2. *The algebras* $\mathcal{A}_0(\mathfrak{g})$ *and* $\mathcal{A}_1(\mathfrak{g})$. Now let $(\mathfrak{g}, d_\mathfrak{g})$ be any DG Lie algebra. Define

$$\mathfrak{a}_0(\mathfrak{g}) = (\mathfrak{g}[\epsilon, u], \delta + u\frac{\partial}{\partial \epsilon}) \tag{2.68}$$

For a DG Lie algebra $\mathfrak{g}$, define

$$\mathrm{Sym}^+(\mathfrak{g}) = \mathrm{Ker}(\epsilon : \mathrm{Sym}(\mathfrak{g}) \to k)$$

This is a DG coalgebra with comultiplication

$$\Delta y = \sum y^{(1)} \otimes y^{(2)} \tag{2.69}$$

DEFINITION 2.3.2. *Let* $\mathcal{A}_0(\mathfrak{g})$ *be the DGA generated by the DG subalgebra* $\mathsf{U}(\mathfrak{g})$ *and by the* $\mathsf{k}[\mathsf{u}]$*-submodule* $\mathrm{Sym}(\mathfrak{g}[1])$*, with the relations*

$$[x,(y)] = (\mathrm{ad}_x(y))$$

for $x \in \mathfrak{g}$*,* $y \in \mathrm{Sym}^+(\mathfrak{g})$ *and the differential*

$$d_{\mathfrak{g}} + \partial_{\mathrm{Cobar}} + uB; \tag{2.70}$$

$$\partial_{\mathrm{Cobar}} : x \mapsto 0,\ x \in \mathfrak{g};\ (y) \mapsto -\sum (-1)^{|y^{(1)}|}(y^{(1)})(y^{(2)}), y \in \mathrm{Sym}^+([\mathfrak{g}) \tag{2.71}$$

$$B : (y) \mapsto y,\ y \in \mathrm{Sym}^1(\mathfrak{g}); (y) \mapsto 0, y \in \mathrm{Sym}^{>1}(\mathfrak{g}) \tag{2.72}$$

There is a natural quasi-isomorphism

$$\mathcal{A}_0(\mathfrak{g}) \to \mathsf{U}(\mathfrak{a}_0(\mathfrak{g})); \tag{2.73}$$

$$x \mapsto x,\ x \in \mathfrak{g};\ (y) \to \epsilon y,\ y \in \mathrm{Sym}^1(\mathfrak{g});\ (y) \to 0,\ y \in \mathrm{Sym}^{>1}(\mathfrak{g}) \tag{2.74}$$

As above, the universal enveloping algebra is defined over the ring of scalars $\mathsf{k}[\mathsf{u}]$.

Now define

$$\mathsf{U}^+(\mathfrak{g}) = \mathrm{Ker}(\epsilon : \mathsf{U}(\mathfrak{g}) \to \mathsf{k})$$

This is a DG coalgebra with the comultiplication denoted by (2.69).

DEFINITION 2.3.3. *Let* $\mathcal{A}_1(\mathfrak{g})$ *be the DG algebra over* $\mathsf{k}[\mathsf{u}]$ *generated by the DG subalgebra* $\mathsf{U}(\mathfrak{g})$ *and by the* $\mathsf{k}[\mathsf{u}]$*-submodule* $\mathsf{U}^+(\mathfrak{g}[1])$*, with the relations*

$$[x,(y)] = (\mathrm{ad}_x(y))$$

for $x \in \mathfrak{g}$*,* $y \in \mathsf{U}^+(\mathfrak{g})$ *and the differential*

$$d_{\mathfrak{g}} + \partial_{\mathrm{Cobar}} + uB; \tag{2.75}$$

$$\partial_{\mathrm{Cobar}} : x \mapsto 0,\ x \in \mathfrak{g};\ (y) \mapsto -\sum (-1)^{|y^{(1)}|}(y^{(1)})(y^{(2)}), y \in \mathsf{U}^+([\mathfrak{g}) \tag{2.76}$$

$$B : (y) \mapsto y \in \mathsf{U}(\mathfrak{g}),\ y \in \mathsf{U}^+(\mathfrak{g}) \tag{2.77}$$

LEMMA 2.3.4. *When* $\mathbb{Q} \subset \mathsf{k}$ *then there is a natural isomorphism of DG algebras*

$$\mathcal{A}_0(\mathfrak{g}) \xrightarrow{\sim} \mathcal{A}_1(\mathfrak{g})$$

PROOF. □

2.4. Extended Cartan calculus, I. We extend the Cartan calculus from the Lie algebra of derivations of A to a bigger DG Lie algebra. Let (A, d_A) be a DG algebra, define the DG Lie algebra $(\mathfrak{g}^\bullet_{A,\mathrm{sh}}, d_A + \delta)$ where

$$\mathfrak{g}^\bullet_{A,\mathrm{sh}} = \mathrm{Der}(A) \oplus A[1];\ \delta(a) = \mathrm{ad}(a);\ \delta(D) = 0 \tag{2.78}$$

for $a \in A$, $D \in \mathrm{Der}(A)$.

Define the action of $\mathfrak{g}^\bullet_{A,\mathrm{sh}}$ on $A^{\otimes \bullet}$ by

$$\lambda_a(a_1 \otimes \ldots \otimes a_N) = \sum_{j=0}^{N} \pm a_1 \otimes \ldots \otimes a_j \otimes a \otimes a_{j+1} \otimes \ldots \otimes a_N \tag{2.79}$$

for $a \in A$ (compare to ***). The sign is computed as follows: a permutation of a and a_j introduces the sign $(-1)^{(|a|+1)(|a_j|+1)}$. For $D \in \mathrm{Der}(A)$, put

$$\lambda_D(a_1 \otimes \ldots \otimes a_N) = \sum_{j=1}^{N} \pm a_1 \otimes \ldots \otimes a_{j-1} \otimes D(a_j) \otimes a_{j+1} \otimes \ldots \otimes a_N \tag{2.80}$$

A permutation of D and a_j introduces the sign $(-1)^{|D|(|a_j|+1)}$.

It is easy to see that this is indeed a DG Lie algebra action. Therefore we may define λ_X for any $X \in U(\mathfrak{g}^\bullet_{A,\mathrm{sh}})$.

Now, for $X \in U(\mathfrak{g}^\bullet_{A,\mathrm{sh}})$ define

$$L_X : CC^-_\bullet(A) \to CC^-_{\bullet-|X|}(A) \tag{2.81}$$

as follows:

$$L_D(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} \pm a_0 \otimes \ldots \otimes a_{j-1} \otimes D(a_j) \otimes a_{j+1} \otimes \ldots \otimes a_n \tag{2.82}$$

for $D \in \mathrm{Der}(A)$;

$$L_a(a_0 \otimes \ldots \otimes a_0) = \sum_{j=0}^{n} \pm a_0 \otimes \ldots \otimes a_j \otimes a \otimes a_{j+1} \otimes \ldots \otimes a_n \tag{2.83}$$

for $a \in A$. The sign rule is the same as above. This defines another action of $\mathfrak{g}^\bullet_{A,\mathrm{sh}}$. We extend it to $U(\mathfrak{g}^\bullet_{A,\mathrm{sh}})$ and get (2.81). Now define

$$I_X : CC^-_\bullet(A) \to CC^-_{\bullet-|X|-1}(A) \tag{2.84}$$

$$I_X = \iota_X + uS_X \tag{2.85}$$

$$S_X(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} \pm 1 \otimes a_j \otimes \ldots \otimes a_n \otimes a_0 \otimes \lambda_X(a_1 \otimes \ldots \otimes a_{j-1}) \tag{2.86}$$

The signs are computed as follows: a permutation of a_i and a_j introduces the sign $(-1)^{(|a_i|+1)(|a_j|+1)}$; a permutation of X and a_j introduces the sign $(-1)^{|X|(|a_j|+1)}$. To define ι_X, first recall the case when $X \in \mathfrak{g}^\bullet_{A,\mathrm{sh}}$ (***Ref).

$$\iota_D(a_0 \otimes \ldots \otimes a_n) = (-1)^{|D||a_0|} a_0 D(a_1) \otimes a_2 \otimes \ldots \otimes a_n \tag{2.87}$$

for $D \in \mathrm{Der}(A)$;

$$\iota_a(a_0 \otimes \ldots \otimes a_n) = (-1)^{|a||a_0|} a_0 a \otimes a_1 \otimes \ldots \otimes a_n \tag{2.88}$$

for $a \in A$. Also, define the operation $\circ$ (or the brace operation) on $\mathfrak{g}^\bullet_{A,\mathrm{sh}}$:

$$D \circ E = DE;\ D \circ a = D(a); a \circ D = a \circ D = 0 \tag{2.89}$$

for $D, E \in \mathrm{Der}(A)$ and $a, b \in A$. Finally, for $X = X_1 \ldots X_m$, $X_j \in \mathfrak{g}^\bullet_{A,\mathrm{sh}}$,

$$\iota_X = \iota_{(\ldots(X_1 \circ X_2)\circ X_3)\circ \ldots \circ X_m)} \tag{2.90}$$

PROPOSITION 2.4.1. *Let* $X \in U(\mathfrak{g}^\bullet_{A,\mathrm{sh}})$ *and* $Y \in U^+(\mathfrak{g}^\bullet_{A,\mathrm{sh}})$. *The assignment* $X \mapsto L_X$ *as in* (2.81) *and* $(Y) \mapsto I_Y$ *as in* (2.85)*defines an action of the DG algebra* $\mathcal{A}_1(\mathfrak{g}^\bullet_{A,\mathrm{sh}})$ *on* $CC^-_\bullet$.

PROOF. □

This makes $\mathrm{CC}^-_\bullet(A)$ an L_∞ module over

$$\mathfrak{a}_0(\mathfrak{g}^\bullet_{A,\mathrm{sh}}) = (\mathfrak{g}^\bullet_{A,\mathrm{sh}}[\epsilon, u], \delta + u\frac{\partial}{\partial\epsilon}).$$

When A is commutative, $\Omega^\bullet_{A/k}[[u]]$ is a module over the same DGA: for $D \in A$, D is the Lie derivative and ϵD acts by the contraction ι_D; for $a \in A$, a acts by multiplication by da, and ϵa acts by multiplication by a.

THEOREM 2.4.2. *Let* A *be a commutative algebra over* k *such that* $\mathbb{Q} \subset k$. *Then* HKR *extends to a natural* L_∞ *morphism of DG modules over* $\mathfrak{a}_0(\mathfrak{g}^\bullet_{A,\mathrm{sh}})$. *This is a morphism over the subalgebra* $\mathfrak{g}^\bullet_{A,\mathrm{sh}}$, *meaning that all its components vanish if at least one of the arguments are in this subalgebra.*

PROOF. □

REMARK 2.4.3. We do not know at the moment whether the same is true for any algebra if we replace HKR by the noncommutative HKR map to noncommutative forms.

3. Extended Cartan calculus, II

Recall that $\mathfrak{g}^\bullet_A$ denotes the DG Lie algebra $C^{\bullet+1}(A, A)$ with the Gerstenhaber bracket. We will extend noncommutative Cartan calculus from $\mathfrak{g}^\bullet_{A,\mathrm{sh}}$ to $\mathfrak{g}^\bullet_A$. (The latter is a DG subalgebra of the former). There are two ways to do that. We present these two ways in Subsections 3.1 and 3.2

3.1. Replacing an algebra by a resolution. Take a semi-free resolution $R \to A$. Then the

$$\mathfrak{g}^\bullet_{R,\mathrm{sh}} \to \mathfrak{g}^\bullet_R \tag{3.1}$$

is a quasi-isomorphism of DG Lie algebras. By a theorem of Keller ***Ref, there is a chain of quasi-isomorphisms of DG Lie algebras connecting $\mathfrak{g}^\bullet_R$ and $\mathfrak{g}^\bullet_A$. Also, the map $\mathrm{CC}^-_\bullet(R) \to \mathrm{CC}^-_\bullet(A)$ is a quasi-isomorphism. This induces on $\mathrm{CC}^-_\bullet(A)$ an L_∞ module structure over $\mathfrak{a}_0(\mathfrak{g}^\bullet_A)$. ***A bit more

3.2. The A_∞ action of $\mathrm{CC}^\bullet_{\mathrm{II}}(U(\mathfrak{g}_A))$. Recall that

$$\mathfrak{g}^\bullet_A \xrightarrow{\sim} \mathrm{Coder}(\mathrm{Bar}(A)).$$

This defines the action of $U(\mathfrak{g}_A)$ on $\mathrm{Bar}(A)$ as well as linear maps

$$\mu_N \colon U(\mathfrak{g}_A)^{\otimes N} \otimes \mathrm{Bar}(A) \to \mathrm{Bar}(A) \tag{3.2}$$

(composition of the above action with the n-fold product on $U(\mathfrak{g}_A)$).

LEMMA 3.2.1. *The above are morphisms of DG coalgebras.*

PROOF. Clear. □

Recall the definition of the cyclic complex of the second kind from Chapter 6. (It uses the definition of the Hochschild complex as the standard complex associated to a bicomplex, i.e. the complex comprised of direct sums over diagonals, as opposed to direct products as would seem more natural if we were doing a dual construction to the case of algebras).

COROLLARY 3.2.2. *The compositions of*

$$\mathrm{CC}^{\bullet}_{\mathrm{II}}(U(\mathfrak{g}_A))^{\otimes N}\otimes \mathrm{CC}^{\bullet}_{\mathrm{II}}(\mathrm{Bar}(A))\longrightarrow \mathrm{CC}^{\bullet}_{\mathrm{II}}(U(\mathfrak{g}_A)^{\otimes N}\otimes \mathrm{Bar}(A))$$

(given by $m(U(\mathfrak{g}_A),\ldots,U(\mathfrak{g}_A),\mathrm{Bar}(A))$ *(cf.* (4.1)*) with the morphism induced by* μ_N *define on* $\mathrm{CC}^{\bullet}_{\mathrm{II}}(\mathrm{Bar}(A))$ *a structure of an* A_∞ *module over the* A_∞ *algebra* $\mathrm{CC}^{\bullet}_{\mathrm{II}}(U(\mathfrak{g}_A))$.

Now put

$$\mathrm{CC}^{-,\mathrm{big}}_{\bullet}(A)=\mathrm{CC}^{\bullet}_{\mathrm{II}}(\mathrm{Bar}(A)) \tag{3.3}$$

We have seen: ***REFS 1) There is a natural quasi-isomorphic embedding

$$\mathrm{CC}^{-}_{\bullet}(A)\xrightarrow{\sim}\mathrm{CC}^{-,\mathrm{big}}_{\bullet}(A) \tag{3.4}$$

2) There is a natural A_∞ quasi-isomorphism defined over the subalgebra $U(\mathfrak{g}^{\bullet}_A)$

$$\mathrm{CC}^{\bullet}_{\mathrm{II}}(U(\mathfrak{g}^{\bullet}_A))\xrightarrow{\sim}\mathcal{A}_1(\mathfrak{g}^{\bullet}_A) \tag{3.5}$$

Recall that $\mathfrak{g}^{\bullet}_A$ acts on $\mathrm{CC}^{-}_{\bullet}(A)$ by Lie derivatives ***Ref. It is easy to see that (3.4) intertwines this action with the one on the right hand side. We therefore obtain

THEOREM 3.2.3. (1) *The action of* $\mathfrak{g}^{\bullet}_A$ *on* $\mathrm{CC}^{-,\mathrm{big}}_{\bullet}(A)$ *naturally extends to a* (u)*-adically continuous* $k[[u]]$*-linear* L_∞ *module structure over* $(\mathfrak{g}^{\bullet}_A[\epsilon][[u]],\delta+u\frac{\partial}{\partial\epsilon})$. *This structure is over* $\mathfrak{g}^{\bullet}_A$, *i.e. all operations* m_k *for* $k>1$ *vanish if at least one element is in the subalgebra.*
(2) *The action of* $\mathfrak{g}^{\bullet}_A$ *on* $\mathrm{CC}^{-}_{\bullet}(A)$ *naturally extends to a* (u)*-adically continuous* $k[[u]]$*-linear* L_∞ *module structure over* $(\mathfrak{g}^{\bullet}_A[\epsilon][[u]],\delta+u\frac{\partial}{\partial\epsilon})$.

PROOF. □

4. Appendix

In 4.1, we show that there is an L_∞ action of $\mathcal{A}_0(\mathfrak{g}^{\bullet}_A)$ on $\mathrm{CC}^{-}_{\bullet}(A)$ if k is over $\mathbb{Q}$. In 4.2 , ***FINISH

***To be shortened, streamlined

4.1. The action of $\mathfrak{g}_A[\epsilon,u]$. The algebra of operations on the negative and periodic cyclic complexes that we used above is a close relative of another, more classical algebra of operations that appears in the usual calculus. It does act on the cyclic complexes but in a less straightforward way. Here we discuss this action and its relation to the above.

Let ϵ be a variable of degree -1 such that $\epsilon^2=0$ and u a variable of (cohomological) degree two. For any DG Lie algebra $(\mathfrak{g},\delta)$ construct a DG Lie algebra $\mathfrak{g}[[u]][\epsilon]$, the differential being $\delta+u\frac{\partial}{\partial\epsilon}$. Consider its universal enveloping algebra over $k[[u]]$. It is a DG algebra over $k[[u]]$.

THEOREM 4.1.1. *Assume that* $\mathbb{Q}\subset k$. *There is a natural* $k[[u]]$*-linear* (u)*-adically continuous* A_∞ *action of the DGA* $(U(\mathfrak{g}_A[\epsilon])[[u]],d+\delta+u\frac{\partial}{\partial\epsilon})$ *on* $\mathrm{CC}^{-}_{\bullet}(A)$ *whose components*

$$\varphi_n:U(\mathfrak{g}_A[\epsilon])[[u]]^{\otimes n}\to \mathrm{End}^{1-n}(\mathrm{CC}^{-}_{\bullet}(A))$$

satisfy the following.

(1) $\varphi_1(D)=(-1)^{|D|}L_D$ *and* $\varphi_1(\epsilon D)=(-1)^{|D|-1}I_D$ *for* $D\in\mathfrak{g}_A$.
(2) $\varphi_n(a_1,\ldots,a_j,D,a_{j+1},\ldots,a_{n-1})=0$ *for* $n>1$, *for any* j, *and* $D\in\mathfrak{g}$.

(cf. (8.0.7) *and* (5.4)*).*

PROOF. To simplify the notation, we will consider an arbitrary DG Lie algebra $(\mathfrak{g}, \delta)$ over $K = k[[u]]$ (so in our example $\mathfrak{g} = \mathfrak{g}_A[[u]]$). Everything will be linear over K.

As usual, for any Lie algebra $\mathfrak{g}$ acting on an associative algebra C by derivations, denote by $U(\mathfrak{g}) \ltimes C$ the algebra generated by two subalgebras $U(\mathfrak{g})$ and C subject to relations $Xc - cX = X(c)$ for $X \in \mathfrak{g}$ and $c \in C$. The multiplication map $U(\mathfrak{g}) \otimes C \to U(\mathfrak{g}) \ltimes C$ is a bijection. Note that

$$U(\mathfrak{g}[\epsilon]) \xrightarrow{\sim} U(\mathfrak{g}) \ltimes U(\epsilon\mathfrak{g}).$$

Recall Proposition 4.2.2. Construct an A_∞ morphism

$$\phi : (U(\mathfrak{g}[\epsilon]), \delta + u\frac{\partial}{\partial\epsilon}) \to (U(\mathfrak{g}) \ltimes \mathrm{Cobar}(\overline{U}(\mathfrak{g})), \delta + b + uB) \tag{4.1}$$

as follows. Notice that both $\mathrm{CobarBar}(U(\epsilon\mathfrak{g}))$ and $\mathrm{Cobar}(\overline{S}(\mathfrak{g}))$ are resolutions of $U(\epsilon\mathfrak{g})$ (where $\overline{S}(\mathfrak{g})$ is the positive part of the symmetric algebra, viewed as a coalgebra). Moreover,

$$\mathrm{Cobar}(\overline{S}(\mathfrak{g})) \to U(\epsilon\mathfrak{g}) \tag{4.2}$$

admits a contracting homotopy that is invariant under the adjoint action of $\mathfrak{g}$ (in fact under all linear endomorphisms of $\mathfrak{g}$). Therefore we can construct an $\mathrm{ad}(\mathfrak{g})$-equivariant morphism of DG algebras

$$\mathrm{CobarBar}(U(\mathfrak{g}\epsilon)) \to \mathrm{Cobar}(\overline{S}(\mathfrak{g})) \tag{4.3}$$

Because (4.3) is $\mathrm{ad}(\mathfrak{g})$-equivariant, we extend it to

$$U(\mathfrak{g}) \ltimes \mathrm{CobarBar}(U(\mathfrak{g}\epsilon)) \to U(\mathfrak{g}) \ltimes \mathrm{Cobar}(\overline{S}(\mathfrak{g})) \tag{4.4}$$

We deform this morphism as follows. Denote by $U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}))$ the same graded algebra with changed differential. Namely, for any $D_1 \ldots D_n \in S^n(\mathfrak{g})$, put

$$\mathrm{PBW}(D_1 \ldots D_n) = \frac{1}{n!} \sum \pm D_{\sigma 1} \ldots D_{\sigma n} \in U(\mathfrak{g})$$

(note that, as usual, PBW is an $\mathrm{ad}(\mathfrak{g})$-equivariant morphism of coalgebras). Define the new differential a free generators of CobarBar to be

$$(D_1 \ldots D_n) \mapsto (\delta + \partial_{\mathrm{Cobar}} + u\mathrm{PBW})(D_1 \ldots D_n).$$

Note that $U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}))$ is still a resolution of $U(\mathfrak{g}[\epsilon])$. In fact, the morphism

$$U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g})) \to U(\mathfrak{g}[\epsilon]) \tag{4.5}$$

defined by

$$(D_1 \ldots D_n) \mapsto \frac{1}{n!} \sum \pm \epsilon D_{\sigma 1} \cdot D_{\sigma 2} \ldots D_{\sigma n} \tag{4.6}$$

admits an $\mathrm{ad}(\mathfrak{g})$-equivariant contracting homotopy (that we construct recursively using the one for (4.2)). We use this homotopy to construct a morphism

$$U(\mathfrak{g}) \ltimes \mathrm{CobarBar}(U(\mathfrak{g}\epsilon)) \to U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g})) \tag{4.7}$$

over $U(\mathfrak{g}[\epsilon])$. This is the same as an A_∞ morphism

$$U(\mathfrak{g}[\epsilon]) \to U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g})) \tag{4.8}$$

satisfying

$$\phi_n(\ldots, D, \ldots) = 0 \tag{4.9}$$

for any $n > 1$. Because of Proposition 4.2.2 and Corollary 3.2.2, we get the morphism as in Theorem 4.1.1. It satisfies property (2). As for (1),

$$\phi_1(D) = D;\ \phi_1(\epsilon D) = (D)$$

Elaborate a bit?* ***FINISH*** □

4.2. A_∞ structure on $C_\bullet(C^\bullet(A))$. The algebra of operations on the negative cyclic complex that was described in Corollary 3.2.2 can be extended as follows. Start by noting that

$$U(\mathfrak{g}_A) \to \mathrm{Bar}(C^\bullet(A)), \tag{4.10}$$

$$D \mapsto (D),\ D \in \mathfrak{g},$$

extends to a morphism of bialgebras. The bialgebra morphisms ***REF

$$\mathrm{Bar}(C^\bullet(A)) \otimes \mathrm{Bar}(C^\bullet(A)) \to \mathrm{Bar}(C^\bullet(A));\ \mathrm{Bar}(C^\bullet(A)) \otimes \mathrm{Bar}(A) \to \mathrm{Bar}(A)$$

induce (because of ***REF) an A_∞ algebra structure on $CC^-_\bullet(C^\bullet(A))$ and an A_∞ module structure on $CC^-_\bullet(A)$.

Below we will construct such a structure explicitly. All the pairings described in 1, 5 are in fact different parts of this structure. *We do not know whether it is the same as described above. ***Seems that we can prove it directly for the Hochschild complex.*

4.2.1. *Explicit construction.* Let A be a differential graded algebra. The complex $C_\bullet(C^\bullet(A))$ contains the Hochschild cochain complex $C^\bullet(A)$ as the subcomplex of zero-chains

$$C^\bullet(A) = C_0(C^\bullet(A)) \xrightarrow{\iota} C_\bullet(C^\bullet(A))$$

and has the Hochschild chain complex $C_\bullet(A)$ as a quotient complex induced by the projection on the zero Hochschild cochains $C^\bullet(A) \to C^0(A)$

$$C_\bullet(C^\bullet(A)) \xrightarrow{\pi} C_\bullet(C^0(A)) = C_\bullet(A).$$

The projection π splits if A is commutative. In general $C_\bullet(A)$ is naturally a graded subspace but not a subcomplex.

THEOREM 4.2.1. *There is an A_∞ structure $\mathbf{m}$ on $C_\bullet(C^\bullet(A))[[u]]$ such that:*

(1) *All m_n are $k[[u]]$-linear, (u)-adically continuous*
(2) $m_1 = b + \delta + uB$
For $x,\ y \in C_\bullet(A)$:
(3) $(-1)^{|x|} m_2(x, y) = (\mathrm{sh} + u\,\mathrm{sh}')(x, y)$
For $D,\ E \in C^\bullet(A)$:
(4) $(-1)^{|D|} m_2(D, E) = D \smile E$
(5) $m_2(1 \otimes D,\ 1 \otimes E) + (-1)^{|D||E|} m_2(1 \otimes E,\ 1 \otimes D) = (-1)^{|D|} 1 \otimes [D,\ E]$
(6) $m_2(D,\ 1 \otimes E) + (-1)^{(|D|+1)|E|} m_2(1 \otimes E,\ D) = (-1)^{|D|+1} [D,\ E]$

THEOREM 4.2.2. *On $C_\bullet(A)[[u]]$, there exists a structure of an A_∞ module over the A_∞ algebra $C_\bullet(C^\bullet(A))[[u]]$ such that:*

(1) *All μ_n are $k[[u]]$-linear, (u)-adically continuous*
(2) $\mu_1 = b + uB$ *on* $C_\bullet(A)[[u]]$
For $a \in C_\bullet(A)[[u]]$:

(3) $\mu_2(a, D) = (-1)^{|a||D|+|a|}(i_D + uS_D)a$
(4) $\mu_2(a, 1 \otimes D) = (-1)^{|a||D|}L_D a$
For $a,\ x \in C_\bullet(A)[[u]]$*:* $(-1)^{|a|}\mu_2(a, x) = (\mathrm{sh} + u\,\mathrm{sh}')(a, x)$

CONSTRUCTION 4.2.3. *The explicit description of the* A_∞ *structure on* $C_\bullet(C^\bullet(A))$*. We define for* $n \geq 2$

$$m_n = m_n^{(1)} + u m_n^{(2)}$$

where, for

$$a^{(k)} = D_0^{(k)} \otimes \ldots \otimes D_{N_k}^{(k)},$$

$$m_n^{(1)}(a^{(1)}, \ldots, a^{(n)}) = \sum \pm m_k\{\underline{\ldots}, D_0^{(0)}\{\underline{\ldots}\}, \underline{\ldots}, D_0^{(n)}\{\underline{\ldots}\}\underline{\ldots}\} \otimes \underline{\ldots}$$

The space designated by $\_$ *is filled with* $D_i^{(j)}$*,* $i > 0$*, in such a way that:*

- *the cyclic order of each group* $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ *is preserved;*
- *any cochain* $D_j^{(i)}$ *may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form* $D_q^{(p)}$ *with* $p < i$*. The sign convention: any permutation contributes to the sign; the parity of* $D_j^{(i)}$ *is always* $|D_j^{(i)}| + 1$*.*

$$m_n^{(2)}(a^{(1)}, \ldots, a^{(n)}) = \sum \pm 1 \otimes \underline{\ldots} \otimes D_0^{(0)}\{\underline{\ldots}\} \otimes \underline{\ldots} \otimes D_0^{(n)}\{\underline{\ldots}\} \otimes \underline{\ldots}$$

The space designated by $\_$ *is filled with* $D_i^{(j)}$*,* $i > 0$*, in such a way that:*

- *the cyclic order of each group* $D_0^{(k)}, \ldots, D_{N_k}^{(k)}$ *is preserved;*
- *any cochain* $D_j^{(i)}$ *may contain some of its neighbors on the right inside the braces, provided that all of these neighbors are of the form* $D_q^{(p)}$ *with* $p < i$*. The sign convention: any permutation contributes to the sign; the parity of* $D_j^{(i)}$ *is always* $|D_j^{(i)}| + 1$*.*

To obtain a structure of an A_∞ *module from Theorem 4.2.2, one has to assume that all* $D_j^{(1)}$ *are elements of* A *and to replace braces* { } *by the usual parentheses* ( ) *symbolizing evaluation of a multi-linear map at elements of* A*.*

PROOF OF THE THEOREM 4.2.1. First let us prove that $m^{(1)}$ is an A_∞ structure on $C_\bullet(C^\bullet(A))$. Decompose it into the sum $\delta + \widetilde{m}^{(1)}$ where δ is the differential induced by the differential on $C^\bullet(A)$. We want to prove that $[\delta, \widetilde{m}^{(1)}] + \frac{1}{2}[\widetilde{m}^{(1)}, \widetilde{m}^{(1)}] = 0$. We first compute $\frac{1}{2}[\widetilde{m}^{(1)}, \widetilde{m}^{(1)}]$. It consists of the following terms:

(1) $m\{\ldots D_0^{(1)} \ldots m\{\ldots D_0^{(i+1)} \ldots D_0^{(j)} \ldots\} \ldots D_0^{(n)} \ldots\} \otimes \ldots$

where the only elements allowed inside the inner $m\{\ldots\}$ are $D_p^{(q)}$ with $i + 1 \leq q \leq j$;

(2) $m\{\ldots D_0^{(1)} \ldots m\{\ldots\} \ldots D_0^{(n)} \ldots\} \otimes \ldots$

where the only elements allowed inside the inner $m\{\ldots\}$ are $D_p^{(q)}$ for one and only q (these are the contributions of the term $\widetilde{m}^{(1)}(a^{(1)}, \ldots, ba^{(q)}, \ldots, a^{(n)})$;

(3) $m\{\ldots D_0^{(1)} \ldots D_0^{(n)} \ldots\} \otimes \ldots \otimes m\{\ldots\} \otimes \ldots$

with the only requirement that the second $m\{\ldots\}$ should contain elements $D_p^{(q)}$ and $D_{p'}^{(q')}$ with $q \neq q'$.(The terms in which the second $m\{\ldots\}$ contains $D_p^{(q)}$

where all q's are the same cancel out: they enter twice, as contributions from $b\widetilde{m}^{(1)}(a^{(1)}, \dots, a^{(q)}, \dots, a^{(n)}$ and from $\widetilde{m}^{(1)}(a^{(1)}, \dots, ba^{(1)}, \dots, a^{(n)})$.

The collections of terms (1) and (2) differ from

(0) $\frac{1}{2}[m, m]\{\dots D_0^{(1)} \dots \dots D_0^{(n)} \dots\} \otimes \dots$

by the sum of all the following terms:

($1'$) terms as in (1), but with a requirement that in the inside $m\{\dots\}$ an element $D_p^{(q)}$ must me present such that $q \leq i$ or $q > j$;

($2'$) terms as in (1), but with a requirement that the inside $m\{\dots\}$ must contain elements $D_p^{(q)}$ and $D_{p'}^{(q')}$ with $q \neq q'$.

Assume for a moment that $D_p^{(q)}$ are elements of a commutative algebra (or, more generally, of a C_∞ algebra, i.e. a homotopy commutative algebra). Then there is no δ and $\widetilde{m}^{(1)} = m^{(1)}$. But the terms ($1'$) and ($2'$) all cancel out, as well as (3). Indeed, they all involve $m\{\dots\}$ with some shuffles inside, and m is zero on all shuffles. (the last statement is obvious for a commutative algebra, and is exactly the definition of a C_∞ algebra).

Now, we are in a more complex situation where $D_p^{(q)}$ are Hochschild cochains (or, more generally, elements of a *brace algebra*). Recall that all the formulas above assume that cochains $D_p^{(q)}$ may contain their neighbors on the right inside the braces. We claim that

(A) the terms ($1'$), ($2'$) and (3), together with (0), cancel out with the terms constituting $[\delta, \widetilde{m}^{(1)}]$.

To see this, recall from [?] the following description of brace operations. To any rooted planar tree with marked vertices one can associate an operation on Hochschild cochains. The operation

$$D\{\dots E_1\{\dots\{Z_{1,1}, \dots, Z_{1,k_1}\}, \dots\} \dots E_n\{\dots\{Z_{n,1}, \dots, Z_{n,k_n}\} \dots\} \dots\}$$

corresponds to a tree where D is at the root, E_i are connected to D by edges, and so on, with Z_{ij} being external vertices. The edge connecting D to E_i is to the left from the edge connecting D to E_j for $i < j$, etc. Furthermore, one is allowed to replace some of the cochains D, E_i, etc. by the cochain m defining the A_∞ structure. In this case we leave the vertex unmarked, and regard the result as an operation whose input are cochains marking the remaining vertices (at least one vertex should remain marked).

For a planar rooted tree T with marked vertices, denote the corresponding operation by $\mathbf{O}_T$. The following corollary from Proposition 7.0.2 was proven in [?]:

$$[\delta, \mathbf{O}_T] = \sum_{T'} \pm \mathbf{O}_{T'}$$

where T' are all the trees from which T can be obtained by contracting an edge. One of the vertices of this new edge of T' inherits the marking from the vertex to which it gets contracted; the other vertex of that edge remains unmarked. There is one restriction: the unmarked vertex of T' must have more than one outgoing edge. Using this description, it is easy to see that the claim (A) is true.

Now let us prove that

$$[\delta, \widetilde{m}^{(2)}] + \widetilde{m}^{(1)} \circ m^{(2)} + m^{(2)} \circ \widetilde{m}^{(1)} = 0$$

The summand $m^{(2)} \circ \widetilde{m}^{(1)}$ *contributes terms of the form:*

(1) $D_0^{(1)} \otimes \dots \otimes D_0^{(2)} \otimes \dots \otimes D_0^{(n)} \otimes \dots$

(2) $D_0^{(n)} \otimes \cdots \otimes D_0^{(1)} \otimes \cdots \otimes D_0^{(n-1)} \otimes \ldots$
(3) $1 \otimes \ldots D_0^{(1)} \otimes \cdots \otimes m\{D_0^{(i+1)} \ldots D_0^{(j)}\} \otimes \ldots \otimes D_0^{(n)} \otimes \ldots$
where $j \geq i$.

The summand $\widetilde{m}^{(1)} \circ m^{(2)}$ *contributes terms of the form:*

(4) Same as (3), but with the only elements allowed inside the $m\{\ldots\}$ being $D_p^{(q)}$ with $i+1 \leq q \leq j$;
(5) $1 \otimes \ldots D_0^{(1)} \otimes \cdots \otimes m\{\ldots\} \otimes \ldots \otimes D_0^{(n)} \otimes \ldots$
where the only elements allowed inside the $m\{\ldots\}$ are $D_p^{(q)}$ for one and only q.

The terms of type (1) and (2) cancel out - indeed, $bm^{(2)}(a^{(1)}, \ldots, a^{(1)})$ contributes both (1) and (2); $\widetilde{m}^{(1)}(a^{(1)}, m^{(2)}(a^{(2)}, \ldots, a^{(n)}))$ contributes (1), and $\widetilde{m}^{(1)}(m^{(2)}(a^{(1)}, \ldots, a^{(n-1)}), a^{(n)})$ contributes (2).

The sum of the terms (3), (4), (5) is equal to zero by the same reasoning as in the end of the proof of $[\widetilde{m}^{(1)}, \widetilde{m}^{(1)}] = 0$. □

THE PROOF OF THE THEOREM 4.2.2 IS THE SAME AS ABOVE. □

REMARK 4.2.4. Let A be a commutative algebra. Then $C_\bullet(A)[[u]]$ is not only a subcomplex but an A_∞ subalgebra of $C_\bullet(C^\bullet(A))[[u]]$. This A_∞ structure on $C_\bullet(A)[[u]]$ was introduced in [?].

The corresponding binary product was defined by Hood and Jones [?]. One can define it for any algebra A, commutative or not. If A is not commutative then this product is not compatible with the differential. Nevertheless, we will use it in ??.

5. Bibliographical notes

Khalkhali; NT

CHAPTER 14

Rigidity and the Gauss-Manin connection for periodic cyclic complexes

1. Introduction

2. Rigidity of periodic cyclic complexes

2.1. Divided powers. Let A be a $\mathbb{Z}$-module without torsion. Let $J^n A$ be a descending filtration on A, $n \geq 0$, $J^0 A = A$. The filtration J^n induces a filtration on any tensor power of A that will be denoted by $J^n A^{\otimes k}$. We will make the following assumption.

0) For every $n \geq 0$ there is $N(n) \geq 0$ such that $N(n) \to \infty$ as $n \to \infty$ and $J^n A^{\otimes k} \subset n! J^N A^{\otimes k}$ for all k.

REMARK 2.1.1. This is the case for the filtration by powers of p where p is an odd prime. Note also that the definition applies to generalizations of the tensor product (such as completed tensor products).

We will consider bilinear products $m(a, b) = ab$ on A with the following properties.

1) $J^m A \cdot J^n A \subset J^{m+n} A$.

2) The induced product on $\mathrm{gr}_J(A)$ is associative.

Due to a technical (*possibly avoidable*) difficulty, we are only able to prove the statements below for *big* periodic cyclic complexes defined as follows:

$$\mathrm{CC}^{\mathrm{PER}}_{\bullet}(A) = \mathrm{CC}^{-\bullet}_{\mathrm{II}}(\mathrm{Bar}(A))[u^{-1}] = \mathrm{C}^{-\bullet}_{\mathrm{II}}(\mathrm{Bar}(A))[[u, u^{-1}] \tag{2.1}$$

Before a product on A is introduced, the complex is defined for the algebra A with zero multiplication. The complex (2.1) is quasi-isomorphic to $\mathrm{CC}^{\mathrm{per}}_{\bullet}(A)$ equipped with the differential uB. The filtration J^n on tensor powers induces a filtration on on $\mathrm{CC}^{\mathrm{PER}}_{\bullet}(A)$. Let $\widehat{\mathrm{CC}}^{\mathrm{PER}}_{\bullet}(A)$ be the completion of $\mathrm{CC}^{\mathrm{PER}}_{\bullet}(A)$ with respect to this filtration.

2.2. The rigidity theorem. For every product m on A as above, we will construct a (continuous) differential D_m on $\widehat{\mathrm{CC}}^{\mathrm{PER}}_{\bullet}(A)$ such that:

1) $D_m^2 = 0$;

2) D_m preserves the filtration;

3) the differential induced by D_m on $\mathrm{gr}_J(\mathrm{CC}^{\mathrm{PER}}_{\bullet}(A))$ is the usual differential $b + uB + \partial_{\mathrm{Bar}}$;

4) if the product m is associative, then D_m is equal to $b + uB + \partial_{\mathrm{Bar}}$.

Then we will prove

THEOREM 2.2.1. *1) If two products m_0 and m_1 induce the same product on gr_J and D_{m_0}, D_{m_1} are the corresponding differentials, then there exists a continuous* *isomorphism of complexes*

$$T_{0,1} : (\widehat{CC}_\bullet^{PER}(A), D_{m_1}) \tilde{\to} (\widehat{CC}_\bullet^{PER}(A), D_{m_0}).$$

2) For every $n+1$ *products* $m_0, \dots, m_n$ *inducing the same product on* gr_J*, there is a homogeneous continuous* $\mathbb{Z}$*-linear map*

$$T_{0,1,\dots,n} : \widehat{CC}_\bullet^{PER}(A) \to (\widehat{CC}_\bullet^{PER}(A)$$

of degree $n-1$ *such that the* A_∞ *relations*

$$D_{m_0}T_{0,1,\dots,n}+(-1)^n T_{0,1,\dots,n}D_{m_n}-\sum_{j=1}^{n-1}(-1)^j T_{0,\dots,\hat{j},\dots,n}+\sum_{j=1}^{n-1}(-1)^j T_{0,\dots,j}T_{j,\dots,n}=0$$

are satisfied.

3. Liftings modulo p of algebras over a field of characteristic p

Let A be a free $\mathbb{Z}$-module. Let $A_0 = A/pA$. Consider an associative unital algebra structure on A_0. Consider a bilinear binary product ab on A for which 1 is a neutral element and whose reduction modulo p is the product in A_0. We call A together with such a product *a lifting modulo* p *of* A_0.

Let $p > 2$. For every lifting m of A_0 to A modulo p we will construct a differential D_m on $\widehat{CC}_\bullet^{PER}(A)$ such that:

1) $D_m^2 = 0$;

2) the reduction of D_m modulo p is the usual differential $b + uB + \partial_{\mathrm{Bar}}$;

3) if m is a lifting (i.e. if it is associative), then D_m is equal to $b + uB + \partial_{\mathrm{Bar}}$.

For these differentials, the following is true.

THEOREM 3.0.1. *1) For every two liftings* m_0 *and* m_1 *and corresponding differentials* D_{m_0}, D_{m_1}*, there is an isomorphism of complexes*

$$T_{0,1} : (\widehat{CC}_\bullet^{PER}(A), D_{m_1}) \tilde{\to} (\widehat{CC}_\bullet^{PER}(A), D_{m_0}).$$

2) For every $n+1$ *liftings* $m_0, \dots, m_n$ *and corresponding differentials* $D_{m_0}, \dots, D_{m_n}$*, there is a homogeneous* $\mathbb{Z}_p$*-linear map*

$$T_{0,1,\dots,n} : (\widehat{CC}_\bullet^{PER}(A), D_{m_0}) \to (\widehat{CC}_\bullet^{PER}(A), D_{m_n})$$

of degree $n-1$*, and the* A_∞ *relations*

$$D_{m_0}T_{0,1,\dots,n}+(-1)^n T_{0,1,\dots,n}D_{m_n}-\sum_{j=1}^{n-1}(-1)^j T_{0,\dots,\hat{j},\dots,n}+\sum_{j=1}^{n-1}(-1)^j T_{0,\dots,j}T_{j,\dots,n}=0$$

4. The Gauss Manin connection

Let S be a scheme over $\mathbb{Z}$. Let $\mathcal{A}$ be an $\mathcal{O}_S$-algebra. Assume that A, and therefore $\mathcal{O}_S$, is without $\mathbb{Z}$-torsion.We also assume that the $\mathcal{O}_S$-module $\mathcal{A}$ has a connection ∇.

Consider the big periodic cyclic complex $CC_\bullet^{PER}(\mathcal{A})$ over $\mathcal{O}_S$ as the ground ring. This means that all tensor products in the definition are taken over $\mathcal{O}_S$. We will construct a flat superconnection

$$\nabla_{GM} = b + uB + \nabla + \mathbf{A} \tag{4.1}$$

where $\mathbf{A} = \sum_{n-1}^{\infty} \mathbf{A}_n$, $\mathbf{A}_N \in \frac{1}{n!}\Omega_S^n \otimes_{\mathcal{O}_S} \mathrm{End}_{\mathcal{O}_S}^{1-n}(CC_\bullet^{PER}(\mathcal{A}))$, such that $\nabla_{GM}^2 = 0$.

More generally, let $\mathcal{J}^n$ be a descending filtration of $\mathcal{A}$ by $\mathcal{O}_S$-submodules. Consider an $\mathcal{O}_S$-bilinear product $m(a,b) = ab$ on $\mathcal{A}$ such that $\mathcal{J}^n\mathcal{J}^m \subset \mathcal{J}^{n+m}$ and the induced product on $\mathrm{gr}_{\mathcal{J}}$ is associative. Assume that for every n there is $N(n)$ such that $\mathcal{J}^n\mathcal{A}^{\otimes k} \subset n!\mathcal{J}^{N(n)}\mathcal{A}^{\otimes k}$ for all k and $N(n) \to \infty$ as $n \to \infty$. Assume also that $\mathcal{A}$ admits a connection ∇ that preserves $\mathcal{J}^n$ and that the induced connection on $\mathrm{gr}_{\mathcal{J}}$ is flat.

Recall D_m and $T_{01\dots n}$ from Theorem 2.2.1.

THEOREM 4.0.1. *There exists*

$$\nabla_{\mathrm{GM}} = \nabla_{\mathrm{GM}}(m) = D_m + uB + \nabla + \mathbf{A} \tag{4.2}$$

where $\mathbf{A} = \sum_{n-1}^{\infty} \mathbf{A}_n$, $\mathbf{A}_N \in \Omega_S^n \otimes_{\mathcal{O}_S} \mathrm{End}_{\mathcal{O}_S}^{1-n}(\widehat{\mathrm{CC}}_{\bullet}^{\mathrm{PER}}(\mathcal{A}))$, *such that* $\nabla_{\mathrm{GM}}^2 = 0$.

For any $m_0, \dots, m_n$ *there exists*

$$\mathsf{T}(m_0,\dots,m_n) = \mathsf{T}_{01\dots n} + \sum_{k=1}^{\infty} \mathbf{T}_k(m_0,\dots,m_n),\ \mathbf{T}_k \in \Omega^k \otimes_{\mathcal{O}_S} \mathrm{End}_{\mathcal{O}_S}^{1-n-k}(\mathrm{CC}_{\bullet}^{\mathrm{PER}}(\mathcal{A})),$$

and

$$\nabla_{\mathrm{GM}}(m_0)\mathsf{T}(m_0,\dots,m_n) + (-1)^n\mathsf{T}(m_0,\dots,m_n)\nabla_{\mathrm{GM}}(m_n) -$$
$$\sum_{j=1}^{n-1}(-1)^j\mathsf{T}(m_0,\dots,\hat{m}_j,\dots,m_n) + \sum_{j=1}^{n-1}(-1)^j\mathsf{T}(m_0,\dots,m_j)\mathsf{T}(m_j,\dots,m_n) = 0$$

COROLLARY 4.0.2. *Let* $p > 2$ *be a prime. Assume that* $\mathcal{A}$ *is without* $\mathbb{Z}_{(2)}$-*torsion and that, as an algebra,* $\mathcal{A}/p\mathcal{A} \xrightarrow{\sim} \overline{A}_0 \otimes_{\mathbb{F}_p} (\mathcal{O}_S/p\mathcal{O}_S)$ *for an* $\mathbb{F}_p$-*algebra* $\overline{A}_0$). *Then the* p-*adically completed big periodic cyclic complex of* $\mathcal{A}$ *carries a flat superconnection.*

4.1. Proof of Theorems 3.0.1 and 4.0.1. .

$$b + uB + \mathcal{J}(-\frac{R}{u}) \tag{4.3}$$

where $\mathcal{J}$ is constructed in Lemma 2.1.3. Let $\mathfrak{a}$ be the graded Lie algebra with the basis R of degree two.

$$x \in \prod_{n=1}^{\infty} u^{-n}(U(\mathfrak{a}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{a})))^{2n} \tag{4.4}$$

Next, let $\mathfrak{g}$ be the graded Lie algebra over $k[[u]]$ generated by three elements λ of degree 1 and $\delta\lambda$, R of degree 2. Define a derivation δ of degree one by

$$\delta : \lambda \mapsto \delta\lambda \mapsto [R,\lambda];\ R \mapsto 0.$$

Assign λ and $\delta\lambda$ weight one, and assign R weight two. Extend the weight to the algebra

$$\mathcal{U} = U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g})) \tag{4.5}$$

multiplicatively. Denote by $\mathfrak{g}(n)$, $\mathcal{U}(n)$, etc. the span of all homogeneous elements of weight at least n. Let

$$\widehat{\mathcal{U}} = \prod_{k=0}^{\infty} \frac{u^{-k}k!}{n!}\mathcal{U}(k)[[u]] \tag{4.6}$$

For any $r \in \mathfrak{g}^2(1)$, define an element

$$x_F(r) \in \widehat{\mathcal{U}} \tag{4.7}$$

by (2.13) with R replaced by r.

Consider the differential

$$\mu(t) = \delta + x(R)t - (-1)^l t x(R - \delta\lambda + \lambda^2) \tag{4.8}$$

for $t \in \mathcal{U}^l$.

We construct an invertible element t_{01} of degree zero in the completion $\widehat{U}$, such that

$$(\mu + \partial_{\text{Cobar}} + uB)t_{01} + t_{01}\lambda = 0 \tag{4.9}$$

We write $t_{01} = 1 + x_1 + x_2 + \dots$ where x_k is in $\mathcal{U}(k)$. We find x_n recursively just as we did above, using the acyclicity of the differential induced by μ on $\mathcal{U}(n)/\mathcal{U}(n+1)$. For example, $x_1 = -\frac{(\lambda)}{u}$ AND***

It remains to prove that $\text{gr}(\mathcal{U})$ is indeed acyclic. Let $\mathfrak{g}_0$ be the graded algebra $\mathfrak{g}$ with the differential δ_0 defined by

$$\delta_0 : \lambda \mapsto \delta\lambda \mapsto 0; \ R \mapsto 0.$$

Define the DGA

$$\mathcal{U}_0 = U(\mathfrak{g}_0) \ltimes_1 \text{Cobar}(\overline{U}(\mathfrak{g}_0)) \tag{4.10}$$

with the differential $\delta_0 + \partial_{\text{Cobar}} + uB$. Then

$$\text{gr}(\mathcal{U}) \xrightarrow{\sim} \oplus \frac{u^{-k}}{k!}\mathcal{U}_0(k)$$

Looking at the differential δ_0 as the leading term of a spectral sequence, we see that the above is quasi-isomorphic to the itself with $\mathfrak{g}_0$ is replaced by $\mathfrak{a}_0$, the latter being the free graded Lie algebra generated by R of degree two (and weight one). Looking at uB as the leading term in a spectral sequence, we see that our complex is indeed acyclic.

More generally, let $\mathfrak{g}_n$ be the free graded algebra with generators λ_{0j} of degree one and $\delta\lambda_{0j}$ of degree two, $1 \le j \le n$, as well as R of degree two. Let the weight of $\delta\lambda_{0j}$ and λ_{0j} be one, and the weight of R be two. Define a derivation δ of degree one by

$$\delta : \lambda_{0j} \mapsto \delta\lambda_{0j} \mapsto [R, \lambda_{0j}]. \tag{4.11}$$

Remark 4.1.1. Consider the graded Lie algebra generated by elements $m_0, \dots, m_n$ of degree one and R of degree two, subject to the relation $[m_0, m_0] = 2R$. Let $\delta = [m_0, \ ]$. The span of all monomials of degree > 1 and of $m_0 - m_j$, $1 \le j \le n$, is a graded subalgebra stable under δ. It maps to $\mathfrak{g}_n$ via $m_0 - m_j \mapsto \lambda_{0j}$; $R \mapsto R$.

To finish the proofs, recall the A_∞ module structure given by Proposition 4.2.2 and Corollary 3.2.2. Denote homogeneous components of this A_∞ module structure by ϕ_n, $n > 0$. Define

$$D_m = \sum_{n=1}^{\infty} \phi_n(\mu, \dots, \mu)$$

Now introduce the following notation. For any $k > 0$, denote by J any collection $0 = j_0 < j_1 < \ldots < j_k = n$. Put $t(i) = t_{j_i \ldots j_{i+1}}$ for $0 \leq i < k$. Also, denote by N any collection $n_0, \ldots, n_k \geq 0$. Put $|N| = \sum n_i + k$. Define

$$T_{01\ldots n} = \sum_{k>0} \sum_J \sum_N \phi_{|N|}(\mu_{j_0}, \ldots, t(0), \mu_{j_1}, \ldots, \mu_{j_{k-1}}, \ldots, t(k-1), \mu_{j_k}, \ldots)$$

The term corresponding to a collection N has n_i arguments μ_{j_i} in front of $t(i)$ or/and after $t(i-1)$.

This proves Theorem 3.0.1. Theorem 4.0.1 is proved exactly the same way with R replaced by $R + \nabla m + \nabla^2$ in the computations.

REMARK 4.1.2. The above argument works for the usual periodic cyclic complex as opposed the big periodic cyclic complex, except at one point. The A_∞ module structure ϕ on the big complex satisfies the property $\phi_n(\ldots, D, \ldots) = 0$ for $D \in \mathfrak{g}$ and for $n > 1$. We do not know whether this can be achieved for the small complex while keeping the denominators under control. Therefore we do not know for sure whether adding more and more arguments $\mu_j = m_j + \ldots$ into ϕ will not lead to series that diverge in the J-adic completion.

4.2. Calculations for $\mathfrak{g}[\epsilon, u]$. Here we show that the image of μ and $t_{01\ldots n}$ from (4.8), (4.9) under the morphism (4.5) becomes very simple.

LEMMA 4.2.1. *Let $(\mathfrak{g}, \delta)$ be a graded Lie algebra over $k((u))$. Assume it has a a decreasing filtration F^n such that for any n there exists k such that $n!F^n \subset F^k$ and k goes to infinity when n does. Let $\widehat{U}(\mathfrak{g}[\epsilon])$ be the completion of the universal enveloping algebra of $(\mathfrak{g}[\epsilon], \delta + u\frac{\partial}{\partial \epsilon})$ over $k((u))$ with respect to the induced filtration. Let λ be an element of $\mathfrak{g}^1$ which is also in F^1. Let δ be a derivation of degree 1 of $\mathfrak{g}$ that preserves F (we do not assume $\delta^2 = 0$). Then*

$$(\delta + u\frac{\partial}{\partial \epsilon} + \lambda + \frac{\epsilon}{u}(\delta\lambda + \frac{1}{2}[\lambda, \lambda]))(e^{-\frac{\epsilon\lambda}{u}}) = 0$$

and

$$(\delta + u\frac{\partial}{\partial \epsilon})(e^{\frac{\epsilon\lambda}{u}}) = e^{\frac{\epsilon\lambda}{u}}(\lambda + \frac{\epsilon}{u}(\delta\lambda + \frac{1}{2}[\lambda, \lambda]))$$

in $\widehat{U}(\mathfrak{g}[\epsilon])$.

PROOF.

$$(\delta + u\frac{\partial}{\partial \epsilon})(e^{-\frac{\epsilon\lambda}{u}}) = \sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} \sum_{k=0}^{n-1} (\epsilon\lambda)^k (\delta + u\frac{\partial}{\partial \epsilon})(\epsilon\lambda)(\epsilon\lambda)^{n-1-k} =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} \sum_{k=0}^{n-1} (\epsilon\lambda)^k (\epsilon\frac{[\lambda, \lambda]}{2} + u\lambda)(\epsilon\lambda)^{n-1-k} =$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} n\epsilon\frac{[\lambda, \lambda]}{2}(\epsilon\lambda)^{n-1} + \sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} \sum_{k=0}^{n-1} uk\epsilon[\lambda, \lambda](\epsilon\lambda)^{n-2} +$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{u^n n!} nu\lambda(\epsilon\lambda)^{n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{u^n (n-1)!} \epsilon\frac{[\lambda, \lambda]}{2}(\epsilon\lambda)^{n-1} +$$

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{u^{n-1}(n-2)!} \epsilon\frac{[\lambda, \lambda]}{2}(\epsilon\lambda)^{n-2} - \lambda e^{-\frac{\epsilon\lambda}{u}} = -(\lambda + \frac{\epsilon}{u}(\delta\lambda + \frac{1}{2}[\lambda, \lambda]))e^{-\frac{\epsilon\lambda}{u}}$$

□

COROLLARY 4.2.2. *Let $\mathfrak{g}$ be the free graded Lie algebra generated by two elements m_0 and m_1 of degree 1. Assign m_0 weight zero and $\lambda_{01} = m_0 - m_1$ weight one. Then in the completion of $U(\mathfrak{g})[\epsilon][[u, u^{-1}]$ with respect to the induced filtration one has*

$$(m_0 - \frac{\epsilon m_0^2}{u} + u\frac{\partial}{\partial\epsilon}) \exp(-\frac{\epsilon\lambda_{01}}{u}) = \exp(-\frac{\epsilon\lambda_{01}}{u})(m_1 - \frac{\epsilon m_1^2}{u})$$

REMARK 4.2.3. We see that, if one replaces the algebra of operations $U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}))$ by a smaller algebra $U(\mathfrak{g})[\epsilon][[u]]$ (to which it is quasi-isomorphic over $\mathbb{Q}$), then one can choose μ and $t_{01\ldots n}$ from (4.8), (4.9) as follows:

$$\mu = m - \frac{\epsilon m^2}{u}; \ \ t_{01} = \exp(-\frac{\epsilon\lambda}{u}); \ \ t_{01\ldots n} = 0, n > 1.$$

Note the resemblance of the formal parameter u to the Planck constant $\hbar$ and of our formulas to BV formalism. Finding the full μ and $t_{01\ldots n}$ in the bigger algebra starting with the above looks similar to finding a solution using WKB approximation.

We also observe that, if $\mathfrak{g}_A[\epsilon][[u, u^{-1}]$ truly acted on the periodic cyclic complex, the Getzler-Gauss-Manin connection

$$\nabla_{\mathrm{GGM}} = b + uB - \frac{1}{u}(\epsilon(\nabla m + \nabla^2)) \tag{4.12}$$

would be a flat superconnection. Since $\mathfrak{g}_A[\epsilon][[u, u^{-1}]$ only acts up to homotopy, ∇_{GGM} only defines a flat connection on homology. Note also that

$$T_{01} = \exp(-\frac{1}{u}\epsilon(m_0 - m_1))$$

represents the monodromy of this flat connection if there is a path from m_0 to m_1. The above results show how to extend ∇_{GGM} to a flat superconnection at the level of complexes.

4.3. More on the action of $\mathfrak{g}[\epsilon, u]$. We finish by a finer version of (4.2) and (4.4). In other words, we will relate $U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}))$ as close as we can to $\mathfrak{g}[\epsilon, u]$ over $\mathbb{Z}$, not just over $\mathbb{Q}$.

Let $K = k[u, u^{-1}]$ where u is a formal parameter of degree 2. All multi-linear operations will be multilinear over K. Let $\mathfrak{g}$ be a DGLA over K, free as a K-module.

We define $S^{\mathrm{pd}}(\mathfrak{g})$ to be the symmetric algebra with divided powers, i.e. the universal graded commutative K-algebra to which $\mathfrak{g}$ maps as a K-module and where $x^{[n]} = \frac{1}{n!}x^n$ are defined and satisfy the standard properties for all elements $x \in \mathfrak{g}$ of even degree (or equivalently: for all generators of even degree in some set of homogeneous generators).

Denote by $U(\mathfrak{g}) \ltimes_0 \mathrm{Cobar}(\overline{S}^{\mathrm{pd}}(\mathfrak{g}))$ the graded algebra $U(\mathfrak{g}) \ltimes \mathrm{Cobar}(\overline{S}^{\mathrm{pd}}(\mathfrak{g}))$ with the new differential $\delta + \partial_{\mathrm{Cobar}} + uB_0$ where the value of B_0 on free generators of Cobar is given by

$$B_0((D_1^{[n_1]} \ldots D_m^{[n_m]})) = 0, \sum n_j > 1;$$

$$B_0((D_1)) = D_1 \in U(\mathfrak{g})$$

Here $D_j \in \mathfrak{g}$.

As above, we consider the DG Lie algebra $\mathfrak{g}[\epsilon]$ with the differential $\delta + u\frac{\partial}{\partial u}$ where ϵ is a formal parameter of degree 1 such that $\epsilon^2 = 0$. We have a morphism

$$U(\mathfrak{g}) \ltimes \mathrm{Cobar}(\overline{S}^{\mathrm{pd}}(\mathfrak{g})) \to U(\mathfrak{g}[\epsilon]) \tag{4.13}$$

that acts on free generators as follows:

$$(D_1^{[n_1]} \dots D_m^{[n_m]}) \mapsto 0, \sum n_j > 1;\ (D) \mapsto \epsilon D;\ D \mapsto D,\ D \in \mathfrak{g}. \tag{4.14}$$

It is easy to see that this is a quasi-isomorphism.

Remark 4.3.1. We have a commutative diagram of morphisms of DG algebras

$$\begin{array}{ccc} \mathrm{CobarBar}(U(\mathfrak{g}[\epsilon])) & \longrightarrow & U(\mathfrak{g}) \ltimes_0 \overline{S}^{\mathrm{pd}}(\mathfrak{g}) \\ & \searrow^{\sim} \quad {}^{\sim}\swarrow & \\ & U(\mathfrak{g}[\epsilon]) & \end{array} \tag{4.15}$$

Diagonal morphisms are quasi-isomorphisms. As far as we know, the horizontal morphism cannot be made $\mathrm{ad}(\mathfrak{g})$-equivariant (unless we are in characteristic zero). We can only make it satisfy

$$\phi(ab_1|\dots|b_n) = a\phi(b_1|\dots|b_n) \tag{4.16}$$

for $a \in U(\mathfrak{g})$ and $b_j \in U(\mathfrak{g}[\epsilon])$.

A computation very close to the one around (2.14) yields a morphism of DG algebras

$$U(\mathfrak{g}) \ltimes_0 \mathrm{Cobar}(\overline{S}^{\mathrm{pd}}(\mathfrak{g})) \longrightarrow U(\mathfrak{g}_{\mathbb{Q}}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g}_{\mathbb{Q}})) \tag{4.17}$$

where the upper horizontal morphism is defined on free generators of Cobar as follows. For $x \in \overline{S}^{n,\mathrm{pd}}(\mathfrak{g})$, define by

$$\Delta_{n-k,k} x = \sum x^{(1)(n-k)} \otimes x^{(2)(k)} \tag{4.18}$$

the component of the coproduct Δx in $\overline{S}^{n-k,\mathrm{pd}}(\mathfrak{g}) \otimes \overline{S}^{k,\mathrm{pd}}(\mathfrak{g})$

$$(x) \mapsto \sum_{k=1}^{n} \frac{k!(n-k)!}{n!} c_k^n \mathrm{PBW}(x^{(1)(n-k)})(x^{(2)(k)}) \tag{4.19}$$

where

$$z(z-1)\dots(z-n+1) = \sum_{k=1}^{n} c_k^n z^k$$

This morphism restricts to

$$(S^{\mathrm{pd},n}(\mathfrak{g})) \to \frac{1}{n!} U(\mathfrak{g}) \ltimes_1 \mathrm{Cobar}(\overline{U}(\mathfrak{g})) \tag{4.20}$$

on free generators of Cobar.

4.4. A $\mathcal{D}$-module analogy. There are several intriguing analogies between the formal parameter u from cyclic theory and a deformation parameter h (or $\hbar$). Here we point out one of them.

Let f be a regular function on an algebraic variety X. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. We start by considering two $\mathcal{D}_X[s]$-modules.

1) *The* $\mathcal{D}_X[s]$*-module* $\mathcal{M}[h^{-1}]e^{-\frac{f}{\hbar}}\delta_0$. Start with a formal parameter ρ. We use the notation $e^{-\frac{f}{\hbar}}\delta_0$ to define a formal generator. Define a $\mathcal{D}_X[s]$-module structure on $\mathcal{M}[h^{-1}]e^{-\frac{f}{\hbar}}\delta_0$ as follows: For a function a on X and for $m \in \mathcal{M}[h^{-1}]$,

$$a \cdot me^{-\frac{f}{\hbar}}\delta_0 = (am)e^{-\frac{f}{\hbar}}\delta_0; \tag{4.21}$$

For a vector field ξ on X and for $m \in \mathcal{M}[h^{-1}]$,

$$\xi \cdot me^{-\frac{f}{\hbar}}\delta_0 = (\xi m - \frac{\xi(f)}{h}m)e^{-\frac{f}{\hbar}}\delta_0; \tag{4.22}$$

$$s^n \cdot me^{-\frac{f}{\hbar}}\delta_0 = (-\rho h\frac{\partial}{\partial h} + \frac{1}{h}f)^n(me^{-\frac{f}{\hbar}}\delta_0) \tag{4.23}$$

Formally, the $\mathcal{D}_X[s]$-module action above is conjugated to the same action for $f = 0$ by $e^{\frac{f}{\hbar}}$.

2) *The* $\mathcal{D}_X[s]$*-module* $\mathcal{M}[s]f^{s+\mathbb{Z}}\delta_1$. Let δ_1 be another formal generator. Consider the module

$$\bigoplus_{j\in\mathbb{Z}} \mathcal{M}[s]f^{s+j}/\sim \tag{4.24}$$

where

$$mf^{s+j+1}\delta_1 \sim (fm)f^{s+j}\delta_1$$

with the following action of $\mathcal{D}_X[s]$:

for a function a on X and for $m \in \mathcal{M}[s]$,

$$a \cdot mf^{s+j}\delta_1 = (am)f^{s+j}\delta_1; \tag{4.25}$$

for a vector field ξ on X and for $m \in \mathcal{M}[s]$,

$$\xi \cdot mf^{s+j}\delta_1 = (\xi(m)f^{s+j} + (s+j)\xi(f)mf^{s+j-1})\delta_1; \tag{4.26}$$

and s acts by multiplication.

The following are morphisms of $\mathcal{D}_X[s]$-modules.

a) When f is invertible:

$$\mathcal{M}[s]f^{s+\mathbb{Z}}\delta_1 \to \mathcal{M}[h^{-1}]e^{-\frac{f}{\hbar}}\delta_0 \tag{4.27}$$

given by

$$s^n f^{s+j}m\delta_1 \mapsto (-\rho h\frac{\partial}{\partial h} + \frac{f}{h})^n\delta_0 \tag{4.28}$$

2) For any f:

$$\mathcal{M}[h^{-1}]e^{-\frac{f}{\hbar}}\delta_0 \to \mathcal{M}[s]f^{s+\mathbb{Z}}\delta_1 \tag{4.29}$$

given by

$$(\frac{1}{h})^n me^{-\frac{f}{\hbar}}\delta_0 \mapsto s(s-\rho)\dots(s-(n-1)\rho)mf^{s-n}\delta_1 \tag{4.30}$$

for $m \in \mathcal{M}$.

If we put

$$s = \frac{y}{h}; \quad \rho = \frac{R}{h} \tag{4.31}$$

then (4.30) implies

$$(\frac{1}{h}f)^n m e^{-\frac{f}{h}}\delta_0 \mapsto \frac{1}{h^n} y(y-R)\dots(y-(n-1)R) m f^s \delta_1 \tag{4.32}$$

Or, informally,

$$\delta_0 \mapsto \sum_{n=0}^{\infty} \frac{h^{-n}}{n!} y(y-R)\dots(y-(n-1)R) f^s \delta_1 = (1+\frac{R}{h})^{\frac{y}{R}} f^s \delta_1 \tag{4.33}$$

Here is an "invariant" interpretation of 1). Let $i : X \to X \times \mathbb{A}^1$ be defined by

$$i(x) = (x, f(x)) \tag{4.34}$$

Consider the $\mathcal{D}_{X\times\mathbb{A}^1}$-module $i_+\mathcal{M}$. If t is the variable on $\mathbb{A}^1$ and $\tau = \frac{\partial}{\partial t}$, then

$$i_+\mathcal{M} \xrightarrow{\sim} \mathcal{M}[\tau] \tag{4.35}$$

Denote

$$\frac{1}{h} = \tau \tag{4.36}$$

Then

$$i_+\mathcal{M} \xrightarrow{\sim} \mathcal{M}[h^{-1}] e^{-\frac{f}{h}} \delta_0 \tag{4.37}$$

when $\rho = 1$.

REMARK 4.4.1. In light of this analogy, it would be interesting to look for a connection between the action of the right hand side of (4.17) and the Connes-Moscovici cocycles discussed in **??**. This would be similar to the connection between the action of the left hand side of of (4.17) and the JLO cocycle (cf. 11).

5. Bibliographical notes

Getzler, Goodwillie,

CHAPTER 15

Noncommutative forms

1. Noncommutative forms

Let $\Omega^\bullet(A)$ be the graded algebra generated by A and by symbols da, $a \in A$, linear in a and subject to relations

a) $d(ab) = da\,b + adb$;

b) the unit of A is the unit of $\Omega^\bullet(A)$.

The grading $|a| = 0$, $|da| = 1$ makes $\Omega^\bullet(A)$ a graded algebra. We define the differential $d : \Omega^\bullet(A) \to \Omega^{\bullet+1}(A)$ as the unique graded derivation sending a to da and da to zero for all a in A. Define also

$$\mathrm{DR}^\bullet(A) = \Omega^\bullet(A)/[\Omega^\bullet(A), \Omega^\bullet(A)] \tag{1.1}$$

$$\overline{\Omega}^\bullet(A) = \Omega^\bullet(A)/k \cdot 1 \tag{1.2}$$

$$\overline{\mathrm{DR}}^\bullet(A) = \mathrm{DR}^\bullet(A)/k \cdot 1 \tag{1.3}$$

1.1. Noncommutative HKR map, I.

PROPOSITION 1.1.1. *The formula*

$$\mathrm{HKR}(a_o \otimes \ldots \otimes a_n) = \frac{1}{n!} a_0 da_1 \ldots da_n \tag{1.4}$$

defines a morphism of complexes

$$(C_\bullet(A)/bC_{\bullet+1}, B) \to (\mathrm{DR}^\bullet(A), d)$$

2. The extended noncommutative De Rham complex

Let t be a formal variable of degree zero. Define

$$\Omega_t^\bullet(A) = \Omega^\bullet(A) * k[t] \tag{2.1}$$

$$\mathrm{DR}_t^\bullet(A) = \Omega_t^\bullet(A)/[\Omega_t^\bullet(A), \Omega_t^\bullet(A)] \tag{2.2}$$

and also

$$\overline{\Omega}_t^\bullet(A) = \Omega^\bullet(A) * k[t]/k[t] \tag{2.3}$$

$$\overline{\mathrm{DR}}_t^\bullet(A) = \overline{\Omega}_t^\bullet(A)/[\overline{\Omega}_t^\bullet(A), \overline{\Omega}_t^\bullet(A)] \tag{2.4}$$

If we put $|t| = 1$, then $\Omega_t^\bullet$ acquires a second grading, as do all the spaces above. Therefore $\Omega_t^\bullet$ is a bi-graded algebra, and all the above spaces are bi-graded. For the first grading, $|d| = 1$ and $|t| = 0$. For the second, $|d| = 0$ and $|t| = 1$. We denote by $\Omega_t^{p,q}$ the component whose degree is p with respect to the first grading and q with respect to the second grading. We get similar decompositions for all the spaces above.

LEMMA 2.0.1.

$$\mathtt{DR}_t^{n,0} = (\Omega/[\Omega,\Omega])^n;\ \mathtt{DR}_t^{n-1,1} \xrightarrow{\sim} \Omega^{n-1}$$

2.1. The derivation ι_t. Let $|\omega|$ be the first grading of ω, *i. e.* $|a| = |t| = 0$ and $|da| = 1$. Define the graded derivation of degree -1 with respect to this grading by

$$\iota_t(a) = \iota_t(t) = 0;\ \iota_t(da) = [t, a]. \tag{2.5}$$

This is a bi-homogeneous map of degree $(-1, 1)$ satisfying

$$\iota_t^2 = 0.$$

LEMMA 2.1.1. *Under the identifications from Lemma 2.0.1, the map* $\mathtt{DR}_t^{n,0} \xrightarrow{\iota_t} \mathtt{DR}_t^{n-1,1}$ *becomes the operator* ι *from Definition 4.1.1 in 4.1.*

We get complexes

$$\mathtt{DR}_t^{n,0} \xrightarrow{\iota_t} \mathtt{DR}_t^{n-1,1} \xrightarrow{\iota_t} \mathtt{DR}_t^{n-2,2} \xrightarrow{\iota_t} \ldots \xrightarrow{\iota_t} \mathtt{DR}_t^{0,n} \tag{2.6}$$

2.2. The extended De Rham complex in terms of the short bar resolutions. Define

$$\mathcal{B}_1^{\mathrm{sh}}(A) = \Omega_A^1 \xrightarrow{\sim} \mathcal{B}_1(A)/\partial\mathcal{B}_2(A) \tag{2.7}$$

$$\mathcal{B}_\bullet^{\mathrm{sh},(0)}(A) = A;$$

$$\mathcal{B}_\bullet^{\mathrm{sh},(n)}(A) = \mathcal{B}_\bullet^{\mathrm{sh}}(A) \otimes_A \ldots \otimes_A \mathcal{B}_\bullet^{\mathrm{sh}}(A)$$

$(n \geq 1)$;

$$\mathcal{B}_\bullet^{\mathrm{sh},(*)}(A) = \bigoplus_{n\geq 0} \mathcal{B}_\bullet^{\mathrm{sh},(n)}(A)$$

$$(\mathrm{DR}_t^\bullet(A), \iota_t) \xrightarrow{\sim} \mathcal{B}_\bullet^{\mathrm{sh},(*)}(A)/[\mathcal{B}_\bullet^{\mathrm{sh},(*)}(A), \mathcal{B}_\bullet^{\mathrm{sh},(*)}(A)] \tag{2.8}$$

This construction will return in Chapter 17, 7.3, where we will follow Waikit Yeung's notation and denote it by $\Upsilon^{(*)}(A)$.

2.3. Noncommutative HKR map extended. We now extend (1.4) to the map from the double complex

$$\begin{array}{ccccccccccc}
A/[A,A] & \xrightarrow{B} & C_1/bC_2 & \xrightarrow{B} & C_2/bC_3 & \xrightarrow{B} & \ldots & \longrightarrow & C_n/bC_{n+1} & \xrightarrow{B} & \\
 & & \downarrow b & & \downarrow b & & & & \downarrow b & & \\
 & & A & \longrightarrow & C_1 & \xrightarrow{B} & \ldots & \xrightarrow{B} & C_{n-1} & \xrightarrow{B} & \\
 & & & & \downarrow & & & & \downarrow b & & \\
 & & & & A & \xrightarrow{B} & \ldots & \xrightarrow{B} & C_{n-2} & \longrightarrow & \\
 & & & & & & & & & & \\
 & & & & & & \ldots & & \ldots & \longrightarrow & \\
 & & & & & & & & \downarrow & & \\
 & & & & & & & & A & \longrightarrow &
\end{array}$$

(where $C_n = C_n(A)$) to the double complex

$$\begin{array}{ccccccccc}
A/[A,A] & \xrightarrow{d} & DR^1(A) & \xrightarrow{d} & DR^2(A) & \xrightarrow{d} \dots \xrightarrow{d} & DR^n(A) & \xrightarrow{d} \\
 & & \downarrow{\iota_t} & & \downarrow{\iota_t} & & \downarrow{\iota_t} & \\
 & & At & \longrightarrow & \Omega^1(A)t & \xrightarrow{d} \dots \xrightarrow{d} & \Omega^{n-1}(A)t & \xrightarrow{d} \\
 & & & & \downarrow & & \downarrow{\iota_t} & \\
 & & & & AtAt & \xrightarrow{d} \dots \xrightarrow{d} & DR_t^{n-2,2}(A) & \longrightarrow \\
 & & & & & \dots & \dots & \longrightarrow \\
 & & & & & & \downarrow & \\
 & & & & & & At\dots At & \xrightarrow{d}
\end{array}$$

In other words: let

$$C_\bullet(A)/F^{HKR}_{n+1}C_\bullet(A) = (C_n(A)/bC_{n+1} \xrightarrow{b} \dots \xrightarrow{b} C_1(A) \xrightarrow{b} C_0(A)) \tag{2.9}$$

and

$$DR^\bullet_{(n)}(A) = (DR^{(n,0)}(A) \xrightarrow{\iota_t} \dots \xrightarrow{\iota_t} DR^{(1,n-1)}(A) \xrightarrow{b} DR^{(0,n)}(A)) \tag{2.10}$$

We will define a morphism of complexes

$$(\prod_{n=0}^{\infty} u^n C_\bullet(A)/F^{HKR}_{n+1}C_\bullet(A), b+uB) \to (\prod_{n=0}^{\infty} u^n DR^\bullet_{(n)}(A), \iota_t + ud) \tag{2.11}$$

Let $\mathcal{A} = \Omega^\bullet_t(A)$ and

$$K = \Omega^\bullet_t(A)/[\Omega^\bullet_t(A), \Omega^\bullet_t(A)] = \mathtt{DR}^\bullet_t(A).$$

Let $\tau : \mathcal{A} \to K$ be the projection.

PROPOSITION 2.3.1. *For a Hochschild chain* c *in* $C_m(A)$, *define*

$$\mathrm{HKR}_t : u^{m+k}c \mapsto u^{(m+k)}\mathrm{HKR}^{(k)}(c) = \frac{1}{k!m!}\chi(t^k, d^m)(c) \tag{2.12}$$

where χ *is the characteristic map from* (2.1). *Then* HKR_t *defines a morphism of complexes* (2.11)

Explicitly,

$$\mathrm{HKR}^{(k)}(a_0 \otimes \dots \otimes a_m) = \frac{1}{(m+k)!} \sum a_0 da_1 \dots da_{j_1} t da_{j_1+1} \dots da_{j_2} t \dots t da_{j_k+1} \dots da_m \tag{2.13}$$

where the sum is over all $0 \le j_1 \le \dots j_k \le m$. The kth row from above in the (b, B) double complex is mapped to the kth complex from above in the (ι_t, b) double complex by means of $\mathrm{HKR}^{(k)}$.

PROOF. Follows from Proposition 2.1.1. □

THEOREM 2.3.2. *The extended HKR map* (2.12) *is a quasi-isomorphism for every column and for the total complex. In particular, the homology of the complex* (2.2) *at* $\mathtt{DR}_t^{n-j,j}$ *is isomorphic to* $H_j(A, A)$.

COROLLARY 2.3.3. *There is a natural filtered quasi-isomorphism*

$$\tau^{B}_{\leq 0}\mathrm{CC}^{-}_{\bullet}(A) \xrightarrow{\sim} \mathrm{DR}^{\bullet}_{t}(A)$$

Here the left hand side is the Beininson truncation (cf. 12) of the negative cyclic complex with respect to the Hodge filtration by powers of u.

PROOF. It follows immediately from the definition that the left hand side of the extended HKR map is isomorphic to the left hand side of the above formula. □

3. Proof of Theorem 2.3.2

The proof consists essentially of two parts: one for homology of the vertical complex $\mathrm{DR}^{n-j,j}_t(A)$ at $j = 0, 1$; the other, for $j \geq 2$. Our plan is the following. We start with the second part. Then we consider the differential $\mathrm{DR}^{n,0}_t(A) \to \mathrm{DR}^{n-1,1}_t(A)$. We establish its properties needed to finish proving the theorem.

After that, we study it as a differential in its own right, defining yet another double complex. Following Ginzburg and Schedler, we compare the homology of this double complex to cyclic homology in its various forms.

3.1. Proof for homology at $\mathrm{DR}^{j,n-j}(A)$ for $j \geq 2$. We have seen in section 2.2 that the extended De Rham complex can be expressed in terms of the bar resolution of the bimodule A, or rather in terms of its short quotient. If we did the same with the full bar resolution, we would obtain the direct sum

$$\bigoplus_{n=1}^{\infty}((\mathcal{B}_{\bullet}(A)\otimes_{A}\ldots\otimes_{A}\mathcal{B}_{\bullet}(A))\otimes_{A\otimes A^{\mathrm{op}}}A)_{C_n} \tag{3.1}$$

which computes

$$\bigoplus_{n=1}^{\infty} H_{\bullet}(A^{\otimes n}, {}_{\alpha}A^{\otimes n})_{C_n} \tag{3.2}$$

where α is the cyclic permutation of tensor factors. But each of the summands is isomorphic to $\mathrm{HH}_{\bullet}(A)$ ***REF

We do not know a proof of the theorem along these lines. However, it turns out that, at least at the level of associated graded quotients of a filtration, the extended De Rham complex is dual to the above.

Namely: consider the following filtration on noncommutative forms. We say that a monomial

$$\alpha_1 t\ldots t\alpha_N t,\ \alpha_j \in \Omega^{\bullet}(A), \tag{3.3}$$

lies in $\mathcal{F}^p$ if at least p forms α_j are in $d\Omega^{\bullet}$.

We claim that $\mathrm{gr}^{*}_{\mathcal{F}}(\mathrm{DR_t}^{\bullet}(A))$ is dual to (3.1) if one replaces the algebra A by the graded coalgebra $T(\overline{A}[1])$, the free coalgebra of $\overline{A}[1]$ where $\overline{A} = A/k\cdot 1$. (This may sound a bit strange since full $\mathcal{B}_{\bullet}$ is larger than $\mathcal{B}^{\mathrm{sh}}_{\bullet}$ and $T(\overline{A}[1])$ is larger than A. Such is life in Hilbert's hotel).

More precisely, for any graded coalgebra C, denote

$$\bar{C}^{\bullet}_{\mathrm{II}}(C)^{(0)} = \overline{\mathrm{CC}}^{\bullet}_{\mathrm{II}}(C) = (\bar{C}^{\otimes(\bullet+1)})_{C_{\bullet+1}} \tag{3.4}$$

with the differential dual to b (or take invariants with the differential dual to b'). For $n > 0$

$$C^{\bullet}_{\mathrm{II}}(C)^{(n)} = (\bar{C}[-1]^{\otimes\bullet}\otimes C\otimes\ldots\otimes\bar{C}[-1]^{\otimes\bullet}\otimes C)^{C_n} \tag{3.5}$$

(the tensor product on the right is n-fold); the differential sends a monomial

$$c_1^{(1)} \otimes \ldots c_{m_1}^{(1)} \otimes x_1 \otimes \ldots \otimes c_1^{(n)} \otimes \ldots c_{m_n}^{(n)} \otimes x_n \tag{3.6}$$

to

$$\sum_{i=1}^{n} \sum_{j-1}^{m_i} \pm \ldots \otimes \Delta(c_j^{(i)}) \otimes \ldots + \sum_{j=1}^{n} (\ldots \otimes \Delta'(x_j) \ldots + \ldots \otimes \Delta''(x_j) \ldots)$$

where we use the following notation. First,

$$\Delta' : C \to \bar{C}[-1] \otimes C;\ \Delta'' : C \to C \otimes \bar{C}[-1] \tag{3.7}$$

is the comultiplication followed by the projection. Second, the second half of the $j = n$ term is by definition

$$\pm x_n^{(2)} \otimes c_1^{(1)} \otimes \ldots c_{m_1}^{(1)} \otimes x_1 \otimes \ldots \otimes c_1^{(n)} \otimes \ldots c_{m_n}^{(n)} \otimes x_n^{(1)}$$

where

$$\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$$

To see this, take a monomial

$$\alpha_1^{(1)} t \ldots \alpha_{m_1}^{(1)} t d\beta_1 t \ldots t \alpha_1^{(n)} t \ldots \alpha_{m_n}^{(n)} t d\beta_n \tag{3.8}$$

in $\mathrm{gr}^n(\mathrm{DR}_t^+)$, and associate to it a monomial (3.6) by the following rule: to a form $a_0 da_1 \ldots da_k$, associate an element $(a_0|\ldots|a_k)$ of $T(\bar{A}[1])$. To see that, put

$$(d_+\Omega)^n = d\Omega^{n-1},\ n > 0;\ (d_+\Omega)^0 = k$$

and observe that:

(1)

$$\iota_t(a_0 da_1 \ldots da_n) = \sum_{k=1}^{n} (-1)^{k-1} (a_0 da_1 \ldots da_{k-1}) t (a_k da_{k+1} \ldots da_n)$$
$$\mathrm{mod}(d_+\Omega t \Omega + \Omega t d_+ \Omega)$$

(2)

$$\iota_t(da_1 \ldots da_n) = \sum_{k=1}^{n} (-1)^k (a_1 da_2 \ldots da_k) t (da_{k+1} da_{k+2} \ldots da_n) +$$
$$\sum_{k=1}^{n} (-1)^{k-1} (da_1 \ldots da_{k-1}) t (a_k da_{k+1} \ldots da_n) \ \mathrm{mod}(d_+\Omega t d_+ \Omega)$$

Now compute the spectral sequence of the filtration $\mathcal{F}$. The first term is the reduced cyclic cohomology of $T(\bar{A}[1])$. Since the coalgebra is cofree,

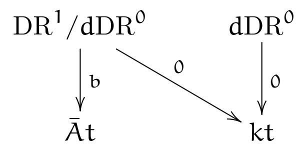
$$\overline{\mathrm{HC}}_{\mathrm{II}}^{\bullet}(T(\bar{A}[1])) \xrightarrow{\sim} \bigoplus_{n=1}^{\infty} (T^n(\bar{A}[1]))^{C_n} \tag{3.9}$$

By the dual version of **ref to subdivisions***, all homologies of $\mathrm{gr}^n_{\mathcal{F}}$ for $n > 0$ are isomophic to the Hochschild cohomology $\mathrm{HH}_{\mathrm{II}}^{\bullet}(T(\bar{A}[1]))$, which is computed by the short Hochschild complex

$$C_{\mathrm{sh}}^{\bullet}(T(\bar{A}[1])) = (T(\bar{A}[1]) \xrightarrow{b} T(\bar{A}[1]) \otimes \bar{A}[1]) \tag{3.10}$$

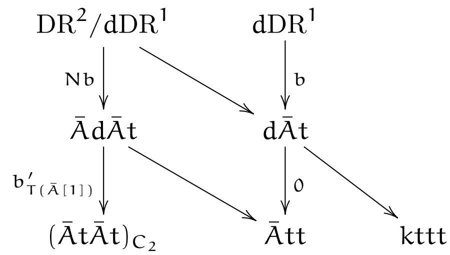
3.2. The spectral sequence of the filtration $\mathcal{F}$. Let us start with examples for small n. In the diagrams below, the vertical differentials are the ones on the E^0 term. the diagonal differentials are the ones on the E^1 term.

3.2.1. *The complex* $\mathrm{DR}^{1,0}(A) \to \mathrm{DR}^{0,1}(A)$. The columns are $\mathrm{gr}^0_{\mathcal{F}}$ and $\mathrm{gr}^1_{\mathcal{F}}$.



The only nonzero differential is $b : a_0 da_1 \mapsto a_0[t, a_1] = (a_1 a_0 - a_0 a_1)t$. This is the first instance of the differential ι_Δ that we will study later.

3.2.2. *The complex* $\mathrm{DR}^{2,0}(A) \to \mathrm{DR}^{1,1}(A) \to \mathrm{DR}^{0,2}(A)$.



The columns are $\mathrm{gr}^0_{\mathcal{F}}$, $\mathrm{gr}^1_{\mathcal{F}}$, and $\mathrm{gr}^2_{\mathcal{F}}$. Let us start with the differential in the left column. One computes

$$a_0 da_1 da_2 \mapsto a_0[t, a_1]da_2 - a_0 da_1[t, a_2] = a_1 da_2 \cdot a_0 - da_2 \cdot a_0 a_1 - a_2 a_0 da_1 + a_0 da_1 \cdot a_2$$

When we put this in the normal form (with the differentials on the right), we get

$$N(b(a_0 da_1 da_2))t - d(a_2 a_0 a_1)t$$

In the first summand, b is the Hochschild differential and N is the cyclic norm and we identify $C_n(A)$ with $\Omega^n(A)$ via $a_0 \otimes \ldots \otimes a_n \mapsto a_0 da_1 \ldots da_n$. Therefore the vertical differential is indeed Nb. (The diagonal one is $a_0 da_1 da_2 \mapsto d(a_2 a_0 a_1)t$).

The next differential in the left column is $a_0 da_1 t \mapsto a_0 a_1$.

$$a_0 da_1 t \mapsto a_0 t a_1 t$$

which is indeed the b' differential in the cyclic complex of the coalgebra $T(\bar{A}[1])$. We also see that the diagonal differential is $a_0 da_1 t \mapsto a_0 a_1$.

The homology of the differential in the left column is: on the bottom, zero; in the middle,

$$N(\bar{A}d\bar{A})/Nb(\bar{A}d\bar{A}d\bar{A}) \tag{3.11}$$

on the top, $\mathrm{Ker}(Nb)$. The diagonal differential from (3.11) is b'; its image is the span of commutators. It is easy to see that the bottom homology is indeed $\mathrm{HH}_0(A)$. As to the top two homologies, we see that they are not straightforward. As we mentioned, we will study them separately later.

3.2.3. *The complex* $\mathrm{DR}^{3,0}(A) \to \mathrm{DR}^{2,1}(A) \to \mathrm{DR}^{1,2}(A) \to \mathrm{DR}^{0,3}(A)$.

$$\begin{array}{ccccccc}
\mathrm{DR}^3/d\mathrm{DR}^2 & & d\mathrm{DR}^2 & & & & \\
{\scriptstyle Nb}\downarrow & \searrow & \downarrow{\scriptstyle b'} & & & & \\
\bar{A}d\bar{A}d\bar{A}t & & d\bar{A}d\bar{A}t & & & & \\
{\scriptstyle b'_{T(\bar{A}[1])}}\downarrow & \searrow & {\scriptstyle b_{T(\bar{A}[1])}}\downarrow & \searrow & & & \\
\bar{A}t\bar{A}d\bar{A}t & & \bar{A}td\bar{A}t + \bar{A}d\bar{A}tt & & d\bar{A}tt & & \\
{\scriptstyle b'_{T(\bar{A}[1])}}\downarrow & \searrow & {\scriptstyle b_{T(\bar{A}[1])}}\downarrow & \searrow & \downarrow{\scriptstyle 0} & \overset{0}{\searrow} & \\
(\bar{A}t\bar{A}t\bar{A}t)_{C_3} & & \bar{A}t\bar{A}tt & & \bar{A}ttt & & kttt
\end{array}$$

This is where the pattern becomes clearer. The differential on the left is Nb on the top (as above; we do not emphasize this now because we are mainly concentrated on the bottom part). In the middle and on the bottom, it is the b' differential of the cyclic complex of the coalgebra $T(\bar{A}[1])$. Namely:

$$a_0 da_1 da_2 t \mapsto a_0 ta_1 da_2 t - a_0 da_1 ta_2 t;\ a_0 ta_1 da_2 t \mapsto a_0 ta_1 ta_2 t$$

The differential of the middle column, in the middle and at the bottom, is indeed the b differential in the Hochschild complex of the coalgebra $T(\bar{A}[1])$. Namely: in the middle,

$$da_1 da_2 t \mapsto -a_1 t da_2 t - da_1 ta_2 t = -a_1 t da_2 t - a_2 t da_1 t$$

At the bottom,

$$a_0 t da_1 t \mapsto a_0 t ta_1 t - a_0 ta_1 tt;\ a_0 da_1 tt \mapsto a_0 ta_1 tt.$$

The vertical differential on the right is zero.

The homology of the vertical differential on the left is: on top, $\mathrm{Ker}(Nb)$; second from above,

$$N(\bar{A}d\bar{A}d\bar{A})/\mathrm{Im}(Nb); \tag{3.12}$$

down from there, zero. The cohomology of the second column from the left is: on top, $\mathrm{Ker}(b')$; the next one is $(d\bar{A}d\bar{A})^{C_2}$ which is isomorphic to

$$(\bar{A} \otimes \bar{A})^{C_2}; \tag{3.13}$$

the second homology group from the bottom is isomorphic to

$$(\bar{A} \otimes \bar{A})_{C_2}; \tag{3.14}$$

the bottom homology is zero.

Now compute the diagonal differential. On (3.13) it is given by b'. Indeed, the full differential acts as

$$da_1 da_2 t \mapsto -a_1 t da_2 t - da_1 ta_2 t + d(a_1 a_2)t^2$$

On (3.14), the diagonal differential is given by b. Indeed, note that a class of $a_0 \otimes a_1$ is given by a cycle

$$a_0 t da_1 t + a_o da_1 tt - a_1 da_0 tt; \tag{3.15}$$

its full differential is $(a_1 a_0 - a_0 a_1)ttt$. This is the pattern of this spectral sequence. In all columns, the complex $\bar{A}^{\otimes j+1} \xrightarrow{1-\tau} \bar{A}^{\otimes j+1}$ appears. All columns but the two on the left are quasi-isomorphic to this complex. The diagonal differentials (computing the E^2 term) act by b' on the left and by b on the right. This allows to conclude that the HKR map is a quasi-isomorphism. The two top homology groups require a separate consideration.

Let us first prove what we just stated above.

3.2.4. *The spectral sequence: the general case.* We have to show that the differential $\mathrm{gr}^k_{\mathcal{F}}\Omega^m_t \to \mathrm{gr}^{k+1}_{\mathcal{F}}\Omega^{m-1}_t$ acts on (3.10) as follows: on $T(\bar{A}[1])$, by b'; on $\bar{A}[1]\otimes T(\bar{A}[1])$, by b. (This is what we saw in the examples). To do that, we have to compare the short complex to the higher full complexes. Namely: we have to compute the compositions

$$C^{\bullet}_{\mathrm{sh}}(T(\bar{A}[1])) \to C^{\bullet}_{\mathrm{II}}(T(\bar{A}[1])) \to C^{\bullet}_{\mathrm{II}}(T(\bar{A}[1]))^{(n)} \tag{3.16}$$

and

$$C_{\mathrm{II}}(T(\bar{A}[1]))^{(n)} \to C^{\bullet}_{\mathrm{II}}(T(\bar{A}[1])) \to C^{\bullet}_{\mathrm{sh}}(T(\bar{A}[1])) \tag{3.17}$$

Let us first look at the more familiar dual picture of algebras. Let $T(V)$ be the tensor algebra of a (graded) vector space V. Consider the following

$$C_{\bullet}(T(V))^{(n)} \to C_{\bullet}(T(V)) \to C^{\mathrm{sh}}_{\bullet}(T(V)) \tag{3.18}$$

and

$$C^{\mathrm{sh}}_{\bullet}(T(V)) \to C_{\bullet}(T(V)) \to C_{\bullet}(T(V))^{(n)} \tag{3.19}$$

The two maps above are induced by maps of resolutions:

$$\mathcal{B}^{\mathrm{sh}}_{\bullet}(A) \to \mathcal{B}_{\bullet}(A);\ a\otimes v\otimes b \mapsto a\otimes v\otimes b, \tag{3.20}$$

$a, b \in T(V)$, $v \in V$;

$$\mathcal{B}_{\bullet}(A) \to \mathcal{B}^{\mathrm{sh}}_{\bullet}(A);\ a\otimes v_1\dots v_m\otimes b \mapsto \sum \pm a v_1\dots v_{j-1}\otimes v_j\otimes v_{j+1}\dots v_m b, \tag{3.21}$$

$a, b \in T(V)$, $v_i \in V$;

$$\mathrm{Id}\otimes\epsilon\otimes\dots\otimes\epsilon : \mathcal{B}_{\bullet}(A)\otimes_A\dots\otimes_A \mathcal{B}_{\bullet}(A) \to \mathcal{B}_{\bullet}(A); \tag{3.22}$$

and

$$\mathcal{B}_{\bullet}(A) \to \mathcal{B}_{\bullet}(A)\otimes_A\dots\otimes_A \mathcal{B}_{\bullet}(A); \tag{3.23}$$

$$a_0\otimes\dots\otimes a_{n+1} \mapsto \sum \pm(a_0\otimes\dots\otimes a_{j_1}\otimes 1)\otimes(1\otimes\dots\otimes 1)\otimes\dots\otimes(1\otimes a_{j_{n-1}+1}\otimes\dots\otimes a_{n+1})$$

(the sum is over $0 \le j_1 \le \dots \le j_{n-1} \le n$). These morphisms induce morphisms of Hochschild complexes. They are homotopy equivalences. We write their dual morphisms and observe that all homotopies also have their dual versions. We recover formulas for (3.16) and (3.17) which are as follows. We use the identification of $C^{\bullet}_{\mathrm{II}}(T(\bar{A}[1]))^{(n)}$ with $\mathrm{gr}^n_{\mathcal{F}}(\mathrm{DR}^{\bullet}_t)$. Also, for the totally ordered set $I = \{i_1, \dots, i_m\}$ and for $\{a_j \in A | i \in I\}$, we write $da_I = da_{i_1}\dots da_{i_m}$. For (3.16):

$$(a_1|\dots|a_N) \mapsto \sum \pm t da_{I_1} t da_{I_2} t \dots t da_{I_n} t \tag{3.24}$$

The sum is taken over all subdivisions of the *cyclically* ordered set $I = \{1, \dots, N\}$ into disjoint sum of intervals (possibly empty) $I_1, \dots, I_n$.

$$(a_1|\dots|a_N)\otimes(a_0) \mapsto \sum \pm t da_{I_1} t da_{I_2} t \dots t da_{I_n} t (da_{I_{n+1}}\cdot a_0 da_{I_0}) t \tag{3.25}$$

The sum is taken over all subdivisions of the *linearly* ordered set $I = \{1, \dots, N\}$ into disjoint sum of intervals (possibly empty) $I_0, \dots, I_{n+1}$.

For (3.17): Consider a monomial

$$\alpha_1 t \dots t\omega_1 t \dots t\alpha_N t \dots \omega_n t \tag{3.26}$$

where $\alpha_i \in \Omega^\bullet(A)/d_+\Omega^{\bullet-1}$ and $\omega_j \in d_+\Omega^\bullet(A)$. We have to specify where such monomial maps to. Answer: it maps to zero unless:

(1) For all but one j, $\beta_j = 1$;
(2) For all but one i, $\alpha_i = 1$, in which case $\alpha_i \in \bar{A}$.

If this is the case, the monomial maps to itself:

$$da_1 \dots da_N t \dots t \mapsto (a_1|\dots|a_N);\;\; tda_1 \dots da_N t \dots ta_0 t \dots t \mapsto (a_1|\dots|a_N) \otimes (a_0)$$

It remains to compute the following composition: apply (3.16), then the differential $\mathrm{gr}^n \to \mathrm{gr}^{n+1}$, then (3.17). When we start with $(a_1|\dots|a_N)$, the only surviving term in (3.24) is when all the intervals but one are empty. Applying the differential to $(a_1 \dots |a_N)t \dots t$ and then applying (3.17), we get $b'(a_1|\dots|a_N)$. When we start with $(a_1|\dots|a_N) \otimes (a_0$, the only surviving terms in (3.25) are:

(1) when all the intervals but one are empty;
(2) $(da_1 \dots |da_{N-1})t \dots t(a_N da_0)$;
(3) $(da_2 \dots |da_N)t \dots t(a_0 da_1)$

Applying the differential to $(a_1 \dots |a_N)t \dots t$ and then applying (3.17), we get $b'(a_1|\dots|a_N) \otimes (a_0)$, $(a_1|\dots|a_{N-1}) \otimes (a_N a_0)$, and $(a_2|\dots|a_n) \otimes (a_0 a_1)$, which sum to the image of $b(a_0 \otimes \dots \otimes a_N)$.

This computes the E^2 term of the spectral sequence for all but two leftmost columns. As soon as we compute it for those two, we will have the theorem proven. Indeed, we will know that the extended HKR map from Proposition 2.3.1 is a quasi-isomorphism. Indeed, the truncated Hochschid complex

$$C_n(A)/bC_{n+1}(A) \to C_{n-1}(A) \to \dots \to C_0(A) \tag{3.27}$$

has its own filtration

$$\mathcal{F}^{n-j}C_j(A) = C_j(A);\; \mathcal{F}^{n-j+1}C_j(A) = 1 \otimes \bar{A}^{\otimes j}; \mathcal{F}^{n-j+2}C_j(A) = 0 \tag{3.28}$$

It is straightforward that the extended HKR map preserves the filtration. This finishes the proof of Theorem 2.3.2 contingent on ***REF below.

4. The (ι_Δ, d) double complex

4.1. The differential ι_Δ.

DEFINITION 4.1.1.

$$\iota_\Delta(a_0 da_1 \dots da_n) = \sum_{j=1}^{n} (-1)^{n(j-1)}[a_j, da_{j+1} \dots da_n a_0 da_1 \dots da_{j-1}]$$

This is just the composition

$$\Omega^n(A) \to \mathrm{DR}^n(A) \xrightarrow{\iota_t} \mathrm{DR}^{n-1,1}(A) \xrightarrow{\sim} \Omega^{n-1}(A)$$

4.2. The main result. Here we summarize the main result that will be proven below. First,

$$\iota_\Delta^2 = [d, \iota_\Delta] = d^2 = 0 \tag{4.1}$$

(This does need a proof and does not directly follow from $\iota_t^2 = 0$, though it is close).

We will use the identification

$$C_\bullet(A) \xrightarrow{\sim} \Omega^\bullet(A); a_0 \otimes \ldots \otimes a_n \mapsto a_0 da_1 \ldots da_n \tag{4.2}$$

(the crude HKR). We get two pairs of commuting differentials on $\Omega^\bullet(A)$: one is (b, B) and the other is (ι_Δ, d). To compare the two, we will introduce the Cuntz-Quillen projections P and $P^\perp$ and prove the following

THEOREM 4.2.1. (1) *The projections* P *and* $P^\perp$ *commute with* b, B, ι_Δ, *and* d.

(2)
$$\Omega^\bullet(A) = P\Omega^\bullet(A) \oplus P^\perp\Omega^\bullet(A)$$

(3) *On* $P\Omega^\bullet(A)$*: Let* $(\mathcal{N}!)^{-1}$ *be the operator whose restriction to* $\Omega^n(A)$ *is* $\frac{1}{n!}\operatorname{Id}$*. Then* $(\mathcal{N}!)^{-1}$ *intertwines* b *with* ι_Δ *and* B *with* d.

(4) *On* $P^\perp\Omega^\bullet(A)$*:* $B = 0$; $\iota_\Delta = 0$; *both* b *and* d *are contractible.*

Another way to express (3): $P \circ \mathrm{HKR}^{(0)}$ intertwines b with ι_Δ and B with d where

$$\mathrm{HKR}^{(0)}(a_0 \otimes \ldots \otimes a_n) = \frac{1}{n!} a_0 da_1 \ldots da_n \tag{4.3}$$

4.3. Comparison between Hochschild and De Rham, I. To start comparing Hochschild to De Rham, start with the "crude HKR map"

$$\mathrm{HKR}^{(0)} : a_0 \otimes a_1 \ldots \otimes a_n \mapsto a_0 da_1 \ldots da_n \tag{4.4}$$

which induces an isomorphism of graded k-modules

$$C_n(A, A) \xrightarrow{\sim} \Omega^n(A) \tag{4.5}$$

therefore one can consider the operator b on $\Omega^\bullet(A)$. One has One has

$$b(a_0 da_1 \ldots da_n) = (-1)^{n-1}[a_0 da_1 \ldots da_{n-1}, a_n] \tag{4.6}$$

Observe that for $n = 1$ $b = \iota_\Delta$;

$$\begin{array}{ccc} C_1(A)/bC_2 & \xrightarrow{\sim} & \mathrm{DR}^1(A) \\ \downarrow b & & \downarrow \iota_\Delta \\ C_0(A) & \xrightarrow{\sim} & \Omega^0(A) \end{array} \tag{4.7}$$

In general, the noncommutative HKR map

$$C_n(A)/bC_{n+1}(A) \to \mathrm{DR}^n(A)$$

is surjective and not an isomorphism. Indeed, (4.6) shows that (4.5) identifies

$$bC_{n+1}(A) \xrightarrow{\sim} [A, \Omega^n(A)] \tag{4.8}$$

which coincides with $[\Omega, \Omega]^n$ only for $n = 1$. But it turns out that there is an operator κ on each $\Omega^n(A)$ that commutes with b (under the identification (4.5)), satisfies $\kappa^n = \mathrm{Id}$ on $C_n(A)/bC_{n+1}$, and for which

(4.9)
$$\begin{array}{ccc} (C_n(A)/bC_{n+1})^\kappa & \xrightarrow{\sim} & \mathrm{DR}^n(A) \\ \downarrow b & & \downarrow \iota_\Delta \\ C_{n-1}(A) & \xrightarrow{\sim} & \Omega^{n-1}(A) \end{array}$$

This is what we are going to describe next.

4.4. The Karoubi operator κ. First look at the component of HKR_t at level one. We get from (2.13)

(4.10)
$$\mathrm{HKR}^{(1)} = \frac{1}{(n+1)!}(1 + \kappa + \ldots \kappa^n)$$

where

(4.11)
$$\kappa(a_0 da_1 \ldots da_n) = (-1)^{n-1} da_n \cdot a_0 da_1 \ldots da_{n-1}$$

LEMMA 4.4.1. *On Ω^n one has*

$$\iota_\Delta = (1 + \kappa + \ldots + \kappa^{n-1})b$$

This follows from HKR_t being a morphism of double complexes.

LEMMA 4.4.2.

$$\kappa^n - \mathrm{Id} = b\kappa^n d$$
$$\kappa^{n+1} d = d$$

PROOF. The first identity follows directly from the definition. To prove the second, note

$$(\kappa^n - \mathrm{Id})(a_0 da_1 \ldots da_n) = [da_1 \ldots da_n, a_0] =$$
$$(-1)^n b(da_1 \ldots da_n da_0) = b\kappa^n d(a_0 da_1 \ldots da_n)$$

□

From (4.13) below, κ commutes with b. We now see that $\kappa^n = \mathrm{Id}$ on $\Omega^n/b\Omega^{n+1}$. Therefore

(4.12)
$$\Omega^n/b\Omega^{n+1} = (\Omega^n/b\Omega^{n+1})^\kappa \oplus (\kappa - \mathrm{Id})(\Omega^n/b\Omega^{n+1})$$

Now recall the diagram (4.9). We have $\mathrm{DR}^n(A) = (\Omega^n/b\Omega^{n+1})/\mathrm{Im}(\mathrm{Id} - \kappa)$. But projection onto coinvariants of κ is an isomorphism on invariants because $\kappa^n = \mathrm{Id}$. Therefore the upper horizontal map is an isomorphism.

4.5. Further properties of the Karoubi operator.

LEMMA 4.5.1.

(4.13)
$$db + bd = \mathrm{Id} - \kappa$$

In particular,

(4.14)
$$[b, \kappa] = [d, \kappa] = 0$$

and

(4.15)
$$\iota_\Delta^2 = 0$$

PROOF. One has

$$db(a_0da_1\dots da_n)=(-1)^n d[a_0da_1\dots da_{n-1},a_n]=$$
$$(-1)^{n-1}([da_0da_1\dots da_{n-1},a_n]+(-1)^{n-1}[a_0da_1\dots da_{n-1},da_n])$$

and

$$bd(a_0da_1\dots da_n)=(-1)^n[da_0da_1\dots da_{n-1},a_n]$$

This, together with Lemma 4.4.1, implies (4.15). □

Now we establish a few additional polynomial identities for κ.

LEMMA 4.5.2.

$$\kappa^{n+1}-\mathrm{Id}=-db \tag{4.16}$$

$$\kappa^{n+1}-\kappa=bd \tag{4.17}$$

$$(\kappa^{n+1}-\mathrm{Id})(\kappa^{n+1}-\kappa)=0 \tag{4.18}$$

PROOF. One has

$$\kappa^n(a_0da_1\dots da_n)=da_1\dots da_n\cdot a_0=da_1\dots d(a_na_0)-$$
$$da_1\dots d(a_{n-1}a_n)da_0+\dots+(-1)^{n-1}d(a_1a_2)\dots da_nda_0$$
$$+(-1)^n a_1da_2\dots da_nda_0;$$
$$\kappa^{n+1}(a_0da_1\dots a_n)=(-1)^{n-1}d(a_na_0)da_1\dots da_{n-1}+$$
$$(-1)^n da_0da_1\dots d(a_{n-1}a_n)+\dots+da_0d(a_1a_2)\dots da_n$$
$$-d(a_0a_1)da_2\dots da_n+a_0da_1da_2\dots da_n=(-db+\mathrm{Id})(a_0da_1\dots da_n)$$

Now (4.17) follows from Lemma 4.13 and (4.16); (4.18) follows from the above and from $d^2=b^2=0$. □

4.6. The projections P and $P^\perp$.

DEFINITION 4.6.1. *Let* P *be the projection onto* $\mathrm{Ker}(\mathrm{Id}-\kappa)^2$ *and let* $P^\perp$ *be the projection onto* $\mathrm{Im}(\mathrm{Id}-\kappa)^2$.

In fact P is a polynomial $F(\kappa)$; namely, F is the image of $(1,0,0,\dots,0)$ under the isomorphism

$$k[\kappa]/(\kappa^{n+1}-1)(\kappa^n-1)=k[\kappa]/(\kappa^1)^2\oplus\oplus_{\lambda\neq1}k$$

where $\lambda\neq1$ are roots of $(\kappa^{n+1}-1)(\kappa^n-1)$. In particular

LEMMA 4.6.2.

$$\Omega=P\Omega\oplus P^\perp\Omega \tag{4.19}$$

Also, operators κ *and* P *commute with* b *and* d

EXAMPLE 4.6.3. For $n=0: P(a_0)=a_0;\ P^\perp(a_0)=0$. For $n=1: P^\perp(a_0da_1)=\frac{1}{2}(a_0da_1+a_1da_0-\frac{1}{2}d(a_0a_1+a_1a_0))$; $P(a_0da_1)=\frac{1}{2}(a_0da_1-a_1da_0+\frac{1}{2}d(a_0a_1+a_1a_0))$.

LEMMA 4.6.4. *The differentials* b *and* d *are both contractible on* $P^\perp\Omega$.

PROOF. Follows from Lemma 4.5.1 and from the invertibility of $\mathrm{Id}-\kappa$ on $P^\perp\Omega$. □

We now see that the embedding of

$$P(C_n(A)/bC_{n+1}) \xrightarrow{b} PC_{n-1}(A) \xrightarrow{b} \dots \xrightarrow{b} PC_0(A) \tag{4.20}$$

into

$$C_n(A)/bC_{n+1} \xrightarrow{b} C_{n-1}(A) \xrightarrow{b} \dots \xrightarrow{b} C_0(A) \tag{4.21}$$

is a quasi-isomorphism. The former is isomorphic to

$$(C_n(A)/bC_{n+1})^\kappa \xrightarrow{b} PC_{n-1}(A) \xrightarrow{b} \dots \xrightarrow{b} PC_0(A) \tag{4.22}$$

because $\kappa^n = \mathrm{Id}$ on $C_n(A)/bC_{n+1}(A)$. Now we conclude from (4.9) that

$$HH_n(A) \xrightarrow{\sim} \mathrm{Ker}(DR^n(A) \xrightarrow{\iota_\Delta} \Omega^{n-1}(A)) \tag{4.23}$$

We can now finish the proof of Theorem 4.2.1 . . We have

(1) Put

$$(\mathcal{N}!)^{-1}|\Omega^n(A) = \frac{1}{n!}\,\mathrm{Id}$$

Then $(\mathcal{N}!)^{-1}$ is:

a) an isomorphism between the complexes $(P\Omega(A), b)$ and $(P\Omega(A), \iota_\Delta)$;

b) an isomorphism between the complexes $(P\Omega(A), B)$ and $(P\Omega(A), d)$.

(2) The complexes $(P^\perp\Omega(A), b)$ and $(P^\perp\Omega(A), d)$ are acyclic.

(3) The differentials of the complexes $(P^\perp\Omega(A), \iota_\Delta)$ and $(P^\perp\Omega(A), B$ are zero.

Indeed, we already proved (2) (Lemma 4.6.4). To prove (1), note that

$$\iota_\Delta = (1 + \dots + \kappa^{n-1})b$$

But $1 + \dots + \kappa^{n-1}$ is invertible on $\ker(\kappa - 1)^2$ (and commutes with b). Moreover, on $P(b(\Omega^{n+1}))$ we have $\kappa = \mathrm{Id}$. Therefore on $P\Omega^n$

$$\iota_\Delta = nb.$$

Similarly, on $P\Omega^n$

$$B = (n+1)d.$$

To prove (3), observe that

$$\iota_\Delta(\kappa - 1)^2 = (\kappa - 1)^2(1 + \kappa + \dots + \kappa^{n-1})b = (\kappa - \mathrm{Id})(\kappa^n - \mathrm{Id})b = 0$$

$$B(\kappa - 1)^2 = (\kappa - 1)^2(1 + \kappa + \dots + \kappa^n)d = (\kappa - \mathrm{Id})(\kappa^{n+1} - \mathrm{Id})d = 0$$

by Lemma 4.4.2.

5. Periodic and negative cyclic homology in terms of d and ι_Δ

Theorem 4.2.1 allows to compute all the versions of cyclic homology in terms of ι_Δ and d. Indeed, on the image of P, the (b, B) and (ι_Δ, d) complexes are isomorphic in all cases. On the image of $P^\perp$, the former is contractible in all cases because b is contractible; the latter is contractible in the periodic case because d is contractible and $\iota_\Delta = 0$. We obtain

THEOREM 5.0.1.

$$HC^{\mathrm{per}}_\bullet(A) \xrightarrow{\sim} H_\bullet(\Omega^\bullet(A)((u)), \iota_\Delta + ud)$$

In the negative case, the image of $P^\perp$ is no longer contractible but becomes so when one factors out the kernel of d in the edge column $u^k\Omega^\bullet$ with $k=0$. In the cyclic case, it becomes contractible if one replace the edge column with the image of d (which is the same as the kernel). This image/kernel can be computer more explicitly:

LEMMA 5.0.2.

$$dP^\perp\Omega = [d\Omega, d\Omega]$$

PROOF. $\square$

Therefore we get

THEOREM 5.0.3.

$$\mathrm{HC}^-_\bullet(A) \xrightarrow{\sim} H_\bullet(\Omega^\bullet(A)[[u]]/[d\Omega, d\Omega], \iota_\Delta + ud)$$

THEOREM 5.0.4.

$$\mathrm{HC}_\bullet(A) \xrightarrow{\sim} H_\bullet(u^{-1}\Omega^\bullet(A)[u^{-1}] + [d\Omega, d\Omega], \iota_\Delta + ud)$$

5.1. The extended De Rham complex and the bar construction. Let $\mathtt{DR}^\bullet_{t,+}(A)$ be the subcomplex of $\mathtt{DR}^\bullet_t(A)$ spanned by elements whose degree with respect to t is positive. This subcomplex can be expressed in the form that we are going to discuss next.

Let us start with any associative unital differential algebra $(\mathcal{A}, \partial)$. View $\mathcal{A}$ as a graded algebra. Introduce a new generator ϵ of degree one and square zero. Consider the cross product algebra

$$\widetilde{\mathcal{A}} = k[\epsilon] \ltimes \mathcal{A} \tag{5.1}$$

generated by ϵ and $\mathcal{A}$ subject to a relation $[\epsilon, a] = \partial a$ for all a in $\mathcal{A}$.

In other words, $\widetilde{\mathcal{A}}$ is generated by the algebra $\mathcal{A}$ and by elements $\underline{a} = \epsilon a$, $a \in \mathcal{A}$, of degree $|a|+1$, linear in a and subject to relations

$$\underline{a}\cdot b = \underline{ab};\ a\cdot \underline{b} = (-1)^{|a|}(\underline{ab} - \partial a \cdot b);\ \underline{a}\cdot\underline{b} = (-1)^{|a|-1}\underline{\partial a \cdot b} \tag{5.2}$$

Now one can consider the reduced cyclic homology $\overline{\mathrm{HC}}_\bullet(\widetilde{\mathcal{A}})$ of the graded algebra $\widetilde{\mathcal{A}}$. More precisely, we will compute it using the following specific complex defined for any A:

$$\overline{\mathrm{CC}}'(A) = (\mathrm{Ker}(1-t), b'); \mathrm{CC}'(A) = \overline{\mathrm{CC}}'(A)/\overline{\mathrm{CC}}'(k) \tag{5.3}$$

where $1-t$ and N are as in the standard $(b, b', 1-t, N)$ double complex. (Recall that

$$(\mathrm{Ker}(1-t), b') = (\mathrm{Im}(N), b') \xrightarrow{\sim} (C(A)/\mathrm{Ker}(N), b) = (C(A)/\mathrm{Im}(1-t), b)$$

and therefore $\overline{\mathrm{CC}}'(A)$ does compute the cyclic homology).

Consider now a dual picture. Let $\mathcal{C}$ be a differential graded counital coalgebra $(\mathcal{C}, \partial)$. For $c \in \mathcal{C}$, let $\underline{c}$ be a formal element of degree $|c|+1$, linear in c. These elements generate the space $\underline{\mathcal{C}}$ which is same as $\mathcal{C}$ but with the grading shifted by one. Let $\widetilde{\mathcal{C}}$ be the graded coalgebra which is a linear direct sum of $\mathcal{C}$ and $\underline{\mathcal{C}}$. The comultiplication is as follows:

$$\Delta c = \sum c^{(1)} \otimes c^{(2)} + \sum (-1)^{|c^{(1)}|}\partial c^{(1)} \otimes \underline{c}^{(2)} \tag{5.4}$$

$$\Delta \underline{c} = \sum \underline{c}^{(1)} \otimes c^{(2)} + (-1)^{|c^{(1)}|} c^{(1)} \otimes \underline{c}^{(2)} + \sum (-1)^{|c^{(1)}|}\underline{\partial c^{(1)}} \otimes \underline{c}^{(2)} \tag{5.5}$$

For any counital DG coalgebra C put

$$\mathrm{CC}'(C) = (\mathrm{Coker}(1-t), b'); \overline{\mathrm{CC}}'(C) = \mathrm{Ker}(\mathrm{CC}'(C) \to \mathrm{CC}'(k)) \tag{5.6}$$

LEMMA 5.1.1. *Let* $A = k + \overline{A}$ *be the algebra obtained from an algebra* $\overline{A}$ *by attaching a unit. Then the complex* $(\mathrm{DR}^{\bullet}_{t,+}(A), \iota_t)$ *is isomorphic to* $\overline{\mathrm{CC}}'(\mathrm{Bar}(\overline{A}))$ *where* Bar *stands for the usual bar construction (which is a DG coalgebra).*

PROOF. Take a monomial $\omega_1 t\omega_2 t \dots \omega_n t$ in $\mathrm{DR}_{t,+}$. Identify it with $\alpha_1 \otimes \alpha_2 \otimes \dots \otimes \alpha_n$ in $\mathrm{CC}'(\mathrm{Bar}(\overline{A}))$ where $\alpha_k = (a_0|a_1|\dots|a_m)$ if $\omega_k = a_0 da_1 \dots da_m$ and $\alpha_k = (|a_1|\dots|a_m)$ if $\omega_k = da_1 \dots da_m$. One checks that this gives an isomorphism of complexes. □

REMARK 5.1.2. The above computation suggests a comparison to the work of Berest , Felder, Patotsky, Ramadoss and Willwacher where the algebra of functions on the *derived* representation scheme is identified with the standard cochain complex of the Lie coalgebra $\mathfrak{gl}_n(\mathrm{Bar}(A))$, *cf.* **??**.

6. HKR maps

The Hochschild-Kostant-Rosenberg map from the Hochschild homology of the algebra of functions to diferential forms is a major motivation and a major tool in noncommutative geometry. It was recently discovered that an HKR map exists with values in noncommutative forms for any algebra, commutative or not. The classical HKR map isn the commutative case is obtained by projection from noncommutative to ordinary forms.

There are two noncommutative HKR maps: one, μ from $C_*(A), b + B$ to $\Omega^*(A), \iota + d$ given by:

$$\mu(a_0 \otimes \dots \otimes a_n) = \frac{1}{(n+1)!} \sum_{i=0}^{n} (-1)^{i(n-1)} da_{i+1} \dots da_n a_0 da_1 \dots da_i;$$

if we identify $C_*(A, A)$ with $\Omega^*(A)$, we can write the other, in reverse direction,

$$\nu(a_0 d \dots da_n) = (n-1)! \sum_{i=0}^{n-1} (-1)^{(i+1)(n-1)} da_{i+1} \dots da_n a_0 da_1 \dots da_i.$$

Recall that

$$b(a_0 da_1 \dots da_n) = (-1)^{(n-1)}[a_0 da_1 \dots da_{n-1}, a_n] \tag{6.1}$$

and

$$\iota(a_0 da_1 \dots da_n) = \sum_{i=0}^{n-1} (-1)^{i(n-1)}[da_{i+1} \dots da_n a_0 da_1 \dots da_{i-1}, a_i] \tag{6.2}$$

We will sometimes drop the index Δ.

In other words, if κ is the Karoubi operator as in (4.11), then $\mu = \frac{1}{(n+2)!}(1 + \kappa + \dots + \kappa^n)$ and $\nu = (n-1)!(1 + \kappa + \dots \kappa^{n-1})$. Both μ and ν are morphisms of complexes; this follows from basic identities (4.13), (4.16), (4.17), and (4.18).

LEMMA 6.0.1. *The isomorphism in Theorem* **??** *is induced by the HKR map* μ.

7. More on operators on noncommutative forms

LEMMA 7.0.1. *One has* $\mathsf{P} = \mathrm{Id}$ *on* $\Omega/[\Omega,\Omega]$.

PROOF. Since $\mathrm{Id} - \kappa$ is invertible on $\mathsf{P}^{\perp}\Omega$, one has

$$\mathsf{P}^{\perp}\Omega = (\mathrm{Id} - \kappa)\mathsf{P}^{\perp}\Omega \subset [\Omega,\Omega] \cap \mathsf{P}^{\perp}\Omega = \mathsf{P}^{\perp}[\Omega,\Omega] \tag{7.1}$$

□

LEMMA 7.0.2. *One has* $\mathsf{P}[\Omega,\Omega] = \iota\Omega$.

PROOF.

$$\mathsf{P}[\Omega,\Omega] = \mathsf{P}[A,\Omega] + \mathsf{P}[dA,\Omega] = b\mathsf{P}\Omega + (\mathrm{Id} - \kappa)\mathsf{P}\Omega.$$

But $\mathrm{Id} - \kappa$ is zero on $\mathsf{P}(\Omega/b\Omega)$ because κ is of finite order on each component and therefore P is the projection to the invariant part. Therefore

$$\mathsf{P}[\Omega,\Omega] = b\mathsf{P}\Omega = bN\mathsf{P}\Omega = \iota\Omega.$$

□

Since $\iota\mathsf{P}^{\perp} = 0$ and $\iota^2 = 0$, the above two lemmas show that $\iota|[\Omega,\Omega] = 0$.

8. Hochschild and cyclic homology in terms of d and ι

THEOREM 8.0.1.

$$\mathrm{HH}_{\bullet}(A) = \mathrm{Ker}(\iota : (\overline{\mathrm{DR}}^{\bullet}(A) \to \Omega^{\bullet -1}(A))$$

PROOF. Note that b is acyclic on $\mathsf{P}^{\perp}\Omega$ because $\mathrm{Id} - \kappa = bd + db$ is invertible there. Therefore $\mathsf{HH}_{\bullet}(A)$ is the homology of $\mathsf{P}\Omega = \Omega/\mathsf{P}^{\perp}\Omega$ with the differential $b\mathsf{P}$ that we can replace by $bN\mathsf{P} = \iota$. One sees that

$$\mathrm{HH}_{\bullet}(A) = \mathrm{Ker}(\iota)/(\mathsf{P}^{\perp}\Omega + \iota\Omega) = \mathrm{Ker}(\iota)/[\Omega,\Omega].$$

□

THEOREM 8.0.2.

$$\overline{\mathrm{HC}}_{\bullet}(A) = \mathrm{Ker}(\iota : \overline{\mathrm{DR}}^{\bullet}(A)/d\overline{\mathrm{DR}}^{\bullet -1}(A) \to \overline{\Omega}^{\bullet -1}(A)/d\overline{\Omega}^{\bullet -2}(A))$$

PROOF. Since b is contractible on $\mathsf{P}^{\perp}\Omega$, the reduced cyclic homology is computed by the complex

$$(\mathsf{P}\overline{\Omega}((u))/u\mathsf{P}\overline{\Omega}[[u]], b + uB) \tag{8.1}$$

and we can replace the differential by $\iota + ud$. Since d is contractible on $\overline{\Omega}$, we can replace this complex by

$$(\mathsf{P}\overline{\Omega}/d\overline{\Omega}, \iota) \tag{8.2}$$

Therefore (recall that the image of ι is contained in the image of P)

$$\overline{\mathrm{HC}}_{\bullet}(A) \xrightarrow{\sim} \mathrm{Ker}(\iota : \overline{\Omega}/(d\overline{\Omega} + \mathsf{P}^{\perp}\overline{\Omega} + \iota\overline{\Omega}) \to \overline{\Omega}/d\overline{\Omega})$$

which is equal to

$$\mathrm{Ker}(\iota : \overline{\Omega}/([\overline{\Omega},\overline{\Omega}] + d\overline{\Omega}) \to \overline{\Omega}/d\overline{\Omega}) = \mathrm{Ker}(\iota : \overline{\mathrm{DR}}/d\overline{\mathrm{DR}} \to \overline{\Omega}/d\overline{\Omega}).$$

□

%

9. On the duality between chains and cochains

10. Chains and cochains

There are maps $\Omega^*(A) \to A * k[\tau] \to C^*(A, A)$ where the first one sends a to a and da to $[\tau, a]$ like in [**292**], and the second sends a to a and τ to $id : A \to A$, the identity one-cochain. The composite map is the universal map of DGAs that sends a to a. Of course it intertwines d with the Hochschild differential δ. In case when A has a trace, there is a map from $C^*(A, A)$ to the dual space of $C_*(A, A)$ induced by the bimodule map $A \to A^*$, $a \mapsto tr(a?)$. Now, the differential dual to B on the right hand side gets intertwined with the differential dual to b in $A * k[\tau] = A^{*+1}$ under the Connes isomorphism between Λ and Λ^{op} (this is a good explanation for the latter). This same differential gets intertwined with ι on Ω^* (which can be viewed as another way to discover ι). So we have morphisms

$$(\Omega^*(A), b + B) \to (\Omega^*(A), \iota + d) \to A * k[\tau], B^{\text{dual}} + b^{\text{dual}}) \to \\ (C^*(A, A), B + \delta) \to (C_*(A, A)^{\text{dual}}) \tag{10.1}$$

The composition of all the maps above can be interpreted in terms of our map χ as follows.

WHAT ABOUT χ? EXPLICIT FORMULA FOR THE PAIRING? PASCHKE?

Consider the map

$$C_*(A, A) \to C_*^{\text{Lie}}(\text{Der}(\mathfrak{gl}(A)) \ltimes \mathfrak{gl}(A)[-1])_{\mathfrak{gl}(k)}$$

given by

$$a_0 \otimes \ldots \otimes a_n \mapsto E_{01}(a_0) \wedge \text{ad}(E_{12}(a_1)) \wedge \ldots \wedge \text{ad}(E_{n0}(a_n))$$

This map intertwines b with the Koszul differential. Now embed A into $M_\infty(A)$ diagonally where $M_\infty(A)$ is the Lie algebra of infinite matrices, say, with finitely many nonzero diagonals. We get an embedding

$$\beta : C_*(A, A) \to C_*(M_\infty(A), M_\infty(A)).$$

Under the composition (10.1), a Hochschild chain c maps to the linear functional $x \mapsto \chi(\alpha(c))(\beta(x))$. Observe that the pairing χ clearly extends to a pairing

$$C_*^{\text{Lie}}(\mathfrak{gl}(A) \ltimes \mathfrak{gl}(A)[-1])_{\mathfrak{gl}(k)} \otimes C_*(M_\infty(A), M_\infty(A))^{\mathfrak{gl}(k)} \to k$$

Note that $C^*(A, A)$ is a brace algebra, and the brace operations are defined on $A * k[\tau]$ so that the map of complexes preserves them (an insertion of $b_0\tau \ldots \tau b_m$ into $a_0\tau \ldots \tau a_n$ acts by taking an appropriate factor τ in the latter and replacing it by the former). Therefore $A * k[\tau]$ is a homotopy BV algebra, with the BV operator being B^{dual}. The brace structure can be also defined on $(\Omega^*(A), d)$. The easiest way to see this is to observe that $\Omega^*(A)$ embeds into $A * k[\tau]$ as the subspace of elements that are annihilated by substituting the unit for any given factor τ.

The above brace structure on $\Omega^*(A)$ is a quantum analog of the Gerstenhaber bracket on $\Omega^*(M)$ where M is a Poisson manifold. The bracket is defined by $[da, b] = \{a, b\}$ and $[da, db] = d\{a, b\}$ for functions a and b. The map $\Omega^*(M) \to \wedge^*(T(M))$ defined by the Poisson structure is a morphism of Gerstenhaber algebras.

When A has a trace such that $\text{tr}(ab)$ is a nondegenerate form, then $C^*(A, A)$ is a homotopy BV algebra, and the morphism $\Omega^*(A) \to C^*(A, A)$ is a morphism of homotopy BV algebras.

QUESTION 10.0.1. The formality theorem for chains says that, for a smooth commutative algebra, $(C_{-*}(A,A)[[u]], b+uB)$ is quasi-isomorphic to the complex $(\Omega^{-*}_{A/k}[[u]], ud)$ as L_∞ modules over the DG Lie algebra $C^{*+1}(A,A)$; the latter acts on the former via the Kontsevich formality morphism to $\wedge^{*+1}T$. If A is a deformation of a smooth commutative algebra corresponding to a formal Poisson structure π, then $(C_{-*}(A,A)[[u]], b+uB)$ is quasi-isomorphic to $(\Omega^{-*}_{A/k}[[u,h]], hL_\pi + ud)$ (as L_∞ modules over $C^{*+1}(A,A)$). What is the correct formality statement that takes into account the brace/BV structure on Hochschild chains?

Note that the homotopy BV algebra $(\Omega^{-*}_{A/k}[[u,h]], hL_\pi + ud)$ has one more property, namely there is an operator ι_π/u whose exponential gives an isomorphism of complexes $(\Omega^*(M)[[h,u]], hL_\pi + ud)$ and $(\Omega^*(M)[[h,u]], ud)$. So it kills b in $b+uB$. Now we see that there are some dualities intertwining b and B. Can it be that, for a smooth compact CY A, there is a similar structure that kills B in $b+uB$?

For example, if A is a deformation quantization of a compact symplectic manifold (localized in $\hbar$), the composition (10.1) is an isomorphism on homology. It coincides with the Poincaré duality $H^{2n-*}(M) \xrightarrow{\sim} H^k(M)^{\text{dual}}$. The Hochschild to cyclic spectral sequence starts with the De Rham cohomology; by rigidity of the periodic cyclic homology, $HP_*(A)$ is the De Rham cohomology; therefore, since its dimension is finite, we know that the spectral sequence degenerates. Maybe there is a similar mechanism for smooth compact CY? Note that this is far from straightforward because (10.1) involves a lot of commutators and seems not to have much chance to be an isomorphism for, say, commutative algebras. Still, maybe there is some more sophisticated version.

11. Hamiltonian actions

Another thing suggested by the above constructions is a definition of a Hamiltonian action of a Hopf algebra H on an algebra A, so that one can define Hamiltonian reduction. Let H be a Hopf algebra acting on an associative algebra A. Put $\overline{H} = \mathrm{Ker}(\epsilon : H \to k)$. The action can be interpreted as an associative algebra morphism $\rho : \overline{H} \to C^1(A,A)$ such that

$$\delta\rho(h) + \sum \rho(h^{(1)}) \smile \rho(h^{(2)}) = 0$$

where $\smile$ is the cup product on Hochschild cochains and

$$1 \otimes h + \sum h^{(1)} \otimes h^{(2)} + h \otimes 1 = \Delta h.$$

The action is recovered from ρ by as $h(a) = \epsilon(h)a + \rho(h - \epsilon(h))(a)$.

Define a Hamiltonian action of H on A as a morphism of associative algebras

$$\rho : \overline{H} \to \Omega^1(A^+)$$

such that

$$d\rho(h) + \sum \rho(h^{(1)})\rho(h^{(2)}) = 0; \tag{11.1}$$

here A^+ is A with the unit adjoined, and the associative product on Ω^1 is given by the brace operation:

$$a_0 da_1 \circ b_0 db_1 = a_0 a_1 b_0 db_1 - a_0 b_0 db_1 a_1.$$

Such ρ defines an action of H on A via the map $\Omega^1(A^+) \to C^*(A^+, A^+) \to C^*(A, A)$. Given ρ as in (11.1), define a reduced algebra by

$$A_{\text{red}} = (A/I)^H$$

Here I is the left ideal of A generated by elements $\sum a_{0,i}(h)xa_{1,i}(h)$ where $\rho(h) = \sum a_{0,i}(h)da_{1,i}(h)$. One observes that the action of H on A descends to an action on A/I. Indeed,

$$h(ya_0(h')xa_1(h')) = a_0(h)[a_1(h), ya_0(h')xa_1(h')] =$$

$$= a_0(h)a_1(h)y(a_0(h')xa_1(h')) - a_0(h)(ya_0(h')xa_1(h'))a_1(h) \in I;$$

as for the product,

$$\sum a_{0,i}(h)xa_{1,i}(h)y \equiv a_{0,i}(h)x[a_{1,i}(h), y] \equiv \sum [a_{0,i}(h), x][a_{1,i}(h), y] \mod I;$$

by (11.1), the latter is equal to

$$\sum a_{0,i}(h^{(1)})[a_{1,i}(h^{(1)}), x]a_{0,i}(h^{(2)})[a_{1,i}(h^{(2)}), x];$$

if y is invariant modulo I, this expression lies in I.

REMARK 11.0.1. For any Hopf algebra H, the tensor algebra $T(H[1])$ is a brace algebra: if $G_i = (g_{i,1}|\ldots|g_{i,n_i}) \in H^{n_i}$, then

$$(h_1|\ldots|h_m)\{G_1, \ldots, G_p\} =$$

$$\sum_{1 \le k_1 < \ldots < k_p \le m} \pm(h_1|\ldots|\Delta^{n_1-1}h_{k_1} \cdot G_1|\ldots|\Delta^{n_p-1}h_{k_p} \cdot G_p|\ldots|h_m).$$

In particular, $(h)\{(g)\} = (gh)$.

An action of H on an algebra A is a morphism of brace algebras $T(H[1]) \to C^*(A, A)$; a Hamiltonian action is a morphism of brace algebras $T(H[1]) \to \Omega^*(A^+)$. By a result of Halbout, if H is an Etingof-Kazhdan quantization of a Lie bialgebra $\mathfrak{g}$, then there is a G_∞ quasi-isomorphism $T(H[1]) \to C_*(\mathfrak{g})$, the right hand side being the Lie algebra chain complex on which the differential is the cochain differential of the Lie algebar $\mathfrak{g}^*$ and the Gerstenhaber bracket is induced by the bracket of $\mathfrak{g}$. This, together with the formality theorem of Kontsevich, should give a classification of Hamiltonian actions of a quantum group on a smooth manifold (though Pavol Ševera seemed to think that some refinement of Halbout's result is needed). Similarly, a correct formality theorem from the previous question should give a classification of Hamiltonian actions.

12. Appendix. Filtered complexes

We recall some standard facts about filtered complexes and the Beilinson t-structure. Let $E_\bullet$ be a complex (the grading is homological) with a decreasing filtration $F^*E_\bullet$. For an integer p, denote

$$\tau^B_{\ge p}E_n = F^{p-n}E_n \cap d^{-1}(F^{p-n+1}E_{n+1}) \tag{12.1}$$

Dually:

$$\tau^B_{\le p}E_n = F^{p-n}E_n/(dF^{p-n}E_{n+1} + F^{p-n+1}E_n) \tag{12.2}$$

Subcomplexes $\tau^B_{\ge p}E_\bullet$ form a decreasing filtration of $E_\bullet$. Dually, quotient complexes of $E_\bullet$ form an inverse system where all the map are epimorphisms.

When $F^0E_\bullet = E_\bullet$ and $F^1E_\bullet = 0$, then we get the usual truncation

$$\tau_{\geq p}E_n = E_n, n > p;\ Z_p, n = p;\ 0, n < p. \tag{12.3}$$

(where Z_p consists of p-cycles) and

$$\tau_{\leq p}E_n = E_n, n < p;\ E_p/B_p, n = p;\ 0, n > p. \tag{12.4}$$

(where B_p consists of p-boundaries).

LEMMA 12.0.1. *The triangle*

$$\tau^B_{\geq p}E_\bullet \to E_\bullet \to \tau^B_{\leq p-1}E_\bullet$$

is distinguished.

LEMMA 12.0.2. *The following are equivalent.*

(1) $\tau^B_{\geq 0}E_\bullet = E_\bullet = \tau^B_{\leq 0}E_\bullet$
(2) *For any* n, $E_n = F^{-n}E_n$ *and* $F^{-n+1}E_n = 0$.

CHAPTER 16

DG categories

****Check; bring into line with sources*

1. Introduction

The contents of this section are taken mostly from [**197**], [**?**], and [**548**].

2. Definition and basic properties

A (small) differential graded (DG) category $\mathcal{A}$ over k is a ***set*** $\mathrm{Ob}(\mathcal{A})$ of elements called objects and of complexes $\mathcal{A}(x, y)$ of k-modules for every $x, y \in \mathrm{Ob}(A)$, together with morphisms of complexes

$$\mathcal{A}(x, y) \otimes \mathcal{A}(y, z) \to \mathcal{M}(x, z),\ a \otimes b \mapsto ab, \tag{2.1}$$

and zero-cycles $\mathbf{1}_x \in \mathcal{A}(x, x)$, such that (2.1) is associative and $\mathbf{1}_x a = a\mathbf{1}_y = a$ for any $a \in \mathcal{A}(x, y)$. For a DG category, its homotopy category is the k-linear category $\mathrm{Ho}(\mathcal{A})$ such that $\mathrm{Ob}(\mathrm{Ho}(\mathcal{A})) = \mathrm{Ob}(\mathcal{A})$ and $\mathrm{Ho}(\mathcal{A})(x, y) = H^0(\mathcal{A}(x, y))$, with the units being the classes of $\mathbf{1}_x$ and the composition induced by (2.1).

A DG functor $\mathcal{A} \to \mathcal{B}$ is a map $\mathrm{Ob}(\mathcal{A}) \to \mathrm{Ob}(\mathcal{B})$, $x \mapsto Fx$, and a collection of morphisms of complexes $F_{x,y} \colon \mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy), x, y \in \mathrm{Ob}(\mathcal{A})$, which commutes with the composition (2.1) and such that $F_{x,x}(\mathbf{1}_x) = \mathbf{1}_{Fx}$ for all x.

The opposite DG category of $\mathcal{A}$ is defined by $\mathrm{Ob}(\mathcal{A}^{\mathrm{op}}) = \mathrm{Ob}(\mathcal{A})$, $\mathcal{A}^{\mathrm{op}}(x, y) = \mathcal{A}(y, x)$, the unit elements are the same as in $\mathcal{A}$, and the composition (2.1) is the one from $\mathcal{A}$, composed with the transposition of tensor factors.

3. Semi-free DG categories

Semi-free DG categories are defined exactly as in 2. For a k-linear graded category $\mathcal{A}$ and a collection of graded k-modules $\{\mathcal{V} = V(x, y) | x, y \in \mathrm{Ob}(\mathcal{A})\}$ one defines by the usual universal property a new k-linear category freely generated by $\mathcal{A}$ and all $\mathcal{V}$. A DG category $\mathcal{R}$ is semi-free over $\mathcal{A}$ if it is freely generated over $\mathcal{A}$ by a collection$\mathcal{V}$ and there is an increasing filtration $\mathcal{V}_n | n \geq -1$, $\mathcal{V}_{-1} = 0$, $d\mathcal{V}_n$ is contained in the subcategory generated by $\mathcal{A}$ and $\mathcal{V}_{n-1}$, and $d|\mathcal{A}$ is the differential of $\mathcal{A}$.

For a set S define the category k_S as follows: $\mathrm{Ob}(k_S) = S$; $k_S(x, y) = 0$ for $x \neq y$; $k_S(x, x) = k\mathbf{1}_x$. A DG category is called semi-free if it is semi-free over the DG category $k_{\mathrm{Ob}(A)}$. Existence and uniqueness up to homotopy equivalence of a semi-free resolution of a DG category is proved as in 2 without any changes. Similarly the relative case 3.2 for a DG functor $\mathcal{A} \to \mathcal{B}$ *which is the identity map on objects.*

4. Quasi-equivalences

A quasi-equivalence [?] between DG categories $\mathcal{A}$ and $\mathcal{B}$ is a DG functor $F : \mathcal{A} \to \mathcal{B}$ such that a) F induces an equivalence of homotopy categories and b) for any $x, y \in \mathrm{Ob}(\mathcal{A})$, $F_{x,y} : \mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ is a quasi-isomorphism.

5. Drinfeld quotient

For a full DG subcategory $\mathcal{A}$ of a DG category $\mathcal{B}$, the quotient of $\mathcal{B}$ by $\mathcal{A}$ is by definition the graded category freely generated by $\mathcal{B}$ and the family $\mathcal{V}(x, y) = k\epsilon_x$, $|\epsilon_x| = -1$, in $x = y \in \mathrm{Ob}(\mathcal{A})$; $\mathcal{V}(x, y) = 0$ in all other cases. The differential on $\mathcal{B}/\mathcal{A}$ extends the one on $\mathcal{B}$ and satisfies $d\epsilon_x = \mathbf{1}_x$.

In other words, it is a DG category $\mathcal{B}/\mathcal{A}$ such that:

(1) $\mathrm{Ob}(\mathcal{B}/\mathcal{A}) = \mathrm{Ob}(\mathcal{B})$;
(2) there is a DG functor $i : \mathcal{B} \to \mathcal{B}/\mathcal{A}$ which is the identity on objects;
(3) for every $x \in \mathrm{Ob}(\mathcal{A})$, there is an element ϵ_x of degree -1 in $\mathcal{B}/\mathcal{A}(x, x)$ satisfying $d\epsilon_x = \mathbf{1}_x$;
(4) for any other DG category $\mathcal{B}'$ together with a DG functor $i' : B \to \mathcal{B}'$ and elements ϵ'_x as above, there is unique DG functor $f : \mathcal{B}/\mathcal{A} \to \mathcal{B}'$ such that $i' = f \circ i$ and $\epsilon_x \mapsto \epsilon'_x$.

One has

$$(\mathcal{B}/\mathcal{A})(x, y) = \bigoplus_{n \geq 0} \bigoplus_{x_1, \ldots, x_n \in \mathrm{Ob}(A)} \mathcal{A}(x, x_1)\epsilon_{x_1}\mathcal{A}(x_1, x_2)\epsilon_{x_2} \ldots \epsilon_{x_n}\mathcal{A}(x_n, y);$$

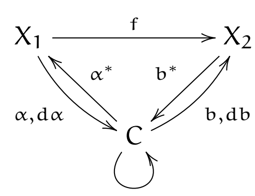
it is easy to define the composition and the differential explicitly. Let $\mathcal{A}$ be any category of complexes (or a DG category equal to its triangulated hull)***REF***, consider a morphism $X_1 \xrightarrow{f} X_2$ and assume that $\mathrm{Cone}(f)$ lies in a full DG subcategory $\mathcal{C}$.

LEMMA 5.0.1. *The morphism f is a strong equivalence in $\mathcal{A}/\mathcal{C}$.*

PROOF. Consider the DG category $\mathcal{A}_0$ with two objects X_1 and X_2 and with

$$\mathcal{A}_0(X_1, X_1) = k\mathbf{1}_{X_1},\ \mathcal{A}_0(X_2, X_2) = k\mathbf{1}_{X_2};\ \mathcal{A}_0(X_1, X_2) = kf,$$

where $df = 0$. Let cA be the full subcategory of the triangulated hull of $\mathcal{A}_0$ generated by objects X_1, X_2, and $C = \mathrm{Cone}(f)$. Then the quotient $\mathcal{A}/\mathcal{C}$ is generated by the following morphisms.



Here

$$|e| = |b| = |b^*| = 0; |\alpha| = -1;\ |\alpha^*| = 1; |\epsilon| = -1.$$

Relations are as follows (recall our convention of writing the composition $x \xrightarrow{f} x \xrightarrow{g} z$ as fg):

$$db^* = 0;\ d\alpha^* = 0;\ d\epsilon = \mathbf{1}_C;$$
$$\alpha^*\alpha = e;\ b^*b = \mathbf{1}_{X_2}; bb^* = \mathbf{1}_C - e;\ eb = 0;\ edb = db;\ b^*e = 0$$

$$e^2 = e;\ ede = de = de(\mathbf{1}_C - e);\ \alpha e = \alpha;\ d\alpha e = 0; \alpha\alpha^* = \mathbf{1}_{X_1};\ e\alpha^* = \alpha^*;\ b^*\alpha^* = 0$$
$$\alpha b = 0;\ d\alpha b = f = -\alpha db;\ fb^* = d\alpha$$

Set

$$\mathcal{P}(X_1, C) = k\alpha + kd\alpha;\ \mathcal{P}(C, X_1) = k\alpha^*;\ \mathcal{P}(X_2, C) = kb^*;\ \mathcal{P}(C, X_2) = kb + kdb;$$
$$\mathcal{P}(C, C) = ke + kde\mathbf{1}_C.$$

We see that there is a short exact sequence

$$0 \to \mathcal{A}_0(X_i, X_j) \to (\mathcal{A}/\mathcal{C})(X_i, X_j) \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} \mathcal{P}(X_i, C)\epsilon(\mathcal{P}(C, C)\epsilon)^{\otimes n}\mathcal{P}(C, X_j) \to 0$$

***[that is when we assume $\epsilon^2 = 0$-check that we do]**. Direct sums of the first n factors, $n \geq 0$, form an increasing filtration of the term on the right in the sequence. For all three cases except $i = 2,\ j = 1$, the associated graded quotients of the filtration are all acyclic (because $\mathcal{P}(X_1, C)$ and $\mathcal{P}(C, X_2)$ are contractible. When $i = 2$ and $j = 1$, all graded factors with $n \geq 1$ are acyclic because $\mathcal{P}(C, C)$ is contractible. We see that $(\mathcal{A}/\mathcal{C})(X_2, X_1)$ is quasi-isomorphic to kg where

$$g = b^*\epsilon\alpha^*. \tag{5.1}$$

The full subcategory of $\mathcal{A}/\mathcal{C}$ generated by X_1 and X_2 is therefore quasi-isomorphic to I_2 and admits a quasi-isomorphism from $\mathcal{I}_2$. □

6. DG modules over DG categories

A DG module over a DG category $\mathcal{A}$ is a collection of complexes of k-modules $\mathcal{M}(x)$, $x \in \mathrm{Ob}(\mathcal{A})$, together with morphisms of complexes

$$\mathcal{A}(x, y) \otimes \mathcal{M}(y) \to \mathcal{M}(x),\ a \otimes m \mapsto am, \tag{6.1}$$

which is compatible with the composition (2.1) and such that $\mathbf{1}_x m = m$ for all x and all $m \in \mathcal{M}(x)$. A DG bimodule over $\mathcal{A}$ is a collection of complexes $\mathcal{M}(x, y)$ together with morphisms of complexes

$$\mathcal{A}(x, y) \otimes \mathcal{M}(y, z) \otimes \mathcal{A}(z, w) \to \mathcal{M}(x, w),\ a \otimes m \otimes b \mapsto amb, \tag{6.2}$$

that agrees with the composition in $\mathcal{A}$ and such that $\mathbf{1}_x m \mathbf{1}_y = m$ for any x, y, m. We put $am = am\mathbf{1}_z$ and $mb = \mathbf{1}_x mb$. A DG bimodule over $\mathcal{A}$ is the same as a DG module over $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$.

6.1. Semi-free DG modules.

6.2. The enveloping pre-triangulated DG category.

7. Hochschild and cyclic complexes of DG categories

7.1. Definitions.

DEFINITION 7.1.1. *For a DG category $\mathcal{A}$ and a DG bimodule $\mathcal{M}$, set*

$$C_\bullet(\mathcal{A}, \mathcal{M}) = \bigoplus_{n \geq 0; x_0, \ldots, x_n \in \mathrm{Ob}(\mathcal{A})} \mathcal{M}(x_0, x_1) \otimes \overline{\mathcal{A}}(x_1, x_2)[1] \otimes \ldots \otimes \overline{\mathcal{A}}(x_n, x_0) \tag{7.1}$$

where

$$\overline{\mathcal{A}}(x, y) = \mathcal{A}(x, y) \text{ when } x \neq y \text{ and } \overline{\mathcal{A}}(x, x) = \mathcal{A}(x, x)/k\mathbf{1}_x.$$

Define also

$$C_\bullet(\mathcal{A}) = C_\bullet(\mathcal{A}, \mathcal{A})$$

the differentials b, d, *and* B *are defined exactly as in 4. Similarly for the non-normalized complex* $\widetilde{C}_\bullet(\mathcal{A})$.

As usual,

$$CC^-(\mathcal{A}) = (C_\bullet(\mathcal{A})[[u]], b + d + uB)$$

and similarly for the cyclic and periodic cyclic complexes.

8. Invariance properties of Hochschild and cyclic complexes

8.1. Passing to matrices. If $\mathcal{A}$ is a DG category then let

$$M(\mathcal{A}) = \varinjlim_{n\to\infty} (\mathcal{A}) = \mathcal{A} \otimes M(k)$$

be the dg category of finite matrices $m_{jk}|j, k \geq 0$ with entries in $\mathcal{A}$. We have an embedding

$$i : \mathcal{A} \to M(\mathcal{A});\ a \mapsto aE_{00} \tag{8.1}$$

Here, as usual, E_{jk} is the elementary matrix with the only nonzero entry 1 that is located in row j and column k.

PROPOSITION 8.1.1. *The embedding* (4) *induces homotopy eqivalences of Hochschild, negative cyclic, cyclic, and periodic cyclic complexes.*

PROOF. As in the case of ordinary algebras, this can be easily deduced from the fact that the Hochschild homology is the derived tensor product. Here we will give an explicit proof that is almost identical to the proofs of the invariance properties in 8.2 below and in 18.

First, observe that

$$\mathrm{tr} : C_\bullet(M(\mathcal{A})) \to C_\bullet(\mathcal{A});\ a_0 \otimes \ldots \otimes a_n \mapsto \sum (a_0)_{i_0 i_1} \otimes \ldots \otimes (a_n)_{i_n i_0}$$

commutes with all the differentials. We have $\mathrm{tr} \circ i = \mathrm{id}$ and $\mathrm{id} - i \circ \mathrm{tr} = [b + d, h]$ where the homotopy h is defined by

$$h(a_0 \otimes \ldots \otimes a_n) = \sum_{p=0}^{n} \sum \pm a_0 E_{i_0 0} \otimes E_{0 i_0} a_1 E_{i_1 0} \otimes E_{0 i_{p-1}} a_p E_{i_p 0} \otimes E_{0 i_p} \otimes a_{p+1} \otimes \ldots \otimes a_n$$

The sign is $(-1)^{\sum_{j=0}^{p} |a_j| + p}$. This proves the statement for the Hochschild complex. the statement for the various versions of the cyclic complex follows because corresponding morphisms preserve the filtration by powers of u and are quasi-isomorphisms on associated graded quotients. □

8.2. Adding idempotents. For a DG category $\mathcal{A}$, consider a new DG category $\mathcal{A}^{\mathrm{id}}$. Its objects are pairs (x, e) where $x \in \mathrm{Ob}(\mathcal{A})$, $e \in \mathcal{A}^0(x, x)$ such that $e^2 = e$ and $de = 0$; morphisms from (x, e) to (y, f) are elements eaf where $a \in \mathcal{A}(x, y)$.

PROPOSITION 8.2.1. *The embedding* $i : \mathcal{A} \to \mathcal{A}^{\mathrm{id}}$ *sending any object* x *to* $(x, \mathbf{1}_x)$ *induces homotopy equivalences of Hochschild, negative cyclic, cyclic, and periodic cyclic complexes.*

PROOF. As in Proposition 8.1.1, it is enough to construct a homotopy inverse for the map of Hochschild complexes. We define it to be

$$P : e_0 a_0 e_1 \otimes \ldots \otimes e_n a_n e_0 \mapsto e_0 a_0 e_1 \otimes \ldots \otimes e_n a_n e_0$$

where in the left hand side $e_j a_j e_{j+1}$ is viewed as an element in $\mathcal{A}^{\mathrm{id}}((x_j, e_j), (x_{j+1}, e_{j+1}))$ and the right hand side as an element in $\mathcal{A}(x, y)$. Therefore $i \circ P$ sends the left hand side to itself where $e_j a_j e_{j+1}$ is viewed as an element in $\mathcal{A}^{\mathrm{id}}((x_j, \mathbf{1}_{x_j}), (x_{j+1}, \mathbf{1}_{x_{j+1}}))$. We have $P \circ i = \mathrm{id}$, while a homotopy between id and $i \circ P$ can be chosen as

$$e_0 a_0 e_1 \otimes \ldots \otimes e_n a_n e_0 \mapsto \sum_{p=0}^{n} \pm e_0 a_0 e_1 \otimes \ldots \otimes e_p a_p e_{p+1} \otimes e_{p+1} \otimes e_{p+1} a_{p+1} e_{p+2} \otimes \ldots \otimes e_n a_n e_0.$$

Here $e_j a_j e_{j+1}$ is viewed as:

1) an element of $\mathcal{A}^{\mathrm{id}}((x_j, e_j), (x_{j+1}, e_{j+1})$ for $j \leq p$;
2) an element of $\mathcal{A}^{\mathrm{id}}((x_j, \mathbf{1}_{x_j}), (x_{j+1}, \mathbf{1}_{x_{j+1}}))$ for $j > p$.

The tensor factor e_{p+1} is viewed as an element $\mathcal{A}^{\mathrm{id}}((x_{p+1}, e_{p+1}), (x_{p+1}, \mathbf{1}_{x_{p+1}}))$. Also, $e_{n+1} = e_0$ and the sign is $(-1)^{\sum_{j=0}^{p} |a_j| + p}$. □

8.3. Invariance up to quasi-equivalence.

THEOREM 8.3.1. *A quasi-equivalence $\mathcal{A} \to \mathcal{B}$ of DG categories induces homotopy equivalences of complexes*

$$C_\bullet(\mathcal{A}) \xrightarrow{\sim} C_\bullet(\mathcal{B});\ CC_\bullet(\mathcal{A}) \xrightarrow{\sim} CC_\bullet(\mathcal{B});\ CC_\bullet^-(\mathcal{A}) \xrightarrow{\sim} CC_\bullet^-(\mathcal{B});\ CC_\bullet^{\mathrm{per}}(\mathcal{A}) \xrightarrow{\sim} CC_\bullet^{\mathrm{per}}(\mathcal{B})$$

PROOF. We will start with proving that an equivalence of graded k-linear categories induces a homotopy equivalence of Hochschild complexes and hence of cyclic complexes of all types. For this it is enough to show that two isomorphic functors induce homotopic maps of Hochschild complexes. If $c : F \xrightarrow{\sim} G$ is an isomorphism of functors $F, G : \mathcal{A} \to \mathcal{B}$, then an explicit homotopy is given by

$$a_0 \otimes \ldots \otimes a_n \mapsto \sum_{j=0}^{n} \pm c_{x_0}^{-1} F a_0 \otimes \ldots \otimes F a_j \otimes c_{x_{j+1}} \otimes G a_{j+1} \otimes \ldots \otimes G a_n \tag{8.2}$$

Here $a_k \in \mathcal{A}(x_k, x_{k+1})$, $x_{n+1} = x_0$, and $c_x : Fx \xrightarrow{\sim} Gx$, $x \in \mathrm{Ob}(\mathcal{A})$, define the isomorphism c. The sign is $(-1)^{\sum_{p \leq j}(|a_p|+1)}$.

The statement for quasi-equivalences follows when we consider the spectral sequences whose first terms are $C_\bullet(H^0(\mathcal{A}))$, $C_\bullet(H^0(\mathcal{B}))$. Functors F and G induce the same morphisms on E_1 terms and therefore on the total compexes. □

8.4. Hochschild and cyclic complexes of Drinfeld quotients.

Let $\mathcal{A}$ be a full DG subvategory of $\mathcal{B}$. Let $\mathcal{B}/\mathcal{A}$ be the Drinfeld quotient.

THEOREM 8.4.1. *(Keller excision theorem).*

$$C_\bullet(\mathcal{A}) \to C_\bullet(\mathcal{B}) \to C_\bullet(\mathcal{B}/\mathcal{A})$$

is a homotopy fibration sequence of complexes.

PROOF. We will deduce the statement from the results of 3.2. Observe first that all these results remain true not only for DG algebras but for DG categories *with a fixed set of objects.* Put $\mathcal{C} = \mathcal{B}/\mathcal{A}$. Observe that $\mathcal{C}$ is semi-free over $\mathcal{B}$. We claim that the complex (3.9)

$$\mathrm{DR}^1(\mathcal{C}/\mathcal{B}) \xrightarrow{b} (\mathcal{C}/\mathcal{B})/[\mathcal{B}, \mathcal{C}/\mathcal{B}] \tag{8.3}$$

is isomorphic to the extended Hochschild complex

$$((\mathcal{A}^{*+1}, d + b') \xrightarrow{1-\tau} (\mathcal{A}^{*+1}, b + d))[1].$$

Indeed, chains on the left, resp. on the right, in the complex above can be identified with the chains of (8.3) as follows:

$$a_0 \otimes \ldots \otimes a_n \mapsto d\epsilon_{x_0} a_0 \epsilon_{x_1} a_1 \ldots \epsilon_{x_n} a_n,$$

resp.

$$a_0 \otimes \ldots \otimes a_n \mapsto \epsilon_{x_0} a_0 \epsilon_{x_1} a_1 \ldots \epsilon_{x_n} a_n.$$

Here x_j are objects of $\mathcal{A}$, $a_j \in \mathcal{A}(x_j, x_{j+1}) = \mathcal{B}(x_j, x_{j+1})$, $x_{n+1} = x_0$. The differential on the left decomes $d + b'$.the one on the right becomes $d + b$, and b in the middle becomes $1 - \tau$. The shift by one occurs because in (8.3) every ϵ_{x_j} contributes -1, while in the Hochschild complex only those with $j \geq 0$ do. ***Maybe a few more words*** □

9. Hochschild cochain complexes

Define the Hochschild cochain complex of a DG category $\mathcal{A}$ in a DG bimodule $\mathcal{M}$ as

(9.1)
$$C^\bullet(\mathcal{A}, \mathcal{M}) = \prod_{n \geq 0; x_0, \ldots, x_n \in \mathrm{Ob}(\mathcal{A})} \underline{\mathrm{Hom}}(\overline{\mathcal{A}}(x_0, x_1)[1] \otimes \ldots \otimes \overline{\mathcal{A}}(x_{n-1}, x_n)[1], \mathcal{M}(x_0, x_n))$$

with the differential $d + \delta$, and similarly $\widetilde{C}^\bullet(\mathcal{A})$.

10. A_∞ categories and A_∞ functors

An A_∞ category is a natural generalization of both a DG category and an A_∞ algebra. We define it as a coderivation of degree one and square zero of Bar($\mathcal{A}$) where $\mathcal{A}$ is a collection graded k-modules $\mathcal{A}(x, y)$ where x, y run through a set Ob($\mathcal{A}$). We view $\mathcal{A}$ as a DG category with zero differential and product.

Remark 10.0.1. Here we only consider the case $m_0 = 0$. In particular, m_1 is a differential. Objects defined the same way but with a possible non-zero m_0 are called *curved* A_∞ algebras or categories. We discuss them in 2.

In other words, start with a DG category $\mathcal{A}$ where the differential and the product are zero. An A_∞ structure is an element m of degree one in $C^\bullet(\mathcal{A}, \mathcal{A})$ such that $m\{m\} = 0$. In addition, we require that the component of m corresponding to $n = 0$ as in (9.1) be zero.

More explicitly, an A_∞ category is a set Ob($\mathcal{A}$) and a collection of complexes $\mathcal{A}(x, y)$, $x, y \in$ Ob($\mathcal{A}$), together with k-linear maps

(10.1)
$$m_n : \mathcal{A}(x_0, x_1) \otimes \ldots \otimes \mathcal{A}(x_{n-1}, x_n) \to \mathcal{A}(x_0, x_n)$$

of degree $2 - n$, satisfying

(10.2)
$$\sum_{j \geq 0; j+k \leq n} (-1)^{\sum_{i=1}^{j} (|a_i|+1)(k+1)} m_{n+1-k}(a_1, \ldots, m_k(a_{j+1}, \ldots, m_{j+k}), \ldots, a_n) = 0$$

We refer the reader, for example, to [?].

10.1. A_∞ bimodules and Hochschild complexes. An A_∞ bimodule $\mathcal{M}$ over $\mathcal{A}$ is a collection of graded k-modules $\mathcal{M}(x, y)$, $x, y \in \mathrm{Ob}(\mathcal{A})$, and an A_∞ category $\mathcal{A} + \mathcal{M}$ with the same objects as $\mathcal{A}$ such that:

a) $\mathcal{A}$ is an A_∞ subcategory;

b) $(\mathcal{A} + \mathcal{M})(x, y) = \mathcal{A}(x, y) \oplus \mathcal{M}(x, y)$ as graded k-modules for all x, y;

c) the operations m_n vanish if more than one argument is in $\mathcal{M}$, and tales values in $\mathcal{M}$ if one argument is in $\mathcal{M}$ and the rest in $\mathcal{A}$.

We define Hochschild and cyclic complexes of an A_∞ category with coefficients in an A_∞ bimodule $\mathcal{M}$ by formulas (7.1) and (9.1). The differentials are

$$b_m = L_m;\ \delta_m = [m, \_] \tag{10.3}$$

11. Bar and cobar constructions for DG categories

The bar construction of a DG category $\mathcal{A}$ is a DG cocategory $\mathrm{Bar}(\mathcal{A})$ with the same objects where

$$\mathrm{Bar}(\mathcal{A})(x, y) = \bigoplus_{n \geq 0} \bigoplus_{x_1, \ldots, x_n} \mathcal{A}(x, x_1)[1] \otimes \mathcal{A}(x_1, x_2)[1] \otimes \ldots \otimes \mathcal{A}(x_n, x)[1]$$

with the differential

$$d = d_1 + d_2;$$

$$d_1(a_1|\ldots|a_{n+1}) = \sum_{i=1}^{n+1} \pm (a_1|\ldots|da_i|\ldots|a_{n+1});$$

$$d_2(a_1|\ldots|a_{n+1}) = \sum_{i=1}^{n} \pm (a_1|\ldots|a_i a_{i+1}|\ldots|a_{n+1})$$

The signs are $(-1)^{\sum_{j<i}(|a_i|+1)+1}$ for the first sum and $(-1)^{\sum_{j \leq i}(|a_i|+1)}$ for the second. The comultiplication is given by

$$\Delta(a_1|\ldots|a_{n+1}) = \sum_{i=0}^{n+1} (a_1|\ldots|a_i) \otimes (a_{i+1}|\ldots|a_{n+1})$$

Dually, for a DG cocategory $\mathcal{B}$ one defines the DG category $\mathrm{Cobar}(\mathcal{B})$. The DG category $\mathrm{CobarBar}(\mathcal{A})$ is a semi-free resolution of $\mathcal{A}$. *****

11.1. Units and counits. It is convenient for us to work with DG (co)categories without (co)units. For example, this is the case $\mathrm{Bar}(\mathcal{A})$ and $\mathrm{Cobar}(\mathcal{B})$ (we sum, by definition, over all tensor products with at least one factor). Let $\mathcal{A}^+$ be the (co)category $\mathcal{A}$ with the (co)units added, i.e. $\mathcal{A}^+(x, y) = \mathcal{A}(x, y)$ for $x \neq y$ and $\mathcal{A}^+(x, x) = \mathcal{A}(x, x) \oplus k\,\mathrm{id}_x$. If $\mathcal{A}$ is a DG category then $\mathcal{A}^+$ is an augmented DG category with units, i.e. there is a DG functor $\epsilon : A^+ \to k_{\mathrm{Ob}(\mathcal{A})}$. The latter is the DG category with the same objects as $\mathcal{A}$ and with $k_I(x, y) = 0$ for $x \neq y$, $k_I(x, x) = k$. Dually, one defines the DG cocategory $k^{\mathrm{Ob}(\mathcal{B})}$ and the DG functor $\eta : k^{\mathrm{Ob}(\mathcal{B})} \to \mathcal{B}^+$ for a DG cocategory B.

11.2. Tensor products. For DG (co)categories with (co)units, define $\mathcal{A}\otimes\mathcal{B}$ as follows: $\mathrm{Ob}(\mathcal{A}\otimes\mathcal{B}) = \mathrm{Ob}(\mathcal{A})\times\mathrm{Ob}(\mathcal{B})$; $(\mathcal{A}\otimes\mathcal{B})((x_1,y_1),(x_2,y_2)) = A(x_1,y_1)\otimes\mathcal{B}(x_2,y_2)$; the product is defined as $(a_1\otimes b_1)(a_2\otimes b_2) = (-1)^{|a_2||b_1|}a_1a_2\otimes b_1b_2$, and the coproduct in the dual way. This tensor product, when applied to two (co)augmented DG (co)categories with (co)units, is again a (co)augmented DG (co)category with (co)units: the (co)augmentation is given by $\epsilon\otimes\epsilon$, resp. $\eta\otimes\eta$.

DEFINITION 11.2.1. *For DG categories $\mathcal{A}$ and $\mathcal{B}$ without units, put*

$$\mathcal{A}\otimes\mathcal{B} = \mathrm{Ker}(\epsilon\otimes\epsilon : \mathcal{A}^+\otimes\mathcal{B}^+ \to k_{\mathrm{Ob}(\mathcal{A})}\otimes k_{\mathrm{Ob}(\mathcal{B})}).$$

Dually, for For DG cocategories $\mathcal{A}$ and $\mathcal{B}$ without counits, put

$$\mathcal{A}\otimes\mathcal{B} = \mathrm{Coker}(\eta\otimes\eta : k^{\mathrm{Ob}(\mathcal{A})}\otimes k^{\mathrm{Ob}(\mathcal{B})} \to \mathcal{A}^+\otimes\mathcal{B}^+).$$

One defines a morphism of DG cocategories

$$\mathrm{Bar}(\mathcal{A})\otimes\mathrm{Bar}(\mathcal{B}) \to \mathrm{Bar}(\mathcal{A}\otimes\mathcal{B}) \tag{11.1}$$

by the standard formula for the shuffle product

$$(a_1|\ldots|a_m)(b_1|\ldots|b_n) = \sum \pm(\ldots|a_i|\ldots|b_j|\ldots) \tag{11.2}$$

The sum is taken over all shuffle permutations of the symbols $(a_1,\ldots,a_m,b_1,\ldots,b_n)$, i.e. over all permutations that preserve the order of the a_i's and the order of the b_j's. The sign is computed as follows: a transposition of a_i and b_j introduces a factor $(-1)^{(|a_i|+1)(b_j|+1)}$. Let us explain the meaning of the factors a_i and b_j in the formula. We assume $a_i\in\mathcal{A}(x_{i-1},x_i)$ and $b_j\in\mathcal{B}(y_{j-1},y_j)$ for $x_i\in\mathrm{Ob}(\mathcal{A})$ and $y_j\in\mathrm{Ob}(\mathcal{B})$, $0\le i\le m$, $0\le j\le m$. Consider a summand $(\ldots|a_i|b_j|b_{j+1}|\ldots|b_k|a_{i+1}|\ldots)$. In this summand, all b_p, $j\le p\le k$, are interpreted as $\mathrm{id}_{x_i}\otimes b_p\in(A\otimes B)((x_i,y_{p-1}),(x_i,y_p))$. Similarly, in the summand $(\ldots|b_i|a_j|a_{j+1}|\ldots|a_k|a_{i+1}|\ldots)$, all a_p, $j\le p\le k$, are interpreted as $a_p\otimes\mathrm{id}_{y_i}\in(\mathcal{A}\otimes\mathcal{B})((x_{p-1},y_i),(x_p,y_i))$. Dually, one defines the morphism of DG cocategories

$$\mathrm{Cobar}(\mathcal{A}\otimes\mathcal{B}) \to \mathrm{Cobar}(\mathcal{A})\otimes\mathrm{Cobar}(\mathcal{B}) \tag{11.3}$$

12. DG category $\mathbf{C}^\bullet(\mathcal{A},\mathcal{B})$

For two DG categories $\mathcal{A}$ and $\mathcal{B}$, define the DG category $\mathbf{C}^\bullet(\mathcal{A},\mathcal{B})$ as follows. Its objects are A_∞ functors $f:\mathcal{A}\to\mathcal{B}$. Define the complex of morphisms as

$$\mathbf{C}^\bullet(\mathcal{A},\mathcal{B})(f,g) = C^\bullet(\mathcal{A},{}_f\mathcal{B}_g) \tag{12.1}$$

where ${}_f\mathcal{B}_g$ is the complex $\mathcal{B}$ viewed as an $\mathcal{A}_\infty$ bimodule on which $\mathcal{A}$ acts on the left via f and on the right via g. The composition is defined by the cup product as in the formula (**??**).

REMARK 12.0.1. Every $\mathcal{A}_\infty$ functor $f:\mathcal{A}\to\mathcal{B}$ defines an A_∞ $(\mathcal{A},\mathcal{B})$-bimodule ${}_f\mathcal{B}$, namely the family of complexes $\mathcal{B}$ on which $\mathcal{A}$ acts on the left via f and $\mathcal{B}$ on the right in the standard way. If for example f, $g:\mathcal{A}\to\mathcal{B}$ are morphisms of algebras then $C^\bullet(\mathcal{A},{}_f\mathcal{B}_g)$ computes $\mathrm{Ext}^\bullet_{\mathcal{A}\otimes\mathcal{B}^{\mathrm{op}}}({}_f\mathcal{B},{}_g\mathcal{B})$. What we are going to construct below does not seem to extend literally to all (A_∞) bimodules. This applies also to related constructions of the category of internal homomorphisms, such as in [**?**] and [**?**]. One can overcome this by replacing $\mathcal{A}$ by the category of A-modules, since every $(\mathcal{A},\mathcal{B})$-bimodule defines a functor between the categories of modules.

Now let us explain how to modify the product $\bullet$ from 8.2 and get a DG functor

$$\bullet : \mathrm{Bar}(\mathbf{C}^\bullet(A,B)) \otimes \mathrm{Bar}(\mathbf{C}^\bullet(B,C)) \to \mathrm{Bar}(\mathbf{C}^\bullet(A,C)) \tag{12.2}$$

12.1. The brace operations on $\mathbf{C}^\bullet(A,B)$. For Hochschild cochains $D \in C^\bullet(B,_{f_0} C_{f_1})$ and $E_i \in \mathbf{C}^\bullet(A,_{g_{i-1}} B_{g_i})$, $1 \le i \le n$, define the cochain

$$D\{E_1,\ldots,E_n\} \in C^\bullet(A,_{f_0 g_0} C_{f_1 g_n})$$

by

$$D\{E_1,\ldots,E_n\}(a_1,\ldots,a_N) = \sum \pm D(\ldots,E_1(\underline{\ldots}),\ldots,E_n(\underline{\ldots}),\ldots) \tag{12.3}$$

where the space denoted by $\underline{\ldots}$ within $E_k(\underline{\ldots})$ stands for $a_{i_k+1},\ldots,a_{j_k}$, and the space denoted by $\ldots$ between $E_k(\underline{\ldots})$ and $E_{k+1}(\underline{\ldots})$ stands for

$$g_k(a_{j_k+1},\ldots,), g_k(\ldots),\ldots,g_k(\ldots,a_{i_{k+1}}).$$

The sum is taken over all possible combinations such that $i_k \le j_k \le i_{k+1}$. The signs are as in (**??**).

12.2. The $\bullet$ product on $\mathrm{Bar}(\mathbf{C}(A,B))$. For Hochschild cochains $D_i \in C^\bullet(B,_{f_{i-1}} C_{f_i})$ and $E_j \in C^\bullet(A,_{g_{j-1}} B_{g_j})$, $1 \le i \le m$, $1 \le j \le n$, we have

$$(D_1|\ldots|D_m) \in \mathrm{Bar}(\mathbf{C}^\bullet(B,C))(f_0,f_m);$$

$$(D_1|\ldots|D_m) \in \mathrm{Bar}(\mathbf{C}^\bullet(A,B))(g_0,g_m);$$

define

$$(D_1|\ldots|D_m) \bullet (E_1|\ldots|E_n) \in \mathrm{Bar}(\mathbf{C}^\bullet(A,C))(f_0 g_0, f_m g_n)$$

by the formula in the beginning of 8.2, with the following modification. The expression $D_i\{E_{j+1},\ldots,E_k\}$ is now in $\mathbf{C}(A,C)(f_{i-1}g_{j+1}, f_i g_j)$, as explained above. The space denoted by $\ldots$ between $D_i\{E_{j+1},\ldots,E_k\}$ and $D_{i+1}\{E_{p+1},\ldots,E_q\}$ contains $f_i(E_{k+1}|\ldots)|f_i(\ldots)|\ldots|f_i(\ldots,E_p)$. Here, for an A_∞ functor f and for cochains $E_1,\ldots,E_k$,

$$f(E_1,\ldots,E_k)(a_1,\ldots,a_N) = \sum f(E_1(a_1,\ldots,a_{i_2-1}),\ldots,E_k(a_{i_k+1},\ldots,a_n)) \tag{12.4}$$

The sum is taken over all possible combinations $1 = i_1 \le i_2 \le \ldots i_k$.

LEMMA 12.2.1. *1) The product $\bullet$ is associative.*
2) It is a morphism of DG cocategories. In other words, one has

$$\Delta \circ \bullet = (\bullet_{13} \otimes \bullet_{24}) \circ (\Delta \otimes \Delta)$$

as morphisms

$$\mathrm{Bar}(C^\bullet(A,B))(f_0,f_1) \otimes \mathrm{Bar}(C^\bullet(B,C))(g_0,g_1) \to$$

$$\mathrm{Bar}(C^\bullet(A,C))(f_0 g_0, fg) \otimes \mathrm{Bar}(C^\bullet(A,C))(fg, f_1 g_1)$$

12.3. Internal $\underline{\mathrm{Hom}}$ of DG cocategories. Following the exposition of [**?**], we explain the construction of Keller, Lyubashenko, Manzyuk, Kontsevich and Soibelman. For two k-modules V and W, let $\mathrm{Hom}(V, W)$ be the set of homomorphisms from V to W, and let $\underline{\mathrm{Hom}}(V, W)$ be the same set viewed as a k-module. The two satisfy the property

$$\mathrm{Hom}(U \otimes V, W) \xrightarrow{\sim} \mathrm{Hom}(U, \underline{\mathrm{Hom}}(V, W)). \tag{12.5}$$

In other words, $\underline{\mathrm{Hom}}(V, W)$ is the internal object of morphisms in the symmetric monoidal category $k-\mathrm{mod}$. The above equation automatically implies the existence of an associative morphism

$$\underline{\mathrm{Hom}}(U, V) \otimes \underline{\mathrm{Hom}}(V, W) \to \underline{\mathrm{Hom}}(U, W) \tag{12.6}$$

If we replace the category of modules by the category of algebras, there is not much chance of constructing anything like the internal object of morphisms. However, if we replace $k - \mathrm{mod}$ by the category of coalgebras, the prospects are much better. For our applications, it is better to consider counital coaugmented coalgebras. In this category, objects $\underline{\mathrm{Hom}}$ do not exist because the equation (12.5) does not agree with coaugmentations. However, as explained in [**?**], the following is true.

PROPOSITION 12.3.1. *The category of coaugmented counital conilpotent cocategories admits internal* $\underline{\mathrm{Hom}}$*s. For two DG categories* A *and* B, *one has*

$$\underline{\mathrm{Hom}}(\mathrm{Bar}(A), \mathrm{Bar}(B)) = \mathrm{Bar}(\mathbf{C}(A, B)) \tag{12.7}$$

****Expand? Delete? Modify?*****Look up in Faonte?***

13. Homotopy and homotopy equivalence for DG categories

DEFINITION 13.0.1. *For every set* X *let* $\mathbf{I}(X)$ *be the DG category defined by* $\mathrm{Ob}(\mathbf{I}(X)) = X$, $\mathbf{I}(x, y) = kf_{xy}$ *for any* $x, y \in X$, $f_{xy}f_{yz} = f_{xz}$, *and* $d = 0$. *Let* $\mathbf{I}_2 = \mathbf{I}(\{0, 1\})$.

DEFINITION 13.0.2. *Denote by* $\mathcal{I}_2$ *the DG category with* $\mathrm{Ob}(\mathcal{I}_2) = \{0, 1\}$, *freely generated by morphisms* $f_{xy}^{(n)}$, $x, y = 0, 1$, *for all nonnegative even* n *when* $x \neq y$ *and for all nonnegative odd* n *when* $x = y$. *We set*

$$|f_{xy}^{(n)}| = -n;$$

$$df_{xy}^{(n)} = \sum_{j+k=n-1} \sum_{z=0,1} (-1)^j f_{xz}^{(j)} f_{zy}^{(k)} - \delta_n^1 \mathbf{1}_x$$

LEMMA 13.0.3. *Define the DG functor* $\mathcal{K} \to \mathbf{I}_2$ *which is identity on objects by the following action on morphisms:* $f_{xy}^{(0)} \mapsto f_{xy}$, $x \neq y$, *and* $f_{xy}^{(0)} \mapsto 0$ *if* $n > 0$. *This DG functor is a quasi-isomorphism.*

In other words, $\mathcal{I}_2$ is a semi-free resolution of $\mathbf{I}_2$.

PROOF. Consider the filtration by the number of factors $f^{(n)}$ in a monomial. The corresponding spectral sequence shows that it is enough to prove the statement for the associated graded, i.e. with the same differential without the last term $\delta_n^1 \mathbf{1}_x$. This algebra is the cobar construction of the DG category with two objects x and y and two morphisms of degree-1 $\xi : x \to y$ and $\eta : y \to x$ subject to $\xi\eta = 0$; $\eta\xi = 0$. ***FINISH*** □

13.1. Strong equivalence of DG categories. Recall the DG category $\mathcal{I}_2$ from Definition 13.0.2.

DEFINITION 13.1.1. *Two objects* x *and* y *of a DG category* $\mathcal{A}$ *are strongly equivalent if there is a DG functor* $\mathcal{I}_2 \to \mathcal{A}$ *sending the object* 0 *to* x *and the object* 1 *to* y.

LEMMA 13.1.2. *Being strongly equivalent is an equivalence relation.*

PROOF. There is a semi-free resolution $\mathcal{I}_3$ of $\mathbf{I}(\{0,1,2\})$ together with $\mathcal{I}_2 \xrightarrow{i_{02}} \mathcal{I}_3$, etc., with ...***** □

REMARK 13.1.3. The above definition coincides with the of homotopic morphisms of DG algebras in 9, with the only difference that the latter assumes that the $n = 0$ component of the zero cochain defining the equivalence is $1 \in B^0$.

DEFINITION 13.1.4. *Two* A_∞ *functors* $f, g\colon \mathcal{A} \to \mathcal{B}$ *between two DG categories are strongly equivalent if they are strongly equivalent as objects of the DG category* $\mathbf{C}(\mathcal{A}, \mathcal{B})$.

DEFINITION 13.1.5. *Two DG categories* $\mathcal{A}$ *and* $\mathcal{B}$ *are strongly equivalent if there are* A_∞ *functors* $f\colon \mathcal{A} \to \mathcal{B}$ *and* $g\colon \mathcal{B} \to \mathcal{A}$ *such that* gf *is strongly equivalent to* $\mathrm{id}_{\mathcal{A}}$ *and* fg *is strongly equivalent to* $\mathrm{id}_{\mathcal{B}}$.

13.2. DG quotients and localization. Drinfeld quotient is sometime called localization. Here we explain why.

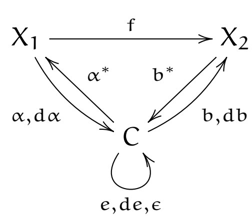
Let $\mathcal{A}$ be any category of complexes (or a DG category equal to its triangulated hull)***REF***, consider a morphism $X_1 \xrightarrow{f} X_2$ and assume that $\mathrm{Cone}(f)$ lies in a full DG subcategory $\mathcal{C}$.

LEMMA 13.2.1. *The morphism* f *is a strong equivalence in* $\mathcal{A}/\mathcal{C}$.

PROOF. Consider the DG category $\mathcal{A}_0$ with two objects X_1 and X_2 and with

$$\mathcal{A}_0(X_1, X_1) = k\mathbf{1}_{X_1},\ \mathcal{A}_0(X_2, X_2) = k\mathbf{1}_{X_2};\ \mathcal{A}_0(X_1, X_2) = kf,$$

where $df = 0$. Let cA be the full subcategory of the triangulated hull of $\mathcal{A}_0$ generated by objects X_1, X_2, and $C = \mathrm{Cone}(f)$. Then the quotient $\mathcal{A}/\mathcal{C}$ is generated by the following morphisms.



Here

$$|e| = |b| = |b^*| = 0; |a| = -1;\ |a^*| = 1; |\epsilon| = -1.$$

Relations are as follows (recall our convention of writing the composition $x \xrightarrow{f} x \xrightarrow{g} z$ as fg):

$$db^* = 0;\ da^* = 0;\ d\epsilon = \mathbf{1}_C;$$

$$a^*a = e;\ b^*b = \mathbf{1}_{X_2}; bb^* = \mathbf{1}_C - e;\ eb = 0;\ edb = db;\ b^*e = 0$$

$$e^2 = e;\ ede = de = de(\mathbf{1}_C - e);\ ae = a;\ dae = 0; aa^* = \mathbf{1}_{X_1};\ ea^* = a^*;\ b^*a^* = 0$$

$$\alpha b = 0;\ d\alpha b = f = -\alpha db;\ fb^* = d\alpha$$

Set

$$\mathcal{P}(X_1, C) = k\alpha + kd\alpha;\ \mathcal{P}(C, X_1) = k\alpha^*;\ \mathcal{P}(X_2, C) = kb^*;\ \mathcal{P}(C, X_2) = kb + kdb;$$

$$\mathcal{P}(C, C) = ke + kde\mathbf{1}_C.$$

We see that there is a short exact sequence

$$0 \to \mathcal{A}_0(X_i, X_j) \to (\mathcal{A}/\mathcal{C})(X_i, X_j) \xrightarrow{\sim} \bigoplus_{n=0}^{\infty} \mathcal{P}(X_i, C)\epsilon(\mathcal{P}(C, C)\epsilon)^{\otimes n}\mathcal{P}(C, X_j) \to 0$$

***[that is when we assume $\epsilon^2 = 0$-check that we do]**. Direct sums of the first n factors, $n \geq 0$, form an increasing filtration of the term on the right in the sequence. For all three cases except $i = 2,\ j = 1$, the associated graded quotients of the filtration are all acyclic (because $\mathcal{P}(X_1, C)$ and $\mathcal{P}(C, X_2)$ are contractible. When $i = 2$ and $j = 1$, all graded factors with $n \geq 1$ are acyclic because $\mathcal{P}(C, C)$ is contractible. We see that $(\mathcal{A}/\mathcal{C})(X_2, X_1)$ is quasi-isomorphic to kg where

$$g = b^*\epsilon\alpha^*. \tag{13.1}$$

The full subcategory of $\mathcal{A}/\mathcal{C}$ generated by X_1 and X_2 is therefore quasi-isomorphic to I_2 and admits a quasi-isomorphism from $\mathcal{I}_2$. □

13.3. Comparison to other notions of equivalence of DG categories. . *******************

We would like to *briefly indicate* relations to:

1) homotopy between DG functors, according to the model structures on DG categories (Tabuada) and A_∞ categories (Lefevre-Hasegawa).

2) the ∞-category structure on DG categories (Toen, Faonte). Also: a bit more about relating this to $\mathbf{C}(\mathcal{A}, \mathcal{B})$ (Faonte).

13.4. Sort of appendix, not sure if needed. We call a DG functor $F : \mathcal{A} \to \mathcal{B}$ *strong quasi-equivalence* if

1) $F : \mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ is a quasi-isomorphism;

2) every object of $\mathcal{B}$ is strongly equivalent to Fx for some object x of $\mathcal{B}$.

Proposition 13.4.1. *Assume that* k *is a field. Then a strong quasi-equivalence is an* A_∞ *homotopy equivalence.*

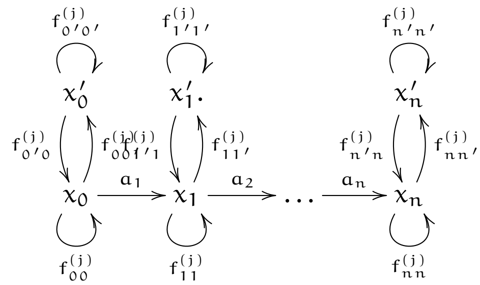
Proof. It is enough to prove the following two statements.

1) Let $F : \mathcal{A} \to \mathcal{B}$ be a DG functor which is a bijection on objects. Assume that $\mathcal{A}(x, y) \to \mathcal{B}(Fx, Fy)$ is a quasi-isomorphism for any x, y in $\mathrm{Ob}(\mathcal{A})$. Then $F_* : C_\bullet(\mathcal{A}) \to C_\bullet(\mathcal{B})$ is a quasi-isomorphism.

2) Let $F : \mathcal{A} \to \mathcal{B}$ be an embedding of a full DG subcategory $\mathcal{A}$ into $\mathcal{B}$. Assume that every object of $\mathcal{A}$ is strongly equivalent to some object of $\mathcal{B}$. Then F is an A_∞ homotopy equivalence.

Statement 1) is proved exactly as for DG algebras. Let us prove 2). For every object x of $\mathcal{B}$ choose an object x' of $\mathcal{A}$ together with a functor $F_x : \mathcal{I}_2 \to \mathcal{B}$ sending 0 to x' and 1 to x. If x is an object of $\mathcal{A}$ then we choose $x' = x$ and F_x as the trivial functor sending both 0 and 1 to x, all morphisms of degree zero to $\mathbf{1}_x$, and the rest to zero We will construct the A_∞ functor G such that $\mathrm{id}_{\mathcal{A}} = GF$, as well as a homotopy between $\mathrm{id}_{\mathcal{B}}$ and FG, on any objects $x_0, \ldots, x_n$ of $\mathcal{B}$ by induction in

n. We will denote the objects in the source category $\mathcal{I}_2$ of F_{x_j} by j' and j instead of 0 and 1.



More precisely, we define $Gx = x'$; on morphisms, we put

$$G(a_1, \dots, a_n) = f^{(0)}_{0'0} a_0 f^{(1)}_{11} a_2 \dots f^{(1)}_{(n-1)(n-1)} a_n f^{(0)}_{nn'} \in \mathcal{B}^{1-n}(x'_0, x'_n) \tag{13.2}$$

We look for all the components in the homotopies in the form

$$\varphi(a_1, \dots, a_n) = \gamma_0 a_0 \gamma_1 a_1 \dots \gamma_{n-1} a_n \gamma_n \tag{13.3}$$

where γ_j are images of morphisms from $\mathcal{I}_2$ under F_j. On each step we get an element in the tensor power of the k-module of morphisms in $\mathcal{I}_2$ whose differential is equal to zero, and we have to find its primitive φ. Since $\mathcal{I}_2$ has no cohomology in nonzero degrees, this is always possible. $\square$

CHAPTER 17

Frobenius algebras, CY DG categories

1. Introduction

We start with a study of the algebraic structure on Hochschild and cyclic complexes of Frobenius algebras, following Tradler-Zeinallian, Kaufmann, Ward, and other works. Frobenius algebras are first examples of noncommutative analogues of functions on a manifold with a volume form, and one might expect similar algebraic operations, namely, an analogue of the divergence operator on multivector fields. Such a structure does indeed exist; it is the homotopy BV structure defined by the dual of the cyclic differential B. After establishing this, we study the Tate Hochschild complex of a Frobenius algebra following Rivera and Wang. We show that the homotopy BV structure extends to it. We also study other operations, such as the Goresky-Hingston coproduct and the Lie bialgebra structure. Following Wang and Rivera-Wang, we also discuss other versions of a Tate-style Hochschild complex: one using Hochschild cochains with values in noncommutative forms, the other defined in terms of the singularity category of bimodules. Unfortunately what we present here is far from a complete treatment.

The case of Frobenius algebras being rather restrictive, we turn to a more general situation. Following Kontsevich and Brav-Dyckerhof, we define smooth and proper DG algebras and categories; we then define a left CY category which is a special case of a smooth one, and a right CY category which is a special case of a proper one. But first we introduce pre-CY algebras and categories as defined by Iyudu, Kontsevich, and Vlassopoulos. We follow in part their work and in part Waikit Yeung's.

As we explain below, this is a peculiar situation because we are dealing with a noncommutative analogue of a volume form which is also a noncommutative analogue of a symplectic form. The reason is that, classically, the latter gives an isomorphism between vector fields and one forms, and the former gives an isomorphism between *all* multivector fields and *all* forms; but in noncommutative geometry there is no good way to isolate one-forms from all forms. Within this framework a pre-CY structure is an analogue of a (shifted) Poisson structure. We also give examples of 3D CY algebras following Ginzburg.

A pre-CY structure is defined in terms of Hochschild cochain complexes

$$C^\bullet(A^{\otimes k}, {}_\alpha A^{\otimes k})$$

where α is the automorphism of the kth tensor power of A given by the cyclic permutation of tensor factors. Note that elsewhere in this book we study an algebraic structure on Hochschild *chain* complexes of the same form; an important part of the structure is that all of those are quasi-isomorphic to the Hochschild chain complex of A. (The name Frobenius plays a crucial role in both places, probably by coincidence). It would be interesting to understand a common structure on

Hochschild chains and cochains, "animated" in the following way. Given algebras A_j, B_j, $1 \leq j \leq n$, and morphisms $f_j : A_j \to B_{j+1}$, $g_j : A_j \to B_{j-1}$ (where for us $0 = n$ and $n+1 = 1$), we can form chain and cochain Hochschild complexes

$$C(A_1 \otimes \ldots \otimes A_n, B_1 \otimes \ldots \otimes B_n)$$

(and some complexes that are part chains and part cochains). For $n = 1$ these complexes can be organized into a 2-category up to homotopy, plus an additional structure when chains are involved. A general structure combining this one with the Kontsevich-Vlassopoulos necklace bracket on Hochschild cochains and with the cyclotomic structure/Frobenius/Cartier morphism on Hochschild chains might be interesting to know. A related question is: what is a full algebraic structure on (higher) Hochschild and cyclic complexes of a (pre-)CY algebra.

2. Frobenius algebras

DEFINITION 2.0.1. *Fix an integer* d. *A Frobenius algebra of degree* d *is a finite dimensional graded algebra over a field* k *together with a nondegenerate scalar product* $\langle\, , \,\rangle$ *of degree* d *on* A *such that* $\langle a, bc\rangle$ *is cyclically invariant, namely*

$$\langle a, bc\rangle = (-1)^{|c|(|a|+|b|)}\langle c, ab\rangle$$

for any homogeneous a, b, c *in* A.

Let A^* be the graded vector space dual to A. The scalar product defines an isomorphism

$$A \xrightarrow{\sim} A^*[d] \tag{2.1}$$

2.1. Cyclic A_∞ algebras.

A cyclic A_∞ algebra of degree $-d$ is a finite dimensional A_∞ algebra over a field k together with a nondegenerate scalar product $\langle\, , \,\rangle$ of degree d on A such that $\langle a_0, m_n(a_1, \ldots, a_n)\rangle$ is cyclically invarint.

3. The Hochschild complex of a Frobenius algebra

The Hochschild cochain complex of a Frobenius algebra can be identified with

$$C^\bullet(A, A) \xrightarrow{\sim} \prod_{n\geq 0} A^*[d] \otimes A^*[-1]^{\otimes n} \tag{3.1}$$

Same for a cyclic A_∞ algebra.

3.1. Operations indexed by black/white ribbon graphs with spines and roots.

A ribbon graph is a finite graph with a choice of a cyclic order on edges incident to every vertex. A black/white (b/w) ribbon graph is a ribbon graph whose vertices are colored in two colors and whose black vertices are all of degree ≥ 3.

For any ribbon graph, its *blow-up graph* is constructed as follows. For any vertex, define the triangulation of a circle whose vertices are indexed by edges incident to this vertex, in their cyclic order. Then the blow-up graph is the union of: a) segments indexed by edges; circles indexed by vertices. For any segment representing an edge, take the two circles corresponding to the two vertices incident to that edge. We identify the two endpoints of every segment with the two corresponding 0-simplices of the triangulation, one for each circle.

The cycles of the blow-up graph that correspond to white vertices are called *inner cycles*.

Now define *outer cycles*. There are two intuitive ways to see what they are. One way is: thicken the vertices and the edges; then the graph becomes a surface with some discs deleted. Outer cycles are boundaries of these discs. Another way is: outer cycles are "minimal" non-inner cycles of the blow-up graph.

More precisely: Define a flag as a pair (v, e) where v is a vertex and e an edge incident to v. Define:

$$\tau(v, e) = (v, e')$$

where e' is the edge that is next to e in the cyclic order;

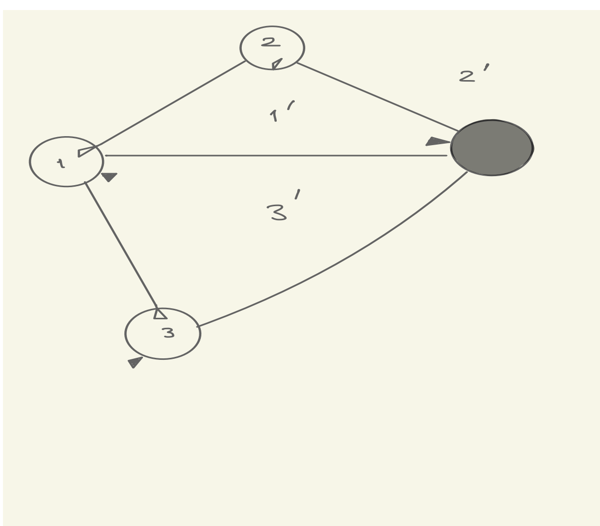
$$\iota(v, e) = (w, e)$$

where w is the other vertex incident to e. Define outer cycles as orbits of the automorphism $\iota\tau$.

A spine is a distinguished point on an inner cycle. We do not distinguish between two points lying on the same open cell of the triangulation of the circle. In other words: a spine is a simplex in this triangulation. It could be of dimension zero or one.

A root is a distinguished point on an outer cycle. More precisely: on the blow-up graph, consider an edge of our outer cycle that lies on some inner cycle (i.e. an edge that comes not from an edge of the original graph). A root is an inner point of this edge. If this edge does not contain a spine, we do not distinguish between any two such points. If the edge does contain a spine, there are three scenarios: the root may coincide with the spine, or lie on either side of it. For each of the three scenarios, we do not distinguish between any two points that realize it.

Definition 3.1.1. *A b/w ribbon graph with spines and roots is a b/w ribbon graph with a spine on every inner cycle and a root on every outer cycle.*



3.1.1. *Operations defined by graphs.* For any b/w ribbon graph with spines and roots, define a linear map

$$C^\bullet(A, A)^{\otimes(\text{Inner cycles})} \to C^\bullet(A, A)^{\otimes(\text{Outer cycles})}$$

1) For every j, consider a Hochschild cochain

$$\alpha^{(j)} = \alpha_0^{(j)} \otimes \ldots \otimes \alpha_{n_j}^{(j)} \tag{3.2}$$

where $\alpha_k^{(j)}$ are elements of A^*. Consider all ways to put the $\alpha_k^{(j)}$ in $n_j + 1$ different points of the jth inner cycle, so that the cyclic order is preserved. Some $\alpha_k^{(j)}$ will fall on special points of the inner cycles, others will not. Here, by special points we mean: vertices, spines, and roots. (A special point may be more than one of those at the same time). We do not distinguish between two results for which same $\alpha_k^{(j)}$ fall on same special points. We require that $\alpha_0^{(j)}$ always fall on the spine.

2) Now consider the cycles of the blow-up graph that correspond to black vertices. Do exactly the same as for white vertices, but use the cochain $m = \sum_{n\geq 2} m_n$ that defines the A_∞ structure. Since m_n are cyclically invariant, it does not matter where to put the spine.

The output of the operation is the sum (with signs) over all such configurations. For every summand:

a) Assume $\alpha_0^{(j)}$ falls on an isolated spine. (A spine is isolated if it lies on an open cell of the circle, and a root does not fall on the same point). Then remove $\alpha_0^{(j)}$ and introduce the factor $\alpha_0^{(j)}(1)$.

b) Look at any edge of the blow-up graph that comes from the original graph. Let α and β be two elements of A^* that fall on its endpoints. Remove them and introduce the factor $\langle\alpha, \beta\rangle$.

c) Now look at any outer cycle. Write the elements of A^* remaining on it in their cyclic order, starting with the element marked with the root. We get an element of $C^\bullet(A, A)$ for every outer cycle. Consider their tensor product, multiplied by the product of factors that we introduced in a) and b)). This is the summand corresponding to a particular configuration (and to a choice of a monomial of m_n for every black vertex).

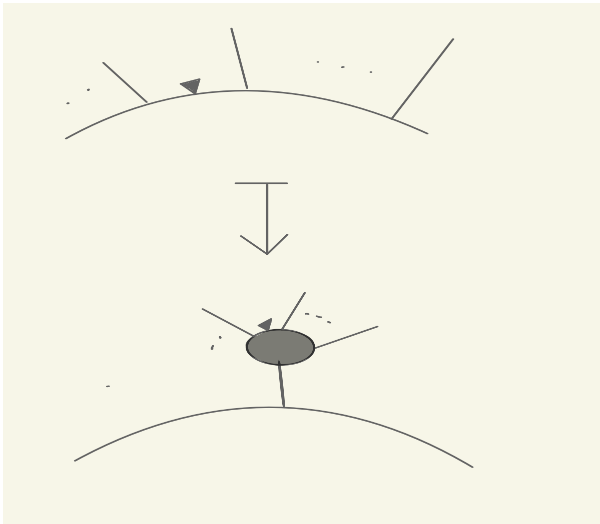
For a ribbon graph with spine and roots, recall that *special points* of inner cycles are: vertices, spines, and roots.

DEFINITION 3.1.2. *An inner edge is a segment between two neighboring special points on an inner cycle.*

LEMMA 3.1.3. *The degree of the operation defined by a b/w ribbon graph with spines and roots is *** ***contribution of an inner cycle:* 1 *- number of inner edges, ... *** contribution of a black vertex: ...*

LEMMA 3.1.4. *The linear span of operations defined by b/w ribbon graphs with spines and roots is closed under the Hochschild differential and under compositions.*

Explicitly: the differential acts by (***Explain***), plus contribution of black cycles just like in A_∞...



3.2. Sullivan chord diagrams. Sullivan chord diagrams are another way to describe b/w ribbon graphs with spines and roots for *associative* Frobenius algebras.

Let Γ be a ribbon graph. Consider a circle S^1_ν for any vertex ν of Γ and a segment I_e for any edge e of Γ. If e has endpoints ν_1 and ν_2, label one endpoint of I_e by ν_1 and the other by ν_2. Let P_ν be the set of endpoints labeled by ν in the disjoint union of I_e over all edges of Γ.

A Sullivan chord diagram is defined by:

i) A ribbon graph Γ;

ii) a triangulation of the circle S^1_ν for any vertex ν of Γ;

iii) for any vertex ν of Γ, a surjective map from the set P_ν to the set of vertices of the triangulation of S_ν that preserves the cyclic order.

The chord diagram defined by these data is the union of:

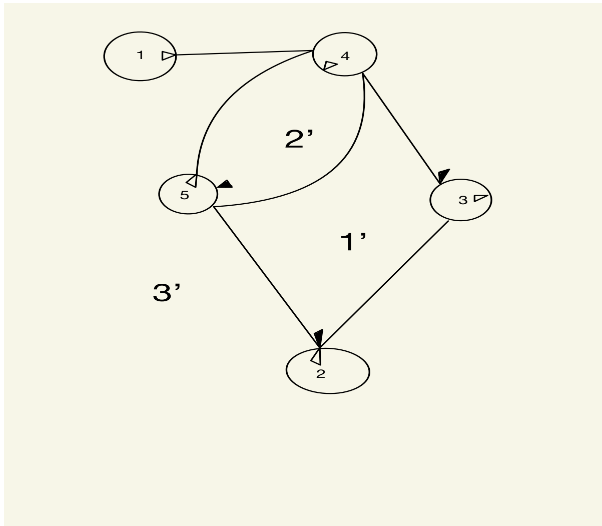
a) disjoint union of circles S^1_ν;

b) disjoint union of segments I_e,

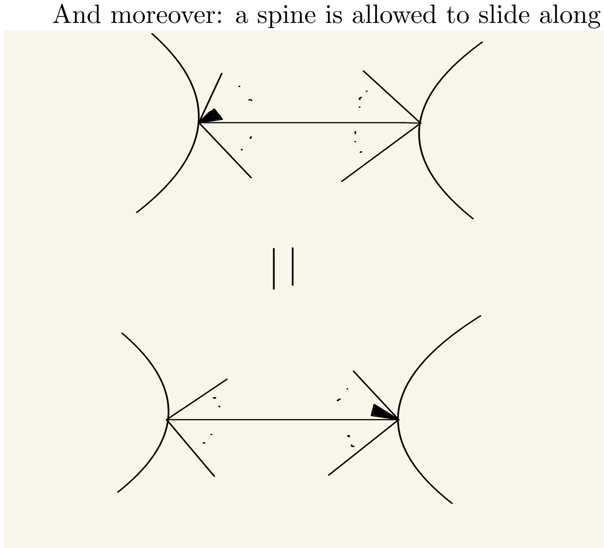
glued according to the maps in iii).

In other words: a Sullivan chord diagram is the same as the blow-up graph of a ribbon graph, but some vertices on inner cycles are allowed to coincide.

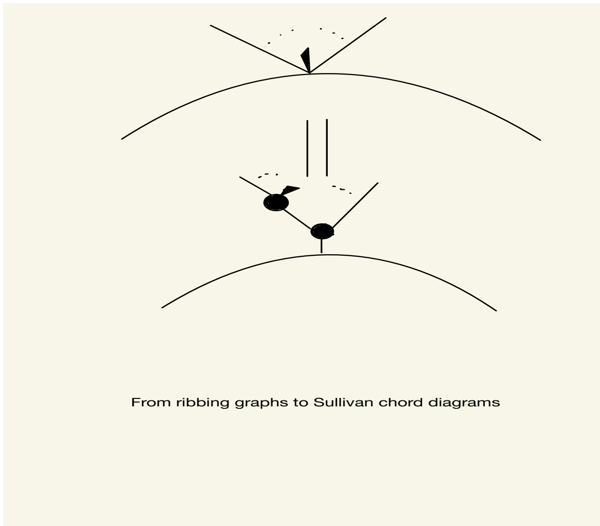
A spine on a Sullivan chord diagram is defined the same way as a spine on a ribbon graph. A root of an outer cycle is a marked point on it. It has to belong to some inner cycle (and therefore cannot be an inner point of an edge of the original graph). We do not distinguish between two such point if they have the same position with respect to all special points of the inner cycle. Furthermore: if the spine falls on a vertex of the triangulation of S^1_ν, we have to specify the two neighboring edges between which it falls. (And: at least one of these edges must come from an edge of Γ).



And moreover: a spine is allowed to slide along an edge of Γ :



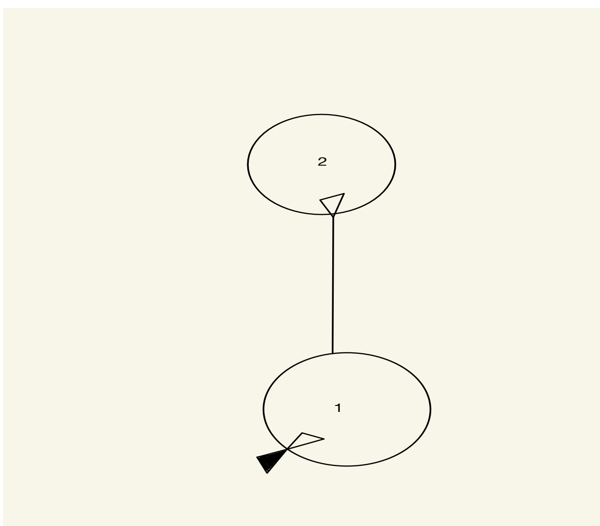
The picture below explains how to pass from a Sullivan chord diagram with spines and roots to a b/w ribbon graph with spines and roots.



3.3. Examples of operations.

3.3.1. *Braces.*

LEMMA 3.3.1. *The brace operation* $D\{E\}$ *is given by the graph*



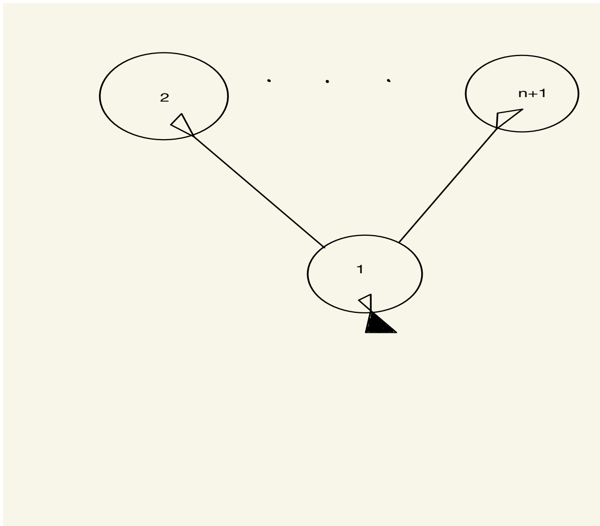
Indeed, by definition, the operation given by this graph is

$$(\alpha_0^{(1)} \otimes \dots \alpha_{n_1}^{(1)}) \otimes (\alpha_0^{(2)} \otimes \dots \alpha_{n_1}^{(2)}) \mapsto$$

$$\sum \pm \langle \alpha_k^{(1)}, \alpha_0^{(2)} \rangle (\alpha_0^{(1)} \otimes \dots \alpha_{k-1}^{(1)} \otimes \alpha_1^{(2)} \otimes \dots \alpha_{n_2}^{(2)} \otimes \alpha_{k+1}^{(1)} \otimes \dots \alpha_{n_1}^{(1)})$$

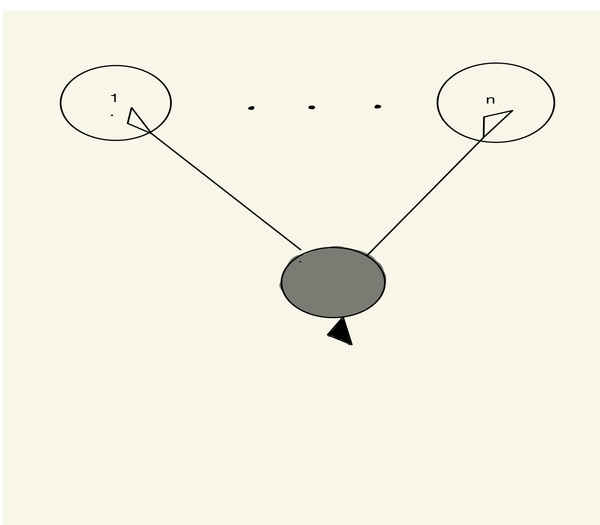
More generally:

LEMMA 3.3.2. *The brace operation* $D_1\{D_2, \dots, D_{n+1}\}$ *is given by the graph*

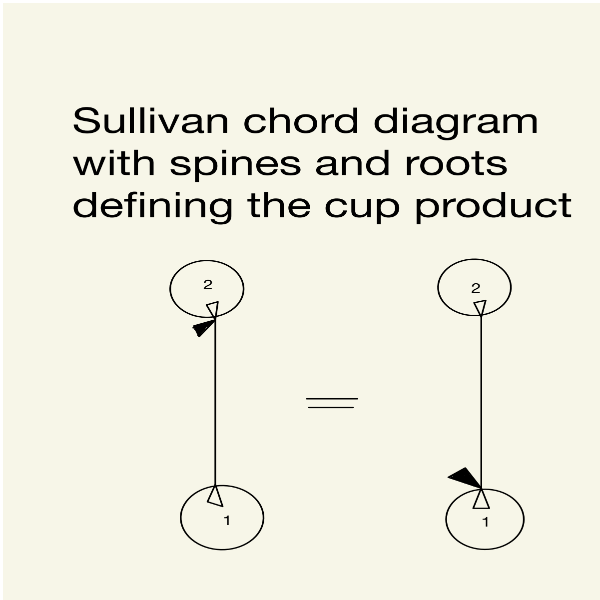


3.3.2. *Cup product.*

LEMMA 3.3.3. *The* n*-ary* A_∞ *cup product* $m\{D_1, \ldots, D_n\}$ *is given by the graph*

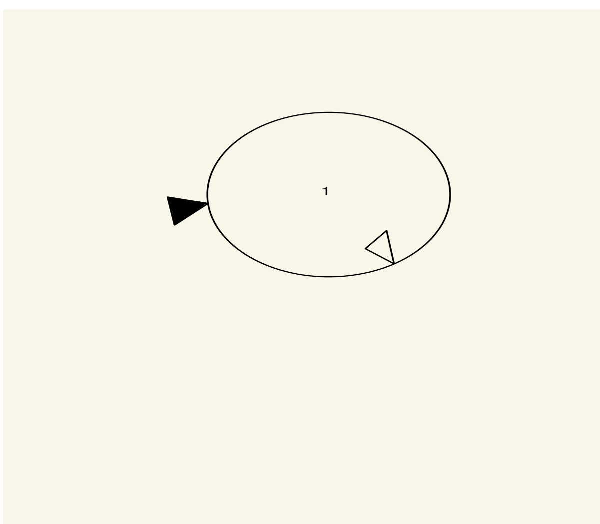


The Sullivan chord diagram corresponding to the binary cup product is



3.3.3. *The cyclic differential* B.

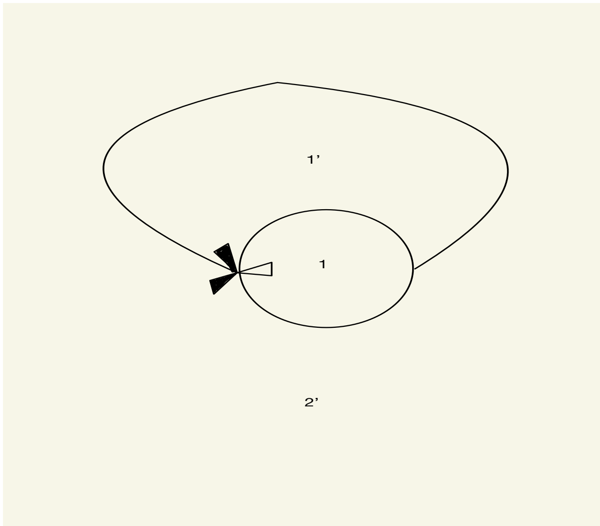
LEMMA 3.3.4. *The dual of the cyclic differential* B *is given by the graph*



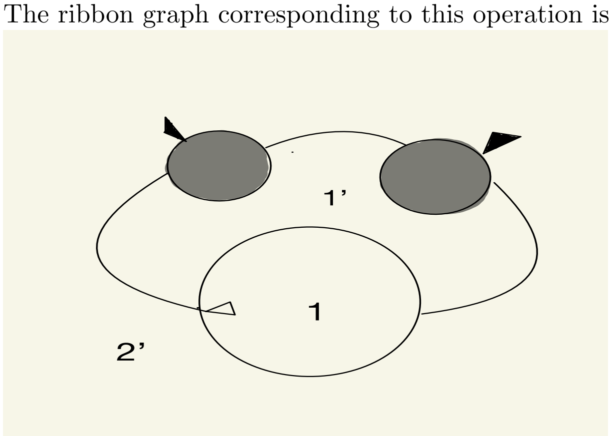
Indeed, the operation defined by this graph acts by

$$(\alpha_0 \otimes \dots \alpha_n) \mapsto \sum \pm \langle \alpha_0, 1 \rangle (\alpha_k \otimes \dots \otimes \widehat{\alpha_0} \otimes \dots \otimes \alpha_{k-1})$$

3.3.4. *The Goresky-Hingston coproduct.* The Goresky-Hingston coproduct is the operation defined by the following Sullivan chord diagram.



The ribbon graph corresponding to this operation is



Ribbon graph with spines and roots for the Goresky-Hingston coproduct

The formula for this operation is

$$(\alpha_0 \otimes \dots \otimes \alpha_n) \mapsto \sum \pm \langle \alpha_0^{[2]}, \alpha_k \rangle (\alpha_0^{[3]} \otimes \alpha_1 \otimes \dots \alpha_{k-1}) \otimes (\alpha_0^{[1]} \otimes \alpha_{k+1} \otimes \alpha_n)$$

4. The cyclic complex of a Frobenius algebra

4.1. The cyclic cochain complex. We assume that k is a field of characteristic zero. There are two isomorphic cyclic cochain complexes of a (finite dimensional) k-algebra. One consists of coinvariants of cyclic groups, the other of invariants:

$$C^\bullet_\lambda(A) = (\prod_{n \geq 0} (A^{*\otimes n+1}[-n])_{C_{n+1}}, b') \overset{N}{\underset{\sim}{\to}} (\prod_{n \geq 0} (A^{*\otimes n+1}[-n])^{C_{n+1}}, b) \tag{4.1}$$

Unless otherwise specified, we will be always using the one on the left.

4.2. Operations indexed by black/white ribbon graphs. Given a b/w ribbon graph, we can define an operation

$$C^{\bullet}_{\lambda}(A)^{\otimes \text{Outer Cycles}} \to C^{\bullet}_{\lambda}(A)^{\otimes \text{Outer Cycles}} \tag{4.2}$$

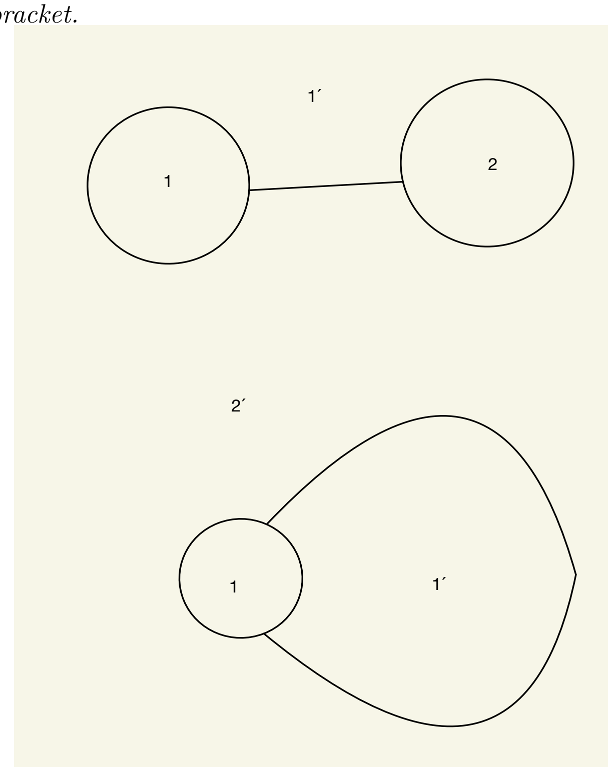
for any Frobenius algebra A (or any cyclic A_∞ algebra). They are defined exactly as in 3.1.1 but without appealing to spines and roots. Namely:

1) Consider *all* ways of putting the $\alpha^{(j)}_k$ on $n_j + 1$ different points of the jth inner cycle for all j so that the cyclic order is preserved. Some of them will fall on vertices of the blow-up graph; others will fall in between. We do not distinguish between any two results for which same $\alpha^{(j)}_k$ fall on the same vertices.

2) For black vertices, do exactly what we did in b) of 3.1.1.

Now do what we did in b) and c) of 3.1.1. We do not have a root on the outer cycle; but, since we are only interested in coinvariants, it is not relevant were we start our tensor product along this cycle. Also, because 1) involves summing over *all* possibilities, the operation is well defined on coinvariants.

4.2.1. *The necklace Lie bracket.*



4.2.2. *The Lie cobracket.*

4.2.3. *The Lie bialgebra structure.*

5. The Tate-Hochschild complex of a Frobenius algebra

For a Frobenius algebra of degree d, we have

(5.1)

$$A \xrightarrow{\sim} A[d]^* \tag{5.2}$$

Under the isomorphism (5.1), the conjugate $A^* \to A^* \otimes A^*$ to the product becomes

$$\Delta : A \to A \otimes A[-d] \tag{5.3}$$

which is a morphism of bimodules if we equip $A \otimes A$ with the inner bimodule structure. Namely, if

$$\Delta(a) = \sum a' \otimes a'' \tag{5.4}$$

then

$$\Delta(xa) = \sum (-1)^{|x||a'|} a' \otimes xa''; \;\; \Delta(ax) = \sum (-1)^{|x||a''|} a'x \otimes a'' \tag{5.5}$$

In particular, $\Delta(1)$ is a central element of $(A \otimes A)^{-d}$ wit respect to the inner bimodule structure.

Explicitly, if e_j and f_j are two bases of A such that

$$e_j \in n_j; \;\; \langle f_i e_j \rangle = \delta_{ij}$$

Then

$$\Delta(1) = \sum_i (-1)^{d(n_i+1)} e_i \otimes f_i = \sum_i (-1)^{d-n_i} f_i \otimes e_i \tag{5.6}$$

Indeed,

$$\langle \Delta(1), f_j \otimes e_k \rangle = \langle 1, f_j e_k \rangle = \langle f_j, e_k \rangle = \delta_{jk};$$

$$\langle \sum_i (-1)^{d(n_i+1)} e_i \otimes f_i, f_j \otimes e_k \rangle = \sum_i (-1)^{d(n_i+1)+(d-n_j)(d-n_k)} \langle e_i, f_j \rangle \langle f_i, e_k \rangle =$$

$$\delta_{jk} (-1)^{d(n_j+1)+d-n_j+n_j(d-n_j)} = \delta_{jk}$$

As a corollary we get

$$\operatorname{tr}(\mu(\Delta(1))) = \sum_n (-1)^n \dim A^{d-n} \tag{5.7}$$

where we write

$$\operatorname{tr}(a) = \langle 1, a \rangle$$

and

$$\mu(a \otimes b) = ab.$$

5.0.1. *The norm operator.* Define

$$N : A \to A[-d]; \; N(a) = \sum_j e_j a f_j \tag{5.8}$$

LEMMA 5.0.1.

$$N([a, b]) = 0 = [a, N(b)]$$

for any a *and* b *in* A.

PROOF. □

We can now define *the Hochschild Tate complex* of a Frobenius algebra A:

(5.9)

$$\ldots \xrightarrow{b} C_1(A, A) \xrightarrow{b} C_0(A, A) \xrightarrow{N} C^0(A, A)[-d] \xrightarrow{\delta} C^1(A, A)[-d] \xrightarrow{\delta} \ldots$$

5.1. Operations on the Hochschild Tate complex.

5.1.1. *The cup product.*

5.1.2. *The Lie bracket.*

5.1.3. *The homotopy BV structure.*

5.1.4. *Other operations.*

5.2. Other definitions.
5.2.1. *The singular Hochschild complex via noncommutative forms.*
5.2.2. *Singular Hochschild cohomology.*

5.3. Comparisons.

6. Pre-CY algebras

6.1. The necklace (Kontsevich-Vlassopoulos) bracket.

6.1.1. *Motivation: The Frobenius double of a finite dimensional algebra.* Given a finite dimensional algebra A over a field k of characteristic zero, let A^* be its linear dual. Fix an integer d. Note that $A \oplus A^*[d-1]$ has the following algebra structure. A is a subalgebra; $A^*[d-1]$ is a two-sided ideal of square zero; the product between A and $A^*[d-1]$ is the bimodule action of A on $A^*[d-1]$ dual to the one defined by the product on A.

This product defines a Frobenius algebra. The inner product is as follows: A and $A^*[d-1]$ are isotropic; the pairing between A and A^* is the obvious one.

Consider the DG Lie bracket of degree $-1-d$ on the (shifted) cyclic complex $C_\lambda^{\bullet-1}(A \oplus A^*[d-1])$ as in 4.2.1. We can identify this complex with the direct product

(6.1)
$$\prod A^*[-1]^{\otimes n}_{C_n} \times \prod_{k\geq 1;\, n_1,\ldots,n_k\geq 0} (A^*[-1]^{\otimes n_1} \otimes A[-d] \otimes \ldots \otimes A^*[-1]^{\otimes n_k} \otimes A[-d])_{C_k}$$

6.1.2. *The higher Hochschild complexes and the KV bracket.* Let us observe that the construction of the DGLA structure on (6.1) can be generalized to any associative algebra A. We note that for a finite dimensional A, (6.1) coincides with

$$C_\lambda^\bullet(A) \times \prod_{k\geq 1} \mathrm{Hom}^\bullet_{A^{\otimes k}\otimes (A^{\otimes k})^{\mathrm{op}}}(\mathcal{B}_\bullet(A) \otimes \ldots \otimes \mathcal{B}_\bullet(A),\ {}_\alpha A^{\otimes k}[-kd])_{C_k} \tag{6.2}$$

Here $\mathcal{B}_\bullet(A)$ is the bimodule bar resolution of A and ${}_\alpha A^{\otimes k}$ is the $A^{\otimes k}$-bimodule twisted by α, the automorphism of the algebra $A^{\otimes k}$ defined by the cyclic permutation of tensor factors.

Definition 6.1.1. *For $k \geq 1$, denote the kth component of* (6.2) *(*before the shift by $-kd$*) by $C^\bullet_{(k)}(A, A)$. We call this complex the kth higher Hochschild complex of A.*

Lemma 6.1.2. *Let A be an algebra over a commutative unital ring k of characteristic zero, free as a k-module. The cohomology of $C^\bullet_{(k)}(A, A)$ is the Hochschild cohomology* $\mathrm{HH}^\bullet(A^{\otimes k}, {}_\alpha A^{\otimes k})$.

Proof. Note that $\mathcal{B}_\bullet(A)\otimes\ldots\otimes\mathcal{B}_\bullet(A)$ is a free resolution of $A^{\otimes k}$. Furthermore, the cyclic group C_k acts trivially on the cohomology. Indeed, ***Finish*** □

Lemma 6.1.3. *The definition of the shifted DGLA from 6.1.1 extends to any algebra, finite dimensional or not. The bracket between $C^\bullet_{(k)}(A, A)$ and $C^\bullet_{(l)}(A, A)$ takes values in $C^\bullet_{(k+l-1)}(A, A)$. The bracket on $C^\bullet_{(1)}(A, A) = C^\bullet(A, A)$ is the Gerstenhaber bracket.*

6.1.3. *Pre-CY structures.*

DEFINITION 6.1.4. *A pre-CY algebra structure on an associative algebra* A *is a Maurer Cartan element in the DGLA*

$$\prod_{k=0}^{\infty} C^{\bullet}_{(k)}(A, A)[-dk + d + 1]$$

*such that .....****

EXAMPLE 6.1.5. Let A be a Frobenius algebra of degree d. Then $A \xrightarrow{\sim} A^*[d]$ and therefore to

$$A^*[d+1] \xrightarrow{\sim} A[1] \tag{6.3}$$

Define the differential on $A \oplus A^*[d+1]$ which is zero on A and given by (6.3) on A^*. This differential turns $A \oplus A^*[d+1]$ as in (**??**) to a cyclic DG algebra.

7. Short higher Hochschild complexes and noncommutative multi-vector fields and forms

7.1. The $\mathfrak{X}$ complex of an algebra. We start with recalling the short bar resolution of an A-bimodule A from (2.7):

$$\mathcal{B}_1^{\mathrm{sh}}(A) = \Omega_A^1 \xrightarrow{\sim} \mathcal{B}_1(A)/\partial\mathcal{B}_2(A) \tag{7.1}$$

(the bimodule of noncommutative forms);

$$\mathcal{B}_1^{\mathrm{sh}} \xrightarrow{\partial} \mathcal{B}_0^{\mathrm{sh}}(A);\ \partial(a_0 da_1 a_2) = a_0 a_1 \otimes a_2 - a_0 \otimes a_1 a_2 \tag{7.2}$$

(we recall that $a_0 da_1 a_2$ corresponds to the class of $a_0 \otimes a_1 \otimes a_2$ modulo $\partial\mathcal{B}_2$). This is denoted by $R^{\min}$ by Iyudu, Kontsevich, and Vlassopoulos in [**?**].

This definition extends to differential graded algebras in the obvious way. The total differential is the sum of ∂ and the differential induced by d_A. When A is a semi-free DGA, the projection $\mathcal{B}_\bullet(A) \to \mathcal{B}_\bullet^{\mathrm{sh}}(A)$ is a quasi-isomorphism.

Now define the dual complex

$$\mathcal{B}_\bullet^{\mathrm{sh}}(A)^\vee = \mathrm{Hom}^\bullet_{A\otimes A^{\mathrm{op}}}(\mathcal{B}_\bullet^{\mathrm{sh}}(A), A \otimes A)$$

The action of $A \otimes A^{\mathrm{op}}$ on $A \otimes A$ that we use in Hom is the inner action. The outer action on $A \otimes A$ induces $\mathcal{B}^\vee$ a A-bimodule structure on $\mathcal{B}_\bullet^{\mathrm{sh}}(A)^\vee$.

For $k \geq 1$, put

$$\mathfrak{X}^{(k)}(A) = ((\mathcal{B}_\bullet^{\mathrm{sh}}(A)^\vee \otimes_A \ldots \otimes_A \mathcal{B}_\bullet^{\mathrm{sh}}(A)^\vee)_{A\otimes A^{\mathrm{op}}} A)_{C_k}[1-k] \tag{7.3}$$

where the number of $\mathcal{B}_\bullet^{\mathrm{sh}}(A)^\vee$ factors is k.

OR, a completed version below. Reconcile with Waikit Y. and I-K-V.

$$\mathfrak{X}^{(k)}(A) = \mathrm{Hom}^\bullet_{A^{\otimes k}\otimes (A^{\otimes k})^{\mathrm{op}}}(\mathcal{B}_\bullet^{\mathrm{sh}}(A) \otimes \ldots \otimes \mathcal{B}_\bullet^{\mathrm{sh}}(A),_\alpha A^{\otimes k})_{C_k}[1-k] \tag{7.4}$$

I-K-V denote this by $\zeta^{(k)}$.

We will denote by δ *the differential which is induced by* ∂.

$$\mathfrak{X}^{(*)}(A, d) = \prod_{k\geq 0} \mathfrak{X}^{(k)}(A)[-kd] \tag{7.5}$$

LEMMA 7.1.1. *For a semifree DG algebra* A*, the embedding of* (7.5) *into* (6.2) *is a quasi-isomorphism.*

We use the following notation. Let A be free as a graded algebra, with generators $x^j, j \in J$. We identify

$$\Omega^1_A \xrightarrow{\sim} \oplus_{j\in J} A dx^j A \tag{7.6}$$

For $a \in A$, under this identification, we write

$$da = \sum_{j\in J} \partial_j^{(1)}(a) dx^j \partial_j^{(2)}(a) \tag{7.7}$$

LEMMA 7.1.2. *Assume that* A *is a semi-free DGA with generators* $x^j, j \in J$. *Then* $\mathfrak{X}^{(*)}(A)$ *is the* completed *quotient by commutators of the complete free algebra*

$$k\langle\langle t^*;\ x^j, \xi_j|_{j\in J}\rangle\rangle \tag{7.8}$$

where t^* *is of degree zero and* ξ_j *is of degree* $1 - |x_j|$.

The differential is as follows. First, define the continuous derivation ∂ *of* (7.8) *that is zero on* A, x^j, *and on* ξ_j, *and for which*

$$\delta t^* = \sum_j [\xi_j, x^j];$$

second, d_A *is the continuous derivation of* (7.8) *induced by the differential in* A*:* $d_A(a)$ *is the same as in* A *for* $a \in A$;

$$d_A(\xi_k) = \sum \pm \partial_k^{(2)}(d_A(x^j))\xi_j \partial_k^{(1)}(d_A(x^j))$$

7.2. The bracket on $\mathfrak{X}^{(*)}(A, d)$. The necklace bracket restricts to a DG Lie bracket on $\mathfrak{X}^{(*)}(A, d)$. Let us describe this bracket more directly.

The differential d_A defines a structure of a cyclic A_∞ coalgebra structure on $\sum_j (kx^j + k\xi_j)$. Therefore it carries a DG Lie algebra structure dual to the one defined ***ref***.

The differential $\delta + d_A$ turns $\sum_j (kx^j + k\xi_j) + kt^*$ into an A_∞ coalgebra which is not cyclic but satisfies a weaker condition (see Remark 7.2.1), so that the cyclic complex still carries a DGLA structure.

Explicitly, the Lie bracket is as follows: represent a monomial as a cyclic word of x^js, ξ_js and t^*s; take two monomials and put each of them on a circle with a marked point in all cyclic orders; if you have an x^j on one marked point and a ξ_j on the other, replace them by 1; sum up the results, with signs (***picture?***)

REMARK 7.2.1. A k-algebra A is a weakly Frobenius algebra if there is a non-degenerate scalar product $\langle\ ,\ \rangle$ such that

$$\langle a, [b, c]\rangle \tag{7.9}$$

is cyclically invariant. A k-coalgebra C is a weakly Frobenius coalgebra if there is a scalar product $\langle\ ,\ \rangle$ on C such that

$$\langle a^{(1)}, b\rangle a^{(2)} \mp \langle a^{(2)}, b\rangle a^{(1)} \tag{7.10}$$

is ***(skew)*** symmetric in a and b.

More generally, a A_∞ coalgebra C is a weakly cyclic A_∞ coalgebra if there is a scalar product $\langle\ ,\ \rangle$ on C such that

***

is...

The cyclic complex of a weakly cyclic A_∞ coalgebra is a DGLA.

Degrees, shift of C

7.3. The Υ complex of an algebra.

Definition 7.3.1.

$$\mathcal{B}^{\mathrm{sh},(n)}_{\bullet}(A) = \mathcal{B}^{\mathrm{sh}}_{\bullet}(A) \otimes_A \ldots \otimes_A \mathcal{B}^{\mathrm{sh}}_{\bullet}(A)$$

($n \geq 1$);

$$\mathcal{B}^{\mathrm{sh},(*)}_{\bullet}(A) = \bigoplus_{n\geq 0} \mathcal{B}^{\mathrm{sh},(n)}_{\bullet}(A);$$

$$\Upsilon^{(n)}_{\bullet}(A) = \mathcal{B}^{\mathrm{sh},(n)}_{\bullet}(A)_{C_n},$$

$n \geq 0$;

$$\Upsilon^{(0)}_{\bullet}(A) = A/[A, A];$$

$$\Upsilon^{(*)}_{\bullet}(A) = \bigoplus_{n\geq 0} \Upsilon^{(n)}_{\bullet}(A)$$

All of the above carry two differentials, one induced by ∂ and the other by d_A. The total differential is their sum. Also, $\mathcal{B}^{\mathrm{sh},(*)}_{\bullet}(A)$ is a DG algebra and

$$\Upsilon^{(*)}_{\bullet}(A) \xrightarrow{\sim} \mathcal{B}^{\mathrm{sh},(*)}_{\bullet}(A)/[\mathcal{B}^{\mathrm{sh},(*)}_{\bullet}(A), \mathcal{B}^{\mathrm{sh},(*)}_{\bullet}(A)].$$

Incorporate/reconcile shifts

We will denote by b *the differential which is induced by* ∂ . Now, put

$$t_* = 1 \otimes 1 \in \mathcal{B}^{\mathrm{sh}}_{0}(A) \tag{7.11}$$

Define $d : \mathcal{B}^{\mathrm{sh},(*)}_{0}(A) \to \mathcal{B}^{\mathrm{sh},(*+1)}_{1}(A)$ as the derivation sending $a \in A$ to da, da to zero, and t_* to zero.

Lemma 7.3.2.

$$[d, d_A] = 0; [d, b] = [t_*, \ ]$$

As a consequence, $d,\ d_A,\ b$ all commute on $\Upsilon^{(*)}_{\bullet}(A)$.

Proposition 7.3.3. *For a semifree DGA* A *and for* $m > 0$, *the complex*

$$(\Upsilon^{(\geq m)}_{\bullet}(A)[[u]], d_A + b + ud)$$

computes the negative cyclic homology of A.

Also note that $\Upsilon^{(0)}(A)/k{\cdot}1$ computes the reduced cyclic homology of A ***Ref***

Lemma 7.3.4. *For a semifree DG algebra* A *with free generators* $x_j | j \in J$, $\mathcal{B}^{\mathrm{sh},(*)}$ *is a free algebra generated by* $x_j | j \in J$, $dx_j | j \in J$, *and* t_*. *The differential* ∂ *is the derivation sending* dx_j *to* $[t_*, x_j]$, x_j *and* t_* *to zero. The differential* d *sends* x_j *to* dx_j, dx_j *and* t_* *to zero.*

8. Smooth and proper DG algebras

8.1. Preliminaries. . Notation, conventions and discussion on $\mathcal{A} \otimes \mathcal{B}$ and $\underline{\mathrm{Fun}}(\mathcal{A}^{\bullet}, \mathcal{B}^{\bullet})$. Also, terminology (cofibrant, etc.)

8.2. Duality for bimodules. Let $\mathcal{A}$ and $\mathcal{B}$ be two DG categories. For a cofibrant ***Here and below, is cofibrant essential?*** (DG) $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$, let

$$\mathcal{M}^{\vee} = \mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{M},\mathcal{B}) \tag{8.1}$$

which is a $(\mathcal{B},\mathcal{A})$-bimodule. The left $\mathcal{B}$-module structure is induced from the one on $\mathcal{B}$; the right $\mathcal{A}$-module structure is dual to the left module structure on $\mathcal{M}$.

DEFINITION 8.2.1. *An $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$ is right dualisable if*

$$\mathcal{N}\otimes^{\mathbb{L}}_{\mathcal{B}}\mathcal{M}^{\vee} \to \mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{M},\mathcal{N})$$

defined by

$$\mathfrak{n}\otimes\varphi \mapsto (\mathfrak{m}\mapsto \mathfrak{n}\varphi(\mathfrak{m}))$$

is a weak equivalence of $\mathcal{A}$-modules for any cofibrant right $\mathcal{B}$-module $\mathcal{N}$.

EXAMPLE 8.2.2. Let $\mathsf{f}:\mathcal{A}\to\mathcal{B}$ be a morphism of algebras. Define the $(\mathcal{A},\mathcal{B})$-bimodule ${}_{\mathsf{f}}\mathcal{B}$, resp. the $(\mathcal{B},\mathcal{A})$-bimodule $\mathcal{B}_{\mathsf{f}}$, to be $\mathcal{B}$ with the action $\mathfrak{a}\mathfrak{b}_0\mathfrak{b} = \mathsf{f}(\mathfrak{a})\mathfrak{b}_0\mathfrak{b}$, resp. $\mathfrak{b}\mathfrak{b}_0\mathfrak{a} = \mathfrak{b}\mathfrak{b}_0\mathsf{f}(\mathfrak{a})$. Then ${}_{\mathsf{f}}\mathcal{B}^{\vee} = \mathcal{B}_{\mathsf{f}}$.

For $\mathcal{M} ={}_{\mathsf{f}}\mathcal{B}$, both sides in the morphism in Definition 8.2.1 are equal to $\mathcal{N}$ on which $\mathcal{A}$ acts on the right via f.

Now, for a (DG) $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$, let

$${}^{\vee}\mathcal{M} = \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{A}) \tag{8.2}$$

As above, this a $(\mathcal{B},\mathcal{A})$-bimodule. The left $\mathcal{B}$-module structure is dual to the right module structure on $\mathcal{M}$; the right $\mathcal{A}$-module structure is induced from the one on $\mathcal{A}$.

DEFINITION 8.2.3. *An $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$ is left dualisable if*

$${}^{\vee}\mathcal{M}\otimes^{\mathbb{L}}_{\mathcal{A}}\mathcal{N} \to \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{M},\mathcal{N})$$

defined by

$$\psi\otimes\mathfrak{n} \mapsto (\mathfrak{m}\mapsto \psi(\mathfrak{m})\mathfrak{n})$$

is a weak equivalence of $\mathcal{A}$-modules for any cofibrant left $\mathcal{A}$-module $\mathcal{N}$.

EXAMPLE 8.2.4. Let $\mathsf{g}:\mathcal{B}\to\mathcal{A}$ be a morphism of algebras. Define the $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{A}_{\mathsf{g}}$, resp. the $(\mathcal{B},\mathcal{A})$-bimodule ${}_{\mathsf{g}}\mathcal{A}$, to be $\mathcal{A}$ with the action $\mathfrak{a}\mathfrak{a}_0\mathfrak{b} = \mathfrak{a}\mathfrak{a}_0\mathsf{g}(\mathfrak{b})$, resp. $\mathfrak{b}\mathfrak{a}_0\mathfrak{a} = \mathsf{g}(\mathfrak{b})\mathfrak{a}_0\mathfrak{a}$. Then ${}^{\vee}\mathcal{A}_{\mathsf{g}} ={}_{\mathsf{g}}\mathcal{A}$.

For $\mathcal{M} = \mathcal{A}_{\mathsf{g}}$, both sides in the morphism in Definition 8.2.3 are equal to $\mathcal{N}$ on which $\mathcal{B}$ acts on the left via g.

LEMMA 8.2.5. *An $(\mathcal{A},\mathcal{B})$-bimodule $\mathcal{M}$ is right dualisable if and only if it is perfect as a right $\mathcal{B}$-module. It is left dualisable if and only if it is perfect as a left $\mathcal{A}$-module.*

PROOF. □

LEMMA 8.2.6. *Let $\mathcal{M}$ be a left dualisable $(\mathcal{A},\mathcal{B})$-bimodule. Then for any left $\mathcal{B}$-module $\mathcal{L}$ and for any left $\mathcal{A}$-module $\mathcal{N}$*

$$\mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{L},{}^{\vee}\mathcal{M}\otimes_{\mathcal{A}}\mathcal{N}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}\otimes_{\mathcal{B}}\mathcal{L},\mathcal{N})$$

PROOF. We have

$$\mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}, {}^{\vee}\mathcal{M} \otimes_{\mathcal{A}} \mathcal{N}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{L}, \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{M} \otimes_{\mathcal{B}} \mathcal{L}, \mathcal{N})$$

□

LEMMA 8.2.7. *Let $\mathcal{M}$ be a right dualisable $(\mathcal{A}, \mathcal{B})$-bimodule. Then for any left $\mathcal{A}$-module $\mathcal{L}$ and for any right $\mathcal{B}$-module $\mathcal{N}$*

$$\mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{N} \otimes_{\mathcal{B}} \mathcal{M}^{\vee}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}, \mathcal{N})$$

PROOF. We have

$$\mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathcal{N} \otimes_{\mathcal{B}} \mathcal{M}^{\vee}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{A}}(\mathcal{L}, \mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{M}, \mathcal{N}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{B}}(\mathcal{L} \otimes_{\mathcal{A}} \mathcal{M}, \mathcal{N})$$

□

Now let $\mathcal{A}$ be a DG category over k. Then $\mathcal{A}$ is a $(\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}, \mathrm{k})$-bimodule. We call this the diagonal bimodule.

The left dual of the diagonal bimodule is

$$\mathcal{A}^{!} = \mathbb{R}\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}(\mathcal{A}, \mathcal{A} \otimes_{\mathrm{in}} \mathcal{A}) \tag{8.3}$$

(This means that we identify $\mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ as a module over itself with $\mathcal{A} \otimes \mathcal{A}$ equipped with the *inner* bimodule structure).

The right dual of the diagonal bimodule is

$$\mathcal{A}^{*} = \mathbb{R}\mathrm{Hom}_{\mathrm{k}}(\mathcal{A}, \mathrm{k}) \tag{8.4}$$

8.3. Smooth DG algebras. A DG category $\mathcal{A}$ is *smooth* if the diagonal bimodule is left dualisable.

Equivalently, $\mathcal{A}$ is smooth if and only if it is perfect as an $\mathcal{A}$-bimodule.

LEMMA 8.3.1. *For a smooth DG algebra $\mathcal{A}$*

$$\mathbb{R}\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}(\mathcal{A}, \mathcal{A}) \xrightarrow{\sim} \mathcal{A}^{!} \otimes^{\mathbb{L}}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}} \mathcal{A}$$

PROOF. Follows from

$$\mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{N}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{C}}(\mathcal{M}, \mathcal{C}) \otimes_{\mathcal{C}} \mathcal{N}$$

for $\mathcal{M}$ perfect over $\mathcal{C}$, applied to $\mathcal{C} = \mathcal{A} \otimes \mathcal{A}^{\mathrm{op}}$ and $\mathcal{M} = \mathcal{N} = \mathcal{A}$. □

8.4. Proper DG algebras. A DG category $\mathcal{A}$ is *proper* if the diagonal bimodule is right dualisable.

Equivalently, $\mathcal{A}$ is proper if it is perfect as a k-module.

LEMMA 8.4.1. *For a proper DG algebra*

$$\mathbb{R}\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}(\mathcal{A}, \mathcal{A}^{*}) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathrm{k}}(\mathcal{A} \otimes^{\mathbb{L}}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}} \mathcal{A}, \mathrm{k})$$

PROOF. Apply Lemma 8.2.7 to $\mathcal{N} = \mathrm{k}$ and $\mathcal{L} = \mathcal{A}$ (the diagonal bimodule). □

9. CY algebras and categories

9.1. Left and right CY algebras.

9.1.1. *Left CY algebras.* Let $\mathcal{A}$ be a smooth DG category. Fix an integer d. Consider a class ω in $\mathrm{HC}^{-}_{d}(\mathcal{A})$. Denote by $\overline{\omega}$ the image of ω in $\mathrm{HH}_{d}(\mathcal{A})$. We have

$$\overline{\omega} \in \mathcal{A} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}} \mathcal{A}[-d] \tag{9.1}$$

Note that the projection

$$\mathcal{A}^{!} = \mathbb{R}\mathrm{Hom}(\mathcal{A}, \mathcal{A}\otimes_{\mathrm{in}}\mathcal{A}) \to \mathbb{R}\mathrm{Hom}(\mathcal{A}\otimes^{\mathbb{L}}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}\mathcal{A}, (\mathcal{A}\otimes\mathcal{A})\otimes_{\mathcal{A}\otimes_{\mathrm{in}}\mathcal{A}^{\mathrm{op}}}\mathcal{A})$$

becomes

$$\mathcal{A}^{!} \to \mathbb{R}\mathrm{Hom}(\mathcal{A}\otimes^{\mathbb{L}}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}\mathcal{A}, \mathcal{A}) \tag{9.2}$$

Combining this with the evaluation at $\overline{\omega}$

$$\mathrm{ev}_{\overline{\omega}} : \mathcal{A}\otimes^{\mathbb{L}}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}\mathcal{A} \to \mathcal{A}[-d],$$

we get a morphism

$$[\overline{\omega}] : \mathcal{A}^{!} \to \mathcal{A}[-d] \tag{9.3}$$

DEFINITION 9.1.1. *A negative cyclic homology class $[\omega]$ of a smooth DG category $\mathcal{A}$ is a left CY structure if $[\overline{\omega}]$ in (9.3) is a weak equivalence.*

9.1.2. *Right CY algebras.* Let $\mathcal{A}$ be a proper DG category. Fix an integer d and consider a cyclic cohomology class

$$\tau : \mathrm{CC}_{\bullet}(\mathcal{A}) \to k[-d] \tag{9.4}$$

Composing τ with the projection

$$\mathcal{A}\otimes^{\mathbb{L}}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}\mathcal{A} = \mathrm{C}_{\bullet}(\mathcal{A}) \to \mathrm{CC}_{\bullet}(\mathcal{A}),$$

we get

$$\overline{\tau} : \mathcal{A}\otimes^{\mathbb{L}}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}\mathcal{A} \to k[-d] \tag{9.5}$$

Note that

$$\mathbb{R}\mathrm{Hom}_{k}(\mathcal{A}\otimes^{\mathbb{L}}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}\mathcal{A}, k) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}}(\mathcal{A}, \mathcal{A}^{*}) \tag{9.6}$$

The image of $\overline{\tau}$ under the morphism (9.6) is denoted by

$$[\overline{\tau}] : \mathcal{A} \to \mathcal{A}^{*}[-d] \tag{9.7}$$

DEFINITION 9.1.2. *A cyclic cohomology class τ of a proper DG category $\mathcal{A}$ is called a right CY structure if $[\overline{\tau}]$ is a weak equivalence.*

EXAMPLE 9.1.3. A Frobenius algebra is a right CY algebra. The cyclic class τ is represented by the trace $\tau(a) = \langle 1, a\rangle$. The statement of Lemma 8.4.1 follows from the fact that the linear dual of the Hochschild chain complex $C_{\bullet}(A, A)$ is isomorphic to the complex of Hochschild cochain complex $C^{\bullet}(A, A^{*})$. In this case, (9.7) is an isomorphism.

9.1.3. *Serre duality for Hochschild (co)homology.* For a left, resp. right, CY algebra $\mathcal{A}$ the Hochschild cochain complex $C^{\bullet}(\mathcal{A}, \mathcal{A})$ is quasi-isomorphic to a shift of the Hochschild chain complex $C_{\bullet}(\mathcal{A}, \mathcal{A})$, resp. a shift of the linear dual $C_{\bullet}(\mathcal{A}, \mathcal{A})^{*}$. [***More***]

9.2. From CY to pre-CY algebras.

9.2.1. *Motivation: symplectic forms and nondegenerate Poisson structures.* We start with a motivation coming from classical geometry. First let us make the following observation:

In the framework we are workig in, the noncommutative analogue of a volume form is obtained by generalizing commutative theory not of volume forms but of (shifted) symplectic forms.

The reason for this is as follows. Start with a symplectic form ω on a manifold X. It defines an isomorphism

$$T_X \to T_X^* \tag{9.8}$$

This can be extended from the sheaf of algebras $\mathcal{O}_X$ to a sheaf of commutative DG algebras $\mathcal{A}$. The derived analogues of T_X^* and T_X are the cotangent complex and its dual. They are the result of the standard construction being applied to a semi-free *commutative* resolution of $\mathcal{A}$. The definition of a shifted symplectic structure involves a quasi-isomorphism between the two, up to a shift. ***Ref***

The cotangent complex does not carry information about differential forms of degrees other than one. For example, it is quasi-isomorphic to T_X^* if X is a smooth algebraic variety.

When we pass to noncommutative algebras, a big change occurs. Noncommutative analogues of the cotangent complex and its dual, provided for example by the complexes $\mathfrak{X}^{(1)}(\mathcal{R})$ and $\Upsilon^{(1)}(\mathcal{R})$ where $\mathcal{R}$ is a semi-free resolution of our DG algebra $\mathcal{A}$, allow to recover *all* Hochschild (co)homology, i.e. the full noncommutative analogue of forms and multi-vectors. Respectively, a noncommutative analogue of (9.8) is now more like Poincaré or Serre duality.

Consistent with that, CY structures on DG algebras are *noncommutative (shifted) symplectic structures* rather than noncommutative volume forms in the following way: they define a shifted symplectic structure on the (derived) representation scheme (cf. 10 below).

9.2.2. *Motivation: a morphism from forms to multivectors given by a (higher) Poisson structure.* Let us go back to (9.8). We can extend this isomorphism multiplicatively to

$$\wedge^\bullet T_X \xrightarrow{\sim} \wedge^\bullet T_X^* \tag{9.9}$$

The image of ω under the inverse isomorphism is $\pi \in \wedge^2 T_X$. The linear equation $d\omega = 0$ becomes a nonlinear equation $[\pi, \pi] = 0$ (or, perhaps less mysteriously, $\pi \mapsto 0$ under the linear operator $[\pi, \ ]$).

The inverse of (9.9) can be constructed in terms of the bivector π only (see below). We denote it by

$$\mu_\pi : \wedge^\bullet T_X^* \xrightarrow{\sim} \wedge^\bullet T_X \tag{9.10}$$

Its two key properties are

$$\mu_\pi(\omega) = \pi \tag{9.11}$$

and

$$[\pi, \ ] \circ \mu_\pi = \mu_\pi \circ d \tag{9.12}$$

These equations allow one to reconstruct ω.

The above is a blueprint for getting a noncommutative Poisson structure from a noncommutative symplectic structure. By this we mean: getting a pre-CY structure and a Poisson bracket on the derived representation scheme from a left CY structure. We carry this construction out, after saying a few more words about the classical definition of μ_π.

Let $m \geq 1$. Given an m-form ω on a manifold X, one defines a m-linear operation on multivectors. Namely, given multivectors $\pi_1, \dots, \pi_m$, if $\omega = \alpha_1 \wedge \dots \wedge \alpha_m$ where α_j are one-forms, the result of the operation is

$$\sum \pm \iota_{\alpha_1}(\pi_{\sigma 1}) \wedge \dots \wedge \iota_{\alpha_m}(\pi_{\sigma m});$$

in general, we extend the operation $\mathcal{O}_X$-multilinearly.

Note that for any multivector π and any form ω

$$\mu_\pi \omega = \sum_{m=0}^{\infty} \omega[\pi, \dots, \pi] \tag{9.13}$$

Below we will define and use a noncommutative analogue of this construction.

9.2.3. *A morphism from Υ to $\mathfrak{X}$ given by an MC element of $\mathfrak{X}$.* Let $\mathcal{A}$ be a DG algebra. For $\omega \in \Upsilon^{(m)}(\mathcal{A})$ and $\pi_j \in \mathfrak{X}^{(k_j)}(\mathcal{A})$, $j = 1, \dots, m$, we define

$$\omega[\pi_1, \dots, \pi_m] \in \mathfrak{X}^{(k_1 + \dots + k_m - m)}(\mathcal{A}) \tag{9.14}$$

as follows. Assume first that $\mathcal{A}$ is free as an algebra, with a set of free generators $x_\alpha | \alpha \in J$.

Consider the planar graph Γ_0^m with the vertex 0 linked to vertices $1, \dots, m$ that are located around the vertex 0 in the counterclockwise order. Note that ω is a cyclic word consisting of letters x_α, dx_α ($\alpha \in J$) and t_*; each of $\pi_1, \dots. \pi_m$ is a cyclic word consisting of letters x_α, ξ^α ($\alpha \in J$), and t^*. The value of $\omega[\pi_1, \dots, \pi_m]$ in (9.14) will be the sum, with signs, over all the ways to put the word ω on the inner cycle of the vertex 0 and the word π_j on the inner cycle of the vertex j of the blown-up graph of Γ_0^m so that the cyclic order of letters in each word is preserved. The corresponding summand is defined as follows. If letters x and x' fall on two vertices of the blown-up graph that are connected by an edge, replace them by $\langle x, x' \rangle$. Here the only non-zero values for $\langle x, x' \rangle$ are:

$$\langle dx_\alpha, \xi^\alpha \rangle = \pm \langle \xi^\alpha, dx_\alpha \rangle = 1; \ \langle t^*, t_* \rangle = \langle t_*, t^* \rangle = 1$$

Now, read the resulting summand from the outer cycle of the blown-up graph.

LEMMA 9.2.1. *For $\pi \in \mathfrak{X}^{(k)}(A)$, let*

$$|\pi|_d = |\pi| - kd + d + 1$$

(in other words, this is the grading that is part of the definition of the DGLA $\mathfrak{X}^{(*)}(A, d)$.) Then

$$|\omega[\pi_1, \dots, \pi|_d - |\omega| - \sum_{j=1}^{m} |\pi_j|_d = m - 1 - d$$

In particular, if $|\pi_j|_d = 1$ for all j then

$$|\omega[\pi_1, \dots, \pi_m]|_d = |\omega| - d + 2m - 1$$

If

$$\omega = \sum_{m=1}^{\infty} u^{m-1} \omega_m$$

where ω_m *is a chain of degree* $d-2m+2$ *in* $\Upsilon^{(m)}(A)$*,* π *a cochain of degree* 1 *in* $\mathfrak{X}^{(\geq 2)}(A,d)$*, and if we write*

$$\mu_\pi(\omega)=\sum_{m=1}^{\infty}\omega_m[\pi,\ldots,\pi],$$

then

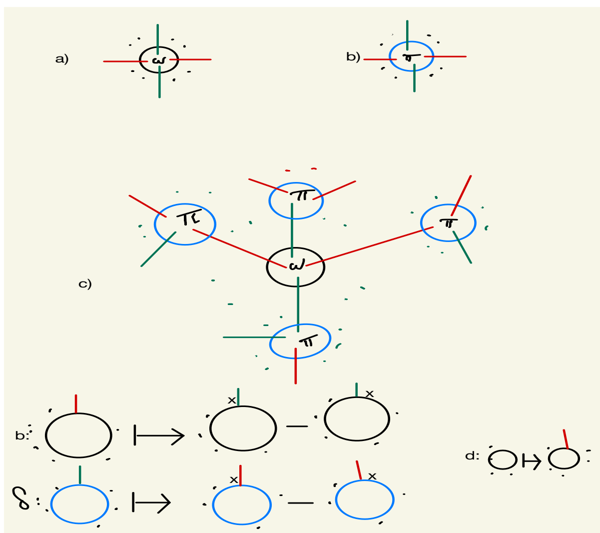
$$|\mu_\pi(\omega)|_d=1.$$

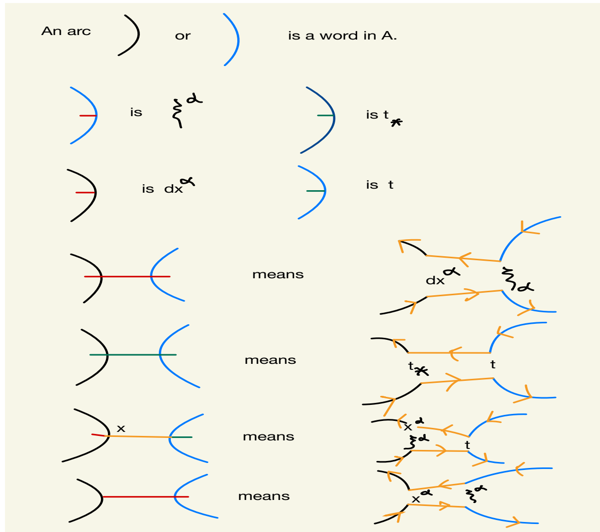
PROOF. □

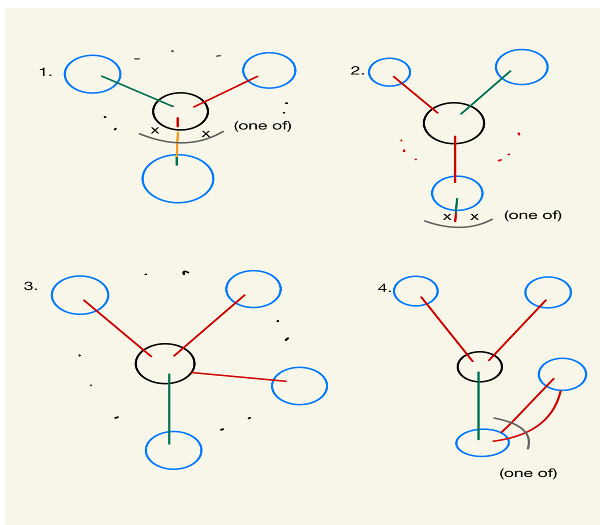
PROPOSITION 9.2.2. *Assume that* $\delta\pi+\frac{1}{2}[\pi,\pi]=0$*. Then*

$$(\delta+[\pi,\;])\circ\mu_\pi=\mu_\pi\circ(b+d)$$

PROOF. □







The above Proposition 9.2.2 implies that a negative cyclic cycle ω has a property which is a noncommutative analogue of (9.12). To formulate an analogue of (9.11), we have to define a $\delta + [\pi, \;]$-cocycle starting from a MC element $\pi \in \mathfrak{X}^{(\geq 2)}(\mathcal{A})$. This can be done by means of rescaling π, namely

$$\pi^{\text{resc}} = \sum_{j=2}^{\infty} (j-1)\pi_j \tag{9.15}$$

where

$$\pi = \sum_{j=2}^{\infty} \pi_j, \; \pi_j \in \mathfrak{X}^{(j)}(\mathcal{A}) \tag{9.16}$$

LEMMA 9.2.3. *If* $\delta\pi + \frac{1}{2}[\pi, \pi] = 0$ *then*

$$\delta\pi^{\text{resc}} + [\pi, \pi^{\text{resc}}] = 0$$

9.2.4. *From left CY structures to pre-CY structures.* Given a left CY algebra $\mathcal{A}$, we can assume it to be semi-free. The CY structure can be represented by

$$\omega \in \Upsilon^{(\geq 1)}(\mathcal{A})$$

such that $(b + d)\omega = 0$ and

$$\text{ev}_\omega : \mathcal{B}_\bullet^{\text{sh}}(\mathcal{A})^\vee \to \mathcal{A} \tag{9.17}$$

is a quasi-isomorphism. Using this non-degeneracy condition, we solve recursively for π such that

$$\delta\pi + \frac{1}{2}[\pi, \pi] = 0;\ \mu_\pi(\omega) = \pi^{\text{resc}} \tag{9.18}$$

9.2.5. *From right CY structures to pre-CY structures.*

10. Higher shifted Poisson structure on the Rep scheme of a pre-CY algebra

Passing to a resolution if needed, we assume that A is semi-free. Using Lemma 4.2.2, we start with a pre-CY structure on A, produce a MC element of the DGLA $\mathfrak{X}^{(*)}(A, d)[d+1]$, and then construct a MC element of the DGLA

$$\mathbb{K}_\bullet(\mathfrak{gl}_m, d; \Theta^\bullet_{\text{Rep}_m(A), d+1})^{\text{GL}_m}[d+1].$$

This gives a derived version of a shifted Poisson structure on $\text{Rep}_m(A)$. ***Explain/write more***

10.1. Double Poisson brackets and Poisson brackets on the Rep scheme. Double Poisson bivectors are MC elements in $\mathfrak{X}^{(2)}(A, d)[d+1]$. Explicitly... ***More***

10.2. Higher shifted symplectic structure the Rep scheme of a CY algebra.

11. ***The work of Brav and Rozenblyum?

12. Examples: CY algebras defined by a potential

12.1. Motivation: the Jacobian *?*** algebra.** For a polynomial $f = f(x_1, \ldots, x_n)$ we define

$$J_f = k[x_1, \ldots, x_n]/(\frac{\partial f}{\partial x_j})$$

This is the algebra of functions on the critical locus $\text{Crit}(f)$.

There is a DG algebra whose zero-degree cohomology is J_f and which is a candidate to be a DG resolution of J_f. It is the algebra

$$R_f = (\Theta^{-\bullet}(\mathbb{A}^n), d_f) \tag{12.1}$$

of multi-vector fields with the differential

$$d_f = [f, \ ]$$

(the Schouten bracket with f).

When R_f is indeed a resolution of A_f (and this is the case if $\frac{\partial f}{\partial x_j}$ form a regular sequence), then

$$\mathbb{L}\Omega^1_{A_f/k} = A_f \otimes_{R_f} \Omega^1_{R_f/k} \tag{12.2}$$

is the cotangent complex of A_f.

Below we will provide a noncommutative analogue of these constructions. As we mentioned in 9.2.2, the noncommutative analogue of the cotangent complex is closely related to the Hochschild complex.

12.2. The algebra A_Φ. Let $F = k\langle x_1, \ldots, x_n\rangle$ be a free algebra. Recall that

$$HH_0(F) = F/[F,F];\ HH_1(F) \xrightarrow{\sim} \bigoplus_{j=1}^{n} F dx_j \tag{12.3}$$

We use the notation

$$B\Phi = \sum_{j=1}^{n} (\frac{\partial \Phi}{\partial x_j}) dx_j \tag{12.4}$$

for $\Phi \in F/[F,F]$. Denote

$$A_\Phi = F/((\frac{\partial \Phi}{\partial x_j})) \tag{12.5}$$

(the quotient of F by the two-sided ideal generated by $\frac{\partial \Phi}{\partial x_j}$, $1 \le j \le n$.

In the notation of [?], $A_\Phi = \mathfrak{A}(F, \Phi)$.

Example 12.2.1. Let $n = 3$ and $\Phi = xyz - yxz$. Then

$$\frac{\partial \Phi}{\partial x} = yz - zy;\ \frac{\partial \Phi}{\partial y} = zx - xz;\ \frac{\partial \Phi}{\partial z} = xy - yx.$$

Therefore

$$A_\Phi \xrightarrow{\sim} k[x, y, z].$$

Example 12.2.2. Let $n = 3$ and $\Phi = xyz - qyxz + f$ where $f \in F/[F,F]$. Then

$$\frac{\partial \Phi}{\partial x} = yz - qzy + \frac{\partial f}{\partial x};\ \frac{\partial \Phi}{\partial y} = zx - qxz + \frac{\partial f}{\partial y};\ \frac{\partial \Phi}{\partial z} = xy - qyx + \frac{\partial f}{\partial z}$$

For example, if

$$\Phi = xyz - qyxz - \frac{1}{2}(x^2 + y^2 + z^2)$$

then

$$A_\Phi \xrightarrow{\sim} U(\mathfrak{so}(3, k)).$$

12.3. The DG algebra $\mathcal{R}_\Phi$ and the $\mathfrak{X}$ complex. Recall that $\mathcal{B}^{\vee,(*)}(F)$ is isomorphic to the (***completed***) graded algebra freely generated by x_j, ξ^j, and t^*, with the differential which is a graded derivation δ sending x_j and ξ^j to zero and such that

$$\delta t^* = \sum_{j=1}^{n} [x_j, \xi^j]$$

Here x_j, t^* are of degree zero and ξ^j of degree one. ***Reconcile with Ginzburg CY***

We change both the grading and the differential on this algebra. First, we put

$$|x_j| = 0;\ |\xi^j| = -1;\ |t^*| = -2 \tag{12.6}$$

Second, the potential Φ introduces an extra differential on this DG algebra. This new differential is a graded derivation that sends x_j and t^* to zero and ξ^j to $\frac{\partial\Phi}{\partial x_j}$. We denote the sum of this differential and δ by $\mathfrak{d} = \mathfrak{d}_\Phi$. In other words, $\mathfrak{d}_\Phi$ is a graded derivation of such that

$$\mathfrak{d}_\Phi(x_j) = 0;\ \mathfrak{d}_\Phi(\xi^j) = \frac{\partial\Phi}{\partial x_j};\ \mathfrak{d}_\Phi(t^*) = \sum_{j=1}^{n}[x_j, \xi^j] \tag{12.7}$$

LEMMA 12.3.1. *a)*

$$\mathfrak{d}_\Phi^2 = 0;$$

*b) The differential induced by $\mathfrak{d}_\Phi$ on the quotient of $k\langle x_j, \xi^j, t^*\rangle$ by the span of commutators is equal to the differential $\delta + [\Phi,\ ]$ on $\mathfrak{X}^{(*)}(F)$ ***Ref****

PROOF. a) follows from

$$\sum_{j=1}^{n}[x_j, \frac{\partial\Phi}{\partial x_j}] = 0$$

(which is a consequence of $[b, B] = 0$); b) is straightforward. □

We write

$$\mathcal{R}_\Phi = (k\langle x_j, \xi^j, t^*\rangle, \mathfrak{d}_\Phi) \tag{12.8}$$

In the notation of [?], $\mathcal{R}_\Phi = \mathfrak{D}(F, \Phi)$.

12.4. The noncommutative cotangemt complex.

$$A_\Phi \otimes_{\mathcal{R}_\Phi} \mathcal{B}^{\mathrm{sh}}_\bullet(\mathcal{R}_\Phi) \otimes_{\mathcal{R}_\Phi} A_\Phi \tag{12.9}$$

Explicitly, (12.9) is a free A_Φ-bimodule with generators dt^*, $d\xi^j$, dx_j, and t_* of degrees $-3, -2, -1, 0$ respectively. The differential

$$A_\Phi dt^* A_\Phi \to \bigoplus_{j=1}^{n} A_\Phi d\xi^j A_\Phi \to \bigoplus_{j=1}^{n} A_\Phi dx_j A_\Phi \to A_\Phi t_* A_\Phi \tag{12.10}$$

is given by

$$dt^* \mapsto \sum_{j=1}^{n}[x_j, d\xi^j]; \tag{12.11}$$

$$d\xi^j \mapsto \sum_{j=1}^{n}(\frac{\partial^2\Phi}{\partial x_j \partial x_k})' dx_k (\frac{\partial^2\Phi}{\partial x_j \partial x_k})'' \tag{12.12}$$

by which we denote $d(\frac{\partial\Phi}{\partial x_j})$;

$$dx_j \mapsto [x_j, t_*] \tag{12.13}$$

*** We hope for (12.10) to be a free bimodule resolution of A_Φ. Unlike the classical case, this has a chance to succeed only when $n = 3$.***

THEOREM 12.4.1. *Let $n = 3$. The following are equivalent:*

1. *The DG algebra $\mathcal{R}_\Phi$ is a DG resolution of A_Φ;*
2. *The complex (12.10) is a free bimodule resolution of A_Φ;*
3. *A_Φ is a left CY algebra.*

The proof is given below.

REMARK 12.4.2. As we did in Chapter 6, by the Chevalley-Eilenberg complexes we understand complexes of the second kind, i.e. the ones defined using the direct sum totalization.

13. Bibliographical notes

Tradler-Zeinallian; Kaufmann; Naef-Willwacher; Ward; Wahl-Westerland; Turaev; Alekseev-Naef;

Rivera-Wang; Keller;

Kontsevich; Ioudiu-Kontsevich-Vlassopoulos; Waikit Yeung; Brav-Dyckerhof;

Van den Bergh; [**560**] [**155**] [**556**] [**555**] [**6**] [**502**] [**500**] [**383**] [**381**] [**380**] [**289**] [**51**], [**52**], [**12**], [**8**], [**10**], [**566**], [**295**] [**325**] [**429**] [**567**]***More refs

CHAPTER 18

Complexes of projective modules; perfect complexes

1. Introduction

We reming the basic facts about projective resolutions of complexes and organize them into A_∞ functors from complexes of modules with cohomology bounded from above to complexes of projective modules. ***Reference for that?** Then we repeat the construction for the case of an algebra A/WA with a central element W.

We specify the above to the case of perfect complexes [Refs, Grothendieck, Thomason?..]

In this chapter, all complexes are complexes of modules over an algebra A.

1.1. Preliminaries.

Lemma 1.1.1. *For a complex $M^\bullet$ whose cohomology is bounded from above, there exists bounded from above complex of projective modules $P^\bullet$ and a quasi-isomorphism $g : P^\bullet \to M^\bullet$.*

Proof. □

Lemma 1.1.2. *Let $g_1 \colon P_1^\bullet \to M_1^\bullet$ and $g_2 \colon P_2^\bullet \to M_2^\bullet$ be two quasi-isomorphisms where $P_1^\bullet$ is a bounded from above complex of projective modules and the cohomology of P_2 is bounded from above. Let $f \colon M_1^\bullet \to M_2^\bullet$ be a morphism. Then there is a quasi-isomorphism $\varphi \colon P_1^\bullet \to P_2^\bullet$ such that $g_2\varphi$ and fg_1 are homotopic.*

Proof. As above, we may assume that the morphism φ and the homotopy h are already constructed on F^k for $k > n$ for some n. One has

$$g_2\varphi - fg_1 = dh + hd \tag{1.1}$$

on P^k, $k > n$. Since $d\varphi d = 0$, φdP_1^n is inside dP_2^n. Since $g_2\varphi d = fg_1 d + dhd$ has its image inside dM_2^n and g_2 is a quasi-isomorphism, $\varphi d|P_1^n$ has its image inside dP_2^n. By projectivity of P_1^n we get $\varphi_0 : P_1^n \to P_2^n$ such that $d\varphi_0 = \varphi d$. We have

$$d(g_2\varphi_0 - fg_1 - hd) = g_2\varphi d - fg_1 d - dhd = (dh + hd)d - dhd = 0$$

and therefore $(g_2\varphi_0 - fg_1 - hd)(P_1^n)$ is inside $dM_2^{n-1} + g_2(\mathrm{Ker}(d|P_2^n))$. By projectivity of P_1^n, there is a map

$$(\varphi_1, h) : P_1^n \to \mathrm{Ker}(d|P_2^n) \oplus M_2^{n-1}$$

such that, for $\varphi = \varphi_0 + \varphi_1$, (1.1) holds. □

Lemma 1.1.3. *Let $P^\bullet \xrightarrow{\varphi} P_1^\bullet \xrightarrow{g} M^\bullet$ where $P^\bullet$ is a complex of projective modules bounded from above, g is a quasi-isomorphism, $g\varphi = [d, h]$ for some homotopy $h \colon P_1^\bullet \to M^{\bullet-1}$. Then there exists $H \colon P^\bullet \to P_1^{\bullet-1}$ such that $\varphi = [d, H]$ and gH is homotopic to h.*

PROOF. Assume that we have, in addition to $h : P^k \to M^{k-1}$ for all k, also$H : P^k \to P_1^{k-1}$ and $s : P^k \to M^{k+2}$ for $k > n$ satisfying

$$dh + hd = g\varphi;\ dH + Hd = \varphi;\ gH - h = ds - sd \tag{1.2}$$

Observe that $d(\varphi - Hd) = 0$ and $g(\varphi - Hd) = dh - dsd$ and therefore the image of $\varphi - Hd$ is inside dP_1^n since g is a quasi-isomorphism. Therefore $\varphi - Hd = dH_0$ for some $H_0 : P^n \to P_1^{n-1}$. Now,

$$d(gH - h + sd) = gdH - dh + dsd = 0$$

and therefore the image of $gH - h + sd : P^n \to M^{n-1}$ is inside $g\ker(d|P_1^{n-1}) + dM^{n-2}$. By projectivity of P^n we have a morphism $(H_1), s) : P^n \to \ker(d|P_1^{n-1}) \oplus M^{n-2}$ such for s and for $H = H_0 + H_1$ (1.2) is true. □

COROLLARY 1.1.4. *1) Any two choices of φ as in Lemma 1.1.2 are homotopic. 2) Any two choices of $P^\bullet$ as in Lemma 1.1.1 are homotopy equivalent.*

PROOF. 1) Apply Lemma 1.1.3 to the difference between the two choices of φ. 2) For the two choices $P_1^\bullet$, and $P_2^\bullet$, consider $\varphi : P_1^\bullet \to P_2^\bullet$ and $\psi : P_2^\bullet \to P_1^\bullet$, then apply 1) to id and $\psi\varphi$, as well as to id and $\varphi\psi$. □

2. Projective resolutions and A_∞ functors

In this section we will explain how the standard constructions of 1.1 can be organized into A_∞ functors that take values in the DG category of bounded from above complexes of projective modules. The latter DG category will be denoted by $\mathrm{Proj}^-(A)$. Let $\mathrm{Com}^-(A)$ be the DG category of complexes of A-modules with bounded cohomology. We will construct two such A_∞ functors:

A. Assume that A is free as a k-module. Denote by $\mathrm{Proj}_f^-(A)$, resp. $\mathrm{Com}_f^-(A)$, full DG subcategories of complexes of modules that are free as k-modules (k is the ground ring). Let i be the inclusion of $\mathrm{Proj}_f^-(A)$ into $\mathrm{Com}_f^-(A)$. We will construct an A_∞ functor

$$P : \mathrm{Com}_f^-(A) \to \mathrm{Proj}_f^-(A) \tag{2.1}$$

together with natural transformations $S\colon iP \to \mathrm{id}$ and $T\colon Pi \to \mathrm{id}$. The latter are, by definition, Hochschild cocycles of degree zero

$$C^\bullet(\mathrm{Com}_f(A), {}_{iP}\mathrm{Com}_f(A)_{\mathrm{id}}),$$

and same but with Pi instead of iP. cf. (12.1).

B. Now view $\mathrm{Com}^-(A)$ as a category (with morphisms being morphisms of complexes). Let $k[\mathrm{Com}^-(A)]$ be the k-linear span of this category. We will construct an A_∞ functor

$$P : k[\mathrm{Com}^-(A)] \to \mathrm{Proj}^-(A) \tag{2.2}$$

Denote by $\pi : k[\mathrm{Com}^-(A)] \to \mathrm{Com}^-(A)$ the DG functor that is the identity on objects and sends every morphism to itself (viewed as a zero-cocycle in the complex of morphisms). Then there is a natural transformation $iP \to \pi$ where i is as above.

Moreover:

LEMMA 2.0.1. *Consider any choice $M^\bullet \mapsto P^\bullet(M^\bullet)$ for all objects in as in Lemma 1.1.1. Assume that this choice associates $P^\bullet$ to itself for every bounded from above complex of projectives. Then there are A_∞ functors as in* (2.1) *and in* (2.2) *whose action on objects is given by $M^\bullet \mapsto P^\bullet(M^\bullet)$. In addition, consider* *any choice* $f \mapsto \varphi(f) : P_1^\bullet(M_1^\bullet) \to P_2^\bullet(M_2^\bullet)$ *as in Lemma 1.1.2 for all morphisms in* Com$^-(A)$. *Then there is an* A_∞ *functor as in* (2.2) *whose action on morphisms is given by* $f \mapsto \varphi(f)$.

2.1. Construction of the A_∞ functor (2.1)**.** It is easy to construct one such A_∞ functor, in fact a DG functor, from the subcategory of complexes that are bounded from above. In fact, for such a complex $M^\bullet$, let Bar$(M^\bullet)$ be its standard bar resolution

$$\mathrm{Bar}(M^\bullet) = (\bigoplus_{n\geq 0} A[1]^{\otimes n} \otimes M^\bullet, \partial_{\mathrm{Bar}} + d_M)$$

Any homogeneous k-linear map $f\colon M_1^\bullet \to M_2^\bullet$ induces a homogeneous k-linear map $B(f)\colon \mathrm{Bar}(M_1^\bullet \to \mathrm{Bar}(M_2^\bullet)$. Therefore we get a DG functor that we denote by P_0.

Now consider any choice $M^\bullet \mapsto P^\bullet(M^\bullet)$ as in Lemma 1.1.1. Extend it to an A_∞ functor P as follows. First consider complexes $M^\bullet$ that are bounded from above. For f_j being k-linear homogemeous maps

$$f_j : M_{j+1}^\bullet \to M_j^\bullet,$$

$j = 1, \ldots, n$, put

$$P_n(f_1, \ldots, f_n) = \varphi_1 B(f_1) h_2 \ldots B(f_{n-1}) h_n B(f_n) \widetilde{\varphi}_{n+1} \tag{2.3}$$

where:

$$\widetilde{g}(M^\bullet) : \mathrm{Bar}(M^\bullet) \to M^\bullet;\ g : P^\bullet(M^\bullet) \to M^\bullet$$

are chosen as in Lemma 1.1.1; for them,

$$\varphi(M^\bullet) : \mathrm{Bar}(M^\bullet) \to P^\bullet(M^\bullet);\ \widetilde{\varphi}(M^\bullet) : P^\bullet(M^\bullet) \to \mathrm{Bar}(M^\bullet)$$

are chosen as in Lemma 1.1.2; $h(M^\bullet)$ is the homotopy between id and $\widetilde{\varphi}(M^\bullet)\varphi(M^\bullet)$ and $\widetilde{h}(M^\bullet)$ is the homotopy between id and $\varphi(M^\bullet)\widetilde{\varphi}(M^\bullet)$ as in Lemma 1.1.3; we put

$$g_j = g(M_j^\bullet);\ \widetilde{g}_j = \widetilde{g}(M_j^\bullet);\ \varphi_j = \varphi(M_j^\bullet);\ \widetilde{\varphi}_j = \widetilde{\varphi}(M_j^\bullet);\ h_j = h(M_j^\bullet)$$

As long as we are restricted to complexes that are bounded from above, we can choose $g(M^\bullet)$ to be surjective. In this case, homotopies in Lemmas 1.1.1 and 1.1.2 can be made zero.

The components of the natural transformation S are defined as follows. On objects,

$$S = \widetilde{g}\widetilde{\varphi}\colon iP(M^\bullet) = P^\bullet(M^\bullet) \to M^\bullet;$$

on morphisms,

$$S_n(f_1, \ldots, f_n) = \widetilde{g}_1 h_1 B(f_1) h_2 \ldots B(f_{n-1}) h_n B(f_n) \widetilde{\varphi}_{n+1} \tag{2.4}$$

Similarly, construct the natural transformation $T : Pi \to \mathrm{id}$. ***FINISH**

Next, consider any choice $M^\bullet \mapsto P^\bullet(M^\bullet)$ for all $M^\bullet$ in $\mathrm{Com}_f^-(A)$ (that is, complexes whose cohomology is bounded from above). Extend P to an A_∞ functor as follows.

Instead of a single complex Bar$(M^\bullet)$, we have an inductive system of quasi-isomorphic embeddings $\mathrm{Bar}(M^\bullet(m)) \to \mathrm{Bar}(M^\bullet(n))$ for $m \leq n$ where m, n are big enough. Here $M^\bullet(n) = \tau_{\leq n} M^\bullet$ is the truncation. In other words,

$$M^\bullet(n) = (\ldots \to M^{n-1} \to M^n \to \mathrm{Ker}(d|M^{n+1}) \to 0 \to \ldots)$$

A homogeneous k-linear map f of complexes induced a morphism of these inductive systems, namely, k-linear homogeneous maps

$$B(f,n) : \mathrm{Bar}(M_1^\bullet(n)) \to \mathrm{Bar}(M_2^\bullet(n+d))$$

compatible with the embeddings, with $d > \deg(f)$. ***FINISH***

2.2. Construction of the A_∞ functor (2.2). For every complex $M^\bullet$ with cohomology bounded from above, make a choice of a bounded from above complex of projective modules $P^\bullet$ and a quasi-isomorphism $g(M^\bullet)\colon F^\bullet \to M^\bullet$ as in Lemma 1.1.1. When $M^\bullet$ is itself a bounded from above complex of projective modules, we choose $g(M^\bullet) = M^\bullet$. Put $P(M^\bullet) = F^\bullet$. Now let $f_i\colon M_i^\bullet \leftarrow M_{i+1}^\bullet$ for $i = 1, \ldots, n$. Denote $g_i = g(M_i^\bullet)$. Let $P(M_i^\bullet) = F_i^\bullet$. We denote the components of the A_∞ functor P by

$$P_n(f_1, \ldots, f_n) \in \underline{\mathrm{Hom}}^{1-n}(P_{n+1}^\bullet, P_1^\bullet)$$

and the components of the natural transformation S by

$$s_n(f_1, \ldots, f_n) \in \underline{\mathrm{Hom}}^{-n}(P_{n+1}^\bullet, M_1^\bullet).$$

Remark 2.2.1. We have two conflicting notations, namely, we denote the composition in any abstract DG category by

$$\mathcal{A}(x,y) \otimes \mathcal{A}(y,z) \to \mathcal{A}(x,z); f \otimes g \mapsto fg \tag{2.5}$$

but denote the composition of morphisms of complexes, as usual, by gf. We resolve this by working not with the category of complexes but with its opposite.

For $n = 0$, we put $s = g(M^\bullet)$ for every $M^\bullet$. For $n = 1$, let $P(f_1)$ be the morphism φ from Lemma 1.1.2. Let $s(f_1)$ be the homotopy from the same Lemma. By inductive hypothesis, assume that all P_m and s_m are already defined for m smaller than n. The maps P and s have to satisfy

$$[d, P(f_1, \ldots, f_n)] = \sum_{k=1}^{n-1} \pm P(f_1, \ldots, f_k) P(f_{k+1}, \ldots, f_n) + \sum_{k=1}^{n-1} \pm P(f_1, \ldots, f_k f_{k+1}, \ldots, f_n) \tag{2.6}$$

and

$$[d, s(f_1, \ldots, f_n)] \pm g_1 P(f_1, \ldots, f_n) = f_1 s(f_2, \ldots, f_n) + \sum_{k=1}^{n-1} \pm s(f_1, \ldots, f_k) P(f_{k+1}, \ldots, f_n) + \sum_{k=1}^{n-1} \pm s(f_1, \ldots, f_k f_{k+1}, \ldots, f_n) \tag{2.7}$$

Denote by R_1 the right hand side of (2.6) and by R_2 the right hand side of (2.7). If we apply (2.7) for all $k < n$, we will get

$$\phi_1 R_1 = [d, R_2] \tag{2.8}$$

By Lemma 1.1.3, there exist the homotopies $P(f_1, \ldots, f_n)$ and $s(f_1, \ldots, s_n)$ satisfying (2.6) and (2.7). This completes the construction of the A_∞ functor P.

2.3. Factoring out quasi-isomorphisms. Recall the definitions of the Drinfeld quotient 5. As usual, by $\mathcal{A}/\mathcal{B}$ we denote the Drinfeld quotient of $\mathcal{A}$ by $\mathcal{B}$.

PROPOSITION 2.3.1. *The* A_∞ *functors* P *as in* (2.1), (2.2) *extend to the quotient of* $\mathrm{Com}^-_f(A)$, *resp.* $k[\mathrm{Com}^-(A)]$, *by the full subcategory of acyclic complexes:*

$$P : \mathrm{Com}^-_f(A)/\mathrm{Acy}_f(A) \to \mathrm{Proj}^-_f(A) \tag{2.9}$$

$$P : k[\mathrm{Com}^-(A)]/k[\mathrm{Acy}(A)] \to \mathrm{Proj}^-(A) \tag{2.10}$$

PROOF. Start with the subcatgories of complexes bounded from above. For an acyclic complex $M^\bullet$, the complex $P^\bullet(M^\bullet)$ is a bounded from above complex of projective modules. Therefore it is contractible. We extend P by sending the element $\epsilon_{M^\bullet}$ to the contracting homotopy. ***FINISH** □

3. Perfect complexes

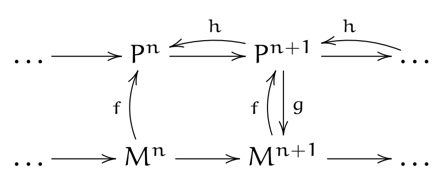
DEFINITION 3.0.1. *Let* A *be an associative algebra. A complex* $¶^\bullet$ *of* A*-modules is* strictly perfect *if each* $¶^k$ *is finitely generated projective and* $¶^k = 0$ *for all but finitely many* k. *A complex* $M^\bullet$ *of* A*-modules is* perfect *if it is quasi-isomorphic to a strictly perfect complex. By* $\mathrm{Perf}(A)$, *resp.* $\mathrm{sPerf}(A)$, *we denote the DG category of perfect, resp. strictly perfect, complexes.*

LEMMA 3.0.2. *For a perfect complex* $M^\bullet$ *there exists a strictly perfect complex* $P^\bullet$ *and a quasi-isomorphism* $g : P^\bullet \to M^\bullet$.

PROOF. It is enough to prove that, if $P^\bullet$ is strictly perfect and $f \colon M^\bullet \to P^\bullet$ is a quasi-isomorphism, that there is a quasi-isomorphism $g \colon P^\bullet \to M^\bullet$ such that fg is homotopic to $\mathrm{id}_{P^\bullet}$. Since $P^\bullet$ is non-zero only in finitely many degrees, we may assume that the morphism g and the homotopy g are already constructed on F^k for $k > n$ for some n. We can assume that we have $g : P^k \to M^k$ and $h : P^k \to P^{k-1}$ for $k > n$ such that

$$\mathrm{id}_F - fg = dh + hd \tag{3.1}$$

on all P^k with $k > n$.



The morphism $gd \colon P^n \to M^{n+1}$ has its image inside $\mathrm{Ker}(d)$; $fgd : P^n \to P^{n+1}$ has its image inside $\mathrm{Im}(d)$ because of (3.1); since f is a quasi-isomorphism, gd has image in dM^n. By projectivity of P^n, we can find $g_0 : P^n \to M^n$ such that $dg_0 = gd$. Now consider the map $\mathrm{id}_{P^n} - fg_0 - hd : P^n \to P^n$. We have $d(\mathrm{id}_{P^n} - fg_0 - hd) = 0$ because of (3.1). Since f is a quasi-isomorphism, the image of $\mathrm{id}_{P^n} - fg_0 - hd$ is in $dP^{n-1} + f(\mathrm{Ker}(d|M^n))$, and therefore the map can be lifted to $(h, g_1) \colon P^n \to P^{n-1} \oplus \mathrm{Ker}(d|M^n)$. This gives us our h. Now define $g = g_0 + g_1$. These h and g satisfy (3.1). □

There is the obvious inclusion functor

$$i \colon \mathrm{sPerf}(A) \to \mathrm{Perf}(A) \tag{3.2}$$

The A_∞ functors from Proposition 2.3.1 restrict to A_∞ functors

$$P : \mathrm{Perf}_f(A)/\mathrm{Acy}_f(A) \to \mathrm{sPerf}_f^-(A) \tag{3.3}$$

$$P : k[\mathrm{Perf}(A)]/k[\mathrm{Acy}(A)] \to \mathrm{sPerf}(A) \tag{3.4}$$

together with natural transformations $S\colon iP \to \mathrm{id}$ and $T\colon Pi \to \mathrm{id}$.

As above, the subscript f refers to the subcategory of complexes of A-modules that are free as k-modules.

Theorem 3.0.3. *The A_∞ functors below, defined as compositions of the embedding and the projection to the Drinfeld quotient, are A_∞ quasi-equivalences of DG categories:*

$$\mathrm{Proj}_f^-(A) \to \mathrm{Com}_f^-(A)/\mathrm{Acy}_f(A)$$

and

$$\mathrm{sPerf}_f^-(A) \to \mathrm{Perf}_f^-(A)/\mathrm{Acy}_f(A)$$

Proof. Since quasi-isomorphisms are isomorphisms in H^0 of the Drinfeld quotient by acyclic complexes, i induces an equivalence of homotopy categories. Looking at the action of i and P, we conclude that i induces a quasi-isomorphism on morphisms ***FINISH** □

4. The trace map $\mathrm{HH}_\bullet(\mathrm{sPerf}(A)) \to \mathrm{HH}_\bullet(A)$

Define the trace map as follows. Let (P_j, d_j) be strictly perfect complexes such that $(P_0, d_0) = (P_{n+1}, d_{n+1})$. For a Hochschild chain $a = a_0 \otimes \ldots \otimes a_n$, $a_j \in \mathrm{Hom}_A(P_j, P_{j+1})$ put

$$\ell_d(a) = \sum \pm a_0 \otimes \ldots \otimes a_{j_1} \otimes d_{j_1+1} \otimes a_{j_1+1} \otimes \ldots \otimes a_n$$

and

$$\mathrm{Tr}(a) = \mathrm{tr}\exp(\ell_d)(a)$$

where tr was defined in 8.1***[need graded version]**(see also 8.2). Note that, despite the exponential involving factorials, in our case ℓ_d^k is divisible by $k!$. The sum is infinite but the trace map is zero on all but finitely many terms. In fact, the trace map has the form

$$(b_0 m_0 \otimes \ldots \otimes b_N m_N) \mapsto \pm\mathrm{tr}(m_0 \ldots m_N)(b_0 \otimes \ldots \otimes b_N)$$

where $b_j \in A$ and m_j are matrices over k. On every P_j there is a grading (just by the degree in the complex). This induces a grading on tensor products of $\mathrm{Hom}_A(P_j, P_{j+1}$. In the infinite sum, there are always $n+1$ factors a_j and a growing number of factors d_j. Therefore the degree of a term in the sum goes to infinity. But the trace is zero on all terms of nonzero degree.

Lemma 4.0.1. *The map* Tr *is a quasi-isomorphism of complexes that commutes with* B.

Proof. Define the completions $C_\bullet^{\mathrm{II}}$ of the Hochschild complexes of sPerf and $\mathrm{sPerf}_{d=0}$ consisting of infinite sums of chains where the degree is allowed to go to infinity. (see Chapter 24). The trace map Tr defines an isomorphism

$$C_\bullet^{\mathrm{II}}(\mathrm{sPerf}_{d=0}(A)) \xrightarrow{\sim} C_\bullet(\mathrm{sPerf}(A)) \tag{4.1}$$

□

We have

$$\mathsf{C}_\bullet(A) \underset{i}{\overset{p}{\leftrightarrows}} \mathsf{C}_\bullet(\mathrm{sPerf}_{d=0}(A)) \circlearrowleft^{s} \xrightarrow{\iota} \mathsf{C}^{\mathrm{II}}_\bullet(\mathrm{sPerf}_{d=0}(A)) \xrightarrow{g} \mathsf{C}^{\mathrm{II}}_\bullet(\mathrm{sPerf}(A)$$

Here i, p, and s are the inclusion, the projection and the homotopy from 8.1 while g is the isomorphism defined in 4. More precisely, g is the map such that $\mathrm{Tr} = \mathrm{tr} \circ g$. One has

$$p \circ i = \mathrm{id}; \ \mathrm{id} - ip = [d + b, s]$$

The easiest thing now is to observe that a) s extends to $\mathsf{C}^{\mathrm{II}}_\bullet(\mathrm{sPerf}_{d=0}(A))$ and b) gsg^{-1} descends to $\mathsf{C}_\bullet(\mathrm{sPerf}(A))$. Therefore $g\iota i$ is a homotopy equivalence of complexes.

We have proved the following theorem of Keller.

THEOREM 4.0.2. *There is a natural quasi-isomorphism*

$$\mathsf{C}_\bullet(\mathrm{Perf}(A)/\mathrm{Perf}^{\mathrm{acyclic}}(A)) \to \mathsf{C}_\bullet(A)).$$

Same for the cyclic complexes of all types.

5. Bibliographical notes

Toledo-Tong; Keller;

CHAPTER 19

What do DG categories form?

1. Introduction

The question in the title of this chapter was asked by Drinfeld in ***REF***. We will give a version of an answer that is related to other versions, such as ***REFS***. Namely, we will show that Hochschild cochains and chains form a category with a trace functor up to homotopy (a variant of the definition that was introduced by Kaledin).

Let us start by noting that categories form a two-category, morphisms being functors and 2-morphisms being natural transformations. A related fact is that rings form a two-category. In fact, for two rings A and B, let $\mathcal{C}(A, B)$ be the category of (A, B)-bimodules. The composition $\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C)$ is given by the tensor product $\otimes_B$. If we restrict ourselves to bimodules which are graphs of morphisms, i.e. for every $f : A \to B$ define ${}_fB$ to be B with the bimodule action $a \cdot b \cdot b_1 = f(a)bb_1$, then the resulting sub-2-category becomes a subcategory of the 2-category of categories above (rings are additive categories with one object).

The 2-category of rings has an extra structure. Namely, there is a functor $\mathrm{TR}_A : \mathcal{C}(A, A) \to \mathbb{Z}$-mod for every ring A, defined by

$$\mathrm{TR}_A : M \mapsto M/[A, M] \xrightarrow{\sim} M \otimes_{A \otimes A^{op}} A \xrightarrow{\sim} HH_0(A, M). \tag{1.1}$$

The functors TR_A have the trace property, namely, for any A and B and for any objects M in $\mathcal{C}(A, B)$ and B in $\mathcal{C}(B, A)$ there is a functorial isomorphism

$$\tau_{AB} : \mathrm{TR}_A(M \otimes_B N) \xrightarrow{\sim} \mathrm{TR}_B(N \otimes_A M).$$

Those isomorphisms satisfy a compatibility condition for every three rings A, B, C and for three objects M in $\mathcal{C}(A, B)$, B in $\mathcal{C}(B, C)$, and $P \in \mathcal{C}(C, A)$:

$$\tau_{AC}\tau_{CB}\tau_{BA} = \mathrm{id} : \mathrm{TR}_A(M \otimes_B N \otimes_C P) \to \mathrm{TR}_A(M \otimes_B N \otimes_C P)$$

For a monoidal category, i.e. a 2-category with one object, we saw such a structure in 2.3.

The goal of this chapter is to describe a derived analog of the above. More precisely, replace morphisms of bimodules by the standard complexes computing $\mathbb{R}\mathrm{Hom}$ and replace the trace of a bimodule by the standard complex computing the derived tensor product, i.e. the Hochschild chain complex. We will actually restrict ourselves to bimodules that are graphs of morphisms (or, more generally, of A_∞ morphisms of DG categories). We will use brace operations on Hochschild cochains, and their analogs on Hochschild cochains and chains, to construct a homotopy version of the structure described above. We will see that much of the structure is actually strict, not up to homotopy, when the morphisms are DG cocategories. The single place where this is not so is precisely where the cyclic differential B appears. We find this significant, together with the fact that bar construction of the algebra of Hochschild cochains of an individual algebra form a Hopf algebra (strict, not up to homotopy).

Let us show how the structure of a two-category up to homotopy (in any reasonable sense) gives rise to a differential B on $\mathrm{TR}_A(\mathrm{id}_A)$ for any A. Start with two morphisms of rings $f : A \to B$ and $g : B \to A$. Then we should have quasi-isomorphisms of complexes

$$\mathrm{TR}_A(gf) \xrightarrow{\tau_{AB}} \mathrm{TR}_B(fg) \xrightarrow{\tau_{BA}} \mathrm{TR}_A(gf)$$

and a homotopy between id and $\tau_{BA}\tau_{AB}$. We denote this homotopy by B_{gf}. Let us also denote the τ_{AB} above by f_* and τ_{BA} by g_*.

Now let $A = B$ and $f = g = \mathrm{id}$. Then $\tau_{BA} = \tau_{AB} = \mathrm{id}$ and the homotopy now commutes with the differential. We get an endomorphism B, or B_A, of (homological) degree one of the complex $\mathrm{TR}_A(\mathrm{id}_A)$.

Why does B define a differential? Note that, in any reasonable definition of a 2-category with a trace functor up to homotopy, any new morphism of our complexes should be homotopic to zero. For example, $f_*B_{gf} - B_{fg}f_*$ is the difference of two homotopies between f_* and $f_*g_*f_*$, it has to be homotopic to zero. Similarly, if we denote by b_A and b_B the differentials in $\mathrm{TR}_A(gf)$ and $\mathrm{TR}_B(fg)$ respectively, then***CONT.; could be a bit subtle.***

Similar considerations show that the cohomologies of $\mathcal{C}(A, A)(\mathrm{id}_A, \mathrm{id}_A)$ and $\mathrm{TR}_A(\mathrm{id}_A)$ carry a structure that is called *a calculus* in ***REF***.

The explicit formulas are as follows. For a morphism $f : A \to A$,

$$\mathrm{TR}_A(f) = C_\bullet(A,_f A);\;\; f_*(a_0 \otimes \ldots \otimes a_n) = f(a_0) \otimes \ldots \otimes f(a_n);$$

$$B_f(a_0 \otimes \ldots \otimes a_n) = \sum_{j=0}^{n} (-1)^{nj} 1 \otimes f(a_j) \ldots \otimes f(a_n) \otimes a_0 \otimes \ldots \otimes a_{j-1}$$

We finish this introduction by describing a structure carried by the pair $C^\bullet(\mathcal{A}, \mathcal{A})$ and $C_\bullet(\mathcal{A}, \mathcal{A})$. This structure is less known or studied than the one involving trace functors.

For a Hopf algebra U over k, denote by U^+ the k-module U equipped with the cobimodule structure

$$U^+ \to U \otimes U^+ \otimes U;\; u \mapsto \sum S((u^{(3)}) \otimes u^{(2)} \otimes S((u^{(1)})$$

where S is the antipode. Note that

$$m^{\mathrm{op}} = m \circ \sigma : U^+ \otimes U^+ \to U^+ \tag{1.2}$$

is a morphism of cobimodules. Here m is the product in U and σ is the transposition.

A di(tetra)module over a Hopf algebra$U$ is a cobimodule M over U together with two cobimodule morphisms

$$\mu_l : U^+ \otimes M \to M;\; \mu^r : M \otimes U \to M \tag{1.3}$$

subject to the following compatibility conditions:

1) (1.3) turn M into a right module over (U, m) and over (U^+, m^{op});

2) the right actions of U and of U^+ commute with each other.

FINISH

It would be interesting to see examples, as well as a quasi-classical analog for Poisson-Lie groups, etc.

2. A category in DG cocategories

Lemma 12.2.1 defines

(1) For every two DG categories $\mathcal{A}$ and $\mathcal{B}$, a DG cocategory

$$\mathbf{B}(\mathcal{A},\mathcal{B}) = \mathrm{Bar}\ \mathbf{C}(\mathcal{A},\mathcal{B}) \tag{2.1}$$

(2) For any three DG categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, a DG functor

$$\mathfrak{m}_{\mathcal{ABC}} : \mathbf{B}(\mathcal{A},\mathcal{B}) \otimes \mathbf{B}(\mathcal{B},\mathcal{C}) \to \mathbf{B}(\mathcal{A},\mathcal{C}) \tag{2.2}$$

that is associative, namely,

$$\mathfrak{m}_{\mathcal{ABD}} \circ \mathfrak{m}_{\mathcal{BCD}} = \mathfrak{m}_{\mathcal{ACD}} \circ \mathfrak{m}_{\mathcal{ABC}} = \mathfrak{m}_{\mathcal{ABCD}} : \mathbf{B}(\mathcal{A},\mathcal{B}) \otimes \mathbf{B}(\mathcal{B},\mathcal{C}) \otimes \mathbf{B}(\mathcal{C},\mathcal{D}) \to \mathbf{B}(\mathcal{A},\mathcal{D})$$

for any four DG categories $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$.

3. A category in DG cocategories with a trace functor

3.1. DG comodules. Our assumptions: DG cocategories and DG comodules are conilpotent and locally finite, i.e.

$$\Delta_{\mathbf{B}} : \mathbf{B}(x,y) \to \mathbf{B}(x,z) \otimes \mathbf{B}(z,y)$$

is equal to zero for all but finitely many z, and

$$\Delta_{\mathbf{M}} : \mathbf{M}(x) \to \mathbf{B}(x,y) \otimes \mathbf{M}(y)$$

is equal to zero for all but finitely many y.

For a DG functor $f : \mathbf{B}_2 \to \mathbf{B}_1$ between two DG cocategories and for a DG comodule $\mathbf{M}$ over $\mathbf{B}_1$, define a DG comodule $f^*\mathbf{M}$ over $\mathbf{B}_2$ as follows. For an object x_2 of $\mathbf{B}_2$, define two maps

$$\bigoplus_{y_1 \in \mathrm{Ob}\mathbf{B}_1} \mathbf{B}_2(x_2, fy_1) \otimes \mathbf{M}(y_1) \rightrightarrows \bigoplus_{y_2 \in \mathrm{Ob}\mathbf{B}_2; z_1 \in \mathrm{Ob}\mathbf{B}_1} \mathbf{B}_2(x_2,y_2) \otimes \mathbf{B}_1(fy_2,z_1) \otimes \mathbf{M}(z_1)$$

One is $\mathrm{id}_{\mathbf{B}_2} \circ \Delta_{\mathbf{M}}$, the other $(\mathrm{id}_{\mathbf{B}_2} \otimes f \otimes \mathrm{id}_{\mathbf{M}}) \circ (\Delta_{\mathbf{B}_1} \circ \mathrm{id}_{\mathbf{M}})$. Define $f^*\mathbf{M}(x_2)$ to be the equalizer of these two maps.

This construction is dual to the construction of $F_!\mathcal{M}$ for a DG functor of DG categories $\mathcal{A}_1 \to \mathcal{A}_2$ and a DG module $\mathcal{M}$ over $\mathcal{A}_1$.

LEMMA 3.1.1. *Let $\mathbf{M}$ be cofreely cogenerated over $\mathbf{B}_1$ by the system of k-modules $M(x_1)$, $x_1 \in \mathrm{Ob}\mathbf{B}_1$. Then $f^*\mathbf{M}$ is cofreely cogenerated over $\mathbf{B}_2$ by the system of k-modules $M(fx_2)$, $x_2 \in \mathrm{Ob}\mathbf{B}_2$.*

PROOF. Define a morphism

$$\bigoplus_{y_2 \in \mathrm{Ob}\mathbf{B}_2} \mathbf{B}_2(x_2,y_2) \otimes M(fy_2) \to f^*\mathbf{M}(x_2) \subset \bigoplus_{z_2,y_1} \mathbf{B}_2(x_2,z_2) \otimes \mathbf{B}_1(fz_2,y_1) \otimes M(y_1)$$

as $(\mathrm{id}_{\mathbf{B}_2} \otimes f \otimes \mathrm{id}_{\mathbf{M}}) \circ (\Delta_{\mathbf{B}} \otimes \mathrm{id}_{\mathbf{M}})$. ***MORE*** □

3.2. The trace functor. We will show that DG categories form a category in DG categories with the following additional structure.

(1) For every DG category $\mathcal{A}$, a DG comodule $\mathrm{TR}_{\mathcal{A}}$ over $\mathbf{B}(\mathcal{A},\mathcal{A})$.
(2) For any two DG categories $\mathcal{A}$ and $\mathcal{B}$, a morphism of DG comodules

$$\tau_{\mathcal{A},\mathcal{B}} : (\mathfrak{m}_{\mathcal{ABA}})^{*}\mathrm{TR}_{\mathcal{A}} \to (\mathfrak{m}_{\mathcal{BAB}} \circ \tau)^{*}\mathrm{TR}_{\mathcal{B}}$$

$$\begin{array}{ccc}
\mathbf{B}(\mathcal{A},\mathcal{B}) \otimes \mathbf{B}(\mathcal{B},\mathcal{A}) & \xrightarrow{\ \tau\ } & \mathbf{B}(\mathcal{B},\mathcal{A}) \otimes \mathbf{B}(\mathcal{A},\mathcal{B}) \\
\downarrow{\scriptstyle \mathfrak{m}_{\mathcal{ABA}}} & & {\scriptstyle \mathfrak{m}_{\mathcal{BAB}}}\downarrow \\
\mathbf{B}(\mathcal{A},\mathcal{A}) & & \mathbf{B}(\mathcal{B},\mathcal{B}) \\
\circlearrowright & & \circlearrowright \\
\mathrm{TR}_{\mathcal{A}} & & \mathrm{TR}_{\mathcal{B}}
\end{array}$$

where τ is the transposition;
(3) a homotopy $\sigma_{\mathcal{ABC}}$ between two morphisms of DG comodules

$$\mathrm{id} : (\mathfrak{m}_{\mathcal{ABCA}})^{*}\mathrm{TR}_{\mathcal{A}} \to (\mathfrak{m}_{\mathcal{ABCA}})^{*}\mathrm{TR}_{\mathcal{B}}$$

and

$$(\tau_{\mathcal{BCA}} \circ \tau^{2}) \circ (\tau_{\mathcal{CAB}} \circ \tau) \circ \tau_{\mathcal{ABC}} : (\mathfrak{m}_{\mathcal{ABCA}})^{*}\mathrm{TR}_{\mathcal{A}} \to (\mathfrak{m}_{\mathcal{ABCA}})^{*}\mathrm{TR}_{\mathcal{B}}$$

$$\begin{array}{ccccc}
\mathbf{B}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{A}) & \xrightarrow{\ \tau\ } & \mathbf{B}(\mathcal{C},\mathcal{A},\mathcal{B},\mathcal{C}) & \xrightarrow{\ \tau\ } & \mathbf{B}(\mathcal{B},\mathcal{C},\mathcal{A},\mathcal{B}) \\
\downarrow{\scriptstyle \mathfrak{m}_{\mathcal{ABCA}}} & & \downarrow{\scriptstyle \mathfrak{m}_{\mathcal{CABC}}} & & \downarrow{\scriptstyle \mathfrak{m}_{\mathcal{BCAB}}} \\
\mathbf{B}(\mathcal{A},\mathcal{A}) & & \mathbf{B}(\mathcal{C},\mathcal{C}) & & \mathbf{B}(\mathcal{B},\mathcal{B}) \\
\circlearrowright & & \circlearrowright & & \circlearrowright \\
\mathrm{TR}_{\mathcal{A}} & & \mathrm{TR}_{\mathcal{C}} & & \mathrm{TR}_{\mathcal{B}}
\end{array}$$

satisfying

$$\tau_{\mathcal{ABC}} \circ \sigma_{\mathcal{ABC}} = (\sigma_{\mathcal{ABC}} \circ \tau) \circ \tau_{\mathcal{ABC}}$$

as two homotopies between $\tau_{\mathcal{ABC}}$ and $\tau_{\mathcal{ABC}} \circ (\tau_{\mathcal{BCA}} \circ \tau^{2}) \circ (\tau_{\mathcal{CAB}} \circ \tau) \circ \tau_{\mathcal{ABC}}$.

Here

$$\mathbf{B}(\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{A}) = \mathbf{B}(\mathcal{A},\mathcal{B}) \otimes \mathbf{B}(\mathcal{B},\mathcal{C}) \otimes \mathbf{B}(\mathcal{C},\mathcal{A})$$

etc.; τ are cyclic permutations of length 3;

$$\tau_{\mathcal{ABC}} : (\mathfrak{m}_{\mathcal{ABCA}})^{*}\mathrm{TR}_{\mathcal{A}} \to (\mathfrak{m}_{\mathcal{CABC}} \circ \tau)^{*}\mathrm{TR}_{\mathcal{C}})$$

is defined as the composition

$$(\mathfrak{m}_{\mathcal{ACA}} \circ \mathfrak{m}_{\mathcal{ABC}})^{*}\mathrm{TR}_{\mathcal{A}} \xrightarrow{\tau_{\mathcal{AC}} \circ \mathfrak{m}_{\mathcal{ABC}}} (\mathfrak{m}_{\mathcal{CAC}} \circ \tau \circ \mathfrak{m}_{\mathcal{ABC}})^{*}\mathrm{TR}_{\mathcal{C}} = (\mathfrak{m}_{\mathcal{CABC}} \circ \tau)^{*}\mathrm{TR}_{\mathcal{C}}.$$

4. A category in DG cocategories with a di(tetra)module

Next we will outline the structure on cochains and chains that are both in coefficients in bimodules ${}_{\mathfrak{f}}\mathcal{B}_{\mathfrak{g}}$ where $\mathfrak{f}, \mathfrak{g} : \mathcal{A} \to \mathcal{B}$ are DG (or more generally A_{∞}) functors.

4.1. Cotrace of a bicomodule. If $\mathcal{M}$ is a DG module over a DG caterory $\mathcal{A}$ then

$$\mathrm{tr}_{\mathcal{A}}(\mathcal{M}) = \mathcal{M} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{op}}} \mathcal{A};$$

explicitly,

$$\mathrm{tr}_{\mathcal{A}}(\mathcal{M}) = \mathrm{coker}(\bigoplus_{x,y\in\mathrm{Ob}(\mathcal{A})} \mathcal{A}(x,y)\otimes\mathcal{M}(y,x) \to \bigoplus_{x\in\mathrm{Ob}(\mathcal{A})} \mathcal{M}(x,x))$$

where the morphism in the right hand side is given by

$$a\otimes m \mapsto am - (-1)^{|a||m|} ma$$

Dually, for a DG bicomodule $\mathbf{M}$ over a DG cocategory $\mathbf{B}$,

$$\mathrm{cotr}_{\mathbf{B}}(\mathbf{M}) = \ker(\bigoplus_{x\in\mathrm{Ob}(\mathbf{B})} \mathbf{M}(x,x) \to \bigoplus_{x,y\in\mathrm{Ob}(\mathbf{B})} \mathbf{B}(x,y)\otimes\mathbf{M}(y,x))$$

where the map in the right hand side is given by

$$m \mapsto \sum b^{(1)}\otimes m^{(2)} - (-1)^{|b^2||m^{(1)}|} b^{(2)}\otimes m^{(1)}$$

Here $\Delta^l : \mathbf{M}(x,y) \to \mathbf{B}(x,z)\otimes\mathbf{M}(z,y)$ is given by $m \mapsto \sum b^{(1)}\otimes m^{(2)}$ and $\Delta^r : \mathbf{M}(x,y) \to \mathbf{M}(x,z)\otimes\mathbf{B}(z,y)$ is given by $m \mapsto \sum m^{(1)}\otimes b^{(2)}$

For a DG functor $F : \mathbf{B}_1 \to \mathbf{B}_2$ and for a DG comodule $\mathbf{M}_2$ over $\mathbf{B}_2$, there is a natural map

$$F_\sharp : \mathrm{cotr}_{\mathbf{B}_1}(F^*\mathbf{M}_2) \to \mathrm{cotr}_{\mathbf{B}_2}\mathbf{M}_2 \tag{4.1}$$

4.2. The di(tetra)module structure. A di(tetra)module over a category $\mathbf{B}$ in DG categories is the following.

(1) A DG cobimodule $\mathbf{M}(\mathcal{A},\mathcal{B})$ over $\mathbf{M}(\mathcal{A},\mathcal{B})$ for every $\mathcal{A}$ and $\mathcal{B}$;
(2) a morphism of DG cobimodules over

$$\mu^r_{\mathcal{ABC}} : \mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C}) \to m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A},\mathcal{C})$$

and
(3) a morphism of DG cobimodules over $\mathbf{B}(\mathcal{B},\mathcal{C})$

$$\mu^l_{\mathcal{ABC}} : \mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})} m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A},\mathcal{C}) \to \mathbf{M}(\mathcal{B},\mathcal{C})$$

for every $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$

such that the following compatibility conditions hold.

1). The composition

$$\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D}) \to m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D}) \xrightarrow{\sim}$$

$$m^*_{\mathcal{ABC}}(\mathbf{M}(\mathcal{A},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})) \to m^*_{\mathcal{ABC}}m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D}) = m^*_{\mathcal{ABCD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

is the same as the composition

$$\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D}) \to \mathbf{M}(\mathcal{A},\mathcal{B})\otimes m^*_{\mathcal{BCD}}\mathbf{B}(\mathcal{B},\mathcal{D}) \xrightarrow{\sim}$$

$$m^*_{\mathcal{BCD}}(\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{D})) \to m^*_{\mathcal{BCD}}m^*_{\mathcal{ABD}}\mathbf{M}(\mathcal{A},\mathcal{D}) = m^*_{\mathcal{ABCD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

2). The composition

$$\mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})}\mathrm{cotr}_{\mathbf{B}(\mathcal{B},\mathcal{C})} m^*_{\mathcal{ABC}}m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D}) \xrightarrow{\sim} \mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})} m^*_{\mathcal{ABC}}m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

$$\xrightarrow{(m_{\mathcal{ABC}})_\sharp} \mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{C})} m^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D}) \to \mathbf{M}(\mathcal{C},\mathcal{D})$$

is the same as the composition

$$\mathrm{cotr}_{\mathbf{B}(\mathcal{B},\mathcal{C})}\mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})}\mathfrak{m}^*_{\mathcal{BCD}}\mathfrak{m}^*_{\mathcal{ABD}}\mathbf{M}(\mathcal{A},\mathcal{D}) \xrightarrow{\sim} \mathrm{cotr}_{\mathbf{B}(\mathcal{B},\mathcal{C})}\mathfrak{m}^*_{\mathcal{BCD}}\mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})}\mathfrak{m}^*_{\mathcal{ABD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

$$\to \mathrm{cotr}_{\mathbf{B}(\mathcal{B},\mathcal{C})}\mathfrak{m}^*_{\mathcal{BCD}}\mathbf{M}(\mathcal{B},\mathcal{D}) \to \mathbf{M}(\mathcal{C},\mathcal{D})$$

Here $(\mathfrak{m}_{\mathcal{ABC}})_\sharp$ is as in (4.1).

3). The composition

$$\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D}) \to \mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathfrak{m}^*_{\mathcal{BCD}}\mathbf{B}(\mathcal{B},\mathcal{D})$$

$$\xrightarrow{\sim} \mathfrak{m}^*_{\mathcal{BCD}}(\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{D})) \to \mathfrak{m}^*_{\mathcal{BCD}}\mathfrak{m}^*_{\mathcal{ABD}}\mathbf{M}(\mathcal{A},\mathcal{D}) \xrightarrow{\sim} \mathfrak{m}^*_{\mathcal{ABCD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

is the same as the composition

$$\mathbf{M}(\mathcal{A},\mathcal{B})\otimes\mathbf{B}(\mathcal{B},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D}) \to \mathfrak{m}^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})$$

$$\mathfrak{m}^*_{\mathcal{ABC}}(\mathbf{M}(\mathcal{A},\mathcal{C})\otimes\mathbf{B}(\mathcal{C},\mathcal{D})) \to \mathfrak{m}^*_{\mathcal{ABC}}\mathfrak{m}^*_{\mathcal{ACD}}\mathbf{M}(\mathcal{A},\mathcal{D}) \xrightarrow{\sim} \mathfrak{m}^*_{\mathcal{ABCD}}\mathbf{M}(\mathcal{A},\mathcal{D})$$

4.3. Di(tetra)modules over Hopf algebras.

5. Constructions and proofs

In this section we define the above structures on Hochschild chains and cochains and provide the needed proofs.

5.1. Operations on chains and cochains. In addition to cup product and braces on cochains, we define three types of pairings between Hochschild cochains and chains. We will refer to them as operations of types $\mathbf{j}$, $\mathbf{L}$, and $\mathbf{B}$. They are, roughly, noncommutative/higher counterparts of: a) contraction $\iota_\pi\alpha$ of a form by a multivector; b) Lie derivative $L_\pi\alpha$ of a form by a multivector; and c) the De Rham differential $d\alpha$ of a form.

5.1.1. *Operations of type* $\mathbf{j}$. For a morphism $f:\mathcal{A}\to\mathcal{A}$, define the chain complex $C_\bullet(\mathcal{A},{}_f\mathcal{A})$; here ${}_f\mathcal{A}$ is viewed as an $\mathcal{A}$-bimodule *via* $a_0\cdot a\cdot a_1 = f(a_0)aa_1$. Define the pairing

$$C^\bullet(\mathcal{A},{}_f\mathcal{A}_g)\otimes C_{-\bullet}(\mathcal{A},{}_g\mathcal{A}) \to C_{-\bullet}(\mathcal{A},{}_f\mathcal{A}) \tag{5.1}$$

by

$$j_\varphi(a_0\otimes\ldots\otimes a_n) = \pm\varphi(a_{n-k+1},\ldots,a_n)a_0\otimes a_1\otimes\ldots\otimes a_k \tag{5.2}$$

for a k-cochain φ.

LEMMA 5.1.1. *The pairing* (5.1) *is a morphism of complexes. Together with the assignment* $f\mapsto C_{-\bullet}(\mathcal{A},{}_f\mathcal{A})$ *it defines a DG module* $\mathbf{C}_\mathcal{A}$ *over the DG category* $\mathbf{C}^\bullet(\mathcal{A},\mathcal{A})$.

5.1.2. *Operations of type* $\mathbf{L}$.

5.1.3. *Notation.* For $\mathbf{a} = a_1 \otimes \ldots \otimes a_n$ and for $f : \mathcal{A} \to \mathcal{B}$, we will write

$$f(\mathbf{a}) = f(a_1) \otimes \ldots \otimes f(a_n)$$

For any partition $1 \le k_1 \le \ldots \le k_{p-1} \le n$, we write

$$\mathbf{a} = \mathbf{a}_1 \otimes \ldots \otimes \mathbf{a}_p \tag{5.3}$$

if $\mathbf{a}_1 = a_1 \otimes \ldots \otimes a_{k_1}$, $\mathbf{a}_2 = a_{k_1+1} \otimes \ldots \otimes a_{k_2}$, ..., $\mathbf{a}_p = a_{k_{p-1}+1} \otimes \ldots \otimes a_n$.

Similarly, for $\Phi = (\varphi_1|\ldots|\varphi_m)$, let $\Phi_1 = (\varphi_1|\ldots|\varphi_{l_1})$, ..., $\Phi_q = (\varphi_{l_{q-1}}|\ldots|\varphi_m)$. Then we will write

$$\Phi = (\Phi_1|\ldots|\Phi_q). \tag{5.4}$$

Remark 5.1.2. If Φ_j is of length 1, i.e. if $\Phi_j = (\varphi_k)$ for some k, we will write **somewhat awkwardly, subject to change?** $\varphi_k = \widetilde{\varphi}_j$ and

$$\Phi = (\Phi_1|\ldots|\widetilde{\varphi}_j|\ldots|\Phi_q).$$

Furthermore, we will write

$$\varphi(\mathbf{a}) = \varphi(a_1, \ldots, a_n)$$

for any n-cochain φ. Let $\varphi_j \in C^\bullet(\mathcal{A},_{f_j} \mathcal{B}_{f_{j+1}})$ and $a_i \in \mathcal{A}$. Then put

$$\lambda_\Phi \mathbf{a} = \sum \pm a_0 \otimes f_1(\mathbf{a}_1) \otimes \varphi_1(\mathbf{a}_2) \otimes f_2(\mathbf{a}_3) \otimes \ldots \otimes \varphi_m(\mathbf{a}_{2m}) \otimes f_{m+1}(\mathbf{a}_{2m+1}) \tag{5.5}$$

The sum is taken over all partitions (5.3).

Lemma 5.1.3. *The above formula defines a morphism of DG coalgebras*

$$\mathrm{Bar}(\mathcal{A}) \otimes \mathrm{Bar}(\mathbf{C}^\bullet(\mathcal{A}, \mathcal{B})) \to \mathrm{Bar}(\mathcal{B})$$

Proof. □

Remark 5.1.4. It is allowed for $\mathbf{a}_j$ or Φ_j to be empty, *i.e.* $k_j = k_{j+1}$ or $l_j = l_{j+1}$. If the latter happens, we put

$$\lambda_{\Phi_j} \mathbf{a} = f_j(\mathbf{a}) \tag{5.6}$$

Finally, for the sake of convenience, we recall the differentials in the bar construction. For $\Phi = (\varphi_1|\ldots|\varphi_m)$, write

$$d\Phi = \sum_{j=1}^{m-1} \pm(\varphi_1 \ldots|\varphi_j \varphi_{j+1}|\ldots \varphi_m) + \sum_{j=1}^{m} \pm(\varphi_1 \ldots|\delta\varphi_j|\ldots \varphi_m) \tag{5.7}$$

For $\mathbf{a} = (a_1|\ldots|a_n)$ (same as $a_1 \otimes \ldots \otimes a_n$), put

$$d\mathbf{a} = \sum_{j=1}^{n-1} \pm(a_1 \ldots|a_j a_{j+1}|\ldots a_n) \tag{5.8}$$

5.1.4. *The operations* $L(\Phi, \Psi)$. For $\varphi_j \in C^\bullet(\mathcal{A},_{f_j} \mathcal{B}_{f_{j+1}})$, $1 \le j \le m$, and $\psi_i \in C^\bullet(\mathcal{B},_{g_i} \mathcal{A}_{g_{i+1}})$, $1 \le i \le n$, and $a_j \in \mathcal{A}$, define

$$L(\Phi, \Psi)(a_0 \otimes \mathbf{a}) = \sum \pm \varphi_1(\lambda_\Psi \lambda_{\Phi_2} \mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda_{\Phi_1}(\mathbf{a}_2) \tag{5.9}$$

if Ψ is not empty, and

$$L(\Phi, \Psi)(a_0 \otimes \mathbf{a}) = \sum \pm \varphi_1(\lambda_{\Phi_2} \mathbf{a}_3, a_0, \mathbf{a}_1) \otimes \lambda_{\Phi_1}(\mathbf{a}_2) + a_0 \otimes \lambda_\Phi \mathbf{a} \tag{5.10}$$

if Ψ is empty. The summation is over all partitions $\Phi = (\varphi_1|\Phi_1|\Phi_2)$.

5.1.5. *Operations of type* $\mathbf{B}$. Let Φ_j be a cochain in $\mathbf{C}^\bullet(\mathcal{A}_{j-1}, \mathcal{A}_j)(f_j, g_j)$, $j = 1, \ldots, p$, and $\mathcal{A}_p = A_0$. Let $a_0 \otimes \mathbf{a}$ be a Hochschild chain in $C_{-\bullet}(\mathcal{A}_{0,g_p \ldots g_1} \mathcal{A}_0)$. Define

$$B(\Phi_1, \ldots, \Phi_p)(a_0 \otimes \mathbf{a}) = \sum \pm 1 \otimes \lambda_{\Phi_1 \bullet \ldots \bullet \Phi_p} \mathbf{a}_2 \otimes a_0 \otimes \mathbf{a}_1 \tag{5.11}$$

which is a chain in $C_{-\bullet}(\mathcal{A}_{0,f_p \ldots f_1} \mathcal{A}_0)$. The sum is over all partitions $\mathbf{a} = \mathbf{a}_1 \otimes \mathbf{a}_2$.

6. Operations on chains and cochains and a trace functor in cocategories

6.0.1. *Operations of type* $\mathbf{j}$ *and the DG comodule* $\mathrm{TR}_\mathcal{A}$.

Definition 6.0.1. *Denote by* $\mathrm{TR}_\mathcal{A}$ *the DG comodule* $\mathrm{Bar}(\mathbf{C}(\mathcal{A}, \mathcal{A}), \mathbf{C}_\mathcal{A})$ *over the DG cocategory* $\mathbf{B}(\mathcal{A}, \mathcal{A}) = \mathrm{Bar}(\mathbf{C}(\mathcal{A}, \mathcal{A}))$ *where* $\mathbf{C}_\mathcal{A}$ *is the DG module from Lemma 5.1.1.*

6.0.2. *Operations of type* $\mathbf{L}$ *and the morphism* $\tau_{\mathcal{A}\mathcal{B}}$.

Lemma 6.0.2. *The DG comodule* $m^*_{\mathcal{A}\mathcal{B}\mathcal{A}} \mathrm{TR}_\mathcal{A}$ *is a cofree graded comodule over* $\mathrm{Bar}(\mathbf{C}^\bullet(\mathcal{A}, B)) \otimes \mathrm{Bar}(\mathbf{C}^\bullet(\mathcal{B}, \mathcal{A}))$ *cogenerated by* $\mathbf{C}_\mathcal{A}(f \otimes g) = C_{-\bullet}(\mathcal{A}_{,gf} \mathcal{A})$, *with the differential*

$$\begin{aligned}
(\varphi_1|\ldots|\varphi_m)(\psi_1|\ldots|\psi_n)c \mapsto \pm(\varphi_1|\ldots|\varphi_m)(\psi_1|\ldots|\psi_n)(b+\partial)c+ \\
\sum_{j=1}^{m-1} \pm(\varphi_1|\ldots|\varphi_j\varphi_{j+1}|\ldots|\varphi_m)(\psi_1|\ldots|\psi_n)c+ \\
\sum_{k=1}^{n-1} \pm(\varphi_1|\ldots|\varphi_m)(\psi_1|\ldots|\psi_k\psi_{k+1}|\ldots|\psi_n)c+ \\
\sum_{j=1}^{m} \pm(\varphi_1|\ldots|(\delta+\partial)\varphi_j|\ldots|\varphi_m)(\psi_1|\ldots|\psi_n)c+ \\
\sum_{k=1}^{n} \pm(\varphi_1|\ldots|\varphi_m)(\psi_1|\ldots|(\delta+\partial)\psi_k|\ldots|\psi_n)c+ \\
(\varphi_1|\ldots|\varphi_{m-1})(\psi_1|\ldots|\psi_n)j_{\varphi_m}c+ \\
\sum_{k=1}^{n+1} \pm(\varphi_1|\ldots|\varphi_k)(\psi_1|\ldots|\psi_{n-1})j_{\psi_n\{\varphi_{k+1},\ldots\varphi_m\}}c
\end{aligned}$$

for a chain c *in* C_A *and for composable morphisms* φ_j *in* $\mathbf{C}^\bullet(A, B)$ *and* ψ_k *in* $\mathbf{C}^\bullet(B, A)$. *In other words:*

$$(\Phi)(\Psi)c \mapsto (d\Phi)(\Psi)c \pm (\Phi)(d\Psi)c \pm (\Phi)(\Psi)(b+\partial)c+$$

$$\sum \pm(\Phi_1)(\Psi)j_{\widetilde{\varphi}_2}c + \sum \pm(\Phi_1)(\Psi_2)j_{\widetilde{\psi}_2\{\Phi_2\}}c$$

Here $b+\partial$ *is the total differential on the Hochschild chain complex and* $\delta+\partial$ *is the total differential on the Hochschild cochain complex.*

(cf. Remark 5.1.2).

Proof. □

PROPOSITION 6.0.3. *The formula*

$$(\Phi)(\Psi)c \mapsto \sum (\Phi_1)(\Psi_1)L(\Phi_2, \Psi_2)c$$

defines a morphism of DG comodules

$$\tau_{\mathcal{AB}} : m^*_{\mathcal{ABA}}\mathrm{TR}_{\mathcal{A}} \to m^*_{\mathcal{BAB}}\mathrm{TR}_{\mathcal{B}}$$

(cf. 2 of 3.2.

PROOF. □

6.0.3. *Operations of type* **B** *and the homotopy* σ.

PROPOSITION 6.0.4. *The formula*

$$(\Phi)(\Psi)(\Theta)c \mapsto \sum (\Phi_1)(\Psi_1)(\Theta_1)B(\Phi_2, \Psi_2, \Theta_2)c$$

defines a morphism of degree +1 graded comodules

$$\sigma_{\mathcal{ABC}} : m^*_{\mathcal{ABCA}}\mathrm{TR}_{\mathcal{A}} \to m^*_{\mathcal{ABCA}}\mathrm{TR}_{\mathcal{A}}$$

satisfying condition 3 of 3.2.

PROOF. □

6.1. Products between chains and cochains. Let φ be a cochain in $C^\bullet(\mathcal{A},_{f_1} \mathcal{B}_{f_2})$ and c a chain in $C_\bullet(\mathcal{A},_{f_2} \mathcal{B}_{f_3})$. Define the chain $j_\varphi c$, or just φc, by

$$j_\varphi(a_0 \otimes \ldots a_n) = \pm\varphi(a_{n-k+1}, \ldots, a_n)a_0 \otimes \ldots \otimes a_{n-k} \tag{6.1}$$

Now let c be a chain in $C_\bullet(\mathcal{A},_{f_1} \mathcal{B}_{f_2})$ and φ a cochain in $C^\bullet(\mathcal{A},_{f_2} \mathcal{B}_{f_3})$. Define the chain $i_\varphi c$, or just $c\varphi$, by

$$i_\varphi(a_0 \otimes \ldots a_n) = \pm a_0\varphi(a_1, \ldots, a_k) \otimes a_{k+1} \ldots \otimes a_n \tag{6.2}$$

LEMMA 6.1.1. *The assignment* $f_1, f_2 \mapsto C_{-\bullet}(\mathcal{A},_{f_1} \mathcal{B}_{f_2})$ *together with the products* (6.1) *and* (6.2) *define a DG bimodule over the DG category* $\mathbf{C}^\bullet(\mathcal{A}, \mathcal{B})$. *We denote this DG bimodule by* $\mathbf{C}_{-\bullet}(\mathcal{A}, \mathcal{B})$.

6.2. Braces between chains and cochains. Let $f_i : \mathcal{A} \to \mathcal{B}$ for $1 \le i \le m+1$; $f'_j : \mathcal{A} \to \mathcal{B}$ for $1 \le j \le n+1$; $g_k : \mathcal{B} \to \mathcal{C}$ for $k = 1, 2$.

Consider cochains $\varphi_i \in C^\bullet(\mathcal{A},_{f_i} \mathcal{B}_{f_{i+1}}), 1 \le i \le m$; $\varphi'_j \in C^\bullet(\mathcal{A},_{f'_j} \mathcal{B}_{f'_{j+1}})$, $1 \le j \le n$; and $\psi \in C^\bullet(\mathcal{B},_{g_1} \mathcal{C}_{g'_1})$. Let c be a chain in $C_{-\bullet}(\mathcal{A},_{f_{m+1}} \mathcal{B}_{f'_1})$. Let $\Phi = (\varphi_1|\ldots|\varphi_m)$ and $\Phi' = (\varphi'_1|\ldots|\varphi'_n)$. Define the chain $\mu_\psi(\Phi, \Phi')c$ in $C_{-\bullet}(\mathcal{A},_{g_1 f_1} \mathcal{C}_{g'_1 f'_{n+1}})$ by

$$\mu_\psi(\Phi, \Phi')(b_0 \otimes \mathbf{a}) = \sum \psi(\lambda_\Phi(\mathbf{a}_3), b_0, \lambda_{\Phi'}(\mathbf{a}_1)) \otimes \mathbf{a}_2 \tag{6.3}$$

(cf. 5.1.3).

Next, let $f_i : \mathcal{A} \to \mathcal{B}$ for $1 \le i \le n+1$; $g_k : \mathcal{B} \to \mathcal{C}$ for $k = 1, 2$. Consider cochains $\varphi_i \in C^\bullet(\mathcal{A},_{f_{i+1}} \mathcal{B}_{f_i}), 1 \le i \le n$; let c be a chain in $C_{-\bullet}(\mathcal{A},_{g_1 f_1} \mathcal{C}_{g_2 f_{n+1}})$. Let $\Phi = (\varphi_1|\ldots|\varphi_n)$. Define the chain $\nu_\Phi c$ in $C_{-\bullet}(\mathcal{B},_{g_1} \mathcal{C}_{g_2})$ by

$$\nu_\Phi(c_0 \otimes \mathbf{a}) = c_0 \otimes \lambda_\Phi(\mathbf{a}) \tag{6.4}$$

LEMMA 6.2.1.

$$\nu_\Phi \nu_\Psi = \pm \nu_{\Psi \bullet \Phi};$$
$$\mu_\theta(\Phi, \Phi')\nu_\Psi = \sum \pm \nu_{\Psi_2} \mu_\theta(\Psi_3 \bullet \Phi, \Psi_1 \bullet \Phi');$$
$$\mu_{\theta\{\Psi\}}(\Phi, \Phi') = \sum \pm \mu_\theta(\Phi \bullet \Psi_1, \Phi' \bullet \Psi_2) + \sum \pm \mu_\theta(\Phi_1 \bullet \Psi_1, \Phi'_2 \bullet \Psi_3)\mu_{\psi_2}(\Phi_2, \Phi'_1)$$
$$[b + \partial, \mu_\psi(\Phi, \Phi')] - \mu_{\delta\psi}(\Phi, \Phi') \mp \mu_\psi(d\Phi, \Phi') \mp \mu_\psi(\Phi, d\Phi') =$$
$$\sum \pm \mu_\psi(\Phi_1, \Phi')j_{g_1\widetilde{\varphi}_2} + \sum \pm j_{g_1\widetilde{\varphi}_1}\mu_\psi(\Phi_2, \Phi') + \sum \pm \mu_\psi(\Phi, \Phi'_2)i_{g'_1\widetilde{\varphi}'_1} + \sum \pm i_{g'_1\widetilde{\varphi}'_2}\mu_\psi(\Phi, \Phi'_1)$$
$$[b + \partial, \nu_\Phi] - \nu(d\Phi) = \sum \pm \nu_{\Phi_2} j_{g_1\widetilde{\varphi}_1} + \sum \pm \nu_{\Phi_1} i_{g_2\widetilde{\varphi}_{n+1}}$$

6.3. Construction of the twisted tetramodule structure.

DEFINITION 6.3.1. *Set*

$$\mathbf{M}(\mathcal{A}, \mathcal{B}) = \mathrm{Bar}(\mathbf{C}^\bullet(\mathcal{A}, \mathcal{B}),\ \mathbf{C}_{-\bullet}(\mathcal{A}, \mathcal{B})), \mathbf{C}^\bullet(\mathcal{A}, \mathcal{B})$$

(the bar construction of the DG bimodule $\mathbf{C}_{-\bullet}(\mathcal{A}, \mathcal{B}))$ *over the DG category* $\mathbf{C}^\bullet(\mathcal{A}, \mathcal{B})$*).*

This is a DG cobimodule over the DG cocategory $\mathbf{B}(\mathcal{A}, \mathcal{B})$.

For $(\Phi)c(\Phi') \in \mathbf{M}(\mathcal{A}, \mathcal{B})$ and $\Psi \in \mathbf{B}(\mathcal{B}, \mathcal{C})$, define

$$(\Phi)c(\Phi') \bullet (\Psi) = \sum \pm ((\Phi_1)(\Psi_1))(\mu_{\widetilde{\psi}_2}(\Phi_2, \Phi'_1)c)((\Phi'_2)(\Psi_3)) \tag{6.5}$$

For $(\Psi) \in \mathbf{B}(\mathcal{B}, \mathcal{C})(g_1, g_2)$, $c \in \mathbf{C}_{-\bullet}(\mathcal{A}, \mathcal{C})(g_2 f_1, g'_1 f_2)$, $\Psi' \in \mathbf{B}(\mathcal{B}, \mathcal{C})(g'_1, g'_2)$, and $\Phi \in \mathbf{B}(\mathcal{A}, \mathcal{B})(f_2, f_1)$, define an element in $\mathbf{M}(g_1, g'_2)$ by

$$(\Phi) \star ((\Psi)c(\Psi')) = \pm(\Psi)(\nu_\Phi c)(\Psi') \tag{6.6}$$

PROPOSITION 6.3.2. *Formula* (6.5) *defines a morphism of DG comodules over* $\mathbf{B}(\mathcal{A}, \mathcal{B}) \otimes \mathbf{B}(\mathcal{B}, \mathcal{C})$

$$\mu^r_{\mathcal{ABC}} : \mathbf{M}(\mathcal{A}, \mathcal{B}) \otimes \mathbf{B}(\mathcal{B}, \mathcal{C}) \to m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A}, \mathcal{C})$$

and formula (6.6) *defines a morphism of DG cobimodules over* $\mathbf{B}(\mathcal{B}, \mathcal{C})$

$$\mu^l_{\mathcal{ABC}} : \mathrm{cotr}_{\mathbf{B}(\mathcal{A},\mathcal{B})} m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A}, \mathcal{C}) \to \mathbf{M}(\mathcal{B}, \mathcal{C})$$

These morphisms satisfy the conditions 1), 2), 3) from 4.2

PROOF. An element of $m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A}, \mathcal{C})$ is a sum of monomials

$$((\Phi)(\Psi))c((\Phi')(\Psi'))$$

where $(\Phi) \in \mathbf{B}(\mathcal{A}, \mathcal{B})(f_1, f_2)$; $(\Phi') \in \mathbf{B}(\mathcal{A}, \mathcal{B})(f'_1, f'_2)$; $(\Psi) \in \mathbf{B}(\mathcal{B}, \mathcal{C})(g_1, g_2)$; $(\Psi') \in \mathbf{B}(\mathcal{A}, \mathcal{B})(g'_1, g'_2)$; $c \in \mathbf{C}_{-\bullet}(\mathcal{A}, \mathcal{C})(g_2 f_2, g'_1 f'_1)$. The differential sends such a monomial to

$$\pm((\Phi)(\Psi))(b + \partial)c((\Phi')(\Psi')) \pm ((d\Phi)(\Psi))c((\Phi')(\Psi'))\pm$$
$$((\Phi)(d\Psi))c((\Phi')(\Psi')) \pm ((\Phi)(\Psi))c((d\Phi')(\Psi')) \pm ((\Phi)(\Psi))c((\Phi')(d\Psi'))+$$
$$\sum \pm((\Phi_1)(\Psi))j_{\widetilde{\varphi}_2}c((\Phi')(\Psi')) + \sum \pm((\Phi_1)(\Psi_1))j_{\widetilde{\Psi}_2\{\Phi_2\}}c((\Phi')(\Psi'))+$$
$$\sum \pm((\Phi)(\Psi))i_{g'_1\widetilde{\varphi'}_1}c((\Phi'_2)(\Psi')) + \sum \pm((\Phi)(\Psi))i_{\widetilde{\Psi}'_1\{\Phi'_1\}}c((\Phi'_2)(\Psi'_2))$$

(compare to Lemma 6.0.2). An element of $\mathrm{cotr}_{\mathcal{ABC}} m^*_{\mathcal{ABC}}\mathbf{M}(\mathcal{A}, \mathcal{C})$ is a sum of monomials

$$(\Phi)((\Psi)c(\Psi'))$$

where $(\Phi) \in \mathbf{B}(\mathcal{A}, \mathcal{B})(f_1, f_2)$; $(\Psi) \in \mathbf{B}(\mathcal{B}, \mathcal{C})(g_1, g_2)$; $(\Psi') \in \mathbf{B}(\mathcal{A}, \mathcal{B})(g'_1, g'_2)$; $c \in \mathbf{C}_{-\bullet}(\mathcal{A}, \mathcal{C})(g_2 f_2, g'_1 f_1)$. The differential sends such a monomial to

$$\pm(\Phi)((\Psi)(b + \partial)c(\Psi')) \pm (d\Phi)((\Psi)c(\Psi')) \pm (\Phi)((d\Psi)c(\Psi')) \pm (\Phi)((\Psi)c(d\Psi'))+$$

$$\sum \pm(\Phi_1)((\Psi)j_{g_2\widetilde{\varphi}_2}c(\Psi')) + \sum \pm(\Phi_1)((\Psi)j_{\widetilde{\psi}_2\{\Phi_2\}}c(\Psi'))+$$
$$+\sum \pm(\Phi_2)(\Psi)i_{g_1'\widetilde{\varphi}_1}c(\Psi_2) + \sum \pm(\Phi_2)(\Psi)i_{\widetilde{\psi}_1'\{\Phi_1\}}(\Psi_2')$$

□

7. A category in DG categories

For any DG categories $\mathcal{A}_0$, ... , $\mathcal{A}_n$, define

$\mathcal{C}(\mathcal{A}_0 \to \ldots \to \mathcal{A}_n) = \mathrm{Cobar}(\mathrm{Bar}(\mathbf{C}(\mathcal{A}_0, \mathcal{A}_1)) \otimes \ldots \otimes \mathrm{Bar}(\mathbf{C}(\mathcal{A}_{n-1}, \mathcal{A}_n)))$

These DG categories carry a structure that we call a homotopy category in DG categories, i.e.

I. DG functors

$$\mu_{j_0 \ldots j_n} : \mathcal{C}(\mathcal{A}_0 \to \ldots \to \mathcal{A}_n) \to \mathcal{C}(\mathcal{A}_{j_0} \to \ldots \to \mathcal{A}_{j_m})$$

for all $m > 0$ and all $0 = j_0 \leq j_1 \leq \ldots \leq j_m = n$;

II. DG functors

$$\Delta_k : \mathcal{C}(\mathcal{A}_0 \to \ldots \to \mathcal{A}_n) \to \mathcal{C}(\mathcal{A}_0 \to \ldots \to \mathcal{A}_k) \otimes \mathcal{C}(\mathcal{A}_k \to \ldots \to \mathcal{A}_n)$$

for $k = 1, \ldots, n-1$

such that DG functors II are:

(1) coassociative;
(2) compatible with DG functors I, namely:

$$(\mu_{j_0 \ldots j_l} \otimes \mu_{j_l \ldots j_n}) \circ \Delta_k = \Delta_{j_l} \circ \mu_{j_0 \ldots j_m};$$

(3) weak equivalences.

Note that in our case DG functors II are bijections on objects, so being weak equivalences just means being quasi-isomorphisms on morphisms.

DG functors I and II are constructed as follows. I are induced on Cobar by the • product ***REF***, whereas II are obtained from the dual EZ product

$$\mathrm{Cobar}(\mathbf{B}_1) \otimes \mathrm{Cobar}(\mathbf{B}_2) \longrightarrow \mathrm{Cobar}(\mathbf{B}_1 \otimes \mathbf{B}_2) \tag{7.1}$$

7.1. The Grothendieck construction. Recal the category Δ from 3. Its objects are $[n]$, $n \geq 0$, and $\Delta'([n],[m])$ consists of transformations

$$(x_0, \ldots x_n) \mapsto (x_{J_0}, \ldots, x_{J_m}), \tag{7.2}$$

(cf.(2)). We write

$$x_{J_k} = x_{j_k+1} \ldots x_{j_{k+1}} \tag{7.3}$$

where $-1 = j_0 \leq \ldots \leq j_k \leq j_{k+1} \ldots \leq j_{m+1} = n$ (and the product of an empty number of x_j is equal to 1).

The category Δ' acts on the ***Set?*** of all symbols

$$\mathcal{A}_0 \to \ldots \to \mathcal{A}_{n+1} \tag{7.4}$$

($n \geq 0$) where $\mathcal{A}_j$ are DG categories. Namely, a morphism (7.3) sends such (7.4) to $\mathcal{A}_{j_0+1} \to \ldots \to \mathcal{A}_{j_{m+1}+1}$.

Definition 7.1.1. *Define the category* Δ'_{Alg} *as follows. Its objects are* $(n, \mathbf{A})$ *where* $n \geq 0$ *and* $\mathbf{A}$ *is as in* (7.4)*; morphisms from* $(n, \mathbf{A})$ *to* $(m, \mathbf{A}')$ *are morphisms in* $\Delta'([n],[m])$ *such that* $\delta\mathbf{A} = \mathbf{A}'$.

For $\mathbf{A}$ as in (7.4), define $s\mathbf{A} = \mathcal{A}_0$ and $t\mathbf{A} = \mathcal{A}_{n+1}$. Define $\Delta^{(N)}_{\mathrm{Alg}}$ to be the full subcategory of $\prod_{i=0}^{N} \Delta'_{\mathrm{Alg}}$ with objects $(\mathbf{A}_1, \dots, \mathbf{A}_N)$ such that $t\mathbf{A}_i = s\mathbf{A}_{i+1}$ for $i = 0, \dots, N-1$. We have the obvious functors

$$D_j : \Delta^{(N+1)} \to \Delta^{(N)}$$

$(0 \leq j \leq N)$ such that

$$(\mathbf{A}_0, \dots, \mathbf{A}_N) \mapsto (\mathbf{A}_0, \dots, \mathbf{A}_j \circ \mathbf{A}_{j+1}, \dots, \mathbf{A}_N)$$

where

$$(\mathcal{A}_0 \to \dots \to \mathcal{A}_{n+1}) \circ (\mathcal{A}_{n+1} \to \dots \to \mathcal{A}_{n+m+1}) = \mathcal{A}_0 \to \dots \to \mathcal{A}_{n+1} \to \dots \to \mathcal{A}_{n+m+1},$$

and

$$S_j : \Delta^{(N)} \to \Delta^{(N+1)},$$

$0 \leq j \leq N+1$, such that

$$(\mathbf{A}_0, \dots, \mathbf{A}_N) \mapsto (\mathbf{A}_0, \dots, \mathbf{A}_{j-1}, (\mathcal{A} \to \mathcal{A}), \mathbf{A}_j, \dots, \mathbf{A}_N)$$

where

$$\mathcal{A} = t\mathbf{A}_{j-1} = s\mathbf{A}_j.$$

Remark 7.1.2. We get a cosimplicial category $\Delta^{(*)}_{\mathrm{Alg}}$ (in other words, a functor from Δ' to categories). The structure of a homotopy category in DG categories that we constructed above can be interpreted as:

(1) a functor $\mathcal{C}^{(N)}$ from $\Delta^{(N)}_{\mathrm{Alg}}$ to DG categories for any $N \geq 0$;;
(2) a natural transformation $\delta^\dagger : \delta^* \mathcal{C}^{(M)} \to \mathcal{C}^{(N)}$ for any $\delta \in \Delta'([N], [M])$ such that
(3) $\delta^\dagger$ is a weak equivalence on every object of $\Delta^{(N)}_{\mathrm{Alg}}$, and
(4)
$$(\delta_1 \delta_2)^\dagger = \delta_2^* (\delta_1^\dagger) \delta_2^\dagger$$
for any composable δ_1 and δ_2 in Δ'.

8. A category in DG categories with a trace functor

8.1. Trace functors in terms of the Grothendieck construction. Recall the definition of a homotopy category in DG categories as in Remark 7.1.2. We will extend it as follows. Let Λ_{Alg} be the category whose objects are $(n, \mathbf{A})$ where $n \geq 0$ and $\mathbf{A}$ is a *cyclic word of length* n, i.e. a symbol

$$\mathcal{A}_0 \to \dots \to \mathcal{A}_n \to \mathcal{A}_0 \tag{8.1}$$

where $\mathcal{A}_j$ are DG categories. A morphism $\lambda \in \Lambda([n], [m])$ transforms cyclic words of length n into cyclic words of length m. A morphism $(n, \mathbf{A}) \to (m, \mathbf{A}')$ in Λ_{Alg} is a morphism $\lambda \in \Lambda([n], [m])$ that transforms $\mathbf{A}$ to $\mathbf{A}'$. The composition is defined by the composition in Λ. We define $\Lambda_{\infty,\mathrm{Alg}}$ in exactly the same way with Λ replaced by Λ_∞.

We will usually write $\mathbf{A}$ instead of $(n, \mathbf{A})$ (of course the length of $\mathbf{A}$ is n.

A homotopy trace functor on a homotopy category $\mathcal{C}$ in DG categories (cf. Remark 7.1.2) is:

(1) A functor from Λ_{Alg} to DG categories that extends the restriction of $\mathcal{C}$ to the full subcategory of Δ_{Alg} whose objects are $(n, \mathbf{A})$ where $\mathbf{A}$ are cyclic words (we denote this functor also by $\mathcal{C}$);

(2) a DG module $\mathrm{TR}_{\mathbf{A}}$ over $\mathcal{C}(\mathbf{A})$ for every cyclic word $\mathbf{A}$;

(3) a weak equivalence $\lambda^\dagger(\mathbf{A}) : \lambda^*\mathrm{TR}_{\lambda\mathbf{A}} \to \mathrm{TR}_{\mathbf{A}}$ for every morphism λ in $\Lambda_{\infty,\mathrm{Alg}}$ such that

$$(\lambda\mu)^\dagger(\mathbf{A}) = \mu^\dagger(\mathbf{A})\mu^*\lambda^\dagger(\mu\mathbf{A})$$

in the diagram

$$\begin{CD}
\mu^*\lambda^*\mathrm{TR}_{\lambda\mu\mathbf{A}} @>{\sim}>> (\lambda\mu)^*\mathrm{TR}_{\lambda\mu\mathbf{A}}\\
@V{\mu^*(\lambda^\dagger(\mu\mathbf{A}))}VV @VV{(\lambda\mu)^*}V\\
\mu^*\mathrm{TR}_{\mathbf{A}} @>{\mu^\dagger}>> \mathrm{TR}_{\mathbf{A}}
\end{CD}$$

(4) a homotopy $\sigma(\mathbf{A})$ between two DG functors id and $(\tau^{n+1})^\dagger(\mathbf{A})$ for any cyclic word $\mathbf{A}$ of length n such that for any $\lambda \in \Lambda_\infty([n],[m])$ one has the equality

$$\lambda^\dagger\lambda^*(\sigma(\lambda\mathbf{A})) = \sigma(\mathbf{A})\lambda^\dagger$$

of the two homotopies between the two DG functors $\lambda^\dagger$ and

$$(\lambda\tau^{m+1})^\dagger = (\tau^{n+1}\lambda)^\dagger$$

as in

$$\begin{array}{ccc}
\lambda^*\mathrm{TR}_{\lambda\mathbf{A}} & \overset{\mathrm{id}}{\underset{\lambda^*(\tau^{m+1})^\dagger}{\rightrightarrows}} & \lambda^*\mathrm{TR}_{\lambda\mathbf{A}}\\
{\scriptstyle\lambda^\dagger}\downarrow & & \downarrow{\scriptstyle\lambda^\dagger}\\
\mathrm{TR}_{\mathbf{A}} & \overset{\mathrm{id}}{\underset{(\tau^{n+1})^\dagger}{\rightrightarrows}} & \mathrm{TR}_{\mathbf{A}}
\end{array}$$

8.2. Higher Hochschild complexes. Now, in addition to cochains described by the "bigon" (**??**), let us consider more general $2k$-gons such as the one below (for k=2)

$$\begin{CD}
A_1 @>{f_{11}}>> B_1\\
@V{g_{12}}VV @AA{g_{21}}A\\
B_2 @<<{f_{22}}< A_2
\end{CD} \tag{8.2}$$

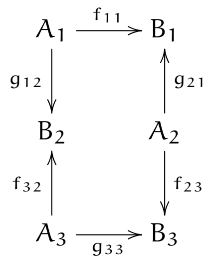
Namely, for any algebras $A_j, B_j, 1 \le j \le k$, and for any morphisms $f_{jj} : A_j \to B_j$ and $g_{j,j+1} : A_j \to B_{j+1}$, $1 \le j \le k$ (the indices are added modulo k, we define the complex

$$C^\bullet(\otimes_{j=1}^k A_j, \ {}_{\otimes f_{jj}}(\otimes_{j=1}^k B_j)_{\otimes g_{j,j+1}}) \tag{8.3}$$

One can generalize the cup product (**??**); namely, for two cochains

$$\begin{array}{ccc}
A_1 & \xrightarrow{f_{11}} & B_1\\
{\scriptstyle g_{12}}\downarrow & & \updownarrow{\scriptstyle g_{21}}\\
B_2 & \underset{g_{22}=f_{22}}{\xleftarrow{f_{22}}} & A_2\\
{\scriptstyle f_{32}}\uparrow & & \downarrow{\scriptstyle f_{23}}\\
A_3 & \xrightarrow[g_{33}]{} & B_3
\end{array} \tag{8.4}$$

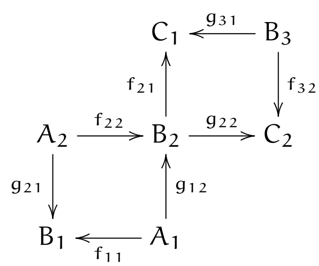
one produces a cochain



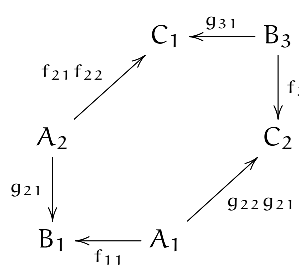
Also, for a morphism $f : A'_j \to A_j$, resp. $B_j \to B'_j$, one defines f^*, resp. g_*.

Furthermore, one can generalize (**??**) and define the $\bullet$ product that takes two cochains as shown below

(8.5)



and produces a cochain described by



As in (**??**), this product is a homotopy between two different ways to compose the two cochains using the cup product and the operations of direct and inverse image. When one takes $A_j = B_j = A$ and $f_{jj} = g_{j,j+1} = \mathrm{id}_A$ for all j, one defines the Kontsevich-Vlassopoulos bracket on

$$\prod_{k=1}^{\infty} C^{\bullet+1}(A^{\otimes k}, {}_{\alpha}(A^{\otimes k}))^{C_k} \tag{8.6}$$

of degree -1 in k. As above, α is the cyclic permutation.

One would expect a generalization of the construction mentioned in **??**. Namely, a strict structure would be as follows. We have defined a complex $C^\bullet(K)$ corresponding to a 2k-gon K. For any picture which is a union of 2k-gons such as (8.4) or (8.5),

$$K = \cup_{j=1}^m K_j \tag{8.7}$$

there should be an operation

$$\mathrm{Op}(K_1, \dots, K_m) : \otimes_{i=1}^m C^\bullet(K_j) \to C^\bullet(K) \tag{8.8}$$

and an associativity condition for $\mathrm{Op}(K_1, \ldots, K_n)$ and $\mathrm{Op}(K_{j1}, \ldots, K_{j,nj})$ for every "double subdivision"

$$K = \cup_{i=1}^{m} K_j;\ K_j = \cup_{i=1}^{n_j} K_{j,i},\ 1 \le j \le m \tag{8.9}$$

More realistically, there should be a version of the notion of an operad (related to and generalizing Batanin's two-operads):

(1) a collection of complexes $\mathcal{O}(K; K_1, \ldots, K_m)$ for any subdivision (8.7);
(2) compositions

$$\mathcal{O}(K;\{K_j\}) \otimes \otimes_{j=1}^{m} \mathcal{O}(K_j;\{K_{ji}\}) \to \mathcal{O}(K;\{K_{ji}\}) \tag{8.10}$$

for any "double subdivision" (8.9);
(3) an associativity condition for any "triple subdivision"

$$K = \cup_{i=1}^{m} K_j;\ K_j = \cup_{i=1}^{n_j} K_{j,i},\ 1 \le j \le m;\ K_{j,i} = \cup_{\ell} K_{\ell,j,i} \tag{8.11}$$

An algebra over such a generalized operad will be

(1) A complex $C^\bullet(K)$ for each K;
(2) A morphism

$$\mathcal{O}(K;\{K_j\}) \otimes \otimes_{j=1}^{m} C^\bullet(K_j) \to C^\bullet(K)$$

for every subdivision (8.7) which is compatible with composition for any (8.9).

We expect the higher Hochschild complexes (8.3) to form an algebra over a generalized operad $\mathcal{O}$ which is homotopically constant, i.e. such that $\mathcal{O}(K;\{K_j\})$ are all weakly homotopy equivalent to the scalar ring k. This would generalize Tamarkin's theorem [**?**].

Furthermore, for a 2k-gon $\{A_j, B_j, f_{jj}, g_{j,j+1}\}$ there is also the chain complex

$$C_\bullet(\otimes_{j=1}^{k} A_j,\ {}_{\otimes f_{jj}}(\otimes_{j=1}^{k} B_j)_{\otimes g_{j,j+1}}) \tag{8.12}$$

(In fact, when $k > 1$, there are also mixed chain-cochain complexes). The above should generalize to this situation, within the context of (generalized) multi-colored operads.

REMARK 8.2.1. This is not quite straightforward because chains have different functoriality properties. For example, given morphisms as in (**??**), at the level of cochains we get morphisms of complexes

$$C^\bullet(B,\ {}_{g_1}C_{g_2}) \to C^\bullet(A,\ {}_{hg_1f}D_{hg_2f})$$

(as we saw earlier); but at the level of chains we get

$$C^\bullet(A,\ {}_{g_1f}C_{g_2f}) \to C^\bullet(B,\ {}_{hg_1}D_{hg_2})$$

In [**?**] we proved two results about the structure of chains and cochains in a two-categorical language. Firstly, we showed that the category in cocategories $\mathrm{Bar}(\mathbf{C}^\bullet(\_,\_))$ admits a trace functor up to homotopy (in a precise sense); this structure involves only the Hochschild chain complexes $\mathrm{TR}_A(f) = C_\bullet(A,{}_f A)$ for endomorphisms f of algebras. The structure on all $C_\bullet(A,\ {}_fB_g)$ is what we call a twisted tetramodule structure over $\mathrm{Bar}(\mathbf{C}^\bullet(\_,\_))$.

CHAPTER 20

Representation schemes

1. Representation schemes of algebras

Let A be an associative algebra. For a natural number n, The nth representation scheme $\mathsf{Rep}_n(A)$ is the scheme whose points are morphisms of algebras $A \to M_n(k)$. More precisely, $\mathcal{O}(\mathsf{Rep}_n(A))$ is the commutative k-algebra with generators $\rho_{jk}(a)$, $a \in A$, $1 \le j, k \le n$, that are k-linear in a and satisfy the relations

$$\rho_{j\ell}(ab) = \sum_{k=1}^{n} \rho_{jk}(a)\rho_{j\ell}(b) \tag{1.1}$$

We will usually fix n and write $\mathsf{Rep}(A)$ instead of $\mathsf{Rep}_n(A)$. We will also denote k^n by V.

There is a morphism of algebras

$$A \to M_n(\mathcal{O}(\mathsf{Rep}(A))); \ a \mapsto (\rho_{jk}(a))_{1\le j,k\le n} \tag{1.2}$$

In fact $\mathcal{O}(\mathsf{Rep}(A))$ is the universal commutative algebra B with a morphism $A \to M_n(B)$. There is also an action of $\mathrm{GL}_n(k)$ by automorphisms: for a matrix $T = (T_{ij})$ in GL_n, the corresponding automorphism acts by

$$\rho_{pq} \mapsto \sum_{j,k} T_{pj}\rho_{jk}(T^{-1})_{kq} \tag{1.3}$$

The morphism (1.2) is actually a morphism

$$A \to M_n(\mathcal{O}(\mathsf{Rep}(A)))^{\mathrm{GL}_n} \tag{1.4}$$

2. Derived representation schemes

Let $(R_\bullet, d_R)$ be a semi-free differential graded resolution of A. We define *the algebra of functions on the derived representation scheme of* A as the differential graded algebra $\mathcal{O}(\mathsf{Rep}_n(R_\bullet))$. More precisely, it is the graded algebra generated by $\rho_{jk}(r)$, $r \in R_p$, of degree p, with relations (1.2); the differential is defined as

$$d\rho(r) = \rho(d_R r) \tag{2.1}$$

We denote the differential graded algebra $\mathcal{O}(\mathsf{Rep}(R_\bullet))$ also by $\mathbb{L}\mathcal{O}(\mathsf{Rep}(A))$.

2.1. Derived representation schemes and the bar construction. Just as for any differential graded algebra $\mathcal{A}$ we can form the differential graded Lie algebra $\mathfrak{gl}_n(\mathcal{A})$, for any differential graded coalgebra $\mathcal{C}$ we can form a we can form the differential graded Lie coalgebra

$$\mathfrak{gl}_n^*(\mathcal{C}) = \mathfrak{gl}_n^* \otimes \mathcal{C} \tag{2.2}$$

And just as we construct the Chevalley-Eilenberg chain complex $C^{CE}_\bullet(\mathcal{L}, k)$ for a DG Lie algebra, we can construct Chevalley-Eilenberg cochain complex $C^\bullet_{CE}(\mathcal{L}, k)$ for a DG Lie coalgebra.

THEOREM 2.1.1. *(Berest , Felder, Patotsky, Ramadoss, Willwacher). There is a quasi-isomorphism of differential graded algebras*

$$\mathbb{L}\mathcal{O}(\mathsf{Rep}_n(A)) \xrightarrow{\sim} C^\bullet_{CE}(\mathfrak{gl}^*_n(\mathrm{Bar}(A)), k)$$

PROOF. This follows directly from applying the definition of $\mathbb{L}\mathcal{O}(\mathsf{Rep})$ when $R_\bullet$ is the standard resolution CobarBar(A). More precisely: the graded commutative algebra $\mathbb{L}\mathcal{O}(\mathsf{Rep}_n(A))$ is freely generated by $\rho_{ij}(c)$ of degree $|c|+1$ where $c \in \mathrm{Bar}(A)$; the differential acts by

$$\partial\rho_{ij}(c) = \sum_k \sum (-1)^{|c^{(1)}|}\rho_{ik}(c^{(1)})\rho_{kj}(c^{(2)})$$

where, as usual, $\Delta c = \sum c^{(1)} \otimes c^{(2)}$. But this is precisely the definition of the right hand side of the formula in the statement of the theorem. □

2.2. The derived tangent space. For a commutative DG algebra R, an R-valued point of the derived representation scheme of A is an A_∞ morphism $\rho : R \otimes A \to M_n(R)$. For such ρ, we denote the corresponding R-valued point by

$$\widetilde{\rho} : \mathbb{L}\mathcal{O}(\mathsf{Rep}_n(A)) \to R \tag{2.3}$$

The derived tangent space at such a point is the complex

$$\mathcal{T}_{\widetilde{\rho}}(\mathbb{L}\mathsf{Rep}_n(A)) = \mathrm{Der}(\mathbb{L}\mathcal{O}(\mathsf{Rep}_n(A)), R_{\widetilde{\rho}}) \tag{2.4}$$

Here $R_{\widetilde{\rho}}$ is R on which $\mathrm{Der}(\mathbb{L}\mathcal{O}(\mathsf{Rep}_n(A))$ acts on both sides through $\widetilde{\rho}$.

THEOREM 2.2.1. *There is a natural quasi-isomorphism*

$$\mathcal{T}_{\widetilde{\rho}}(\mathbb{L}\mathsf{Rep}_n(A)) \xrightarrow{\sim} \mathbb{R}\mathrm{Hom}_{R\otimes A}(R_{\widetilde{\rho}}, R_{\widetilde{\rho}})$$

PROOF. Choose the standard resolution Cobar(Bar(A)) of A.. A derivation as in the right hand side of (2.4) is a linear map $\varphi : \mathrm{Bar}(A)[-1] \to R$. Identify the space of those with the standard complex for computing $\mathbb{R}\mathrm{Hom}_{R\otimes A}(R_{\widetilde{\rho}}, R_{\widetilde{\rho}})$. The differential in $\mathcal{T}_{\widetilde{\rho}}$ becomes the differential in this standard complex. □

3. Cyclic homology and representation varieties, I

3.1. The stabilization theorem. Define

$$\mathbb{L}\mathcal{O}(\mathsf{Rep}_\infty(A)) = \varinjlim \mathbb{L}\mathcal{O}(\mathsf{Rep}_n(A)) \tag{3.1}$$

Define also

$$\mathrm{GL}_\infty = \varinjlim \mathrm{GL}_n$$

THEOREM 3.1.1. *Let A be a DG albebra concentrated in non-positive degrees (with respect to the cohomological grading). There is a natural quasi-isomorphism of commutative DG algebras*

$$\mathbb{L}\mathcal{O}(\mathsf{Rep}_\infty(A))^{\mathrm{GL}_\infty} \xrightarrow{\sim} \mathrm{Sym}(CC_\bullet(A))$$

PROOF. Again, choose the standard resolution Cobar(Bar(A)) of A. An argument identical to the proof of Theorem 4.0.2, combined with Theorem 2.1.1, shows that there is an isomorphism

$$\mathbb{L}\mathcal{O}(\mathsf{Rep}_\infty(A))^{\mathrm{GL}_\infty} \xrightarrow{\sim} \mathrm{Sym}(C^\bullet_{\mathrm{II},\lambda}(\mathrm{Bar}(A))[-1]) \tag{3.2}$$

Here for a DG coalgebra $\mathcal{C}$

$$C^\bullet_{\mathrm{II},\lambda}(\mathcal{C}) = \bigoplus_{n\geq 0}(\mathcal{C}\otimes\mathcal{C}[-1]^{\otimes n})^{\mathbb{Z}/(n+1)\mathbb{Z}}, \tag{3.3}$$

the differential being the Hochschild differential b plus the differential induced by the one of A. (Compare to Chapter 6). In general, this is quasi-isomorphic to the *direct product totalization* of the two-periodic double complex

$$(\bigoplus_{n\geq 0}\mathcal{C}^{\otimes n+1}[-n], b') \xleftarrow{1-\tau} (\bigoplus_{n\geq 0}\mathcal{C}^{\otimes n+1}[-n], b) \xleftarrow{N} \ldots \tag{3.4}$$

Because of our assumption on A, the direct product totalization coincides with the direct sum totalization. Now, for $\mathcal{C} = \mathrm{Bar}(A)$ this complex can be replaced by the direct sum totalization of the two-periodic complex

$$C'^{\mathrm{sh}}_{\mathrm{II}}(\mathcal{C}) \xleftarrow{1-\tau} C^{\mathrm{sh}}_{\mathrm{II}}(\mathcal{C}) \xleftarrow{N} \ldots \tag{3.5}$$

Here

$$C'^{\mathrm{sh}}_{\mathrm{II}}(\mathcal{C}) = (\mathcal{C} \xrightarrow{b'} \mathrm{Ker}(b' : (\mathcal{C}\otimes\mathcal{C})[-1] \to (\mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C})[-2])$$

$$C^{\mathrm{sh}}_{\mathrm{II}}(\mathcal{C}) = (\mathcal{C} \xrightarrow{b} \mathrm{Ker}(b : (\mathcal{C}\otimes\mathcal{C})[-1] \to (\mathcal{C}\otimes\mathcal{C}\otimes\mathcal{C})[-2])$$

Finally, the complex (3.5) is quasi-isomorphic to $\mathrm{CC}_\bullet(A)[1]$. To see this, note that the first complex becomes

$$\mathrm{Cone}((A^{\otimes\geq 1}, b') \to (A^{\otimes\geq 2}, b')) \tag{3.6}$$

The second complex becomes

$$\mathrm{Cone}((A^{\otimes\geq 1}, b') \xrightarrow{1-\tau} (A^{\otimes\geq 2}, b)) \tag{3.7}$$

The morphism $1-\tau$ in (3.5) becomes as follows: on $(A^{\otimes\geq 2}, b)$ it is N; on $(A^{\otimes\geq}, b')$, it is zero. And he morphism N in (3.5) is as follows: on $(A^{\otimes\geq 2}, b)$ it is $1-\tau$; on $(A^{\otimes\geq}, b')$, it is the identity. It is clear that the resulting complex compputes $\mathrm{HC}_\bullet(A)$. (MAYBE A FEW MORE WORDS)

□

4. Cyclic homology and representation varieties, II

We compare the Hochschild (resp. periodic cyclic) homology of an algebra A to forms (resp. equivariang De Rham cohomology) of the variety of representations of A. We provide two constructions. First, we compare the extended De Rham complex from Chapter 15 to equivariant forms on the representation scheme; second, we compare the complexes from Chapter 17 to equivariant forms and multivectors on the (derived) representation scheme. Hopefully, these methods can be extended to compare cyclic homology of DG categories to the De Rham cohomology of their moduli spaces as studied by Toën and Vezzosi.

4.1. From the extended De Rham complex to equivariant forms on $\mathsf{Rep}(A)$**.** Let

$$\Omega^\bullet(\mathsf{Rep}(A)) = \Omega^\bullet_{\mathcal{O}(\mathsf{Rep}(A))/k} \tag{4.1}$$

be the algebra of Kähler differentials. For $X \in \mathfrak{gl}_n$ let ν_X be the derivation of $\mathcal{O}(\mathsf{Rep}(A))$ defined by

$$\mathbf{v}(X)(\rho_{pq}) = \sum_{k=1}^{n} (\rho_{pj} X_{jq} - X_{pj}\rho_{jq}) \tag{4.2}$$

This is the infinitesimal form of the action (1.3) of GL_n . Denote

$$G = \mathrm{GL}_n(k);\ \mathfrak{g} = \mathfrak{gl}_n(k).$$

Consider a differential

$$\iota_{\mathbf{v}} : \mathrm{Hom}(S^j(\mathfrak{g}), \Omega^p(\mathsf{Rep}(A)))^G \to \mathrm{Hom}(S^{j+1}(\mathfrak{g}, \Omega^{p-1}(\mathsf{Rep}(A)))^G \tag{4.3}$$

defined by

$$(\iota_{\mathbf{v}} f)(X) = \iota_{\mathbf{v}(X)} f(X) \tag{4.4}$$

where we view $\mathrm{Hom}(S^j\mathfrak{g}, \Omega)$ as the space of homogeneous maps $\mathfrak{g} \to \Omega$ of degree j.

Define the map of differential graded algebras

$$\Omega^\bullet_t(A) \to \mathrm{Hom}(S^\bullet\mathfrak{g}, M_n(\Omega^\bullet(\mathsf{Rep}(A))))^G \tag{4.5}$$

as follows:

$$a \mapsto (\rho_{jk}(a));\ da \mapsto d(\rho_{jk}(a))$$

(we view $M_n(\Omega)$ as $\mathrm{Hom}(S^0\mathfrak{g}, M_n(\Omega))$;

$$t \mapsto \mathrm{id} : S^1\mathfrak{g} \to M_n(k) \subset M_n(\Omega^\bullet(\mathsf{Rep}(A))).$$

Composing (4.5) with the ordinary matrix trace tr, we observe that the result is equal to zero on all commutators. It is immediate taht the above map intertwines ι_t with $\iota_{\mathbf{v}}$. Tensoring with $k((u))$, we get a morphism

$$(\mathtt{DR}^\bullet_t(A)((u)), \iota_t + ud) \to (\mathrm{Hom}(S^\bullet\mathfrak{g}, \Omega^\bullet(\mathsf{Rep}(A)))^G, \iota_{\mathbf{v}} + ud) \tag{4.6}$$

We have obtained

PROPOSITION 4.1.1. *The map* (4.6) *is a morphism of complexes.*

4.2. The $\mathfrak{X}$ complex and equivariant multivector fields on the Rep **scheme.** Consider the diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{\ \delta\ } & (\Omega^1_A)^\vee \\ \downarrow & & \downarrow \\ (\mathfrak{gl}_n \otimes \mathcal{O}_{\mathrm{Rep}_n(A)})^{\mathrm{GL}_n} & \longrightarrow & T_{\mathrm{Rep}_n(A)} \end{array} \tag{4.7}$$

The bottom morphism is as follows. Denote by ξ_{jk} the fundamental vector field of the element E_{jk} of $\mathfrak{gl}_n$ on $\mathrm{Rep}_n(A)$. The bottom morphism is as follows.

$$E_{jk} \otimes f \mapsto f\xi_{jk} \tag{4.8}$$

The vertical morphism on the left is defined as

$$a \otimes c \mapsto \sum_{j,k} E_{jk} \otimes \rho_{kj}(ca) \tag{4.9}$$

for $a, c \in A$. The vertical morphism on the right is as follows. Recall that an element of $(\Omega^1_A)^\vee$ is a derivation $D : A \to (A \otimes A)_{\text{in}}$ (i.e. with values in $A \otimes A$ viewed as a bimodule with the inner structure). Put

$$D(a) = \sum D'(a) \otimes D''(a) \tag{4.10}$$

Then

$$D \mapsto (\rho_{jk}(a) \mapsto \sum \rho_{jk}(D''(a)D'(a))) \tag{4.11}$$

LEMMA 4.2.1. *The diagram* (4.7) *is commutative.*

PROOF. □

The role of a Cartan model of equivariant multivectors is played by the invariant part of the Koszul complex:

$$\mathbb{K}_\bullet(\mathfrak{g}, d, \mathrm{Rep}_n(A))^{\mathrm{GL}_n} = (S^\bullet(\mathfrak{gl}_n[-d]) \otimes \Theta^\bullet_{\mathrm{Rep}_n(A), d+1})^{\mathrm{GL}_n} \tag{4.12}$$

where for a scheme X

$$\Theta^\bullet_{X,d+1} = S_{\mathcal{O}_X} T_X[-1-d] \tag{4.13}$$

The differential on the Koszul complex is

$$\sum_{j,k} \frac{\partial}{\partial E_{jk}} \otimes (\xi_{jk} \cdot) \tag{4.14}$$

(In other words,

$$\mathbb{K}_\bullet(\mathfrak{g}, d, \mathrm{Rep}_n(A))^{\mathrm{GL}_n} = C_\bullet(\mathfrak{gl}_n[\epsilon_d], \mathfrak{gl}_n; \Theta^\bullet_{\mathrm{Rep}_n(A), d+1}) \tag{4.15}$$

where ϵ_d is of degree $d+1$, $\epsilon_d^2 = 0$, and $X + \epsilon_d Y$ acts by $L_{\xi_X} + (\xi_Y \wedge)$ for $X, Y \in \mathfrak{gl}_n$).

We extend (4.17) multiplicatively to a morphism

$$\mathfrak{X}^{(*)}(A, d) \to \mathbb{K}_\bullet(\mathfrak{g}, d, \mathrm{Rep}_n(A))^{\mathrm{GL}_n} \tag{4.16}$$

We discuss various complexes related to a group action on a scheme in 5.

LEMMA 4.2.2. *The shift of* (4.16) *by* $d + 1$ *is a DGLA morphism.*

PROOF. □

4.3. The Υ complex and equivariant forms on the Rep **scheme.** Now consider the diagram

$$\begin{array}{ccc} \Omega^1_A & \xrightarrow{\ b\ } & A \otimes A \\ \downarrow & & \downarrow \\ \Omega^1_{\mathrm{Rep}_n(A)} & \longrightarrow & (\mathfrak{gl}_n^* \otimes \mathcal{O}_{\mathrm{Rep}_n(A)})^{\mathrm{GL}_n} \end{array} \tag{4.17}$$

The vertical morphism on the left is

$$a \cdot db \cdot c \mapsto \rho_{ij}(a) \cdot d\rho_{jk}(b) \cdot \rho_{ki}(c); \tag{4.18}$$

The vertical morphism on the right is

$$a \otimes c \mapsto \sum_{j,k} E^*_{jk} \otimes \rho_{jk}(ca); \tag{4.19}$$

the bottom morphism is

$$\omega \mapsto \sum_{jk} \iota_{\xi_{jk}} \omega \tag{4.20}$$

for $\omega \in \Omega^1_A$.

LEMMA 4.3.1. *The diagram* (4.17) *is commutative.*

PROOF. ∎

As in ***ref***, define the Cartan model of equivariant forms as

$$\Omega^\bullet_{\mathrm{Rep}_n(A),\mathrm{GL}_n} = (\widehat{S}^\bullet(\mathfrak{gl}_n^*)\widehat{\otimes}\Omega^\bullet_{\mathrm{Rep}_n(A)})^{\mathrm{GL}_n} \tag{4.21}$$

with the differential

$$d + \sum_{j,k} E^*_{jk} \otimes \iota_{\xi_{jk}} \tag{4.22}$$

(In other words,

$$\Omega^\bullet_{\mathrm{Rep}_n(A),\mathrm{GL}_n} = C^\bullet(\mathfrak{gl}_n[\epsilon], \mathfrak{gl}_n; \Omega^\bullet_{\mathrm{Rep}_n(A)}) \tag{4.23}$$

where $X + \epsilon Y$ acts by $L_{\xi_X} + \iota_{\xi_Y}$ for $X, Y \in \mathfrak{gl}_n$).

We extend (4.17) multiplicatively to a morphism

$$(\Upsilon^{(*)}_\bullet(A), d_A + b + d) \to \Omega^\bullet_{\mathrm{Rep}_n(A),\mathrm{GL}_n} \tag{4.24}$$

check the LHS

4.4. The noncommutative cotangent complex and the Rep **scheme.** Recall that $\Phi \in F/[F,F]$ defines a regular GL_m-invariant regular function on $\mathrm{Rep}_m(F)$

$$\mathrm{Tr}\widehat{\Phi}(\rho) = \sum_{j=1}^{m} \rho_{jj}(\Phi) \tag{4.25}$$

LEMMA 4.4.1.

$$\mathrm{Rep}_m(A_\Phi) = \mathrm{Crit}(\mathrm{Tr}\widehat{\Phi})$$

PROOF. ∎

Note that for any scheme X and any regular function ϕ on X the map $d \circ \iota_{d\phi} : T_X \to T^*_X$ descends to

$$d \circ \iota_{d\phi} : T_X|\mathrm{Crit}_\phi \to T^*_X|\mathrm{Crit}_\phi \tag{4.26}$$

Let us intruduce the following notation.

$$\mathfrak{R}_F = \mathrm{Rep}_m(F);\ \mathfrak{R}_\Phi = \mathrm{Rep}_m(A_\Phi) \tag{4.27}$$

$$(\Omega^1_{F|A_\Phi})^\vee = A_\Phi \otimes_F (\Omega^1_F)^\vee \otimes_F A_\Phi; \tag{4.28}$$

$$\Omega^1_{F|A_\Phi} = A_\Phi \otimes_F \Omega^1_F \otimes_F A_\Phi; \tag{4.29}$$

The diagrams (4.7) and (4.17) extend to the following
(4.30)

$$\begin{array}{ccccccc} A_\Phi \otimes A_\Phi & \xrightarrow{\delta} & (\Omega^1_{F|A_\Phi})^\vee & \longrightarrow & \Omega^1_{F|A_\Phi} & \xrightarrow{b} & A_\Phi \otimes A_\Phi \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ (\mathfrak{gl}_m \otimes \mathcal{O}_{\mathfrak{R}_\Phi})^{\mathrm{GL}_m} & \longrightarrow & T_{\mathfrak{R}_F}|_{\mathfrak{R}_\Phi} & \xrightarrow{d \circ \iota_{d \mathrm{Tr} \widehat{\Phi}}} & \Omega^1_{\mathfrak{R}_F}|_{\mathfrak{R}_\Phi} & \longrightarrow & (\mathfrak{gl}^*_m \otimes \mathcal{O}_{\mathfrak{R}_\Phi})^{\mathrm{GL}_m} \end{array}$$

5. Appendix. Complexes associated to a group action

5.1. The Koszul complex of a Hamiltonian action. Let an algebraic group G act on a Poisson algebra A. We assume the action to be Hamiltonian, i.e. that there is a G-equivariant Lie algebra morphism

$$\mathfrak{g} \to A;\ X \mapsto H_X \tag{5.1}$$

so that the action of $X \in \mathfrak{g}$ is given by

$$a \mapsto \{H_X, a\},\ \ a \in A. \tag{5.2}$$

Define

$$\mathbb{K}_\bullet(\mathfrak{g}, A) = \wedge^\bullet(\mathfrak{g}) \otimes A \tag{5.3}$$

with the differential

$$\kappa : \mathbb{K}_\bullet(\mathfrak{g}, A) \to \mathbb{K}_{\bullet-1}(\mathfrak{g}, A) \tag{5.4}$$

$$\kappa(X_1 \wedge \ldots \wedge X_k \otimes a) = \sum_{j=1}^{m} (-1)^{k-j} X_1 \wedge \ldots \wedge \widehat{X}_j \wedge \ldots \wedge X_k \otimes H_{X_j} a$$

In other words,

$$\mathbb{K}_\bullet(\mathfrak{g}, A) = C_\bullet(\mathfrak{g}_{\mathrm{comm}}, A), \tag{5.5}$$

the chain complex of $\mathfrak{g}$ viewed as an Abelian Lie algebra that acts on A by multiplication.

Next, fix an integer d and assume that $\mathcal{A}$ is a graded commutative algebra with a shifted Poisson bracket of degree $-1-d$. Then $\mathcal{A}_{d+1}$ is a Lie algebra. A Hamiltonian action is an action of G together with a G-equivariant morphism of Lie algebras

$$\mathfrak{g} \to \mathcal{A}_{d+1};\ X \mapsto H_X \tag{5.6}$$

such that the action of $X \in \mathfrak{g}$ is given by (5.2). Define the Koszul complex

$$\mathbb{K}_\bullet(\mathfrak{g}, d; \mathcal{A}) = (S(\mathfrak{g}[-d]) \otimes \mathcal{A}, \kappa) = C_\bullet(\mathfrak{g}[-d-1], \mathcal{A}) \tag{5.7}$$

Here $\mathfrak{g}[-d-1]$ is viewed as an Abelian graded Lie algebra acting on $\mathcal{A}$ by multiplication by H_X, $X \in \mathfrak{g}$. The differential κ is given by

$$\kappa = \sum \frac{\partial}{\partial e_j} H_{e_j}. \tag{5.8}$$

5.2. The derived reduction of a Hamiltonian action. For a Hamiltonian action of $\mathfrak{g}$ on a Poisson algebra A, define

$$A^\bullet_{\mathrm{red}} = C^\bullet(\mathfrak{g}, \mathbb{K}_\bullet(\mathfrak{g}, A)) \tag{5.9}$$

As a graded k-module,

$$A^\bullet_{\mathrm{red}} = \wedge^\bullet(\mathfrak{g}^*) \otimes \wedge^{-\bullet}(\mathfrak{g}) \otimes A$$

which is a Poisson algebra. Namely, it is the tensor product of two Poisson algebras, one being A and the other $\wedge^\bullet(\mathfrak{g}^*) \otimes \wedge^{-\bullet}(\mathfrak{g})$. The Poisson bracket on the latter is determined by its restriction to $\mathfrak{g}[1]^* \oplus \mathfrak{g}[1]$ which is the usual duality pairing with values in $k \cdot 1$.

We observe that the differential is the bracket with the homogeneous element

$$h = \frac{1}{2}\sum_{i,j,k} f^i_{jk} e_i e^j e^k + \sum_j e^j H_{e_j} \tag{5.10}$$

where e_j is a basis of $\mathfrak{g}$, e^j is the dual basis, and f^i_{jk} are the structure constants.

In particular, $A^\bullet_{\mathrm{red}}$ is a graded Poisson algebra.

When $\mathcal{A}$ has a Poisson bracket of degree $-1-d$ and $X \mapsto H_X$ is a Hamiltonian action of $\mathfrak{g}$ then

$$\mathcal{A}^\bullet_{\mathrm{red}} = C^\bullet(\mathfrak{g}, \mathbb{K}_\bullet(\mathfrak{g}, d; \mathcal{A})) = (S(\mathfrak{g}^*[-1]) \otimes S(\mathfrak{g}[-d]) \otimes \mathcal{A}, \partial^{\mathrm{Lie}} + \kappa) \tag{5.11}$$

The right hand side is a graded commutative algebra with a Poisson bracket of degree $-1-d$. The differential is the bracket with the homogeneous element (5.10) of degree $d+2$. The above construction for a Poisson algebra is a partial case when $d=-1$.

5.2.1. *Koszul and BRST complexes for multivector fields.* Let X be a scheme over k with an action of an algebraic group G. For $e \in \mathfrak{g}$ let e_X be the corresponding vector field on X. We get a Hamiltonian action of G on the graded algebra $\Theta_X = \wedge T_X$ of multivector fields. We apply the above with $d=0$ and the shifted Poisson bracket being the Schouten bracket. In particular, we have the Koszul complex

$$\mathbb{K}_\bullet(\mathfrak{g}, \Theta_X) = (S(\mathfrak{g}) \otimes \Theta_X, \kappa) \tag{5.12}$$

and the derived reduced algebra

$$D^{\mathrm{BRST}}(\mathfrak{g}, X) = \Theta^\bullet_{X,\mathrm{red}} = C^\bullet(\mathfrak{g}, \mathbb{K}_\bullet(\mathfrak{g}, \Theta_X)) \tag{5.13}$$

The differential is of degree $+1$. More generally, we have for any d

$$\mathbb{K}_\bullet(\mathfrak{g}, d, \Theta_{X,d+1}) = (S(\mathfrak{g}[-d]) \otimes \Theta_{X,d+1}, \kappa) \tag{5.14}$$

and

$$D^{\mathrm{BRST}}(\mathfrak{g}, d, X) = \Theta^\bullet_{X,\mathrm{red}} = C^\bullet(\mathfrak{g}, \mathbb{K}_\bullet(\mathfrak{g}, d, \Theta_{X,d+1})) \tag{5.15}$$

(We recall that $\Theta^\bullet_{X,d+1} = S_{\mathcal{O}_X}(T_X[-1-d])$).

5.3. Koszul and BRST complexes for multivector fields with a potential. Now assume that we are in the situation of 5.2.1 and a G-invariant regular function ϕ on X is given. Then

$$\iota_{d\phi} = \{\phi, \ \} : \Theta^\bullet_X \to \Theta^{\bullet-1}_X \tag{5.16}$$

extends $S(\mathfrak{g})$-linearly to $\mathbb{K}_\bullet(\mathfrak{g}, \Theta_X)$. This is a differential of degree -1 that commutes with κ.

We denote by $\mathbb{K}_\bullet(\mathfrak{g}, \phi, \Theta_X))$ the mixed complex $\mathbb{K}_\bullet(\mathfrak{g}, \Theta_X))$ with two differentials κ of degree $+1$, ι_ϕ of degree -1; or change the grading like in [?]***

$$D^{\mathrm{BRST}}(\mathfrak{g}, \phi, X) = C^\bullet(\mathfrak{g}, \mathbb{K}_\bullet(\mathfrak{g}, \phi, \Theta_X)) \tag{5.17}$$

as a mixed complex with the differentials $\partial^{\mathrm{Lie}} + \kappa$ of degree 1, ι_ϕ of degree -1

We identify $S(\mathfrak{g})$ with $k[e_1, \ldots, e_n]$ where $\{e_1, \ldots, e_n\}$ is a basis of $\mathfrak{g}$ and $|e_j| = 0$ for each j. Then

$$\kappa = \sum_{j=1}^{n} \frac{\partial}{\partial e_j} \otimes (e_{j,X} \wedge) \tag{5.18}$$

Here, as above, e_X is a vector field on X defined by the action of $e \in \mathfrak{g}$.

5.4. BV complexes. Recall that, given a Gerstentaber algebra $\mathcal{A}$, a BV operator is by definition

$$\Delta : \mathcal{A}^\bullet \to \mathcal{A}^{\bullet - 1} \tag{5.19}$$

satisfying

$$\Delta^2 = 0;\ \Delta(ab) = \Delta(a)b + (-1)^{|a|} a\Delta(b) + (-1)^{|a|}\{a, b\} \tag{5.20}$$

As a consequence, Δ is a graded Lie algebra derivation. Assume that:

- there is a Hamiltonian action of G on $\mathcal{A}$ such that $\Delta H_e = 0$ for any e in $\mathfrak{g}$;
- A G-invariant element $\phi \in \mathcal{A}^0$ is given such that $\Delta\phi + \frac{1}{2}\{\phi, \phi\} = 0$.

LEMMA 5.4.1. *Let*

$$\Delta_\phi = \Delta + \iota_{d\phi};$$

then

$$\Delta_\phi^2 = [\Delta_\phi, \kappa] = \kappa^2 = 0$$

on $(S(\mathfrak{g}) \otimes \mathcal{A})^G$.

PROOF. □

5.4.1. *The big BV complex.* In the generality of 5.4, we can extend Lemma 5.4.1 to the full derived reduced complex as follows.

Recall the modular character of a Lie algebra $\mathfrak{g}$:

$$\chi(e) = \mathrm{tr}(\mathrm{ad}(e));\ \chi = \sum_{j,k} f_{jk}^j e^k \in \mathfrak{g} \tag{5.21}$$

Define

$$\Delta_{\mathfrak{g}} = \sum_j \pm \frac{\partial}{\partial e^j} \frac{\partial}{\partial e_j} \tag{5.22}$$

LEMMA 5.4.2. *Put*

$$\Delta_\phi^{\mathrm{big}} = \Delta_{\mathfrak{g}} + \Delta + \iota_{d\phi}$$

Then

$$(\Delta_\phi^{\mathrm{big}})^2 = [\Delta_\phi^{\mathrm{big}}, \partial^{\mathrm{Lie}} + \kappa + \chi\cdot] = (\partial^{\mathrm{Lie}} + \kappa + \chi\cdot)^2 = 0$$

PROOF. For h as in (5.10), one has

$$\Delta(h) = 0;\ \Delta_{\mathfrak{g}}(h) = \chi; \{h, h\} = 0 \tag{5.23}$$

as well as

$$\{h, \chi\} = \chi^2 = 0 \tag{5.24}$$

□

5.5. Equivariant multivector fields and Koszul complexes. In our discussions here, the role of the Cartan model of equivariant multivector fields was played by the invariant part of the Koszul complex. However, a dual, and in some sense more direct, construction is also possible, and the two are related. We discuss this here.

5.5.1. *Equivariant multivector fields.* Let G act on a scheme X over k. Put

$$\Theta^{\bullet}_{X,G} = (\widehat{S}(\mathfrak{g}[2]^*)\widehat{\otimes}\Theta^{\bullet}_X)^G \tag{5.25}$$

The Schouten bracket on $\Theta^{\bullet+1}_X$ extends to $\Theta^{\bullet+1}_{X,G}$ continuously and $\widehat{S}(\mathfrak{g}[2]^*)$-linearly. Put

$$\lambda = \sum_{j=1}^{n} e^j \otimes e_{j,X} \tag{5.26}$$

LEMMA 5.5.1. (1) λ *is a central element in* $\Theta^3_{X,G}$.

(2) *There is a bijection between elements* Π *of* $\Theta^2_{X,G}$ *such that* $\frac{1}{2}[\Pi, \Pi] = \lambda$ *and pairs* $\{\pi, H\}$ *where* π *is a* G*-invariant Poisson structure on* X *and* H *is a Hamiltonian action compatible with the given action of* G*. The bijection is implemented as follows. Given a Poisson structure* π *and a Hamiltonian action* $e \mapsto H_e$*, the element* Π *is defined as*

$$\Pi = \pi + \sum_{j=1}^{n} e^j \otimes H_{e_j}$$

PROOF. □

5.5.2. *Relation to the Koszul complex.* For G acting on X, we consider $M = \mathfrak{g}^* \times X$ with the product action. We identify $\mathcal{O}(\mathfrak{g}^*)$ with $S(\mathfrak{g}) \xrightarrow{\sim} k[e_1, \ldots, e_n]$. Let

$$\pi^{\mathfrak{g}^*}_{\text{KKS}} = \frac{1}{2} f^i_{jk} e_i \frac{\partial}{\partial e_j} \frac{\partial}{\partial e_k}$$

be the Kirillov-Kostant-Souriau Poisson bivector lifted to $\mathfrak{g}^* \times X$. Define

$$\pi_{\text{KKS}} = \pi^{\mathfrak{g}^*}_{\text{KKS}} + \sum_{j=1}^{n} \frac{\partial}{\partial e_j} e_{j,X} \tag{5.27}$$

Define also

$$J_e = e,\ e \in \mathfrak{g} \tag{5.28}$$

LEMMA 5.5.2. *The bivector field* π_{KKS} *is Poisson;* J *defines a Hamiltonian action compatible with the product action of* G *on* $\mathfrak{g}^* \times X$.

PROOF. □

In particular, if

$$\Pi_{\mathrm{KKS}} = \pi_{\mathrm{KKS}} + \sum_{j=1}^{n} e^j \otimes e_j \tag{5.29}$$

then

$$[\Pi_{\mathrm{KKS}}, \Pi_{\mathrm{KKS}}] = \lambda = \sum_{j=1}^{n} e^j \otimes ej, \mathfrak{g}^* \times X$$

which is central. Therefore $[\Pi_{\mathrm{KKS}}, \ ]$ is a differential on $\Theta^\bullet_{\mathfrak{g}^* \times X}$.

Recall the Weil algebra of a Lie algebra $\mathfrak{g}$ which is by definition

$$W(\mathfrak{g}) = (S(\mathfrak{g}[2]^*) \otimes S(\mathfrak{g}[1]^*), \partial^{\mathrm{Lie}} + \partial_1 \tag{5.30}$$

Here ∂^{Lie} is the Chevalley-Eilenberg differential in $C^\bullet(\mathfrak{g}, S(\mathfrak{g}[2]^*)$ and ∂_1 is the graded derivation of degree 1 and square zero that sends $\mathfrak{g}[1]^*$ identically to $\mathfrak{g}[2]^*$.

LEMMA 5.5.3. *The complex* $\Theta^\bullet_{\mathfrak{g}^* \times X}$ *with the differential* $[\Pi_{\mathrm{KKS}}, \ ]$ *is isomorphic to*

$$(\widehat{W}(\mathfrak{g}) \widehat{\otimes} \mathbb{K}_\bullet(\mathfrak{g}, X))^G.$$

PROOF. Identify $\Theta^\bullet_{\mathfrak{g}^* \times X}$ with

$$\Theta^\bullet_X[e_1, \ldots, e_n, \xi^1, , \ldots, \xi^n][[e^1, \ldots, e^n]]^G \tag{5.31}$$

with $|e_j| = 0$, $|\xi^j| = 1$, $|e^j| = 2$. Here e_j are coordinates on $\mathfrak{g}^*$ (i.e. a basis of $\mathfrak{g}$); e^j are dual coordinates on $\mathfrak{g}$; $\xi^j = \frac{\partial}{\partial e_j}$. The only non-zero brackets involving the new generators are

$$\{\xi^j, e_j\} = 1 \tag{5.32}$$

We have

$$\Pi_{\mathrm{KKS}} = \frac{1}{2} \sum_{i,j,k} f^i_{jk} e_i \xi^j \xi^k + \sum_{j=1}^{n} \xi^j e_{j,X} + \sum_{j=1}^{n} e^j e_j \tag{5.33}$$

Therefore $\{\Pi_{\mathrm{KKS}}, \ \}$ becomes

$$\frac{1}{2} f^i_{jk} \xi^j \xi^k \frac{\partial}{\partial \xi^i} \pm \xi^k \sum f^i_{jk} e_i \frac{\partial}{\partial e_j} \pm \sum \xi^j \{e_{j,X}, \ \} \pm \sum (e_{j,X} \wedge \ ) \frac{\partial}{\partial e_j} \pm \sum e^j \frac{\partial}{\partial \xi^j} \tag{5.34}$$

Since we are applying this to a G-invariant expression, the second and third terms of (5.34) can be replaced by

$$\pm \sum \xi^j f^i_{jk} e^k \frac{\partial}{\partial e^i} \pm \sum \xi^j f^i_{jk} \xi^k \frac{\partial}{\partial \xi^i} \tag{5.35}$$

This, combined with the first term of (5.34), becomes ∂^{Lie} from W. The fourth term of (5.34) becomes the Koszul differential κ and the fifth term becomes ∂_1 from W. □

6. Bibliographical notes

Berest-Felder-Ramadoss; Berest-Ramadoss; Berest , Felder, Patotsky, Ramadoss, Willwacher; Kontsevich-Rosenberg; Toën-Vezzosi; Ginzburg; Ginzburg-Schedler; Etingof-Ginzburg; Esposito-Kraft-Schnitzer;

CHAPTER 21

Basics of noncommutative Hodge theory

1. Introduction

In [**378**], Katzarkov, Kontsevich, and Pantev suggested the following approach to defining a noncommutative analogue of the pure Hodge structure on the cohomology of a smooth proper variety. In this chapter we follow [**378**] and Shklyarov's work [**510**].

Consider a smooth and proper DG category A. The role of the De Rham cohomology is played by the periodic cyclic homology. The two components of a Hodge structure are: a) an integral (or perhaps rational) lattice and b) a filtration (the Hodge filtration). One can hope to get the rational lattice as the image of the Chern character from a suitable K-theory. As for the filtration, the idea is to define it as the filtration by generalized eigenvalues of a u-connection. A u-connection is an algebraic object that we define in Section 2.

How to construct a u-connection? The Getzler-Gauss-Manin connection is defined on the periodic cyclic homology of a family of algebras. Any DG algebra comes in a one-parameter family; namely, one can multiply its product and its differential by a parameter t. The formulas for the connection get somewhat complicated because the unit changes; in fact, at the value t, the unit is $t^{-1}1$. To deal with this, replace A by a bigger algebra $A^+ = A + k\mathbb{1}$. Recall Definition 2.0.2 and Lemma 2.0.3. We have

$$\mathbf{CC}^{\mathrm{per}}_{\bullet}(A) \xrightarrow{\sim} \mathrm{Ker}(\mathrm{CC}^{\mathrm{per}}_{\bullet}(A^+) \to \mathrm{CC}^{\mathrm{per}}_{\bullet}(k\mathbb{1})) \tag{1.1}$$

Here $\mathbf{CC}^{\mathrm{per}}_{\bullet}$ in the right hand side stands for the periodization of the $(b, b', 1-\tau, N)$ complex.

One gets a t-connection on a $k[t]((u))$-module. Now, there is an action of the multiplicative group; an appropriate reduction by this group eliminates the variable t andproduces what we call the u-connection.

We carry out this construction of a u-connection in subsections 4.1 and 4.2 of Section 4.

Remark 1.0.1. As we see in Example 2.0.3, there is an easily defined u-connection on the periodic cyclic complex of a differential $\mathbb{Z}$-graded category. The u-connection constructed in ection 4 is defined for a differential $\mathbb{Z}/2$-graded category. In the $\mathbb{Z}$-graded case the two constructions are equivalent, as we show in Section 5.

There is one important feature of the noncommutative version of the Hodge filtration. The usual Hodge filtration of a pure Hodge structure produces a u-connection; we recall the construction in Section 3). This connection is *regular*, i.e. of the form $\frac{\partial}{\partial u} + \frac{1}{u}A$. The u-connection that we get from the Getzler-Gauss-Manin connection is *irregular*; it is of the form $\frac{\partial}{\partial u} + \frac{1}{u^2}A$. As it is well known, such connections have additional invariants called the Stokes data. The definition of a noncommutative Hodge structure includes a requirement that the Stokes data should agree with the rational structure.

REMARK 1.0.2. The irregularity of the u-connection can be understood from the following observation: the isomorphisms playing the role of its monodromy are of the form $\exp(\frac{S}{u})$ where S is some operation on the negative cyclic complex. Cf., e.g., ***Ref

2. u-connections

DEFINITION 2.0.1. *A u-connection is a $\mathbb{Z}/2$-graded $k[[u]]$-module $\mathcal{H}$ with an odd $k[[u]]$-linear operator D and an even linear operator ∇_u such that:*

(1)
$$D^2 = 0;$$

(2)
$$[\nabla_u, u] = \mathrm{id};$$

(3)
$$[\nabla_u, D] = \frac{1}{2u} D$$

REMARK 2.0.2. Note that the equation in 3 becomes

$$(u^{-\frac{1}{2}} D + \nabla_u du)^2 = 0 \tag{2.1}$$

over the algebra $k((u^{\frac{1}{2}}))[du]$ of differential forms on the formal punctured disc.

EXAMPLE 2.0.3. For a complex manifold X, consider the mixed complex $\mathcal{H} = \Omega^{\bullet,\bullet}(X)[[u]]$ with $D = \bar{\partial} + u\partial$. Define

$$\Gamma' = q - p \text{ on } \Omega^{p,q}(X).$$

Then

$$\nabla_u = \frac{\partial}{\partial u} + \frac{\Gamma'}{2u}$$

is a u-connection. This follows from

$$[\Gamma', \bar{\partial}] = \bar{\partial};\ [\Gamma', \partial] = -\partial.$$

EXAMPLE 2.0.4. For a mixed complex $(C_\bullet, b, B)$, let

$$\mathcal{H} = C_\bullet[[u]];\ D = b + uB$$

Define

$$\Gamma' = -n \text{ on } C_n$$

Then

$$\nabla_u = \frac{\partial}{\partial u} + \frac{\Gamma'}{2u}$$

is a u-connection. This follows from

$$[\Gamma', b] = b;\ [\Gamma', B] = -B.$$

EXAMPLE 2.0.5. Let W be a function on a variety Y. Let

$$\mathcal{H} = \Omega^{\bullet}(Y)[[u]]$$

$$D = -dW + ud$$

$$\nabla_u = \frac{\partial}{\partial u} + \frac{W}{u^2} - \frac{1}{2u}\Gamma$$

where

$$\Gamma = p \text{ on } \Omega^p(Y)$$

3. From a Hodge structure to a u-connection

Given a vector space V over $\mathbb{C}$ with a pure Hodge structure of weight w, let $F^{\bullet}$ denote the Hodge filtration. Consider the flat connection

$$\nabla_u = \frac{\partial}{\partial u} - \frac{w}{2u}$$

on $V((u))$. Let

$$\mathcal{H} = \sum u^{-j} F^j V[[u]] \tag{3.1}$$

with $b = B = 0$.

4. From the Gauss-Manin connection to a u-connection

4.1. Dilating the product. For an algebra A, consider a family of algebra structures on A^+:

$$(a + \alpha \mathbb{1}) \cdot_t (b + \beta \mathbb{1}) = tab + a\beta + \alpha b + \alpha\beta\mathbb{1} \tag{4.1}$$

Denote by A_t^+ the $k[t]$-module $A^+[t]$ with the product as above. We will be using the identification (1.1). Let is introduce the formal variable ν of homological degree -1; we will write $u = \nu^2$. We will use the identification

$$\mathbf{CC}^{\mathrm{per}}_{\bullet}(A) = (A^{\otimes(\bullet+1)})((u)) \oplus \nu(A^{\otimes(\bullet+1)})((u)) \tag{4.2}$$

with the differential

$$(b, b') + \nu(N, 1-\tau) \tag{4.3}$$

This means the following. The map (b, b') is b on $(A^{\otimes(\bullet+1)})((u))$ and b' on $\nu(A^{\otimes(\bullet+1)})((u))$. The map $\nu(N, 1-\tau)$ is νN on $(A^{\otimes(\bullet+1)})((u))$ and $\nu(1-\tau)$ on $\nu(A^{\otimes(\bullet+1)})((u))$. Now, under the identification (1.1), the differential $b + uB$ for the algebra A_t^+ becomes

$$t(b, b') + \nu(N, 1-\tau) \tag{4.4}$$

In general, we will denote by (L_1, L_2) the operator that acts by L_1 on $(A^{\otimes(\bullet+1)})((u))$ and by L_2 on $\nu(A^{\otimes(\bullet+1)})((u))$. Using this convention, the connection form of the Getzler-Gauss-Manin connection is given by

$$\frac{1}{u} I_{t\frac{\partial m_t}{\partial t}} \frac{dt}{t} = \left(\left(\frac{t^2}{u}\iota_m, \frac{t}{\nu}\eta_m\right) + \left(\frac{t}{\nu}S_m, 0\right)\right)\frac{dt}{t} \tag{4.5}$$

Here

$$\iota_m(a_0 \otimes \ldots . \otimes a_n) = (-1)^{|a_0|}(a_0 a_1 a_2 \otimes \ldots . \otimes a_n) \tag{4.6}$$

$$\eta_m(a_0 \otimes \ldots . \otimes a_n) = a_0 a_1 \otimes \ldots . \otimes a_n \tag{4.7}$$

(4.8) $$S_m(a_0 \otimes \dots . \otimes a_n) = \sum_{j=0}^{n} \sum_{k=1}^{j-2} \pm a_j \otimes \dots a_n \otimes a_0 \otimes \dots \otimes a_k a_{k+1} \otimes \dots \otimes a_{j-1}$$

The sign rule is: there is an overall factor of

$$-(-1)^{(\sum_{p=j}^{n} + \sum_{p=0}^{k})(|a_p|+1)};$$

and a permutation of a_p and a_q introduces the factor $(-1)^{(|a_p|+1)(a_q|+1)}$. There is no need for $1\otimes$ because the result is in the b' column, and the identifications involve tensoring those columns by $\mathbb{1}$.

4.2. Dilating the differential. Let A be a DG algebra. We denote the differential by δ. We have a family of DG algebras $(A_t^+, t\delta)$. The component of the connection form coming from the differential is

(4.9) $$\frac{1}{u} I_{t\frac{\partial(t\delta)}{\partial t}} \frac{dt}{t} = ((\frac{t^2}{u}\iota_\delta, \frac{t}{v}\eta_\delta) + (\frac{t}{v}S_\delta, 0))\frac{dt}{t}$$

(4.10) $$\iota_\delta(a_0 \otimes \dots . \otimes a_n) = (-1)^{|a_0|}(a_0\delta(a_1) \otimes a_2 \otimes \dots . \otimes a_n)$$

(4.11) $$\eta_\delta(a_0 \otimes \dots . \otimes a_n) = \delta(a_0) \otimes a_1 \otimes \dots . \otimes a_n$$

(4.12) $$S_\delta(a_0 \otimes \dots . \otimes a_n) = \sum_{j=0}^{n} \sum_{k=1}^{j-1} \pm a_j \otimes \dots a_n \otimes a_0 \otimes \dots \otimes \delta(a_k) \otimes \dots \otimes a_{j-1}$$

The sign rule is: there is an overall factor of

$$-(-1)^{(\sum_{p=j}^{n} + \sum_{p=0}^{k-1})(|a_p|+1)};$$

and a permutation of a_p and a_q introduces the factor $(-1)^{(|a_p|+1)(a_q|+1)}$.

The Getzler-Gauss-Manin connection for the family $(A_t^+, t\delta)$ is given by

(4.13) $$\nabla^{\mathrm{GM}} = t(b, b') + v(N, 1-\tau) + (t\frac{\partial}{\partial t} + \frac{1}{u} I_{t\frac{\partial m_t}{\partial t}} + \frac{1}{u} I_{t\frac{\partial(t\delta)}{\partial t}})\frac{dt}{t}$$

This is a a connection on a $k[t]$-module on a k-module which is also a $k((u))$-module. We would like to get a connection on a $k((u))$-module. To do that, we will consider the action of the multiplicative group $\mathbb{G}_m$ given by

(4.14) $$\mu(t, v) = (\mu t, \mu v)$$

whose generator is

(4.15) $$\Lambda = t\frac{\partial}{\partial t} + v\frac{\partial}{\partial v} = t\frac{\partial}{\partial t} + 2u\frac{\partial}{\partial u}$$

We have

(4.16) $$\nabla^{\mathrm{GM}} = \mathcal{D} + \mathcal{A}$$

where $\mathcal{D}$ is of degree zero and $\mathcal{A}$ is of degree one with respect to the grading by degree of forms on the t space $\mathbb{A}^1$, and

(4.17) $$[\Lambda, \mathcal{D}] = \mathcal{D};\ [\Lambda, \mathcal{A}] = 0; [\mathcal{D}, \mathcal{A}] = \mathcal{A}^2 = 0$$

The operator

(4.18) $$\frac{1}{2u}(\Lambda - \iota_{t\frac{\partial}{\partial t}}\mathcal{A})$$

is a differential operator in two variables t and u. It is of order one and its principal symbol is $u\frac{\partial}{\partial u}$. This operator satisfies

$$[\frac{1}{2u}(\Lambda - \iota_{t\frac{\partial}{\partial t}}\mathcal{A}), \mathcal{D}] = \frac{1}{2u}\mathcal{D}$$

Denote by ∇_u its restriction to the line $t = 1$. We see that

$$\nabla_u = \frac{\partial}{\partial u} - \frac{1}{2u^2}((\iota_m, \nu\eta_m) + (\nu S_m, 0) + (\iota_\delta, \nu\eta_\delta) + (\nu S_\delta, 0)) \tag{4.19}$$

is a u-connection.

5. Morphisms and equivalences of u-connections

6. The rational structure

7. Definition of a noncommutative Hodge structure

8. Bibliographical notes

Kontsevich-Soibelman; Katzarkov-Kontsevich-Pantev; Shklyarov; Kaledin; Mochizuki; Sabbah; Simpson;

CHAPTER 22

Cyclic homology in characteristic p and over p-adics

1. Homology of $\mathbb{F}_p$-algebras over $\mathbb{Z}_p$

For a graded commutative unital ring K and for a K-algebra A, we denote by $HH_\bullet(A/K)$ the homology of the Hochschild complex of $\widetilde{A}$ with the ring of scalars K where $\widetilde{A}$ is any DG resolution of A over K which is flat as a K-module. Similarly for HC, etc.

Proposition 1.0.1.

$$HH_\bullet(\mathbb{F}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathbb{F}_p\{\sigma\}$$

where the right hand side denotes the divided power polynomials in one variable σ *of homological degree two.*

Proof. Take $\widetilde{A} = (\mathbb{Z}_p[\xi], p\frac{\partial}{\partial \xi})$. The basis of the Hochschild complex is:

$$\xi \otimes \xi^{\otimes n},\ n \geq 0,\ \text{of degree } 2n+1;\ 1 \otimes \xi^{\otimes n},\ n \geq 0,\ \text{of degree } 2n \tag{1.1}$$

The differentials are as follows:

$$b = 0;\ p\frac{\partial}{\partial \xi}(1 \otimes \xi^{\otimes n}) = 0;\ p\frac{\partial}{\partial \xi}(\xi \otimes \xi^{\otimes n}) = p \cdot 1 \otimes \xi^{\otimes n} \tag{1.2}$$

Therefore the basis of $HH_\bullet(\mathbb{F}_p/\mathbb{Z}_p)$ consists of $1 \otimes \xi^{\otimes n}$. The shuffle product is

$$(1 \otimes \xi^{\otimes n})(1 \otimes \xi^{\otimes m}) = \frac{(n+m)!}{n!m!} 1 \otimes \xi^{\otimes n+m} \tag{1.3}$$

The statement follows. □

Proposition 1.0.2. *for* j *odd,*

$$HC^-_j(\mathbb{F}_p/\mathbb{Z}_p) = 0$$

For $j \geq 0$,

$$HC^-_{2j}(\mathbb{F}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{Coker}(p\mathrm{Id} + E_j)$$

where

$$E_j : \prod_{n=j}^{\infty} \mathbb{Z}_p e_n \to \prod_{n=j}^{\infty} \mathbb{Z}_p e_n;\ E_j : e_n \mapsto (n+1)e_{n+1}$$

For $j < 0$,

$$HC^-_{2j}(\mathbb{F}_p/\mathbb{Z}_p) \xrightarrow{\sim} HC_0(\mathbb{F}_p/\mathbb{Z})$$

This follows immediately from (1.2) and from

$$B(1 \otimes \xi^{\otimes n}) = 0;\ B(\xi \otimes \xi^{\otimes n}) = (n+1)1 \otimes \xi^{\otimes n+1} \tag{1.4}$$

1.1. Completed periodic cyclic homology of $\mathbb{F}_p$. For a $\mathbb{Z}_p$-algebra A denote by $\widetilde{\mathrm{CC}}^{\mathrm{per}}_\bullet(A/\mathbb{Z}_p)$ the p-adic completion of $\mathrm{CC}^{\mathrm{per}}_\bullet(A/\mathbb{Z}_p)$. The homology of this complex is denoted by $\widetilde{\mathrm{HC}}^{\mathrm{per}}_\bullet(A/\mathbb{Z}_p)$.

Recall that the periodic cyclic homology $\mathrm{HC}^{\mathrm{per}}_\bullet(\mathbb{F}_p/\mathbb{Z}_p)$ is given by Proposition 1.0.2 and is equal to $\mathrm{HC}^-_0(\mathbb{F}_p/\mathbb{Z}_p)$.

PROPOSITION 1.1.1. *For j even and $p > 2$,*

$$\widehat{\mathrm{HC}}^{\mathrm{per}}_j(\mathbb{F}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathbb{Z}_p \times \prod_{n=0}^{\infty}(\mathbb{Z}/(n+1)\mathbb{Z})$$

For j odd,

$$\widehat{\mathrm{HC}}^{\mathrm{per}}_j(\mathbb{F}_p/\mathbb{Z}_p) = 0$$

PROOF. The operator

$$J: \prod_{n=0}^{\infty}\mathbb{Z}_p e_n \to \prod_{n=0}^{\infty}\mathbb{Z}_p e_n;\ e_n \mapsto e_{n-1} \tag{1.5}$$

satisfies the equation

$$[J, E_0] = \mathrm{Id}$$

where, as in Proposition 1.0.2 above,

$$E_0: \prod_{n=0}^{\infty}\mathbb{Z}_p e_n \to \prod_{n=0}^{\infty}\mathbb{Z}_p e_n;\ E_0: e_n \mapsto (n+1)e_{n+1}$$

Therefore the operator $\exp(pJ)$ intertwines E_0 with $p\mathrm{Id} + E_0$. Therefore

$$\widehat{\mathrm{HC}}^{\mathrm{per}}_j(\mathbb{F}_p/\mathbb{Z}_p) \xrightarrow{\sim} \mathrm{Coker}(E_0)$$

□

REMARK 1.1.2. Proposition 1.0.2 establishes the isomorphism

$$\widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(\mathbb{Z}_p[\xi], p\frac{\partial}{\partial\xi} \xrightarrow{\sim} \widehat{\mathrm{CC}}^{\mathrm{per}}_\bullet(\mathbb{Z}_p[\xi], 0)$$

REMARK 1.1.3. The above is a partial case of Theorem 3.0.1.

2. Hochschild-Witt homology

Here we follow Kaledin's works [**349**], [**348**].

2.1. Classical Witt vectors.

LEMMA 2.1.1. *(Dwork's lemma). Let x and y be two elements of a commutative algebra R over $\mathbb{Z}$. Then*

$$(x + py)^{p^n} - x^{p^n} \in p^{n+1}R$$

in $V^{\otimes p^n}$.

PROOF. Induction by n and the binomial formula. □

2.2. Noncommutative Witt vectors. For a $\mathbb{Z}$-module V, consider the action of the cyclic group C_{p^n} on $V^{\otimes p^n}$ by permutations. We denote the generator of C_{p^n} by σ and write

$$N = 1 + \sigma + \ldots + \sigma^{p^n - 1} \tag{2.1}$$

We denote the image of N by Norms.

LEMMA 2.2.1. *Let x and y be two elements of V. Then*

$$(x + y)^p - x^p - y^p \in \text{Norms}$$

COROLLARY 2.2.2. *In an associative algebra A over $\mathbb{F}_p$, for any x and y in A*

$$(x + y)^p = x^p + y^p$$

in $A/[A, A]$.

LEMMA 2.2.3. *(Noncommutative Dwork's lemma). Let x and y be two elements of V. Then*

$$(x + py)^{p^n} - x^{p^n} \in p\text{Norms}$$

in $V^{\otimes p^n}$.

PROOF. Let V be a free $\mathbb{Z}$-module with a basis $\{x_j | j \in J\}$. We will denote a monomial in $V^{\otimes p^n}$ (with respect to this basis) by X. We say a monomial is primitive if it is not a pth power of another monomial. Any monomial is uniquely of the form

$$X = Y^{p^{n-k}}$$

where y is a primitive monomial in $V^{\otimes p^k}$. This happens if and only if the C_{p^n}-orbit of X is of order p^k.

Let M_k be a set of representatives of all primitive monomials in $V^{\otimes p^k}$ (two monomials are equivalent if one is obtained from the other by a permutation from C_{p^k}).

Let V be a free $\mathbb{Z}$-module with the basis $\{x, y\}$. Then

$$(x + y)^{p^n} = \sum_{k=0}^{n} \sum_{Y \in M_k} N_{p^k}(Y^{p^{n-k}}) \tag{2.2}$$

where

$$N_{p^k} = 1 + \sigma + \ldots + \sigma^{p^k - 1} \tag{2.3}$$

(in particular $N = N_{p^n}$). Now let y be divisible by p. Then, unless $k = 0$ and $Y = x$, $N_{p^k}(Y^{p^{n-k}})$ is in the image of $p^{p^{n-k}} N_{p^k}$. But the image of $p^{n-k+1} N_{p^k}$ is in the image of pN. Indeed, $n - k + 1 \leq p^{n-k}$. This proves Lemma 2.2.3. Lemma 2.2.1 follows from (2.2) when $n = 1$. $\square$

DEFINITION 2.2.4. *Let V be a free $\mathbb{Z}$-module. Put*

$$W_n(V) = (V^{\otimes p^n})^{C_{p^n}} / \text{Norms}$$

In other words,

$$W_n(V) = \check{H}^0(C_{p^n}, V^{\otimes p^n})$$

(the Tate cohomology of degree zero). Put also

$$W'_n(V) = (V^{\otimes p^n})^{C_{p^n}} / p\text{Norms}$$

LEMMA 2.2.5.

$$W_n(V) = \bigoplus_{k=0}^{n-1} \bigoplus_{Y \in M_k} (\mathbb{Z}/p^{n-k}\mathbb{Z}) N_{p^k}(Y^{p^{n-k}})$$

$$W'_n(V) = \bigoplus_{k=0}^{n} \bigoplus_{Y \in M_k} (\mathbb{Z}/p^{n-k+1}\mathbb{Z}) N_{p^k}(Y^{p^{n-k}})$$

(Recall that M_k is a set of representatives of primitive monomials of length p^k up to cyclic permutation).

The proof is clear: one only has to compute $M^{C_{p^n}}/N(M)$ and $M^{C_{p^n}}/pN(M)$ for a C_{p^n}-module M induced from a trivial representation of C_{p^k}. And in this case, $M^{C_{p^n}} = N_{p^{n-k}}(M)$ and $N(M) = p^k N_{p^{n-k}}(M)$.

LEMMA 2.2.6. *Let* f *and* g *be two linear maps* $V_1 \to V_2$ *that differ modulo* p. *Then* $f^{\otimes p^n}$ *and* $g^{\otimes p^n}$ *define the same maps* $W_n(V_1) \to W_n(V_2)$ *and* $W'_n(V_1) \to W'_n(V_2)$.

This follows from noncommutative Dwork's lemma 2.2.3.

COROLLARY 2.2.7. *For a vector space* E *over* $\mathbb{F}_p$ *choose a free* $\mathbb{Z}$*-module* $\widetilde{E}$ *together with an isomorphism* $\widetilde{E}/p\widetilde{E} \xrightarrow{\sim} E$. *For any* $n \geq 0$, $E \mapsto W_n(\widetilde{E})$ *is a well-defined functor from vector spaces over* $\mathbb{F}_p$ *to modules over* $\mathbb{Z}/p^n\mathbb{Z}$.

LEMMA 2.2.8. *There is a natural isomorphism*

$$W'_n(V) \xrightarrow{\sim} W_{n+1}(V)$$

PROOF. First observe that the two sides become isomorphic if one identifies the terms corresponding to the same primitive monomial Y in the decomposition from Lemma 2.2.5. It remains to see that this isomorphism is natural. We call a linear map $(V \to V^{\otimes p})^{C_p}$ standard if the induced map

$$V/pV \to (V^{\otimes p})^{C_p}/\text{Norms}$$

is the isomorphism sending each v to v^p. From Lemma 2.2.6 we see that any standard map defines the same map $W'_n(V) \to W_{n+1}(V)$. On the other hand, the map $x_j \mapsto x_j^p$, $j \in J$, induces precisely the isomorphism above. □

2.2.1. *Restriction and Verschiebung.*

DEFINITION 2.2.9. *Define the natural transformation*

$$R : W_{n+1}(V) \to W_n(V)$$

by

$$W_{n+1}(V) \xleftarrow{\sim} W'_n(V) \to W_n(V)$$

where the isomorphism on the left is from Lemma 2.2.8 and the map on the right is the obvious projection.

In terms of the decomposition from Lemma 2.2.5, R is the projection

$$(\mathbb{Z}/p^{n+1-k}\mathbb{Z}) N_{p^k}(Y^{p^{n-k}}) \to (\mathbb{Z}/p^{n-k}\mathbb{Z}) N_{p^k}(Y^{p^{n-k}})$$

for every primitive monomial Y. If Y is of length p^n then it maps to zero.

Definition 2.2.10. *Define the natural transformation*

$$V : W_n(V^{\otimes p}) \to W_{n+1}(V)$$

by

$$N_p : ((V^{\otimes p})^{\otimes p^n})^{C_{p^n}} \xrightarrow{\sim} (V^{\otimes p^{n+1}})^{C_{p^n}} \to (V^{\otimes p^{n+1}})^{C_{p^{n+1}}}$$

(Recall that

$$N_p = 1 + \sigma + \ldots + \sigma^{p-1};$$

note that V takes norms to norms. Indeed, on the left hand side the norm is given by

$$N = 1 + \sigma^p + \ldots + \sigma^{p(p^n-1)};$$

therefore its composition with N_p is the norm on the right).

2.3. Trace functors. Following Kaledin, we define the trace functor from a monoidal category $(\mathcal{A}, \otimes)$ to a category $\mathcal{K}$ as a functor $\mathrm{Tr} : \mathcal{A} \to \mathcal{K}$ together with a natural transformation

$$\tau_{M,N} : \mathrm{Tr}(M \otimes N) \xrightarrow{\sim} \mathrm{Tr}(N \otimes M) \tag{2.4}$$

such that

$$\tau_{M\otimes N,L}\tau_{N\otimes L,M}\tau_{L\otimes M,N} = \mathrm{id}_{\mathrm{Tr}(L\otimes M\otimes N)} \tag{2.5}$$

and

$$\tau_{M,1} = \tau_{1,M} = \mathrm{id}_{\mathrm{Tr}(M)} \tag{2.6}$$

Given a trace functor Tr from k-modules to a category $\mathcal{K}$, Kaledin defines a cyclic object $\mathrm{Tr}^\natural(A)$ of $\mathcal{K}$ for any k-algebra A. Namely, we put

$$\mathrm{Tr}^\natural(A)[n] = \mathrm{Tr}(A^{\otimes(n+1)}) \tag{2.7}$$

The face maps $d_0, \ldots, d_{n-1}$ are induced by the ones on $A^{\otimes(n+1)}$ and so are the degeneracy maps. The action of the cyclic permutation is by $\tau_{A^{\otimes n},A}$.

More generally, for a k-algebra A with an authomorphism α of order p one defines a p-cyclic object $\mathrm{Tr}^\natural(A, \alpha)$ of $\mathcal{K}$.

2.4. The construction.

Lemma 2.4.1. *The cyclic permutation*

$$\sigma : (M \otimes N)^{\otimes p^n} \xrightarrow{\sim} (N \otimes M)^{\otimes p^n},$$

$$v_1 \otimes w_1 \otimes \ldots \otimes v_{p^n} \otimes w_{p^n} \mapsto w_1 \otimes v_{p^n} \otimes \ldots \otimes w_{p^n} \otimes v_1,$$

$v_i \in M$, $w_i \in N$, *turns* W_n *into a trace functor.*

Now for an $\mathbb{F}_p$-algebra A define the cyclic $\mathbb{Z}/p^n\mathbb{Z}$-module

$$W_n^\natural(A)[k] = W_n(A^{\otimes k+1}) \tag{2.8}$$

Definition 2.4.2.

$$W_n\mathrm{HH}_\bullet(A) = \mathrm{HH}_\bullet(W_n^\natural(A));\ W_n\mathrm{HC}_\bullet(A) = \mathrm{HC}_\bullet(W_n^\natural(A))$$

The following theorems are from [**349**] and [**348**].

THEOREM 2.4.3. *For a finitely generated smooth commutative algebra over* $\mathbb{F}_p$ *there is a natural isomorphism*

$$W_n HH_\bullet(A) \xrightarrow{\sim} W_n \Omega^\bullet_A$$

where the right hand side denotes De Rham -Witt forms of Deligne-Illusie [**185**]. *This isomorphism intertwines the cyclic differential* B *with the De Rham differential.*

THEOREM 2.4.4. *For any algebra over* $\mathbb{F}_p$ *there is a natural isomorphism*

$$W_n HH_0(A) \xrightarrow{\sim} W_n^H(A)$$

where the right hand side denotes Hesselholt's generalized Witt vectors [**314**] .

3. Noncommutative Frobenius and Cartier morphisms

3.1. The Kaledin resolution and (co)invariants. We have seen Proposition 5.4.1 that the ℓ-cyclic module $(A^{\otimes \ell})^\natural_\alpha$ has the same Hochschild and cyclic homology as the cyclic module $A^\natural$. It is important to have more information about the two *cyclic* modules that it gives rise to, namely, $\pi_{\ell !}(A^{\otimes \ell})^\natural_\alpha$ (the coinvariants of the cyclic group C_ℓ) and the $\pi_{\ell *}(A^{\otimes \ell})^\natural_\alpha$ (the invariants).

LEMMA 3.1.1. *For an* ℓ*-cyclic module* M *one has an exact sequence of cyclic modules*

$$0 \to \pi_{\ell *} M \to \pi_{\ell !}\mathbb{K}_1(M) \xrightarrow{\partial} \pi_{\ell !}\mathbb{K}_0(M) \to \pi_{\ell !} M \to 0$$

PROOF. The homology of ∂ acting on coinvariants of $\mathbb{Z}/\ell\mathbb{Z}$ is the homology of the circle with coefficients in the local syslem with fiber M on which the monodromy acts via the action of $\mathbb{Z}/\ell\mathbb{Z}$ on M. □

We get a short exact sequence of complexes of cyclic modules

$$0 \to \pi_{\ell *} M[1] \to \pi_{\ell !}\mathbb{K}(M) \to \pi_{\ell !} M \to 0 \tag{3.1}$$

We now extend it to the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_{\ell *} M[1] & \longrightarrow & \mathbb{K}(\pi_{\ell *} M) & \longrightarrow & \pi_{\ell *} M & \longrightarrow & 0 \\
 & & \big\downarrow = & & \big\downarrow \varphi_\ell & & \big\downarrow \mathbf{e}_\ell & & \\
0 & \longrightarrow & \pi_{\ell *} M[1] & \longrightarrow & \pi_{\ell !}\mathbb{K}(M) & \longrightarrow & \pi_{\ell !} M & \longrightarrow & 0 \\
 & & \big\downarrow \mathbf{e}_\ell & & \big\downarrow \nu_\ell & & \big\downarrow = & & \\
0 & \longrightarrow & \pi_{\ell !} M[1] & \longrightarrow & \mathbb{K}(\pi_{\ell !} M) & \longrightarrow & \pi_{\ell !} M & \longrightarrow & 0
\end{array}$$

The vertical morphisms on the left and on the right are either the identity or the norm map from coinvariants to invariants. The lines are short exact sequences. The vertical maps in the middle are as follows.

First observe that for a cyclic object E one has

$$\mathbb{K}(E) \xrightarrow{\sim} \pi_{\ell !}\mathbb{K}(\pi_\ell^* E) \xrightarrow{\sim} \pi_{\ell *}\mathbb{K}(\pi_\ell^* E) \tag{3.2}$$

Apply this to $E = \pi_{\ell !} M$ and $E = \pi_{\ell *} M$. We get

$$\mathbb{K}(\pi_{\ell !} M) \xrightarrow{\sim} \pi_{\ell !}\mathbb{K}(\pi_\ell^* \pi_{\ell !} M) \longleftarrow \pi_{\ell !}\mathbb{K}(M)$$

and

$$\mathbb{K}(\pi_{\ell *} M) \xrightarrow{\sim} \pi_{\ell !}\mathbb{K}(\pi_\ell^* \pi_{\ell *} M) \longrightarrow \pi_{\ell *}\mathbb{K}(M)$$

We obtain natural morphisms

$$\nu_\ell : \pi_{\ell!}\mathbb{K}(M) \to \mathbb{K}(\pi_{\ell!}M);\ \varphi_\ell : \mathbb{K}(\pi_{\ell*}M) \to \mathbb{K}(\pi_{\ell*}(M)) \tag{3.3}$$

Explicitly: given a chain

$$\sum_{j=0}^{n} m_j \otimes x_j$$

in $\mathbb{K}(\pi_{\ell*}M)$ where x_j are either vertices of edges of the triangulation $[n]$ and e_j are $\mathbb{Z}/\ell\mathbb{Z}$-invariant elements of M_n, φ_ℓ sends it to the chain

$$\sum_{j=0}^{n}\sum_{i=0}^{\ell-1} m_j \otimes x_j^{(i)}$$

And given a chain

$$\sum_{j=0}^{n}\sum_{i=0}^{\ell-1} m_j^{(i)} \otimes x_j^{(i)}$$

in $\pi_{\ell!}\mathbb{K}(M)$, ν_ℓ sends it to the chain

$$\sum_{j=0}^{n}\sum_{i=0}^{\ell-1} \tau^i(m_j^{(i)}) \otimes x_j^{(i)}$$

4. The Frobenius map

In this section k is a perfect field of characteristic $p > 0$. As usual, for $a \in k$, $Fa = a^p$ is the Frobenius morphism. For any vector space V over k we will consider $V^{\otimes p}$ as a $\mathbb{Z}/p\mathbf{Z}$-module (with the action by cyclic permutations). As usual, for every module M over a finite group G, its Tate cohomology in degree zero is defined by

$$\check{H}^0(G, V) = V^G/\mathrm{im}(N) \tag{4.1}$$

where

$$N = \sum_{g\in G} g : V \to V.$$

4.1. Frobenius map for vector spaces.

LEMMA 4.1.1. *For a vector space* V *over* p, *the map* $x \mapsto x^{\otimes p}$ *induces an* F-*linear isomorphism*

$$V \tilde{\to} \check{H}^0(\mathbb{Z}/p\mathbb{Z}, V^{\otimes p}) \tag{4.2}$$

PROOF. Consider a basis $\mathbf{B}$ of V over k. Then the following two sets of vectors form a basis of $V^{\otimes p}$: a) $v^{p\otimes}$, $v \in \mathbf{B}$ and b) $v_1 \otimes \ldots \otimes v_p$ where v_j are all in $\mathbf{B}$ and not all the same. The subset a) generates a constant $\mathbb{Z}/p\mathbb{Z}$-module that coincides with its degree zero Tate cohomology. The subset b) generates a free $\mathbb{Z}/p\mathbb{Z}$-module, therefore its Tate cohomology vanishes. Furthermore, the map $x \mapsto x^p$ is additive because

$$(x + y)^p = x^p + y^p + Nz$$

for some z. □

4.2. Frobenius map for cyclic objects.

PROPOSITION 4.2.1. *Let* A *be an algebra over* k. *There is a natural isomorphism of cyclic objects*

$$\varphi_p : A^\sharp \xrightarrow{\sim} \check{H}^0(\mathbb{Z}/p\mathbb{Z}, i_p^*(A^\sharp)) \tag{4.3}$$

where the action of $\mathbb{Z}/p\mathbb{Z}$ *on the* p*-cyclic vector space* $i_p^*(A^\sharp)$ *is via the group generated by* σ *from* (**??**).

PROOF. Note that $i_p^*(A^\sharp)_n \xrightarrow{\sim} (A_n^\sharp)^{\otimes p}$ ((5.5)); it is straightforward that the map $x \mapsto x^{\otimes p}$ is a morphism of cyclic vector spaces. □

CHAPTER 23

Coperiodic cyclic homology

Notation: For a double complex $C_{\bullet,*}$ there are two ways to totalize it:

$$\mathrm{tot}_n(C_{\bullet,*}) = \bigoplus_{j+k=n} C_{j,k} \tag{0.1}$$

and

$$\mathrm{Tot}_n(C_{\bullet,*}) = \prod_{j+k=n} C_{j,k} \tag{0.2}$$

1. Coperiodic cyclic complex of a complex of an algebra

We start by defining the coperiodic cyclic complex of an associative algebra. Below we extend this definition to DG algebras and categories. This will be done in three different ways (that give the same answer for an algebra).

For an algebra A over k define

$$\mathrm{CC}^{\mathrm{pol}}_{\bullet}(A) = \mathrm{CC}^{\mathrm{coper},f}_{\bullet}(A) = \mathrm{CC}^{\mathrm{coper}}_{\bullet}(A) = (C_{\bullet}(A)[u^{\pm 1}], b + uB) \tag{1.1}$$

LEMMA 1.0.1. *If* $\mathrm{char}(k) = 0$ *then* $\mathrm{CC}^{\mathrm{pol}}_{\bullet}(A)$ *is acyclic.*

PROOF. □

EXAMPLE 1.0.2. Assume that k is torsion-free.

2. Coperiodic cyclic complex of a complex of cyclic modules

For a mixed complex $(V_{\bullet}, b, B)$ define the polynomial, coperiodic, and coperiodic cyclic complexes of $V_{\bullet}$ by

$$\mathrm{CC}^{\mathrm{pol}}_{\bullet}(V_{\bullet}) = (V_{\bullet}[u^{\pm 1}], b + uB) \tag{2.1}$$

$$\mathrm{CC}^{\mathrm{per}}_{\bullet}(V_{\bullet}) = (V_{\bullet}((u)), b + uB) \tag{2.2}$$

$$\mathrm{CC}^{\mathrm{coper}}_{\bullet}(V_{\bullet}) = (V_{\bullet}((u^{-1})), b + uB) \tag{2.3}$$

There are natural embeddings

$$\mathrm{CC}^{\mathrm{coper}}_{\bullet}(V_{\bullet}) \longleftarrow \mathrm{CC}^{\mathrm{pol}}(V_{\bullet}) \longrightarrow \mathrm{CC}^{\mathrm{per}}(V_{\bullet}) \tag{2.4}$$

If $V_{\bullet}$ is bounded from below (in homological grading), i.e. if $V_j = 0$ for $j << 0$, then the morphism on the left of (2.4) is a quasi-isomorphism. If in addition $\mathrm{char}(k) = 0$ then both polynomial and coperiodic complexes are acyclic.

If $(V_{\bullet}, b)$ is acyclic then so is the periodic cyclic complex (but not always the other two).

Let (M, d) be a complex of cyclic k-modules. Let

$$V_{\bullet}(M) = (\mathrm{tot}(C_{\bullet}(M)), b + d, B) \tag{2.5}$$

Define

$$\mathrm{CC}^{\mathrm{pol}}_{\bullet}(\mathsf{M}) = \mathrm{CC}^{\mathrm{pol}}(\mathsf{V}_{\bullet}(\mathsf{M})) \tag{2.6}$$

$$\mathrm{CC}^{\mathrm{per}}_{\bullet}(\mathsf{M}) = \mathrm{CC}^{\mathrm{per}}(\mathsf{V}_{\bullet}(\mathsf{M})) \tag{2.7}$$

$$\mathrm{CC}^{\mathrm{coper}}_{\bullet}(\mathsf{M}) = \mathrm{CC}^{\mathrm{coper}}_{\bullet}(\mathsf{V}_{\bullet}(\mathsf{M})) \tag{2.8}$$

2.1. The $(\mathsf{b}, \mathsf{b}', 1 - \mathsf{t}, \mathsf{N})$ version. Let $\widetilde{\mathsf{C}}_{\bullet}(\mathsf{M}_*)$ be as in (4.25). Define

$$\widetilde{\mathrm{CC}}^{?}_{\bullet}(\mathsf{M}) = \mathrm{CC}^{?}(\widetilde{\mathsf{V}}_{\bullet}(\mathsf{M})) \tag{2.9}$$

where

$$\mathsf{V}_{\bullet}(\mathsf{M}) = (\mathrm{tot}(\widetilde{\mathsf{C}}_{\bullet}(\mathsf{M})), \mathsf{b} + \mathsf{d}, \mathsf{B}) \tag{2.10}$$

and ? is pol, coper, and or per.

Recall the quasi-isomorphism

$$\widetilde{\mathsf{C}}_{\bullet}(\mathsf{M}) \to \mathsf{C}_{\bullet}(\mathsf{M}) \tag{2.11}$$

REF? It commutes intertwines the B differentials and therefore defines a morphism

$$\widetilde{\mathrm{CC}}^{?}_{\bullet}(\mathsf{M}) \to \mathrm{CC}^{?}_{\bullet}(\mathsf{M}) \tag{2.12}$$

LEMMA 2.1.1. *The morphism* (2.12) *is a quasi-isomorphism.*

PROOF. This follows from the Hochschild-to-cyclic spectral sequence argument in the periodic case. In the other two cases the spectral sequence does not converge. The statement follows from Lemma 2.1.2 below. □

LEMMA 2.1.2. *Let* $\mathsf{V}_{\bullet}$ *be a mixed complex together with* $\mathsf{h} : \mathsf{V}_{\bullet} \to \mathsf{V}_{\bullet+1}$ *such that* $[\mathsf{b}, \mathsf{h}] = \mathrm{id}$. *Assume that*

$$[\mathsf{h}, \mathsf{B}]^{\mathsf{n}} = 0$$

for some n. *Then* $\mathrm{CC}^{?}(\mathsf{V}_{\bullet})$ *is acyclic for* ? = per, pol, *and* coper.

PROOF. □

2.2. Restricted (co)periodic cyclic complex. As above, let M be a complex of cyclic objects in the category of k-modules. We use the homological notation and write

$$[\mathsf{m}] \mapsto \mathsf{M}_{\bullet}[\mathsf{m}] = \mathsf{M}^{-\bullet}[\mathsf{m}] \tag{2.13}$$

for an object $[\mathsf{m}]$ of Λ.

For every $\mathsf{m} \geq 0$ consider the double complex

$$\widetilde{\mathsf{C}}_{\bullet}(\mathsf{M})[\mathsf{m}] = (\mathsf{M}_{\bullet}[\mathsf{m}] \xrightarrow{1-\tau} \mathsf{M}_{\bullet}[\mathsf{m}]) \tag{2.14}$$

The norm N induces a morphism of complexes

$$\mathsf{B} : \widetilde{\mathsf{C}}_{\bullet}(\mathsf{M})[\mathsf{m}] \to \widetilde{\mathsf{C}}_{\bullet+1}(\mathsf{M}_{\bullet})[\mathsf{m}] \tag{2.15}$$

Now define

$$\widetilde{\mathrm{CC}}^{\mathrm{per},\mathsf{f}}_{\bullet}(\mathsf{M})[\mathsf{m}] = (\widetilde{\mathsf{C}}_{\bullet}(\mathsf{M}_{\bullet})[\mathsf{m}]((\mathsf{u})), \mathsf{b} + \mathsf{u}\mathsf{B}) \tag{2.16}$$

$$\widetilde{\mathrm{CC}}^{\mathrm{coper},\mathsf{f}}_{\bullet}(\mathsf{M})[\mathsf{m}] = (\widetilde{\mathsf{C}}_{\bullet}(\mathsf{M}_{\bullet})[\mathsf{m}]((\mathsf{u}^{-1})), \mathsf{b} + \mathsf{u}\mathsf{B}) \tag{2.17}$$

The differential

$$(b', b) : \widetilde{C}_\bullet[m] \to \widetilde{C}_\bullet[m-1]$$

turns

$$\widetilde{CC}_\bullet^{?,f}(M)[*]$$

into a double complex.

DEFINITION 2.2.1.

$$\widetilde{CC}_\bullet^{\mathrm{per},f}(M) = \mathrm{tot}(\widetilde{CC}_\bullet^{\mathrm{per},f}(M)[*])$$
$$\widetilde{CC}_\bullet^{\mathrm{coper},f}(M) = \mathrm{tot}(\widetilde{CC}_\bullet^{\mathrm{coper},f}(M)[*])$$

We now have five versions of a periodic cyclic complex of a complex of cyclic k-modules M:

(2.18)

$$\widetilde{CC}_\bullet^{\mathrm{coper}}(M) \longleftarrow \widetilde{CC}_\bullet^{\mathrm{coper},f}(M) \longleftarrow \widetilde{CC}_\bullet^{\mathrm{pol}}(M) \longrightarrow \widetilde{CC}_\bullet^{\mathrm{per},f}(M) \longrightarrow \widetilde{CC}_\bullet^{\mathrm{per}}(M)$$

In concrete terms:

(1) An element of degree N of $\widetilde{CC}_\bullet^{\mathrm{coper}}$ is

$$\sum_{j=-m}^{\infty} u^{-j} \sum_{k=0}^{\infty} a_{N-k-2j}^{(j)}[k] \tag{2.19}$$

for some m.

(2) An element of degree N $\widetilde{CC}_\bullet^{\mathrm{coper},f}$ is

$$\sum_{j=-m}^{\infty} u^{-j} \sum_{k=0}^{q} a_{N-k-2j}^{(j)}[k] \tag{2.20}$$

for some q and m. In both cases

$$a_{N-k-2j}^{(j)}[k] \in M_{N-k-2j}[k]$$

(3) An element of degree N of $\widetilde{CC}_\bullet^{\mathrm{per}}$ is

$$\sum_{j=m}^{\infty} u^{j} \sum_{k=0}^{\infty} a_{N-k+2j}^{(j)}[k] \tag{2.21}$$

for some m.

(4) An element of degree N of $\widetilde{CC}_\bullet^{\mathrm{per},f}$ is

$$\sum_{j=m}^{\infty} u^{j} \sum_{k=0}^{q} a_{N-k+2j}^{(j)}[k] \tag{2.22}$$

for some m and q. In both cases

$$a_{N-k+2j}^{(j)}[k] \in M_{N-k+2j}[k]$$

(5) An element of $\widetilde{CC}_\bullet^{\mathrm{pol}}$ is a sum like in any of the above that has finitely many nonzero terms.

LEMMA 2.2.2. *If* $\mathrm{char}(k) = 0$ *then* $\widetilde{CC}_\bullet^{\mathrm{coper}}$ *and* $\widetilde{CC}_\bullet^{\mathrm{coper},f}$ *are acyclic.*

PROOF. The complex $(1-t, N)$ is acyclic. Given a $d+b+uB$-cycle, we construct a chain of which it is a boundary. ***More □

3. Coperiodic cyclic complex of a DG algebra

DEFINITION 3.0.1. *For a DG algebra* $\mathcal{A}^\bullet$

$$\mathrm{CC}^?_\bullet(\mathcal{A}^\bullet) = \mathrm{CC}^?_\bullet(\mathsf{M})$$

where $\mathsf{M} = \mathsf{C}_*(\mathcal{A}^\bullet)$ *viewed as a complex of cyclic* k*-modules (or a cyclic object on complexes of* k*-modules). Here ? stands for any of the five versions of the periodic cyclic complex in* (2.18).

Namely,

4. Conjugate spectral sequence

5. Invariance under quasi-isomorphisms

6. Bibliographic references

Kaledin; Beilinson-Bhatt;

CHAPTER 24

Hochschild and cyclic complexes of the second kind

1. Introduction

Here we define the Hochschild and cyclic chain complexes of the second kind. The Hochschild complex of the second kind is defined as in Chapter 2 but using direct products instead of direct sums in the total complex. Since direct sums map to direct products, Hochschild homology of the first kind maps to the Hochschild homology of the second kind. We prove a theorem giving a sufficient condition for this map to be an isomorphism. Our main reference is the article [**481**] by Positselski and Polishchuk.

Complexes of the second kind are well suited for curved DG algebras and categories. First, note that complexes of the first kind are invariant under quasi-isomorphisms and quasi-equivalences. But a curved DGA is not even a complex, so quasi-isomorphism cannot be defined. In any case, a morphism of curved DG algebras (which of course *can* be defined) only induces a morphism of complexes of the second kind. The reason is that the relevant morphisms are infinite sums that require working with direct products but not with direct sums. Another key property of Hochschild and cyclic complexes is their invariance when we replace the DG category by its DG category of (perfect) modules. That invariance still holds for complexes of the second kind.

We have already seen the importance of complexes of the second kind in the dual context of coalgebras. Not being invariant under quasi-isomorphisms is actually a good feature, since one of the most often used DG coalgebras, the bar construction, is contractible (for unital algebras).

2. Curved DG algebras and categories

DEFINITION 2.0.1. *A curved DG algebra is a graded algebra $\mathcal{A}$ with a derivation d of degree one and an element $R \in \mathcal{A}^2$ such that*

1) $d^2 a = [R, a]$ *for all* a *in* $\mathcal{A}$*;*

2) $dR = 0$.

In other words, a curved DG algebra is a curved A_∞ algebra with the only non-zero operations m_0, m_1, m_2 (cf. Remark 10.0.1). A DG algebra structure on $\mathcal{A}$ can be twisted by an element $\alpha \in \mathcal{A}^1$: for such an element, define

$$d_\alpha = d + \mathrm{ad}(\alpha);\ R_\alpha = R + d_\alpha + \alpha^2. \tag{2.1}$$

DEFINITION 2.0.2. *A morphism of curved DG algebras $\mathcal{A}, d_{\mathcal{A}}, R_{\mathcal{A}}$ and $\mathcal{B}, d_{\mathcal{B}}, R_{\mathcal{B}}$ is a pair (F, β) where $F : \mathcal{A} \to \mathcal{B}$ is a linear map of degree zero, $\beta \in \mathcal{B}^1$,*

$$d_{\mathcal{B}} F(a) - F(d_{\mathcal{A}} a) = [\beta, F(a)],$$

and

$$F(R_{\mathcal{A}}) = R_{\mathcal{B}} + d_{\mathcal{B}}\beta + \beta^2.$$

(Note: the first equation implies that the difference of the left and right hand sides in the second equation commutes with $F(a)$ for all a).

In other words, a morphism is a strict morphism (i.e. a map preserving all the structures) from $\mathcal{A}$ to a twist of $\mathcal{B}$. Alternatively, a morphism can be defined as a curved A_∞ morphism (P_n) with the only non-zero components P_0 and P_1.

2.1. Modules over curved DG algebras. When it comes to modules, there are two possibilities. One is given by Definition 2.1.1 below. While natural and useful (for example for the theory of matrix factorizations), it is sometimes too strict. Indeed, if we want to interprete a module $\mathcal{A}$ over $\mathcal{A}$ as a morphism from $\mathcal{A}$ to $\mathrm{End}_k(\mathcal{V})$, then an element β of degree one should be involved. But it is no longer natural to start with a complex $\mathcal{V}$; instead, we can view it as a pre-complex, i.e. endow it with a "differential" $d_{\mathcal{V}}$ without requiring $d_{\mathcal{V}}^2 = 0$. Precomplexes do indeed form a curved DG category (see below). But, to define a morphism from $\mathcal{A}$ to it, one needs another element of degree one in addition to $d_{\mathcal{V}}$, which seems unreasonable. This suggests Definition 2.1.2.

DEFINITION 2.1.1. *A strict DG module over a curved DG algebra* $(d_{\mathcal{A}}, d, R)$ *is a graded module* $\mathcal{V}$ *over the graded algebra* $\mathcal{A}$ *together with an element* $d_{\mathcal{V}} \in \mathrm{End}^1_k(\mathcal{V})$ *such that* $d_{\mathcal{V}}^2 = R$ *and*

$$d_{\mathcal{V}}(av) = (da)v + (-1)^{|a|}a \cdot d_{\mathcal{V}}v$$

for a *in* $\mathcal{A}$ *and* v *in* $\mathcal{V}$

A more general definition is as follows.

DEFINITION 2.1.2. *A QDG module over a curved DG algebra* $(d_{\mathcal{A}}, d, R)$ *is a graded module* $\mathcal{V}$ *over the graded algebra* $\mathcal{A}$ *together with an element* $d_{\mathcal{V}} \in \mathrm{End}^1_k(\mathcal{V})$ *such that*

$$d_{\mathcal{V}}(av) = (da)v + (-1)^{|a|}a \cdot d_{\mathcal{V}}v$$

for a *in* $\mathcal{A}$ *and* v *in* $\mathcal{V}$.

In other words, a module structure on $\mathcal{V}$ is an endomorphism $d_{\mathcal{V}}$ of degree one and a strict morphism from $\mathcal{A}$ to the twist of $\mathrm{End}(\mathcal{V})$ by $d_{\mathcal{V}}$, *but without the condition on curvatures.* Polishchuk and Possitselsky call such morphisms quasi-curved morphisms (or, more generally, quasi-curved DG functors).

Let $(\mathcal{V}, d_{\mathcal{V}}, \beta)$ be a QDG module over $\mathcal{A}$. Let

$$R_{\mathcal{V}} = d_{\mathcal{V}}^2 - R. \tag{2.2}$$

Observe that

1) $[d_{\mathcal{V}}, \_]$ sends $\mathrm{End}_{\mathcal{A}}(\mathcal{V})$ to itself. Indeed, for $f \in \mathrm{End}(\mathcal{V})$,

$$[a, [d_{\mathcal{V}}, f]] = \pm[d_{\mathcal{V}}, [a, f]] \pm [[d_{\mathcal{V}}, a], f] \pm = \pm[d_{\mathcal{V}}, [a, f]] \pm [da, f] = 0$$

if $f \in \mathrm{End}_{\mathcal{A}}(\mathcal{V})$.

2) $R_{\mathcal{V}} \in \mathrm{End}^2_{\mathcal{A}}(\mathcal{V})$.

Therefore $(\mathrm{End}_{\mathcal{A}}(\mathcal{V}), [d_{\mathcal{V}},], R_{\mathcal{V}})$ is a curved DG algebra.

3. Curved DG categories

The definition of a DG category is a straightforward generalization.

DEFINITION 3.0.1. *A curved DG category is a graded category $\mathcal{A}$ with a derivation* d_x, $x \in \mathrm{Ob}(\mathcal{A})$, *of degree one and an element* $R_x \in \mathcal{A}^2(x, x)$ *for each object* x *of* $\mathcal{A}$, *such that*

1) $d^2 a = R_x a - a R_y$ *for all* a *in* $\mathcal{A}(x, y)$*;*

2) $dR_x = 0$.

(Remember: our convention for composition is $\mathcal{A}(x, y) \otimes \mathcal{A}(y, z) \to \mathcal{A}(x, z)$; $f \otimes g \mapsto fg$).

DEFINITION 3.0.2. *A curved DG functor between curved DG algebras* $\mathcal{A}, d_{\mathcal{A}}, R_{\mathcal{A}}$ *and* $\mathcal{B}, d_{\mathcal{B}}, R_{\mathcal{B}}$ *is a pair* (F, β) *where* $F : \mathcal{A} \to \mathcal{B}$ *is a (*k*-linear) functor* $F : \mathcal{A} \to \mathcal{B}$ *preserving the grading, together with* $\beta_x \in \mathcal{B}^1(Fx, Fx)$ *for every* $x \in \mathrm{Ob}(\mathcal{A})$, *such that*

$$d_{\mathcal{B}} F(a) - (-1)^{|a|} F(d_{\mathcal{A}} a) = \beta_x F(a) - F(a) \beta_y$$

for $a \in \mathcal{A}(x, y)$ *and*

$$F(R_{\mathcal{A},x}) = R_{\mathcal{B},Fx} + d_{\mathcal{B}} \beta_x + \beta_x^2$$

for all x.

3.1. Modules over curved DG categories.

DEFINITION 3.1.1. *A QDG module over a curved DG category* $\mathcal{A}$ *is a graded module* $\mathcal{V} = (\mathcal{V}_x,\ x \in \mathrm{ob}(\mathcal{A}))$, *over the graded category* $\mathcal{A}$ *together with an element* $d_{\mathcal{V},x} \in \mathrm{End}^1_k(\mathcal{V}_x)$ *for any* x*, such that*

$$d_{\mathcal{V},x}(av) = (da)v + (-1)^{|a|}\, a \cdot d_{\mathcal{V},y} v$$

for a *in* $\mathcal{A}(x, y)$ *and* v *in* $\mathcal{V}_y$*.*

DEFINITION 3.1.2. *A strict DG module over a curved DG category* $\mathcal{A}$ *is a QDG module* $(\mathcal{V}, d_{\mathcal{V},x}|_{x \text{ in } \mathrm{Ob}(\mathcal{A})})$ *such that* $d^2_{\mathcal{V},x} = R_x$ *for any object* x.

EXAMPLE 3.1.3. The category of precomplexes of k-modules. An object is a graded k-module V together with an endomorphism d_V of degree one. The graded k-module of morphisms $\mathrm{Hom}(V, W)$ is the usual one.The differential is $f \mapsto d_V f - (-1)^{|f|} f d_V$; $R_V = d_V^2$. ***check conventions on composition?

Under this definition, a QDG module over $\mathcal{A}$ is a strict DG functor from $\mathcal{A}$ to pre-complexes but without the condition on curvatures (or a quasi-curved DG functor).

EXAMPLE 3.1.4. The category of QDG modules over a DG algebra $\mathcal{A}$ is a curved DG category. For such a module $\mathcal{V}$, the curvature element is $R_{\mathcal{V}}$ as in (2.2).

EXAMPLE 3.1.5.

The definition of the DG category $\mathbf{C}(A, B)$ can be extended...

4. Hochschild and cyclic complexes of the second kind of curved DG categories

4.1. Definitions.

DEFINITION 4.1.1. *For a curved DG category $\mathcal{A}$ set*

$$C^{II}_\bullet(\mathcal{A}) = \prod_{n\geq 0; x_0,\ldots,x_n\in \mathrm{Ob}(\mathcal{A})} \mathcal{A}(x_0,x_1)\otimes \overline{\mathcal{A}}(x_1,x_2)[1]\otimes\ldots\otimes\overline{\mathcal{A}}(x_n,x_0) \tag{4.1}$$

where

$$\overline{\mathcal{A}}(x,y)=\mathcal{A}(x,y)\,\mathrm{when}\; x\neq y\; and \overline{\mathcal{A}}(x,x)=\mathcal{A}(x,x)/k\mathbf{1}_x.$$

The differentials b*,* d*, and* B *are defined exactly as in 4, and there is an extra differential* ℓ_R *which we define here for any* $c=(c_x), x\in \mathrm{Ob}(\mathcal{A})$ *:*

$$\ell_c(a_0\otimes\ldots\otimes a_n)=\sum_{j=0}^{n}(-1)^{(\sum_{k=0}^{j}(|a_k|+1)+1)(|c|+1)}a_0\otimes\ldots\otimes a_j\otimes c\otimes\ldots\otimes a_n$$

Similarly for the non-normalized complex $\widetilde{C}_\bullet(\mathcal{A})$*.*

In particular, a morphism of curved DG algebras defines a morphism of Hochschild complexes of the second kind. As usual,

$$CC^{-,II}(\mathcal{A})=(C^{II}_\bullet(\mathcal{A})[[u]], b+d+uB)$$

and similarly for the cyclic and periodic cyclic complexes.

LEMMA 4.1.2. *The Hocschild and negative cyclic complexes of the second kind of $\mathcal{A}$ are isomorphic to corresponding complexes of any twist of $\mathcal{A}$.*

PROOF. For an odd element β of a curved DGA (or, more generally, for a collection of elements of degree one ($\beta_x\in\mathcal{A}(x,x)$) for a curved DG category $\mathcal{A}$), define

$$\exp(\ell_\beta)(a_0\otimes\ldots\otimes a_n)=\sum a_0\otimes\ldots\otimes\beta\otimes\ldots\otimes\beta\otimes\ldots$$

where $\ldots$ stands for tensor factors $a_1,\ldots,a_n$ in their natural order and the number of β factors varies from 0 to ∞. It is easy to show that $\exp(\ell_\beta)$ commutes with B and is an isomorphism of Hochschild complexes of $\mathcal{A}$ and its twist by β. ☐

LEMMA 4.1.3. *For curved DG categories $\mathcal{A}$, $\mathcal{B}$ let $\mathcal{A}_0$, $\mathcal{B}_0$ be $\mathcal{A}$ and $\mathcal{B}$ viewed as graded categories. There is a spectral sequence starting from the Hochschild complex of the second kind of $\mathrm{HH}^{II}_\bullet(\mathcal{A}_0)$ and converging to $\mathrm{HH}^{II}_\bullet(\mathcal{A})$. If a DG morphism induces an isomorphism*

$$\mathrm{HH}^{II}_\bullet(\mathcal{A}_0)\to\mathrm{HH}^{II}_\bullet(\mathcal{B}_0)$$

then it induces an isomorphism

$$\mathrm{HH}^{II}_\bullet(\mathcal{A})\to\mathrm{HH}^{II}_\bullet(\mathcal{B}).$$

Same for cyclic, negative cyclic, and periodic cyclic homology of the second kind.

PROOF. The spectral sequence is defined by the filtration by number of tensor factors. It converges because we are using direct products and not direct sums. ☐

4.2. The trace map. Let $\mathcal{A}$ be a curved DG algebra. A strict DGmodule (resp. a QDG module) over $\mathcal{A}$ is *strictly perfect* if it is free of finite type as a graded module. [projective...] Denote the curved DG category of these modules by $\mathrm{sPerf}^{\mathrm{str}}(\mathcal{A})$, resp. $\mathrm{sPerf}^{\mathrm{Q}}(\mathcal{A})$.

LEMMA 4.2.1. *For a curved DG algebra $\mathcal{A}$,*

$$C^{II}_\bullet(\mathcal{A})) \xrightarrow{\sim} C^{II}_\bullet(\mathrm{sPerf}^{\mathrm{Q}}(\mathcal{A}))$$

PROOF. This is a direct generalization of Lemma 4.0.1. Let $\mathrm{Free}(\mathcal{A})$ be the category of free $\mathcal{A}$-modules; its Hochschild complex of the second kind is defined exactly the same as the one of $\mathcal{A}$, but with a_j being (rectangular) matrices. One has an isomorphism of complexes

$$\exp(\ell_{d_\nu}) : C^{II}_\bullet(\mathrm{Free}(\mathcal{A})) \xrightarrow{\sim} C^{II}_\bullet(\mathrm{sPerf}^{\mathrm{Q}}(\mathcal{A}))$$

The trace map

$$C^{II}_\bullet(\mathrm{Free}(\mathcal{A}) \to C^{II}_\bullet(\mathcal{A})$$

is a quasi-isomorphism, which is proven exactly as in the non-curved case. ***Extends to projectives as in...*** □

LEMMA 4.2.2. *The embedding* $\mathrm{sPerf}^{\mathrm{str}}(\mathcal{A})) \to \mathrm{sPerf}^{\mathrm{Q}}(\mathcal{A}))$ *induces a quasi-isomorphism of Hochschild complexes of the second kind. Same for cyclic, negative cyclic, and periodic cyclic complexes of the second kind.*

PROOF. By Lemma 4.1.3, it is enough to prove the statement for the underlying graded categories

$$\mathrm{sPerf}^{\mathrm{str}}(\mathcal{A}))_0 \to \mathrm{sPerf}^{\mathrm{Q}}(\mathcal{A}))_0.$$

On them, our functor becomes a fully faithful embedding. It is not surjective on objects, but every object of $\mathrm{sPerf}^{\mathrm{Q}}(\mathcal{A}))_0$ is isomorphic to a direct summand of an object of $\mathrm{sPerf}^{\mathrm{str}}(\mathcal{A}))_0$. Indeed, ***FINISH □

5. Curved DG algebras and Bar/Cobar construction

For a curved DG algebra $(\mathcal{A}, d, R)$, define

$$\mathrm{Bar}_+(\mathcal{A}) = (\bigoplus_{n\geq 0} \mathcal{A}[1]^{\otimes n} \partial_{\mathrm{Bar}} + d + \ell_R) \tag{5.1}$$

where

$$\ell_R(a_1|\dots|a_n) = \sum_{j=1}^{n+1} (-1)^{\sum_{i<j}(|a_i|+1)}(a_1|\dots|R|a_j|\dots|a_n)$$

This is a DG coalgebra. For any curved DG module $\mathcal{M}$ over $\mathcal{A}$,

$$\mathrm{Bar}_+(\mathcal{A}, \mathcal{M}) = (\bigoplus_{n\geq 0} \mathcal{A}[1]^{\otimes n} \otimes \mathcal{M}, d + \partial_{\mathrm{Bar}} + \ell_R) \tag{5.2}$$

is a DG comodule.

The bar construction itself, the sum over $n \geq 1$, has the following structure.

1) it is a graded coalgebra.

2) The differential

$$d = \partial_{\mathrm{Bar}} + d + \ell_R$$

satisfies $d^2 = 0$. However, it is not a coderivation. Instead, its failure to be one is the cocommutator with $\rho = (R)$ of degree one; namely,

$$\Delta(dc) - (D \otimes 1 - 1 \otimes \Delta)(c) : (\partial_{\text{Bar}} + d + \ell_R)^2 : c \mapsto \rho \otimes c - c \otimes \rho$$

3) $d\rho = 0; \Delta\rho = 0$.

We call a graded coalgebra $\mathcal{B}$ together with d and ρ satisfying 1), 2), 3) a ***curved DG coalgebra? or not?**

Given such $(\mathcal{B}, d, \rho)$, define a graded derivation d on Cobar$(\mathcal{B})$ as the sum of ∂_{Cobar} and the derivation that acts on free generators via $(b) \mapsto (db)$ for $b \in \mathcal{B}$. Then Cobar$(\mathcal{B})$ is a curved DG algebra with the curvature element $R = (\rho)$.

The following is a direct generalization of ***REF, together with a dual statement.

PROPOSITION 5.0.1. *For a curved DG algebra $\mathcal{A}$,*

$$C_\bullet(\mathcal{A}) \xrightarrow{\sim} C^\bullet_{\text{II}}(\text{Bar}(\mathcal{A})).$$

*For a ***curved DG coalgebra $\mathcal{B}$,*

$$C^\bullet(\mathcal{B}) \xrightarrow{\sim} C^{\text{II}}_\bullet(\text{Cobar}(\mathcal{B})).$$

6. Koszul duality and comparison of complexes of the first and second kind

PROPOSITION 6.0.1. ****Under conditions***

$$C_\bullet(\mathcal{A}) \xrightarrow{\sim} C^{\text{II}}_\bullet(\mathcal{A})$$

***This is in the end of [**481**]. Hopefully, will write a short proof by using the above.**

7. Bibliographical notes

Positselski-Polishchuk, Getzler-Jones,

CHAPTER 25

Matrix factorizations

1. Introduction

Let A be an algebra with a central element W. A matrix factorization is a finitely generated $\mathbb{Z}/2\mathbb{Z}$-graded module with an odd operator D such that

$$D^2 = W.$$

There is a notion of a trivial matrix factorization, such as $M = A \oplus A[1]$ with

$$D = \begin{pmatrix} 0 & 1 \\ W & 0 \end{pmatrix} \tag{1.1}$$

A U-matrix factorization is a $\mathbb{Z}$-graded finitely gererated A-module M with a $R[[U]]$-linear, (U)-adically complete operator of degree $+1$

$$D : M[[U]] \to M[[U]];\ D^2 = UW.$$

Here U is a formal parameter of degree two. Again, there is a notion of a trivial U-matrix factorization.

Assume that W is not a zero divisor in A. Then there is a straight link between matrix factorizations and A/WA-modules. Indeed, in this case A/WA can be replaced by its DG resolution $A[\xi], W\frac{\partial}{\partial \xi}$. Consider a DG module over this resolution which is a finitely generated projective A-module. Let ∂ be the differential. Put

$$D = \partial + U\xi$$

Then one has $D^2 = UW$.

In general, given a bounded from above complex of A/WA-modules which is perfect as an A-module, we do have a quasi-isomorphism between it and a finitely generated graded A-module with an action of a bigger resolution of A/WA; this produces a U-matrix factorization where higher powers of U have non-zero coefficients.

Now, if we start with a free A/WA-module (say, of rank one), we obtain the U-matrix factorization $M = A \oplus A[-1]$ with

$$D = \begin{pmatrix} 0 & 1 \\ UW & 0 \end{pmatrix} \tag{1.2}$$

This suggests that a (U)-matrix factorization up to trivial matrix factorizations is an invariant of an A-perfect complex of A/WA-modules up to A/WA-perfect complexes. Below we show that the constructions mentioned above are organized into an A_∞ functor between Drinfeld quotients of DG categories.

***MORE

2. Algebras with central elements

As above, let A be an algebra with a central element W. Then the results of Chapter 18 can be generalized as follows. First, observe that, if we take a A-projective resolution $(P^\bullet, \partial)$ of an A/WA-module, W acts on it homotopically trivially. Let $[W]$ be a homotopy. Then $[W]^2$ is a cocycle; because $P^\bullet$ is a resolution, it is cohomologous to zero. Continuing the process, we get a $k[[U]]$-linear operator $D = \partial + U[W] + \dots$ on $P^\bullet[[U]]$ such that

$$D^2 = UW \tag{2.1}$$

Alternatively, one can observe that such a D exists for the bar resolution, and then transfer it to any resolution. As above, this construction extends to an A_∞ functor.

Definition 2.0.1. *Let A be an algebra with a central element W. Let U be a formal parameter of cohomological degree 2. We use the capital letter, not to confuse U with a similar but different variable u in cyclic theory). Let $\mathrm{Proj}^-_{f,u}(A, W)$ be the following DG category: an object is a bounded from above graded projective module $P^\bullet$ over A that is free as a k-module, together with a $k[[U]]$-linear (U)-adically continuous operator*

$$D = \sum_{n=0}^{\infty} U^n D_n : P^\bullet[[U]] \to P^\bullet[[U]]$$

of degree one, such that

$$D^2 = UW;$$

a morphism from $(P_1^\bullet, D_1)$ to $(P_2^\bullet, D_2)$ is a homogeneous $k[[U]]$-linear (U)-adically continuous map

$$F = \sum_{n=0}^{\infty} U^n P_n : P_1^\bullet[[U]] \to P_2^\bullet[[U]];$$

the differential on morphisms is given by

$$F \mapsto D_2 F - (-1)^{|F|} F D_1.$$

We drop the subscript f when we do not require our modules to be free over k.

Proposition 2.0.2. *The restriction of the A_∞ functors P (Proposition 2.3.1) to complexes of A/WA-modules lifts to A_∞ functors*

$$P : \mathrm{Com}^-_f(A/WA)/\mathrm{Acy}_f(A/WA) \to \mathrm{Proj}^-_{f,u}(A, W) \tag{2.2}$$

$$P : k[\mathrm{Com}^-(A/WA)]/k[\mathrm{Acy}(A/WA)] \to \mathrm{Proj}^-_u(A, W) \tag{2.3}$$

Definition 2.0.3. *Denote by $\mathrm{Perf}^A(A/WA)$ the DG category of complexes of A/WA-modules that are perfect as complexes of A-modules. Also, denote by $\mathrm{MF}_u(A, W)$ the full DG subcategory of $\mathrm{Proj}^-_u(A, W)$ whose objects are strictly perfect complexes. As usual, we add the subscript f if we require them to be free over k.*

An object of $\mathrm{MF}_u(A, W)$ is called a U-matrix factorization.

Proposition 2.0.4. *The restriction of the A_∞ functors P (Proposition 2.0.2) to A-perfect complexes of A/WA-modules defines A_∞ functors*

$$P : \mathrm{Perf}^A_f(A/WA)/\mathrm{Acy}_f(A/WA) \to \mathrm{MF}_{f,u}(A, W) \tag{2.4}$$

$$P : k[\mathrm{Perf}^A(A/WA)]/k[\mathrm{Acy}(A/WA)] \to \mathrm{MF}_u(A, W) \tag{2.5}$$

PROOF. The construction of these A_∞ functors goes exactly as for the ones in Proposition 2.3.1 but with one modification. We will carry it out only for the case of modules that are free as k-modules. First construct D on any projective resolution $P^\bullet$ of a bounded from above complex of A/WA-modules $M^\bullet$. Start with the bar resolution $\mathrm{Bar}(M^\bullet)$. For $a_1, \ldots, a_n \in A$ and $a_{n+1} \in M^\bullet$, define

$$[W](a_1 \otimes \ldots \otimes a_{n+1}) = \sum_{j=0}^{n} (-1)^j a_1 \otimes \ldots \otimes W \otimes a_{j+1} \otimes \ldots \otimes a_{n+1} \tag{2.6}$$

We have

$$[\partial, [W]] = W; \ [W]^2 = 0.$$

For any choice of a projective resolution $P^\bullet(M^\bullet)$, in the notation of 2.1,

$$[W]^{(n)} = \varphi[W]h \ldots h[W]\widetilde{\varphi};$$

$$D = \partial + \sum_{n=1}^{\infty} u^n [W]^{(n)}$$

Then we have

$$D^2 = uW$$

$$P_n(f_1, \ldots, f_n) = \sum_{N \geq 0} u^N P_n^{(N)}(f_1, \ldots, f_n); \tag{2.7}$$

$$P_n^{(N)}(f_1, \ldots, f_n) = \sum \varphi_1 W(k_1) B(f_1) W(k_2) h_2 \ldots B(f_{n-1}) W(k_n) h_n B(f_n) W(k_{n+1}) \widetilde{\varphi}_{n+1}$$

where $W(k_j) = (h_j W[1])^{k_j}$ and the sum is taken over all $k_1, \ldots, k_{n+1} \geq 0$ such that $\sum k_j = N$. Similarly for the components of S and T ***? □

We have already defined u-matrix factorizations (Definition 2.0.3). Here we define the $\mathbb{Z}/2$-graded DG category of matrix factorizations and compute its Hochschild and cyclic homology following Efimov.

3. Hochschild and cyclic homology of matrix factorizations

Let A be a commutative algebra and W an element of A. A matrix factorization is a $\mathbb{Z}/2$-graded finitely generated projective A-module P together with an odd A-module morphism $d : P \to P$ such that $d^2 = W \cdot \mathrm{id}_A$.

Observe that everything we proved about (curved) DG categories works if we replace $\mathbb{Z}$-graded algebras, categories, and modules by $\mathbb{Z}/2$-graded. The Hochschild and cyclic complexes, both standard and of the second kind, are defined without any difference, but now they are $\mathbb{Z}/2$-graded k-modules with an odd k-linear endomorphism of square zero. Under these definitions A, concentrated in degree zero, with $d = 0$ and $R = W$, can be viewed as a curved DG algebra that we denote by (A, W). A matrix factorization is the same as a strict strictly perfect curved DG module over this algebra.

Matrix factorizations with given A and W form a $\mathbb{Z}/2$-graded DG category which we denote by $\mathrm{MF}(A, W)$.

THEOREM 3.0.1. *Let* A *be a regular Noetherian algebra over a field* k *of characteristic zero. Then*

$$C_\bullet(\mathrm{MF}(A, W)) \xrightarrow{\sim} (\Omega^\bullet_{A/k}, dW\wedge)$$

$$CC^-_\bullet(\mathrm{MF}(A, W)) \xrightarrow{\sim} (\Omega^\bullet_{A/k}[[u]], dW \wedge + ud)$$

$$\mathrm{CC}^{\mathrm{per}}_{\bullet}(\mathrm{MF}(A,W)) \xrightarrow{\sim} (\Omega^{\bullet}_{A/k}((u)), dW\wedge + ud)$$

PROOF. By Lemmas 4.2.1 and 4.2.2, we have

$$C^{\mathrm{II}}_{\bullet}(\mathrm{sPerf}^{\mathrm{str}}(A,W)) \xrightarrow{\sim} C^{\mathrm{II}}_{\bullet}(\mathrm{sPerf}^{Q}(A,W)) \xrightarrow{\sim} C^{\mathrm{II}}_{\bullet}(A,W)$$

The HKR morphism intertwines ℓ_W with $dW\wedge$. It is a quasi-isomorphism to $(\Omega^{\bullet}_{A/k}, dW\wedge)$ from both $C_{\bullet}(A,W)$ and $C^{\mathrm{II}}_{\bullet}(A,W)$. Also,

$$C_{\bullet}(\mathrm{MF}(A,W)) = C_{\bullet}(\mathrm{sPerf}^{\mathrm{str}}(A,W)).$$

Now our statement follows from

PROPOSITION 3.0.2. ****Under conditions***

$$C_{\bullet}(\mathrm{MF}(A,W) \xrightarrow{\sim} C^{\mathrm{II}}_{\bullet}(\mathrm{MF}(A,W)$$

This is a consequence of Proposition 6.0.1. Similarly for cyclic complexes. □

3.1. The Gauss-Manin u-connection. Let us recall and modify the u-connection defined in Section 4. Let $\mathcal{A}$ be a curved DG algebra. The t-dependent curved DG algebra structure on $\mathcal{A}$ will now be the constant multiplication which is same as the original one; the differential td; and the curvature element t^2R. The Getzler-Gauss-Manin connection is now

$$\nabla^{\mathrm{GM}}_{\frac{\partial}{\partial t}} = \frac{\partial}{\partial t} + \frac{2t}{u}\iota_R + 2tS_R + \frac{1}{u}i_d + S_d$$

which becomes

$$\nabla^{\mathrm{GM}}_{\frac{\partial}{\partial t}} = \frac{\partial}{\partial t} + \mathcal{N} + \frac{2t}{u}\iota_R + 2S_R + \frac{t}{u}i_d + t^{-1}S_d$$

after conjugating with

$$\mathcal{N}(a_0\otimes\ldots\otimes a_n) = n(a_0\otimes\ldots\otimes a_n) \tag{3.1}$$

Therefore

$$\nabla_u = \frac{\partial}{\partial u} - \frac{\mathcal{N}}{2u} - \frac{1}{u^2}\iota_R - \frac{1}{u}S_R - \frac{1}{2u^2}i_d - \frac{1}{2u}S_d \tag{3.2}$$

Now let $\mathcal{A} = (A,W)$ as in 3. Then we get

$$\nabla_u = \frac{\partial}{\partial u} - \frac{\mathcal{N}}{2u} - \frac{1}{u^2}\iota_W - \frac{1}{u}S_W \tag{3.3}$$

The HKR map intertwines it with the following operator on $\Omega^{\bullet}_{A/k}((u))$:

$$\alpha \mapsto (\frac{\partial}{\partial u} - \frac{\mathcal{N}}{2u} - \frac{1}{u^2}W)\alpha - \frac{1}{2u}dW\wedge d\alpha \tag{3.4}$$

But the last summand is homotopic to zero (the homotopy is $\frac{1}{2u^2}d$). We see that the HKR map

$$(\mathrm{CC}^{\mathrm{per}}(A,W), b+\ell_R+uB) \to (\Omega^{\bullet}_{A/k}((u)), ud)$$

intertwines ∇_u with

$$\frac{\partial}{\partial u} - \frac{\mathcal{N}}{2u} - \frac{1}{u^2}W \tag{3.5}$$

Note that

$$[\frac{\partial}{\partial u} - \frac{\mathcal{N}}{2u} - \frac{1}{u^2}W, dW\wedge + ud] = \frac{1}{2u}(dW\wedge + ud)$$

THEOREM 3.1.1. *For a regular Noetherian algebra* A *over a field* k *of characteristic zero, there is a natural chain of quasi-isomorphisms of* $\mathbb{Z}/2$*-graded mixed complexes with* u*-connections*

$$(CC^{per}(MF(A, W)), b + \ell_R + uB, \nabla_u) \xrightarrow{\sim} (\Omega^{\bullet}_{A/k}.dW \wedge +uB, \frac{\partial}{\partial u} - \frac{\mathcal{N}}{2u} - \frac{1}{u^2}W)$$

PROOF. We will prove the theorem using Theorem 2.4.2 from Chapter 13 . □

***Double check; reconcile with Chapter 21

4. Bibliographical notes

Buchweitz; Orlov; Efimov; Efimov-Possitselsky; Preygel; Blanc-Robalo-Toën-Vezzosi;

CHAPTER 26

Category of singularities and Tate cohomology

1. Introduction

For a regular Noetherian commuttive algebra, any complex of finitely generated modules with cohomology bounded from above is perfect (***specify; reference). This is not so in general. For example, for the algebra of dual numbers $k[x]/(x^2)$ the module k has infinite homological dimension and therefore cannot be perfect. Therefore the quotient by the subcategory of perfect complexes can be viewed as a measure of how singular the spectrum of our algebra is.

In this chapter we define a DG version of the category of singularities. Our constructions are mainly extracted from the book **??** by Buchweitz. Let A be an algebra with a central element W which is not a zero divisor. We connect the category of singularities to the category of acyclic complexes when the algebra A is Gorenstein (i.e. when the left and right module A has finite injective dimension). When the algebra is A/WA with A regular, we connect the category of singularities to the category of matrix factorizations; we prove (***?) a DG version of Orlov's theorem stating a precise connection between the two.

2. Category of singularities and acyclic complexes

Let A be a k-algebra. Denote by $\mathrm{Proj}^{-,b}(A)$ the DG category of complexes of projective finitely generated A-modules whose cohomology is bounded (i.e. concentrated in finitely many degrees). We recall that $\mathrm{sPerf}(A)$ is the category of strictly perfect complexes, i.e. finite complexes of finitely generated projective A-modules. Let

$$\mathrm{Sg}(A) = \mathrm{Proj}^{-,b}(A)/\mathrm{sPerf}(A), \tag{2.1}$$

the Drinfeld quotient.

We are assuming that A *is Noetherian (?) and of finite injective dimension as both left and right module over* A.

Let $\mathrm{ACP}(A)$ be the DG category of (unbounded) complexes of finitely generated projective modules over A.

As usual, we add the lower index f if we consider full subcategories of complexes of modules that are free over k.

Construct A_∞ functors

$$F : \mathrm{Sg}_f(A) \rightleftarrows \mathrm{ACP}_f(A) : G \tag{2.2}$$

as follows.

To construct F: First let us do it on objects. Let P be an object in $\mathrm{Proj}^{-,b}(A)$. Put

$$P^* = \mathrm{Hom}_A(P, A)$$

This is a right A-module. Let $\mathcal{P} \xrightarrow{\sim} P^*$ be a quasi-isomorphism to P^* from an object of $\mathrm{Proj}^{-,b}(A^{\mathrm{op}})$. It exists because A has finite injective dimension and therefore the cohomology of P^* is bounded from above (it is isomorphic to $\mathrm{Ext}^\bullet(P, A)$ which has a spectral sequence that converges to it and whose second term is $\mathrm{Ext}^j(H^k(P), A)$). Now let

$$\mathcal{P}^* = \mathrm{Hom}_A(\mathcal{P}, A)$$

which is a complex of left A-modules. Consider the composition

$$P \xrightarrow{\sim} P^{**} \to \mathcal{P}^* \tag{2.3}$$

We claim that it is a quasi-isomorphism and therefore its mapping cone is acyclic. To prove that, note that, for n big enough, P^* is quasi-isomorphic to the complex

$$\mathcal{A}(n) = (P^*_m \to \ldots \to P^*_n \to K^{n+1}) \tag{2.4}$$

where $P_j = 0$ for $j < m$ and $K^{n+1} = \mathrm{im}(P^*_n \to P^*_{n+1})$. (We will use homological grading $P_k = P^{-k}$, mainly to avoid its interfering with the superscript $*$). Choose $\mathcal{P}$ to be a projective resolution of (2.4). The complex $\mathcal{A}(n)$ is concentrated in degrees $m \le j \le n+1$. The cohomology of $\mathbb{R}\mathrm{Hom}_A(\mathcal{A}(n), A)$ is concentrated in degrees $-1-n \le k \le d-m$ where d is the injective dimension of A. Now replace n by a big enough N. The cohomology will not change. The cohomology $\mathbb{R}\mathrm{Hom}_A(\mathcal{A}(N), A)$ is the limit of a spectral sequence whose only non-zero E^1 terms are:

1) P_j (in total degree j) and

2) $\mathrm{Ext}^k_A(K^{N+1}, A)$ (in total degree $k-1-N$).

The terms 2) survive to the E_2 term, the rest of which is the cohomology of

$$0 \to \mathrm{Ext}^0_A(K^{N+1}, A) \to P_N \to P_{N-1} \to \ldots \to P_m \tag{2.5}$$

The cohomology of this complex is the cohomology of P, plus possibly the cohomology in the first two places on the left. Those contribute to the cohomology of total degree $-1-N$ and $-N$. Therefore, for N big, these two cohomology groups vanish.

When N is big enough,

$$\mathrm{Ext}^k_A(K^{N+1}, A) \xrightarrow{\sim} 0$$

for all $k > 0$. Indeed, for N big enough, either $k-1-N < -2-n$ or $k > d$. We see that the E_2 term of the spectral sequence is the cohomology of P, and the spectral sequance collapses.

We define $F(P)$ as the acyclic complex which is the cone of the morphism (2.3). To define F on morphisms, we use the A_∞ functor (2.1) ***FINISH

To construct G: On objects, an acyclic complex $P^\bullet$ goes to $G(P)^j = P^j$ for $j \le 0$ and $G(P)_j = 0$ for $j > 0$. On morphisms, let $f : P^j \to Q^k$ be a morphism of A-modules. Define $G(f) : G(P) \to G(Q)$ of degree $k - j$. If $k > 0$, set $G(f) = 0$. If $k \le 0$ and $j \le 0$, set $G(f) = f$. If $j > 0$ and $k \le 0$, then define $G(f)$ as a composition

$$G(P^\bullet) \xrightarrow{d} (P^1) \xrightarrow{\epsilon_{(P^1)}} (P^1) \xrightarrow{d} \ldots \xrightarrow{d} (P^j) \xrightarrow{\epsilon_{(P^j)}} (P^j) \xrightarrow{f} G(Q^\bullet)$$

Here (P^i) is the complex P^i concentrated in degree zero.

EXAMPLE 2.0.1. Let $A = k[x]/(x^2)$ and M an A-module (free over k). Choose $P^\bullet$ to be

$$\ldots \xrightarrow{x} A \otimes M \xrightarrow{x} \ldots \xrightarrow{x} A \otimes M \to 0$$

which is a projective resolution of the A-module k. Then P^* is

$$0 \to A \otimes M \xrightarrow{x} \ldots \xrightarrow{x} A \otimes M \xrightarrow{x} \ldots$$

and $G(P^\bullet)$ is

$$\ldots \xrightarrow{x} A \otimes M \xrightarrow{x} \ldots \xrightarrow{x} A \otimes M \xrightarrow{x} \ldots$$

3. The Tate cohomology

LEMMA 3.0.1. *Let* $A = k[x]/(x^2)$ *and* M *is an* A*-module, free over* k*. Then*

$$\underline{\mathrm{Hom}}^\bullet(F(k), F(M))$$

is homotopy equivalent to

$$\ldots \xrightarrow{x} M \xrightarrow{x} M \xrightarrow{x} \ldots$$

PROOF. If we take $F(k)$ and $F(M)$ as in Example 2.0.1 then $\underline{\mathrm{Hom}}^\bullet(F(k), F(M))$ becomes the following. Let $C^{j,k} = A \otimes M$ for all $j, k \in \mathbb{Z}$. Define

$$d' : C^{j,k} \to C^{j+1,k}$$

to be multiplication by x and

$$d' : C^{j,k} \to C^{j,k-1}$$

to be multiplication by $(-1)^k x$. Let

$$C^n = \prod_{j-k=n} C^{j,k};\ d = d' + d'' : C^n \to C^{n+1}.$$

Consider the subcomplex $C^{j>0,k}$ and the quotient complex $C^{j\leq 0,k}$. The former is acyclic because the spectral sequence starting with the acyclic differential d'' converges. The former is quasi-isomorphic to $C^{0,\bullet}/d'C^{-1,\bullet}$ because the spectral sequence starting with the differential d' converges. But $C^{0,\bullet}/d'C^{-1,\bullet}$ is exactly the complex in the statement of the lemma. □

Similarly:

LEMMA 3.0.2. *Let* $A = k[x]/(x^n - 1)$ *and* M *is an* A*-module, free over* k*. Then*

$$\underline{\mathrm{Hom}}^\bullet(F(k), F(M))$$

is homotopy equivalent to

$$\ldots \xrightarrow{x-1} M \xrightarrow{N} M \xrightarrow{x-1} \ldots$$

where

$$N = 1 + x + \ldots + x^{n-1}$$

*** Still needed: a) an analog for cyclic objects, using Kaledin resolutions. b) F and G are "quasi-inverse"

4. Singularities and matrix factorizations

4.1. From an A/WA-module to a U-matrix factorization. Assume that W is a central element of A that is not a zero divisor. Let M be any complex of A/WA modules that is perfect as an A-module. Choose a quasi-isomorphism $P \xrightarrow{\sim} M$ where P is a strictly perfect complex of A-modules. Take $D = \partial + \sum_{n=1}^{\infty} U^n D_n$ (note that the sum is finite).

Proposition 4.1.1. *The* A_∞ *functor* (2.4) *extends to an* A_∞ *functor*

$$P : \mathrm{Perf}_f^A(A/WA)/\mathrm{Perf}_f(A/WA) \to \mathrm{MF}_{f,u}(A, W)$$

Proof. We will show that perfect complexes map to contractible U-matrix factorizations. We call a U-matrix factorization trivial if it is of the following form: $P^0 = P^{-1} = P$ for some projective module P of finite type; $P^j = 0$ for $j \neq 0, -1$; for $(p^0, p^{-1}) \in P^0 \oplus P^{-1}$, $D(p^0, p^1) = (Wp^{-1}, up^0)$. We call a U- matrix factorization "trivial" if it has a finite filtration whose graded quotients are trivial.

In other words, a trivial u-matrix factorization is of the form

$$(P[\xi], W\frac{\partial}{\partial \xi} + \xi W) \tag{4.1}$$

where ξ is a formal variable of degree -1 such that $\xi^2 = 0$ and P is a finitely generated projective module. □

Lemma 4.1.2. *Let* V *be a "trivial"* u*-matrix factorization. Then* id_V *is a cocycle in* $\mathrm{MF}_u(A, W)(V, V)$.

Proof. For a trivial matrix factorization (4.1) define η as follows: η sends an even morphism *** to ***. and an odd morphism *** to ***. Then $d\eta = \mathrm{id}_P$. For a "trivial" matrix factorization, consider a filtration with trivial graded quotients and define η by induction. ***FINISH □

Lemma 4.1.3. *For a perfect complex of* A/WA*-modules there is an* A*-perfect resolution* P *such that the corresponding matrix factorization is "trivial".*

Proof. Choose a strictly perfect resolution $P^\bullet$ of $M^\bullet$. Let $P^m = (A/WA)^{k_m} e_m$ where e_m are idempotents in A/WA. First assume that P^m are free, i.e., $e_m = 1$. Extend the differential ∂ to an A-module morphism $\widetilde{\partial}$ of degree -1 such that $\widetilde{\partial}^2 = WE$. Introduce an extra variable ξ of degree -1 such that $\xi^2 = 0$. Put $\widetilde{P}^m = A^{k_m}$. Then

$$(\widetilde{P}^\bullet[\xi], \partial + W\frac{\partial}{\partial \xi} + \xi E) \tag{4.2}$$

is a projective resolution of $M^\bullet$; if we put

$$D = \partial + W\frac{\partial}{\partial \xi} + \xi E + U\xi$$

then we have $D^2 = uW$. The corresponding matrix factorization is filtered by trivial matrix factorizations.

***For resolutions that are not necessarily free** □

4.2. The $\mathbb{Z}/2$-graded case. We modify the construction from 4.1 for matrix factorizations as defined in 3.

4.2.1. *From a A/WA-module to a matrix factorization.* If P is a U-matrix factorization associated to a complex of modules in the beginning of 4.1, then $(P[U]/(U-1), D)$ is a matrix factorization. In other words, it is P viewed as a $\mathbb{Z}/2$-graded module, with

$$D = \partial + \sum_{n=1}^{\infty} U^n D_n$$

(the sum is actually finite).

4.2.2. *From a matrix factorization to an A/WA-module.*

5. Comparison between 2 and 4.1

Consider a strictly A-perfect complex $(P^\bullet, \partial)$ which is quasi-isomophic to a complex $M^\bullet$ of A/WA-modules. Here we will relate the U-matrix factorization associated to $P^\bullet$ in4.1 to the acyclic complex of projective modules assocaited to it in 2. We assume that W is a central element of A that is not a zero divisor. All our modules are free over k.

Start with the DG algebra $(A[\xi], W\frac{\partial}{\partial\xi})$ where ξ is a formal variable of degree -1. As above, let U be a formal variable of degree 2. Let D be the operator on $P^\bullet[U]$ constructed in 4.1. Then

$$(P^\bullet[\xi, U], D + W\frac{\partial}{\partial\xi} - U\xi) \tag{5.1}$$

is a DG module over $(A[\xi], W\frac{\partial}{\partial\xi})$ quasi-isomorphic to $M^\bullet$. Then

$$(A/WA) \otimes_{A[\xi]} P^\bullet[\xi, U] \tag{5.2}$$

is a resolution of $M^\bullet$ which is two-periodic below certain degree. It is acyclic below certain degree because $M^\bullet$ has bounded cohomology.

LEMMA 5.0.1. *The two-periodic complex obtained from* (5.2) *is homotopy equivalent to the acyclic complex* $F(P^\bullet)$ *constructed in 2.*

PROOF. □

EXAMPLE 5.0.2. Let $A = k[x]$ and $W = x^2$. Let M be a $k[x]/(x^2)$-module. ***FINISH**

Bibliography

[1]
[2]
[3]
[4]
[5]
[6] H. Abbaspour, T. Tradler, and M. Zeinalian, *Algebraic string bracket as a Poisson bracket*, J. Noncommut. Geom., 4 (2010), pp. 331–347.
[7] A. Adem and M. Karoubi, *Periodic cyclic cohomology of group rings*, C. R. Acad. Sci. Paris Sér. I Math., 326 (1998), pp. 13–17.
[8] A. Alekseev, N. Kawazumi, Y. Kuno, and F. Naef, *Higher genus Kashiwara-Vergne problems and the Goldman-Turaev Lie bialgebra*, C. R. Math. Acad. Sci. Paris, 355 (2017), pp. 123–127.
[9] ———, *The Goldman-Turaev Lie bialgebra in genus zero and the Kashiwara-Vergne problem*, Adv. Math., 326 (2018), pp. 1–53.
[10] ———, *Goldman-Turaev formality implies Kashiwara-Vergne*, Quantum Topol., 11 (2020), pp. 657–689.
[11] A. Alekseev and A. Lachowska, *Invariant $*$-products on coadjoint orbits and the Shapovalov pairing*, Comment. Math. Helv., 80 (2005), pp. 795–810.
[12] A. Alekseev and F. Naef, *Goldman-Turaev formality from the Knizhnik-Zamolodchikov connection*, C. R. Math. Acad. Sci. Paris, 355 (2017), pp. 1138–1147.
[13] A. Alekseev and C. Torossian, *The Kashiwara-Vergne conjecture and Drinfeld's associators*, Ann. of Math. (2), 175 (2012), pp. 415–463.
[14] A. Alekseev and P. Ševera, *Equivariant cohomology and current algebras*, Confluentes Math., 4 (2012), pp. 1250001, 40.
[15] B. Antieau, B. Bhatt, and A. Mathew, *Counterexamples to Hochschild-Kostant-Rosenberg in characteristic* p, Forum Math. Sigma, 9 (2021), pp. Paper No. e49, 26.
[16] L. L. Avramov and M. Vigué-Poirrier, *Hochschild homology criteria for smoothness*, Internat. Math. Res. Notices, (1992), pp. 17–25.
[17] D. Ayala, J. Francis, and H. L. Tanaka, *Factorization homology of stratified spaces*, Selecta Math. (N.S.), 23 (2017), pp. 293–362.
[18] D. Bar-Natan, *On associators and the Grothendieck-Teichmuller group. I*, Selecta Math. (N.S.), 4 (1998), pp. 183–212.
[19] M. A. Batanin and C. Berger, *The lattice path operad and Hochschild cochains*, in Alpine perspectives on algebraic topology, vol. 504 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2009, pp. 23–52.
[20] P. Baum and V. Nistor, *Periodic cyclic homology of Iwahori-Hecke algebras*, C. R. Acad. Sci. Paris Sér. I Math., 332 (2001), pp. 783–788.
[21] ———, *Periodic cyclic homology of Iwahori-Hecke algebras*, K-Theory, 27 (2002), pp. 329–357.
[22] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, and D. Sternheimer, *Deformation theory and quantization. I. Deformations of symplectic structures*, Ann. Physics, 111 (1978), pp. 61–110.
[23] ———, *Deformation theory and quantization. II. Physical applications*, Ann. Physics, 111 (1978), pp. 111–151.
[24] K. Behrend and P. Xu, *Differentiable stacks and gerbes*, J. Symplectic Geom., 9 (2011), pp. 285–341.

[25] A. Beilinson, *Relative continuous K-theory and cyclic homology*, Münster J. Math., 7 (2014), pp. 51–81.

[26] D. Ben-Zvi, J. Francis, and D. Nadler, *Integral transforms and Drinfeld centers in derived algebraic geometry*, J. Amer. Math. Soc., 23 (2010), pp. 909–966.

[27] M.-T. Benameur, J. Brodzki, and V. Nistor, *Cyclic homology and pseudodifferential operators, a survey*, in Aspects of boundary problems in analysis and geometry, vol. 151 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 2004, pp. 239–264.

[28] M.-T. Benameur and V. Nistor, *Homology of complete symbols and noncommutative geometry*, in Quantization of singular symplectic quotients, vol. 198 of Progr. Math., Birkhäuser, Basel, 2001, pp. 21–46.

[29] ———, *Homology of complete symbols and noncommutative geometry*, in Quantization of singular symplectic quotients, vol. 198 of Progr. Math., Birkhäuser, Basel, 2001, pp. 21–46.

[30] M.-T. Benameur and V. Nistor, *Residues and homology for pseudodifferential operators on foliations*, Math. Scand., 94 (2004), pp. 75–108.

[31] Y. Berest, X. Chen, F. Eshmatov, and A. Ramadoss, *Noncommutative Poisson structures, derived representation schemes and Calabi-Yau algebras*, in Mathematical aspects of quantization, vol. 583 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2012, pp. 219–246.

[32] Y. Berest, G. Felder, S. Patotski, A. C. Ramadoss, and T. Willwacher, *Representation homology, Lie algebra cohomology and the derived Harish-Chandra homomorphism*, J. Eur. Math. Soc. (JEMS), 19 (2017), pp. 2811–2893.

[33] Y. Berest, G. Felder, and A. Ramadoss, *Derived representation schemes and noncommutative geometry*, in Expository lectures on representation theory, vol. 607 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2014, pp. 113–162.

[34] Y. Berest, G. Khachatryan, and A. Ramadoss, *Derived representation schemes and cyclic homology*, Adv. Math., 245 (2013), pp. 625–689.

[35] A. J. Berrick and L. Hesselholt, *Topological Hochschild homology and the Bass trace conjecture*, J. Reine Angew. Math., 704 (2015), pp. 169–185.

[36] A. A. Beĭ linson, *Higher regulators and values of L-functions*, in Current problems in mathematics, Vol. 24, Itogi Nauki i Tekhniki, Akad. Nauk SSSR, Vsesoyuz. Inst. Nauchn. i Tekhn. Inform., Moscow, 1984, pp. 181–238.

[37] R. Bezrukavnikov and V. Ginzburg, *On deformations of associative algebras*, Ann. of Math. (2), 166 (2007), pp. 533–548.

[38] R. Bezrukavnikov and D. Kaledin, *Fedosov quantization in algebraic context*, Mosc. Math. J., 4 (2004), pp. 559–592, 782.

[39] B. Bhatt, M. Morrow, and P. Scholze, *Integral p-adic Hodge theory—announcement*, Math. Res. Lett., 22 (2015), pp. 1601–1612.

[40] A. Blanc, M. Robalo, B. Toën, and G. Vezzosi, *Motivic realizations of singularity categories and vanishing cycles*, J. Éc. polytech. Math., 5 (2018), pp. 651–747.

[41] P. Blanc and J.-L. Brylinski, *Cyclic homology and the Selberg principle*, J. Funct. Anal., 109 (1992), pp. 289–330.

[42] S. Bloch, *Algebraic K-theory and crystalline cohomology*, Inst. Hautes Études Sci. Publ. Math., (1977), pp. 187–268 (1978).

[43] ———, *The dilogarithm and extensions of Lie algebras*, in Algebraic K-theory, Evanston 1980 (Proc. Conf., Northwestern Univ., Evanston, Ill., 1980), vol. 854 of Lecture Notes in Math., Springer, Berlin-New York, 1981, pp. 1–23.

[44] J. Block and E. Getzler, *Equivariant cyclic homology and equivariant differential forms*, Ann. Sci. École Norm. Sup. (4), 27 (1994), pp. 493–527.

[45] J. Block, E. Getzler, and J. D. S. Jones, *The cyclic homology of crossed product algebras. II. Topological algebras*, J. Reine Angew. Math., 466 (1995), pp. 19–25.

[46] J. Block, N. Higson, and J. Sanchez, Jr., *On Perrot's index cocycles*, in Cyclic Cohomology at 40: Achievements and Future Prospects, vol. 105 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, [2023] ©2023, pp. 29–62.

[47] M. Bökstedt, W. C. Hsiang, and I. Madsen, *The cyclotomic trace and the K-theoretic analogue of Novikov's conjecture*, Proc. Nat. Acad. Sci. U.S.A., 86 (1989), pp. 8607–8609.

[48] ———, *The cyclotomic trace and algebraic K-theory of spaces*, Invent. Math., 111 (1993), pp. 465–539.

[49] M. Bökstedt and I. Madsen, *Topological cyclic homology of the integers*, Astérisque, (1994), pp. 7–8, 57–143. K-theory (Strasbourg, 1992).

[50] F. Bonechi, A. S. Cattaneo, and M. Zabzine, *Towards equivariant Yang-Mills theory*, J. Geom. Phys., 189 (2023), pp. Paper No. 104836, 17.

[51] C. Brav and T. Dyckerhoff, *Relative Calabi-Yau structures*, Compos. Math., 155 (2019), pp. 372–412.

[52] ———, *Relative Calabi-Yau structures II: shifted Lagrangians in the moduli of objects*, Selecta Math. (N.S.), 27 (2021), pp. Paper No. 63, 45.

[53] L. Breen and W. Messing, *Differential geometry of gerbes*, Adv. Math., 198 (2005), pp. 732–846.

[54] P. Bressler, A. Gorokhovsky, R. Nest, and B. Tsygan, *Deligne groupoid revisited.*

[55] ———, *Deformation quantization of gerbes*, Adv. Math., 214 (2007), pp. 230–266.

[56] ———, *Deformation quantization of gerbes*, Adv. Math., 214 (2007), pp. 230–266.

[57] ———, *Deformations of Azumaya algebras*, in Proceedings of the XVIth Latin American Algebra Colloquium (Spanish), Bibl. Rev. Mat. Iberoamericana, Rev. Mat. Iberoamericana, Madrid, 2007, pp. 131–152.

[58] ———, *Chern character for twisted complexes*, in Geometry and dynamics of groups and spaces, vol. 265 of Progr. Math., Birkhäuser, Basel, 2008, pp. 309–324.

[59] ———, *Chern character for twisted complexes*, in Geometry and dynamics of groups and spaces, vol. 265 of Progr. Math., Birkhäuser, Basel, 2008, pp. 309–324.

[60] ———, *Deformations of gerbes on smooth manifolds*, in K-theory and noncommutative geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 349–392.

[61] ———, *Deformations of gerbes on smooth manifolds*, in K-theory and noncommutative geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 349–392.

[62] ———, *Algebraic index theorem for symplectic deformations of gerbes*, in Noncommutative geometry and global analysis, vol. 546 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2011, pp. 23–38.

[63] ———, *Algebraic index theorem for symplectic deformations of gerbes*, in Noncommutative geometry and global analysis, vol. 546 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2011, pp. 23–38.

[64] ———, *Deformations of algebroid stacks*, Adv. Math., 226 (2011), pp. 3018–3087.

[65] ———, *Deformations of algebroid stacks*, Adv. Math., 226 (2011), pp. 3018–3087.

[66] ———, *Formality theorem for gerbes*, Adv. Math., 273 (2015), pp. 215–241.

[67] P. Bressler, R. Nest, and B. Tsygan, *A Riemann-Roch type formula for the microlocal Euler class*, Internat. Math. Res. Notices, (1997), pp. 1033–1044.

[68] ———, *A Riemann-Roch type formula for the microlocal Euler class*, Internat. Math. Res. Notices, (1997), pp. 1033–1044.

[69] ———, *Riemann-Roch theorems via deformation quantization. I, II*, Adv. Math., 167 (2002), pp. 1–25, 26–73.

[70] ———, *Riemann-Roch theorems via deformation quantization. I, II*, Adv. Math., 167 (2002), pp. 1–25, 26–73.

[71] J. Brodzki, *Simplicial normalization in the entire cyclic cohomology of Banach algebras*, K-Theory, 9 (1995), pp. 353–377.

[72] ———, *An introduction to* K*-theory and cyclic cohomology*, Advanced Topics in Mathematics, PWN—Polish Scientific Publishers, Warsaw, 1998.

[73] ———, *Cyclic cohomology after the excision theorem of Cuntz and Quillen*, J. K-Theory, 11 (2013), pp. 575–598.

[74] ———, *Cyclic cohomology after the excision theorem of Cuntz and Quillen*, J. K-Theory, 11 (2013), pp. 575–598.

[75] J. Brodzki, S. Dave, and V. Nistor, *The periodic cyclic homology of crossed products of finite type algebras*, Adv. Math., 306 (2017), pp. 494–523.

[76] J. Brodzki and Z. A. Lykova, *Excision in cyclic type homology of Fréchet algebras*, Bull. London Math. Soc., 33 (2001), pp. 283–291.

[77] J. Brodzki and R. Plymen, *Periodic cyclic homology of certain nuclear algebras*, C. R. Acad. Sci. Paris Sér. I Math., 329 (1999), pp. 671–676.

[78] ———, *Entire cyclic cohomology of Schatten ideals*, Homology Homotopy Appl., 7 (2005), pp. 37–52.

[79] J.-L. Brylinski, *Cyclic homology and equivariant theories*, Ann. Inst. Fourier (Grenoble), 37 (1987), pp. 15–28.

[80] ———, *Some examples of Hochschild and cyclic homology*, in Algebraic groups Utrecht 1986, vol. 1271 of Lecture Notes in Math., Springer, Berlin, 1987, pp. 33–72.

[81] ———, *Some examples of Hochschild and cyclic homology*, in Algebraic groups Utrecht 1986, vol. 1271 of Lecture Notes in Math., Springer, Berlin, 1987, pp. 33–72.

[82] ———, *A differential complex for Poisson manifolds*, J. Differential Geom., 28 (1988), pp. 93–114.

[83] ———, *A differential complex for Poisson manifolds*, J. Differential Geom., 28 (1988), pp. 93–114.

[84] ———, *Central localization in Hochschild homology*, J. Pure Appl. Algebra, 57 (1989), pp. 1–4.

[85] ———, *Loop spaces, characteristic classes and geometric quantization*, Modern Birkhäuser Classics, Birkhäuser Boston, Inc., Boston, MA, 2008. Reprint of the 1993 edition.

[86] J.-L. Brylinski and E. Getzler, *The homology of algebras of pseudodifferential symbols and the noncommutative residue*, K-Theory, 1 (1987), pp. 385–403.

[87] ———, *The homology of algebras of pseudodifferential symbols and the noncommutative residue*, K-Theory, 1 (1987), pp. 385–403.

[88] J.-L. Brylinski and V. Nistor, *Cyclic cohomology of étale groupoids*, K-Theory, 8 (1994), pp. 341–365.

[89] D. Burghelea, *The cyclic homology of the group rings*, Comment. Math. Helv., 60 (1985), pp. 354–365.

[90] ———, *The cyclic homology of the group rings*, Comment. Math. Helv., 60 (1985), pp. 354–365.

[91] ———, *Cyclic homology and the algebraic* K*-theory of spaces. I*, in Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), vol. 55 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1986, pp. 89–115.

[92] ———, *Cyclic theory for commutative differential graded algebras and* S*-cohomology*, in Quanta of maths, vol. 11 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2010, pp. 85–105.

[93] D. Burghelea and Z. Fiedorowicz, *Cyclic homology and algebraic* K*-theory of spaces. II*, Topology, 25 (1986), pp. 303–317.

[94] D. Burghelea, Z. Fiedorowicz, and W. Gajda, *Adams operations in Hochschild and cyclic homology of de Rham algebra and free loop spaces*, K-Theory, 4 (1991), pp. 269–287.

[95] ———, *Erratum: "Adams operations in Hochschild and cyclic homology of de Rham algebra and free loop space"*, K-Theory, 5 (1991), p. 293.

[96] D. Burghelea and C. Ogle, *The Künneth formula in cyclic homology*, Math. Z., 193 (1986), pp. 527–536.

[97] D. Burghelea and M. Vigué-Poirrier, *Cyclic homology of commutative algebras. I*, in Algebraic topology—rational homotopy (Louvain-la-Neuve, 1986), vol. 1318 of Lecture Notes in Math., Springer, Berlin, 1988, pp. 51–72.

[98] H. Bursztyn and S. Waldmann, *The characteristic classes of Morita equivalent star products on symplectic manifolds*, Comm. Math. Phys., 228 (2002), pp. 103–121.

[99] D. Calaque and F. Naef, *A trace formula for the quantization of coadjoint orbits*, Int. Math. Res. Not. IMRN, (2015), pp. 11236–11252.

[100] A. L. Carey, S. Neshveyev, R. Nest, and A. Rennie, *Twisted cyclic theory, equivariant* KK*-theory and KMS states*, J. Reine Angew. Math., 650 (2011), pp. 161–191.

[101] H. Cartan and S. Eilenberg, *Homological algebra*, Princeton University Press, Princeton, N. J., 1956.

[102] P. Cartier, *Homologie cyclique: rapport sur des travaux récents de Connes, Karoubi, Loday, Quillen...*, Astérisque, (1985), pp. 123–146. Seminar Bourbaki, Vol. 1983/84.

[103] A. Cattaneo, B. Keller, C. Torossian, and A. Bruguières, *Introduction*, in Déformation, quantification, théorie de Lie, vol. 20 of Panor. Synthèses, Soc. Math. France, Paris, 2005, pp. 1–9, 11–18. Dual French-English text.

[104] A. S. Cattaneo, G. Felder, and T. Willwacher, *On* L_∞*-morphisms of cyclic chains*, Lett. Math. Phys., 90 (2009), pp. 85–101.

[105] A. S. Cattaneo, G. Felder, and T. Willwacher, *The character map in deformation quantization*, Adv. Math., 228 (2011), pp. 1966–1989.

[106] B. Cenkl and M. Vigué-Poirrier, *The cyclic homology of* $P(G)$, in The Proceedings of the Winter School Geometry and Topology (Srní, 1992), no. 32, 1993, pp. 195–199.

[107] A. H. Chamseddine, A. Connes, and V. Mukhanov, *Quanta of geometry: noncommutative aspects*, Phys. Rev. Lett., 114 (2015), pp. 091302, 5.

[108] M. Chas and D. Sullivan, *Closed string operators in topology leading to Lie bialgebras and higher string algebra*, in The legacy of Niels Henrik Abel, Springer, Berlin, 2004, pp. 771–784.

[109] P. Chen and V. Dolgushev, *A simple algebraic proof of the algebraic index theorem*, Math. Res. Lett., 12 (2005), pp. 655–671.

[110] A. Connes, *Cohomologie cyclique et foncteurs* Ext^n, C. R. Acad. Sci. Paris Sér. I Math., 296 (1983), pp. 953–958.

[111] ———, *Noncommutative differential geometry*, Inst. Hautes Études Sci. Publ. Math., (1985), pp. 257–360.

[112] ———, *Cyclic cohomology and noncommutative differential geometry*, in Géométrie différentielle (Paris, 1986), vol. 33 of Travaux en Cours, Hermann, Paris, 1988, pp. 33–50.

[113] A. Connes, *Entire cyclic cohomology of Banach algebras and characters of* θ*-summable Fredholm modules*, K-Theory, 1 (1988), pp. 519–548.

[114] ———, *Entire cyclic cohomology of Banach algebras and characters of* θ*-summable Fredholm modules*, K-Theory, 1 (1988), pp. 519–548.

[115] A. Connes, *Noncommutative geometry*, Academic Press, Inc., San Diego, CA, 1994.

[116] ———, *Noncommutative geometry year 2000 [MR1826266 (2003g:58010)]*, in Highlights of mathematical physics (London, 2000), Amer. Math. Soc., Providence, RI, 2002, pp. 49–110.

[117] ———, *Cyclic cohomology, noncommutative geometry and quantum group symmetries*, in Noncommutative geometry, vol. 1831 of Lecture Notes in Math., Springer, Berlin, 2004, pp. 1–71.

[118] ———, *Cyclic cohomology, quantum group symmetries and the local index formula for* $\mathrm{SU}_q(2)$, J. Inst. Math. Jussieu, 3 (2004), pp. 17–68.

[119] ———, *An essay on the Riemann hypothesis*, in Open problems in mathematics, Springer, [Cham], 2016, pp. 225–257.

[120] A. Connes and C. Consani, *Cyclic homology, Serre's local factors and* λ*-operations*, J. K-Theory, 14 (2014), pp. 1–45.

[121] ———, *The cyclic and epicyclic sites*, Rend. Semin. Mat. Univ. Padova, 134 (2015), pp. 197–237.

[122] ———, *Cyclic structures and the topos of simplicial sets*, J. Pure Appl. Algebra, 219 (2015), pp. 1211–1235.

[123] ———, *Projective geometry in characteristic one and the epicyclic category*, Nagoya Math. J., 217 (2015), pp. 95–132.

[124] ———, *Absolute algebra and Segal's* Γ*-rings: au dessous de* $\overline{\mathrm{Spec}(\mathbb{Z})}$, J. Number Theory, 162 (2016), pp. 518–551.

[125] ———, *Geometry of the arithmetic site*, Adv. Math., 291 (2016), pp. 274–329.

[126] ———, *The scaling site*, C. R. Math. Acad. Sci. Paris, 354 (2016), pp. 1–6.

[127] A. Connes and J. Cuntz, *Quasi homomorphismes, cohomologie cyclique et positivité*, Comm. Math. Phys., 114 (1988), pp. 515–526.

[128] A. Connes, M. Flato, and D. Sternheimer, *Closed star products and cyclic cohomology*, Lett. Math. Phys., 24 (1992), pp. 1–12.

[129] A. Connes and M. Karoubi, *Caractère multiplicatif d'un module de Fredholm*, C. R. Acad. Sci. Paris Sér. I Math., 299 (1984), pp. 963–968.

[130] ———, *Caractère multiplicatif d'un module de Fredholm*, K-Theory, 2 (1988), pp. 431–463.

[131] A. Connes and M. Marcolli, *A walk in the noncommutative garden*, in An invitation to noncommutative geometry, World Sci. Publ., Hackensack, NJ, 2008, pp. 1–128.

[132] A. Connes and H. Moscovici, *The local index formula in noncommutative geometry*, Geom. Funct. Anal., 5 (1995), pp. 174–243.

[133] ———, *Hopf algebras, cyclic cohomology and the transverse index theorem*, Comm. Math. Phys., 198 (1998), pp. 199–246.

[134] A. Connes and H. Moscovici, *Cyclic cohomology and Hopf algebra symmetry*, Lett. Math. Phys., 52 (2000), pp. 1–28. Conference Moshé Flato 1999 (Dijon).

[135] ———, *Differentiable cyclic cohomology and Hopf algebraic structures in transverse geometry*, in Essays on geometry and related topics, Vol. 1, 2, vol. 38 of Monogr. Enseign. Math., Enseignement Math., Geneva, 2001, pp. 217–255.

[136] A. Connes and G. Skandalis, *The longitudinal index theorem for foliations*, Publ. Res. Inst. Math. Sci., 20 (1984), pp. 1139–1183.
[137] G. Cortiñas, L*-theory and dihedral homology*, Math. Scand., 73 (1993), pp. 21–35.
[138] ———, L*-theory and dihedral homology. II*, Topology Appl., 51 (1993), pp. 53–69.
[139] ———, *Infinitesimal* K*-theory*, J. Reine Angew. Math., 503 (1998), pp. 129–160.
[140] ———, *On the cyclic homology of commutative algebras over arbitrary ground rings*, Comm. Algebra, 27 (1999), pp. 1403–1412.
[141] ———, *Periodic cyclic homology as sheaf cohomology*, K-Theory, 20 (2000), pp. 175–200. Special issues dedicated to Daniel Quillen on the occasion of his sixtieth birthday, Part II.
[142] ———, *The obstruction to excision in* K*-theory and in cyclic homology*, Invent. Math., 164 (2006), pp. 143–173.
[143] ———, *Cyclic homology, tight crossed products, and small stabilizations*, J. Noncommut. Geom., 8 (2014), pp. 1191–1223.
[144] G. Cortiñas, J. Guccione, and O. E. Villamayor, *Cyclic homology of* $\mathsf{K}[\mathbf{Z}/p \cdot \mathbf{Z}]$, in Proceedings of Research Symposium on K-Theory and its Applications (Ibadan, 1987), vol. 2, 1989, pp. 603–616.
[145] G. Cortiñas, J. A. Guccione, and J. J. Guccione, *Decomposition of the Hochschild and cyclic homology of commutative differential graded algebras*, J. Pure Appl. Algebra, 83 (1992), pp. 219–235.
[146] G. Cortiñas, C. Haesemeyer, M. Schlichting, and C. Weibel, *Cyclic homology, cdh-cohomology and negative* K*-theory*, Ann. of Math. (2), 167 (2008), pp. 549–573.
[147] G. Cortiñas, C. Haesemeyer, and C. Weibel, K*-regularity,* cdh*-fibrant Hochschild homology, and a conjecture of Vorst*, J. Amer. Math. Soc., 21 (2008), pp. 547–561.
[148] G. Cortiñas, C. Haesemeyer, and C. A. Weibel, *Infinitesimal cohomology and the Chern character to negative cyclic homology*, Math. Ann., 344 (2009), pp. 891–922.
[149] G. Cortiñas and C. Valqui, *Excision in bivariant periodic cyclic cohomology: a categorical approach*, K-Theory, 30 (2003), pp. 167–201. Special issue in honor of Hyman Bass on his seventieth birthday. Part II.
[150] G. Cortiñas and C. Weibel, *Homology of Azumaya algebras*, Proc. Amer. Math. Soc., 121 (1994), pp. 53–55.
[151] ———, *Relative Chern characters for nilpotent ideals*, in Algebraic topology, vol. 4 of Abel Symp., Springer, Berlin, 2009, pp. 61–82.
[152] G. H. Cortiñas, *On the derived functor analogy in the Cuntz-Quillen framework for cyclic homology*, Algebra Colloq., 5 (1998), pp. 305–328.
[153] G. H. Cortiñas and O. E. Villamayor, *Cyclic homology of* $\mathsf{K}[\mathbf{Z}/2\mathbf{Z}]$, Rev. Un. Mat. Argentina, 33 (1987), pp. 55–61 (1990).
[154] G. Cortinas, *De Rham and infinitesimal cohomology in Kapranov's model for noncommutative algebraic geometry*, Compositio Math., 136 (2003), pp. 171–208.
[155] K. Costello, *Topological conformal field theories and Calabi-Yau categories*, Adv. Math., 210 (2007), pp. 165–214.
[156] ———, *Topological conformal field theories and gauge theories*, Geom. Topol., 11 (2007), pp. 1539–1579.
[157] ———, *A geometric construction of the Witten genus, I*, in Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 942–959.
[158] ———, *Renormalization and effective field theory*, vol. 170 of Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2011.
[159] W. Crawley-Boevey, P. Etingof, and V. Ginzburg, *Noncommutative geometry and quiver algebras*, Adv. Math., 209 (2007), pp. 274–336.
[160] J. Cuntz, *A new look at* KK*-theory*, K-Theory, 1 (1987), pp. 31–51.
[161] ———, *Universal extensions and cyclic cohomology*, C. R. Acad. Sci. Paris Sér. I Math., 309 (1989), pp. 5–8.
[162] ———, *Cyclic cohomology and* K*-homology*, in Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 969–978.
[163] ———, *Quantized differential forms in noncommutative topology and geometry*, in Representation theory of groups and algebras, vol. 145 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1993, pp. 65–78.

[164] J. Cuntz, *A survey of some aspects of noncommutative geometry*, Jahresber. Deutsch. Math.-Verein., 95 (1993), pp. 60–84.

[165] J. Cuntz, *Excision in periodic cyclic theory for topological algebras*, in Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), vol. 17 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1997, pp. 43–53.

[166] ———, *Morita invariance in cyclic homology for nonunital algebras*, K-Theory, 15 (1998), pp. 301–305.

[167] ———, *Cyclic theory and the bivariant Chern-Connes character*, in Noncommutative geometry, vol. 1831 of Lecture Notes in Math., Springer, Berlin, 2004, pp. 73–135.

[168] ———, *Cyclic theory, bivariant* K*-theory and the bivariant Chern-Connes character*, in Cyclic homology in non-commutative geometry, vol. 121 of Encyclopaedia Math. Sci., Springer, Berlin, 2004, pp. 1–71.

[169] ———, *Quillen's work on the foundations of cyclic cohomology*, J. K-Theory, 11 (2013), pp. 559–574.

[170] J. Cuntz and C. Deninger, *An alternative to Witt vectors*, Münster J. Math., 7 (2014), pp. 105–114.

[171] J. Cuntz and D. Quillen, *On excision in periodic cyclic cohomology*, C. R. Acad. Sci. Paris Sér. I Math., 317 (1993), pp. 917–922.

[172] ———, *On excision in periodic cyclic cohomology. II. The general case*, C. R. Acad. Sci. Paris Sér. I Math., 318 (1994), pp. 11–12.

[173] ———, *Algebra extensions and nonsingularity*, J. Amer. Math. Soc., 8 (1995), pp. 251–289.

[174] ———, *Operators on noncommutative differential forms and cyclic homology*, in Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995, pp. 77–111.

[175] ———, *Excision in bivariant periodic cyclic cohomology*, Invent. Math., 127 (1997), pp. 67–98.

[176] J. Cuntz, G. Skandalis, and B. Tsygan, *Cyclic homology in non-commutative geometry*, vol. 121 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, 2004. Operator Algebras and Non-commutative Geometry, II.

[177] A. D'Agnolo and P. Polesello, *Stacks of twisted modules and integral transforms*, in Geometric aspects of Dwork theory. Vol. I, II, Walter de Gruyter, Berlin, 2004, pp. 463–507.

[178] ———, *Deformation quantization of complex involutive submanifolds*, in Noncommutative geometry and physics, World Sci. Publ., Hackensack, NJ, 2005, pp. 127–137.

[179] ———, *Morita classes of microdifferential algebroids*, Publ. Res. Inst. Math. Sci., 51 (2015), pp. 373–416.

[180] Y. L. Daletskii and B. L. Tsygan, *Operations on Hochschild and cyclic complexes*, Methods Funct. Anal. Topology, 5 (1999), pp. 62–86.

[181] B. H. Dayton and C. A. Weibel, *Module structures on the Hochschild and cyclic homology of graded rings*, in Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991), vol. 407 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1993, pp. 63–90.

[182] M. De Wilde and P. B. A. Lecomte, *Existence of star-products and of formal deformations of the Poisson Lie algebra of arbitrary symplectic manifolds*, Lett. Math. Phys., 7 (1983), pp. 487–496.

[183] P. Deligne, *Letter to L. Breen*, (1994).

[184] ———, *Déformations de l'algèbre des fonctions d'une variété symplectique: comparaison entre Fedosov et De Wilde, Lecomte*, Selecta Math. (N.S.), 1 (1995), pp. 667–697.

[185] P. Deligne and L. Illusie, *Relèvements modulo* p^2 *et décomposition du complexe de de Rham*, Invent. Math., 89 (1987), pp. 247–270.

[186] V. Dolgushev, *A formality theorem for Hochschild chains*, Adv. Math., 200 (2006), pp. 51–101.

[187] V. Dolgushev and P. Etingof, *Hochschild cohomology of quantized symplectic orbifolds and the Chen-Ruan cohomology*, Int. Math. Res. Not., (2005), pp. 1657–1688.

[188] V. Dolgushev, D. Tamarkin, and B. Tsygan, *The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal*, J. Noncommut. Geom., 1 (2007), pp. 1–25.

[189] ———, *The homotopy Gerstenhaber algebra of Hochschild cochains of a regular algebra is formal*, J. Noncommut. Geom., 1 (2007), pp. 1–25.
[190] ———, *Formality theorems for Hochschild complexes and their applications*, Lett. Math. Phys., 90 (2009), pp. 103–136.
[191] ———, *Formality theorems for Hochschild complexes and their applications*, Lett. Math. Phys., 90 (2009), pp. 103–136.
[192] V. A. Dolgushev, *Hochschild cohomology versus de Rham cohomology without formality theorems*, Int. Math. Res. Not., (2005), pp. 1277–1305.
[193] V. A. Dolgushev, D. E. Tamarkin, and B. L. Tsygan, *Noncommutative calculus and the Gauss-Manin connection*, in Higher structures in geometry and physics, vol. 287 of Progr. Math., Birkhäuser/Springer, New York, 2011, pp. 139–158.
[194] ———, *Noncommutative calculus and the Gauss-Manin connection*, in Higher structures in geometry and physics, vol. 287 of Progr. Math., Birkhäuser/Springer, New York, 2011, pp. 139–158.
[195] ———, *Noncommutative calculus and the Gauss-Manin connection*, in Higher structures in geometry and physics, vol. 287 of Progr. Math., Birkhäuser/Springer, New York, 2011, pp. 139–158.
[196] ———, *Proof of Swiss cheese version of Deligne's conjecture*, Int. Math. Res. Not. IMRN, (2011), pp. 4666–4746.
[197] V. Drinfeld, *DG quotients of DG categories*, J. Algebra, 272 (2004), pp. 643–691.
[198] ———, *DG quotients of DG categories*, J. Algebra, 272 (2004), pp. 643–691.
[199] ———, *On the notion of geometric realization*, Mosc. Math. J., 4 (2004), pp. 619–626, 782.
[200] N. Dupont and M. Vigué-Poirrier, *Formalité des espaces de lacets libres*, Bull. Soc. Math. France, 126 (1998), pp. 141–148.
[201] ———, *Finiteness conditions for Hochschild homology algebra and free loop space cohomology algebra*, K-Theory, 21 (2000), pp. 293–300.
[202] A. I. Efimov, *A proof of the Kontsevich-Soibelman conjecture*, Mat. Sb., 202 (2011), pp. 65–84.
[203] A. I. Efimov, *Generalized non-commutative degeneration conjecture*, Proc. Steklov Inst. Math., 290 (2015), pp. 1–10.
[204] ———, *Cyclic homology of categories of matrix factorizations*, Int. Math. Res. Not. IMRN, (2018), pp. 3834–3869.
[205] G. A. Elliott, T. Natsume, and R. Nest, *Cyclic cohomology for one-parameter smooth crossed products*, Acta Math., 160 (1988), pp. 285–305.
[206] G. A. Elliott, T. Natsume, and R. Nest, *The Atiyah-Singer index theorem as passage to the classical limit in quantum mechanics*, Comm. Math. Phys., 182 (1996), pp. 505–533.
[207] ———, *The Atiyah-Singer index theorem as passage to the classical limit in quantum mechanics*, Comm. Math. Phys., 182 (1996), pp. 505–533.
[208] G. A. Elliott, R. Nest, and M. Rø rdam, *The cyclic homology of algebras with adjoined unit*, Proc. Amer. Math. Soc., 113 (1991), pp. 389–395.
[209] I. Emmanouil, *Cyclic homology and de Rham homology of commutative algebras*, C. R. Acad. Sci. Paris Sér. I Math., 318 (1994), pp. 413–417.
[210] ———, *Cyclic homology andde Rham homology of affine algebras*, ProQuest LLC, Ann Arbor, MI, 1994. Thesis (Ph.D.)–University of California, Berkeley.
[211] ———, *The cotangent complex of complete intersections*, C. R. Acad. Sci. Paris Sér. I Math., 321 (1995), pp. 21–25.
[212] ———, *The cyclic homology of affine algebras*, Invent. Math., 121 (1995), pp. 1–19.
[213] ———, *The periodic cyclic cohomology of a tensor product*, C. R. Acad. Sci. Paris Sér. I Math., 320 (1995), pp. 263–267.
[214] ———, *The Künneth formula in periodic cyclic homology*, K-Theory, 10 (1996), pp. 197–214.
[215] ———, *On the excision theorem in bivariant periodic cyclic cohomology*, C. R. Acad. Sci. Paris Sér. I Math., 323 (1996), pp. 229–233.
[216] ———, *Traces and idempotents in group algebras*, Math. Z., 245 (2003), pp. 293–307.
[217] ———, *On the trace of idempotent matrices over group algebras*, Math. Z., 253 (2006), pp. 709–733.
[218] I. Emmanouil and I. B. S. Passi, *Group homology and Connes' periodicity operator*, J. Pure Appl. Algebra, 205 (2006), pp. 375–392.

[219] M. Engeli and G. Felder, *A Riemann-Roch-Hirzebruch formula for traces of differential operators*, Ann. Sci. Éc. Norm. Supér. (4), 41 (2008), pp. 621–653.
[220] P. Etingof and C.-H. Eu, *Hochschild and cyclic homology of preprojective algebras of* ADE *quivers*, Mosc. Math. J., 7 (2007), pp. 601–612, 766.
[221] P. Etingof and V. Ginzburg, *Noncommutative del Pezzo surfaces and Calabi-Yau algebras*, J. Eur. Math. Soc. (JEMS), 12 (2010), pp. 1371–1416.
[222] P. Etingof and T. Schedler, *Poisson traces and* D*-modules on Poisson varieties*, Geom. Funct. Anal., 20 (2010), pp. 958–987. With an appendix by Ivan Losev.
[223] ———, *Traces on finite* $\mathcal{W}$*-algebras*, Transform. Groups, 15 (2010), pp. 843–850.
[224] F. Fathizadeh and M. Khalkhali, *Weyl's law and Connes' trace theorem for noncommutative two tori*, Lett. Math. Phys., 103 (2013), pp. 1–18.
[225] B. Fedosov, *Deformation quantization and index theory*, vol. 9 of Mathematical Topics, Akademie Verlag, Berlin, 1996.
[226] B. Feigin, G. Felder, and B. Shoikhet, *Hochschild cohomology of the Weyl algebra and traces in deformation quantization*, Duke Math. J., 127 (2005), pp. 487–517.
[227] ———, *Hochschild cohomology of the Weyl algebra and traces in deformation quantization*, Duke Math. J., 127 (2005), pp. 487–517.
[228] ———, *Hochschild cohomology of the Weyl algebra and traces in deformation quantization*, Duke Math. J., 127 (2005), pp. 487–517.
[229] G. Felder and B. Shoikhet, *Deformation quantization with traces*, Lett. Math. Phys., 53 (2000), pp. 75–86.
[230] Y. Felix, J.-C. Thomas, and M. Vigué-Poirrier, *The Hochschild cohomology of a closed manifold*, Publ. Math. Inst. Hautes Études Sci., (2004), pp. 235–252.
[231] P. Feng and B. Tsygan, *Hochschild and cyclic homology of quantum groups*, Comm. Math. Phys., 140 (1991), pp. 481–521.
[232] B. L. Feĭ gin and B. L. Tsygan, *Additive* K*-theory and crystalline cohomology*, Funktsional. Anal. i Prilozhen., 19 (1985), pp. 52–62, 96.
[233] ———, *Additive* K*-theory*, in K-theory, arithmetic and geometry (Moscow, 1984–1986), vol. 1289 of Lecture Notes in Math., Springer, Berlin, 1987, pp. 67–209.
[234] ———, *Cyclic homology of algebras with quadratic relations, universal enveloping algebras and group algebras*, in K-theory, arithmetic and geometry (Moscow, 1984–1986), vol. 1289 of Lecture Notes in Math., Springer, Berlin, 1987, pp. 210–239.
[235] ———, *Riemann-Roch theorem and Lie algebra cohomology. I*, in Proceedings of the Winter School on Geometry and Physics (Srní, 1988), no. 21, 1989, pp. 15–52.
[236] B. L. Feĭgin and B. L. Tsygan, *Cohomology of Lie algebras of generalized Jacobi matrices*, Funktsional. Anal. i Prilozhen., 17 (1983), pp. 86–87.
[237] Z. Fiedorowicz and J.-L. Loday, *Crossed simplicial groups and their associated homology*, Trans. Amer. Math. Soc., 326 (1991), pp. 57–87.
[238] ———, *Crossed simplicial groups and their associated homology*, Trans. Amer. Math. Soc., 326 (1991), pp. 57–87.
[239] P. Fillmore and M. Khalkhali, *Entire cyclic cohomology of Banach algebras*, in Nonselfadjoint operators and related topics (Beer Sheva, 1992), vol. 73 of Oper. Theory Adv. Appl., Birkhäuser, Basel, 1994, pp. 256–263.
[240] J. Francis, *The tangent complex and Hochschild cohomology of* $\mathcal{E}_n$*-rings*, Compos. Math., 149 (2013), pp. 430–480.
[241] D. Gaitsgory and N. Rozenblyum, *Crystals and D-modules*, Pure Appl. Math. Q., 10 (2014), pp. 57–154.
[242] T. Geisser and L. Hesselholt, *Topological cyclic homology of schemes*, in Algebraic K-theory (Seattle, WA, 1997), vol. 67 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 1999, pp. 41–87.
[243] ———, *Bi-relative algebraic* K*-theory and topological cyclic homology*, Invent. Math., 166 (2006), pp. 359–395.
[244] S. I. Gelfand and Y. I. Manin, *Methods of homological algebra*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, second ed., 2003.
[245] S. Geller, L. Reid, and C. Weibel, *The cyclic homology and* K*-theory of curves*, J. Reine Angew. Math., 393 (1989), pp. 39–90.
[246] S. Geller and C. Weibel, *Hochschild and cyclic homology are far from being homotopy functors*, Proc. Amer. Math. Soc., 106 (1989), pp. 49–57.

[247] S. C. Geller and C. A. Weibel, *Hodge decompositions of Loday symbols in* K*-theory and cyclic homology*, K-Theory, 8 (1994), pp. 587–632.
[248] I. M. Gel′fand, Y. L. Daletskiĭ, and B. L. Tsygan, *On a variant of noncommutative differential geometry*, Dokl. Akad. Nauk SSSR, 308 (1989), pp. 1293–1297.
[249] M. Gerstenhaber, *The cohomology structure of an associative ring*, Ann. of Math. (2), 78 (1963), pp. 267–288.
[250] ———, *The cohomology structure of an associative ring*, Ann. of Math. (2), 78 (1963), pp. 267–288.
[251] ———, *On the deformation of rings and algebras*, Ann. of Math. (2), 79 (1964), pp. 59–103.
[252] ———, *On the deformation of rings and algebras. II*, Ann. of Math., 84 (1966), pp. 1–19.
[253] ———, *On the deformation of rings and algebras. III*, Ann. of Math. (2), 88 (1968), pp. 1–34.
[254] ———, *On the deformation of rings and algebras. IV*, Ann. of Math. (2), 99 (1974), pp. 257–276.
[255] M. Gerstenhaber and A. Giaquinto, *On the cohomology of the Weyl algebra, the quantum plane, and the* q*-Weyl algebra*, J. Pure Appl. Algebra, 218 (2014), pp. 879–887.
[256] M. Gerstenhaber and S. D. Schack, *On the deformation of algebra morphisms and diagrams*, Trans. Amer. Math. Soc., 279 (1983), pp. 1–50.
[257] M. Gerstenhaber and S. D. Schack, *On the cohomology of an algebra morphism*, J. Algebra, 95 (1985), pp. 245–262.
[258] ———, *Relative Hochschild cohomology, rigid algebras, and the Bockstein*, J. Pure Appl. Algebra, 43 (1986), pp. 53–74.
[259] M. Gerstenhaber and S. D. Schack, *A Hodge-type decomposition for commutative algebra cohomology*, J. Pure Appl. Algebra, 48 (1987), pp. 229–247.
[260] M. Gerstenhaber and A. A. Voronov, *Homotopy* G*-algebras and moduli space operad*, Internat. Math. Res. Notices, (1995), pp. 141–153.
[261] E. Getzler, *Cyclic homology and the Beĭlinson-Manin-Schechtman central extension*, Proc. Amer. Math. Soc., 104 (1988), pp. 729–734.
[262] ———, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, in Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), vol. 7 of Israel Math. Conf. Proc., Bar-Ilan Univ., Ramat Gan, 1993, pp. 65–78.
[263] ———, *Cartan homotopy formulas and the Gauss-Manin connection in cyclic homology*, in Quantum deformations of algebras and their representations (Ramat-Gan, 1991/1992; Rehovot, 1991/1992), vol. 7 of Israel Math. Conf. Proc., Bar-Ilan Univ., Ramat Gan, 1993, pp. 65–78.
[264] ———, *Cyclic homology and the Atiyah-Patodi-Singer index theorem*, in Index theory and operator algebras (Boulder, CO, 1991), vol. 148 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1993, pp. 19–45.
[265] ———, *The odd Chern character in cyclic homology and spectral flow*, Topology, 32 (1993), pp. 489–507.
[266] E. Getzler, *Batalin-Vilkovisky algebras and two-dimensional topological field theories*, Comm. Math. Phys., 159 (1994), pp. 265–285.
[267] E. Getzler, *A Darboux theorem for Hamiltonian operators in the formal calculus of variations*, Duke Math. J., 111 (2002), pp. 535–560.
[268] ———, *Lie theory for nilpotent* L_∞*-algebras*, Ann. of Math. (2), 170 (2009), pp. 271–301.
[269] E. Getzler and J. D. S. Jones, *Operads, homotopy algebra and iterated integrals for double loop spaces*, hep-th9403055.
[270] ———, A_∞*-algebras and the cyclic bar complex*, Illinois J. Math., 34 (1990), pp. 256–283.
[271] ———, A_∞*-algebras and the cyclic bar complex*, Illinois J. Math., 34 (1990), pp. 256–283.
[272] ———, *The cyclic homology of crossed product algebras*, J. Reine Angew. Math., 445 (1993), pp. 161–174.
[273] E. Getzler, J. D. S. Jones, and S. Petrack, *Differential forms on loop spaces and the cyclic bar complex*, Topology, 30 (1991), pp. 339–371.
[274] E. Getzler, J. D. S. Jones, and S. B. Petrack, *Loop spaces, cyclic homology and the Chern character*, in Operator algebras and applications, Vol. 1, vol. 135 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1988, pp. 95–107.

[275] E. Getzler and M. M. Kapranov, *Cyclic operads and cyclic homology*, in Geometry, topology, & physics, Conf. Proc. Lecture Notes Geom. Topology, IV, Int. Press, Cambridge, MA, 1995, pp. 167–201.

[276] E. Getzler and A. Szenes, *On the Chern character of a theta-summable Fredholm module*, J. Funct. Anal., 84 (1989), pp. 343–357.

[277] ———, *On the Chern character of a theta-summable Fredholm module*, J. Funct. Anal., 84 (1989), pp. 343–357.

[278] G. Ginot, *Caractère de Chern et opérations d'Adams en homologie cyclique, algèbres de Gerstenhaber et théorème de formalité*, vol. 2002/41 of Prépublication de l'Institut de Recherche Mathématique Avancée [Prepublication of the Institute of Advanced Mathematical Research], Université Louis Pasteur, Département de Mathématique, Institut de Recherche Mathématique Avancée, Strasbourg, 2002. Dissertation, Université de Strasbourg I (Louis Pasteur) Strasbourg, 2002.

[279] ———, *Formules explicites pour le caractère de Chern en* K*-théorie algébrique*, Ann. Inst. Fourier (Grenoble), 54 (2004), pp. 2327–2355 (2005).

[280] ———, *Higher order Hochschild cohomology*, C. R. Math. Acad. Sci. Paris, 346 (2008), pp. 5–10.

[281] ———, *On the Hochschild and Harrison (co)homology of* C_∞*-algebras and applications to string topology*, in Deformation spaces, Aspects Math., E40, Vieweg + Teubner, Wiesbaden, 2010, pp. 1–51.

[282] ———, *Notes on factorization algebras, factorization homology and applications*, in Mathematical aspects of quantum field theories, Math. Phys. Stud., Springer, Cham, 2015, pp. 429–552.

[283] G. Ginot and G. Halbout, *A formality theorem for Poisson manifolds*, Lett. Math. Phys., 66 (2003), pp. 37–64.

[284] G. Ginot, T. Tradler, and M. Zeinalian, *Higher Hochschild homology, topological chiral homology and factorization algebras*, Comm. Math. Phys., 326 (2014), pp. 635–686.

[285] G. Ginot, T. Tradler, and M. Zeinalian, *Higher Hochschild homology, topological chiral homology and factorization algebras*, Comm. Math. Phys., 326 (2014), pp. 635–686.

[286] V. Ginzburg, I. Gordon, and J. T. Stafford, *Differential operators and Cherednik algebras*, Selecta Math. (N.S.), 14 (2009), pp. 629–666.

[287] V. Ginzburg and D. Kaledin, *Poisson deformations of symplectic quotient singularities*, Adv. Math., 186 (2004), pp. 1–57.

[288] V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math. J., 76 (1994), pp. 203–272.

[289] V. Ginzburg and T. Schedler, *Moyal quantization and stable homology of necklace Lie algebras*, Mosc. Math. J., 6 (2006), pp. 431–459, 587.

[290] ———, *Moyal quantization and stable homology of necklace Lie algebras*, Mosc. Math. J., 6 (2006), pp. 431–459, 587.

[291] ———, *Differential operators and BV structures in noncommutative geometry*, Selecta Math. (N.S.), 16 (2010), pp. 673–730.

[292] ———, *Free products, cyclic homology, and the Gauss-Manin connection*, Adv. Math., 231 (2012), pp. 2352–2389.

[293] ———, *Free products, cyclic homology, and the Gauss-Manin connection*, Adv. Math., 231 (2012), pp. 2352–2389.

[294] ———, *A new construction of cyclic homology*, Proc. Lond. Math. Soc. (3), 112 (2016), pp. 549–587.

[295] W. M. Goldman, *Invariant functions on Lie groups and Hamiltonian flows of surface group representations*, Invent. Math., 85 (1986), pp. 263–302.

[296] T. G. Goodwillie, *Cyclic homology, derivations, and the free loopspace*, Topology, 24 (1985), pp. 187–215.

[297] ———, *On the general linear group and Hochschild homology*, Ann. of Math. (2), 121 (1985), pp. 383–407.

[298] ———, *Correction to: "On the general linear group and Hochschild homology" [Ann. of Math. (2)* **121** *(1985), no. 2, 383–407; MR0786354 (86i:18013)]*, Ann. of Math. (2), 124 (1986), pp. 627–628.

[299] ———, *Relative algebraic* K*-theory and cyclic homology*, Ann. of Math. (2), 124 (1986), pp. 347–402.

[300] A. Gorokhovsky, *Chern classes in Alexander-Spanier cohomology*, K-Theory, 15 (1998), pp. 253–268.

[301] ———, *Characters of cycles, equivariant characteristic classes and Fredholm modules*, Comm. Math. Phys., 208 (1999), pp. 1–23.

[302] ———, *Secondary characteristic classes and cyclic cohomology of Hopf algebras*, Topology, 41 (2002), pp. 993–1016.

[303] ———, *Bivariant Chern character and longitudinal index*, J. Funct. Anal., 237 (2006), pp. 105–134.

[304] A. Gorokhovsky and J. Lott, *Local index theory over étale groupoids*, J. Reine Angew. Math., 560 (2003), pp. 151–198.

[305] ———, *Local index theory over foliation groupoids*, Adv. Math., 204 (2006), pp. 413–447.

[306] A. L. Gorokhovsky, *Explicit formulae for characteristic classes in noncummutative geometry*, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–The Ohio State University.

[307] R. Grady and O. Gwilliam, L_∞ *spaces and derived loop spaces*, New York J. Math., 21 (2015), pp. 231–272.

[308] O. Gwilliam, *Factorization Algebras and Free Field Theories*, ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Northwestern University.

[309] O. Gwilliam and R. Grady, *One-dimensional Chern-Simons theory and the* $\hat{A}$ *genus*, Algebr. Geom. Topol., 14 (2014), pp. 2299–2377.

[310] D. Halpern-Leistner and D. Pomerleano, *Equivariant hodge theory and noncommutative geometry*, Geometry & Topology, 24 (2020), pp. 2361–2433.

[311] M. Hassanzadeh and M. Khalkhali, *Cup coproducts in Hopf cyclic cohomology*, J. Homotopy Relat. Struct., 10 (2015), pp. 347–373.

[312] M. Hassanzadeh, D. Kucerovsky, and B. Rangipour, *Generalized coefficients for Hopf cyclic cohomology*, SIGMA Symmetry Integrability Geom. Methods Appl., 10 (2014), pp. Paper 093, 16.

[313] M. Hassanzadeh and B. Rangipour, *Equivariant Hopf Galois extensions and Hopf cyclic cohomology*, J. Noncommut. Geom., 7 (2013), pp. 105–133.

[314] L. Hesselholt, *Witt vectors of non-commutative rings and topological cyclic homology*, Acta Math., 178 (1997), pp. 109–141.

[315] ———, *Algebraic* K*-theory and trace invariants*, in Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002), Higher Ed. Press, Beijing, 2002, pp. 415–425.

[316] ———, *Topological Hochschild homology and the de Rham-Witt complex for* $\mathbb{Z}_{(p)}$*-algebras*, in Homotopy theory: relations with algebraic geometry, group cohomology, and algebraic K-theory, vol. 346 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2004, pp. 253–259.

[317] ———, *On the topological cyclic homology of the algebraic closure of a local field*, in An alpine anthology of homotopy theory, vol. 399 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2006, pp. 133–162.

[318] ———, *The big de Rham-Witt complex*, Acta Math., 214 (2015), pp. 135–207.

[319] L. Hesselholt and I. Madsen, *On the De Rham-Witt complex in mixed characteristic*, Ann. Sci. École Norm. Sup. (4), 37 (2004), pp. 1–43.

[320] N. Higson, *The local index formula in noncommutative geometry*, in Contemporary developments in algebraic K-theory, ICTP Lect. Notes, XV, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2004, pp. 443–536.

[321] N. Higson and V. Nistor, *Cyclic homology of totally disconnected groups acting on buildings*, J. Funct. Anal., 141 (1996), pp. 466–495.

[322] W. Hong and P. Xu, *Poisson cohomology of del Pezzo surfaces*, J. Algebra, 336 (2011), pp. 378–390.

[323] C. E. Hood and J. D. S. Jones, *Some algebraic properties of cyclic homology groups*, K-Theory, 1 (1987), pp. 361–384.

[324] L. Illusie, *Complexe de de Rham-Witt et cohomologie cristalline*, Ann. Sci. École Norm. Sup. (4), 12 (1979), pp. 501–661.

[325] N. Iyudu, M. Kontsevich, and Y. Vlassopoulos, *Pre-Calabi-Yau algebras as noncommutative Poisson structures*, J. Algebra, 567 (2021), pp. 63–90.

[326] A. Jaffe, A. Lesniewski, and K. Osterwalder, *Quantum* K*-theory. I. The Chern character*, Comm. Math. Phys., 118 (1988), pp. 1–14.

[327] J. D. S. Jones, *Cyclic homology and equivariant homology*, Invent. Math., 87 (1987), pp. 403–423.

[328] J. D. S. Jones and C. Kassel, *Bivariant cyclic theory*, K-Theory, 3 (1989), pp. 339–365.

[329] J. D. S. Jones and J. McCleary, *Hochschild homology, cyclic homology, and the cobar construction*, in Adams Memorial Symposium on Algebraic Topology, 1 (Manchester, 1990), vol. 175 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 1992, pp. 53–65.

[330] D. Kaledin, *Hochschild homology and Gabber's theorem*, in Moscow Seminar on Mathematical Physics. II, vol. 221 of Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 2007, pp. 147–156.

[331] ———, *Hochschild homology and Gabber's theorem*, in Moscow Seminar on Mathematical Physics. II, vol. 221 of Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 2007, pp. 147–156.

[332] D. Kaledin, *Beilinson conjectures in the non-commutative setting*, in Higher-dimensional geometry over finite fields, vol. 16 of NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., IOS, Amsterdam, 2008, pp. 78–91.

[333] ———, *Beilinson conjectures in the non-commutative setting*, in Higher-dimensional geometry over finite fields, vol. 16 of NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., IOS, Amsterdam, 2008, pp. 78–91.

[334] D. Kaledin, *Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie*, Pure Appl. Math. Q., 4 (2008), pp. 785–875.

[335] ———, *Non-commutative Hodge-to-de Rham degeneration via the method of Deligne-Illusie*, Pure Appl. Math. Q., 4 (2008), pp. 785–875.

[336] D. Kaledin, *Cartier isomorphism and Hodge theory in the non-commutative case*, in Arithmetic geometry, vol. 8 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2009, pp. 537–562.

[337] ———, *Cartier isomorphism and Hodge theory in the non-commutative case*, in Arithmetic geometry, vol. 8 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2009, pp. 537–562.

[338] D. Kaledin, *Cyclic homology with coefficients*, in Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, vol. 270 of Progr. Math., Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 23–47.

[339] ———, *Cyclic homology with coefficients*, in Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, vol. 270 of Progr. Math., Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 23–47.

[340] ———, *Geometry and topology of symplectic resolutions*, in Algebraic geometry—Seattle 2005. Part 2, vol. 80 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2009, pp. 595–628.

[341] ———, *Geometry and topology of symplectic resolutions*, in Algebraic geometry—Seattle 2005. Part 2, vol. 80 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2009, pp. 595–628.

[342] ———, *Motivic structures in non-commutative geometry*, in Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 461–496.

[343] ———, *Motivic structures in non-commutative geometry*, in Proceedings of the International Congress of Mathematicians. Volume II, Hindustan Book Agency, New Delhi, 2010, pp. 461–496.

[344] ———, *Universal Witt vectors and the "Japanese cocycle"*, Mosc. Math. J., 12 (2012), pp. 593–604, 669.

[345] ———, *Universal Witt vectors and the "Japanese cocycle"*, Mosc. Math. J., 12 (2012), pp. 593–604, 669.

[346] D. Kaledin, *Beilinson conjecture for finite-dimensional associative algebras*, in The influence of Solomon Lefschetz in geometry and topology, vol. 621 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2014, pp. 77–88.

[347] D. Kaledin, *Trace theories and localization*, in Stacks and categories in geometry, topology, and algebra, vol. 643 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2015, pp. 227–262.

[348] ———, *Witt vectors as a polynomial functor*, Selecta Math. (N.S.), 24 (2018), pp. 359–402.

[349] ———, *Witt vectors, commutative and non-commutative*, Uspekhi Mat. Nauk, 73 (2018), pp. 3–34.

[350] D. B. Kaledin, *Cyclotomic complexes*, Izv. Ross. Akad. Nauk Ser. Mat., 77 (2013), pp. 3–70.

[351] ———, *Cyclotomic complexes*, Izv. Ross. Akad. Nauk Ser. Mat., 77 (2013), pp. 3–70.

[352] ———, *Cartier isomorphism for unital associative algebras*, Proc. Steklov Inst. Math., 290 (2015), pp. 35–51.

[353] M. Karoubi, *Connexions, courbures et classes caractéristiques en* K*-théorie algébrique*, in Current trends in algebraic topology, Part 1 (London, Ont., 1981), vol. 2 of CMS Conf. Proc., Amer. Math. Soc., Providence, R.I., 1982, pp. 19–27.

[354] ———, *Homologie cyclique des groupes et des algèbres*, C. R. Acad. Sci. Paris Sér. I Math., 297 (1983), pp. 381–384.

[355] ———, *Homologie cyclique et* K*-théorie algébrique. I*, C. R. Acad. Sci. Paris Sér. I Math., 297 (1983), pp. 447–450.

[356] ———, *Homologie cyclique et* K*-théorie algébrique. II*, C. R. Acad. Sci. Paris Sér. I Math., 297 (1983), pp. 513–516.

[357] ———, *Homologie cyclique et régulateurs en* K*-théorie algébrique*, C. R. Acad. Sci. Paris Sér. I Math., 297 (1983), pp. 557–560.

[358] ———, *Formule de Künneth en homologie cyclique. I*, C. R. Acad. Sci. Paris Sér. I Math., 303 (1986), pp. 527–530.

[359] ———, *Formule de Künneth en homologie cyclique. II*, C. R. Acad. Sci. Paris Sér. I Math., 303 (1986), pp. 595–598.

[360] ———, K*-théorie multiplicative et homologie cyclique*, C. R. Acad. Sci. Paris Sér. I Math., 303 (1986), pp. 507–510.

[361] ———, *Homologie cyclique et* K*-théorie*, Astérisque, (1987), p. 147.

[362] ———, *Cyclic homology and characteristic classes of bundles with additional structures*, in Algebraic topology (Arcata, CA, 1986), vol. 1370 of Lecture Notes in Math., Springer, Berlin, 1989, pp. 235–242.

[363] ———, *Sur la* K*-théorie multiplicative*, in Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), vol. 17 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1997, pp. 59–77.

[364] ———, K*-theory*, Classics in Mathematics, Springer-Verlag, Berlin, 2008. An introduction, Reprint of the 1978 edition, With a new postface by the author and a list of errata.

[365] M. Karoubi and O. E. Villamayor, *Homologie cyclique d'algèbres de groupes*, C. R. Acad. Sci. Paris Sér. I Math., 311 (1990), pp. 1–3.

[366] C. Kassel, *Algèbres enveloppantes et homologie cyclique*, C. R. Acad. Sci. Paris Sér. I Math., 303 (1986), pp. 779–782.

[367] ———, *A Künneth formula for the cyclic cohomology of* $\mathbf{Z}/2$*-graded algebras*, Math. Ann., 275 (1986), pp. 683–699.

[368] ———, *Cyclic homology, comodules, and mixed complexes*, J. Algebra, 107 (1987), pp. 195–216.

[369] ———, K*-théorie algébrique et cohomologie cyclique bivariantes*, C. R. Acad. Sci. Paris Sér. I Math., 306 (1988), pp. 799–802.

[370] ———, *L'homologie cyclique des algèbres enveloppantes*, Invent. Math., 91 (1988), pp. 221–251.

[371] ———, *Caractère de Chern bivariant*, K-Theory, 3 (1989), pp. 367–400.

[372] ———, *Le résidu non commutatif (d'après M. Wodzicki)*, Astérisque, (1989), pp. Exp. No. 708, 199–229. Séminaire Bourbaki, Vol. 1988/89.

[373] ———, *Quand l'homologie cyclique périodique n'est pas la limite projective de l'homologie cyclique*, in Proceedings of Research Symposium on K-Theory and its Applications (Ibadan, 1987), vol. 2, 1989, pp. 617–621.

[374] ———, *Homologie cyclique, caractère de Chern et lemme de perturbation*, J. Reine Angew. Math., 408 (1990), pp. 159–180.

[375] ———, *Cyclic homology of differential operators, the Virasoro algebra and a* q*-analogue*, Comm. Math. Phys., 146 (1992), pp. 343–356.

[376] ———, *A Künneth formula for the decomposition of the cyclic homology of commutative algebras. Appendix to: "Decomposition of the bivariate cyclic cohomology of commutative algebras" [Math. Scand.* **70** *(1992), no. 1, 5–26; MR1174200 (93j:18011)] by P. Nuss*, Math. Scand., 70 (1992), pp. 27–33.

[377] C. Kassel and A. B. Sletsjø e, *Base change, transitivity and Künneth formulas for the Quillen decomposition of Hochschild homology*, Math. Scand., 70 (1992), pp. 186–192.

[378] L. Katzarkov, M. Kontsevich, and T. Pantev, *Hodge theoretic aspects of mirror symmetry*, in From Hodge theory to integrability and TQFT tt*-geometry, vol. 78 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2008, pp. 87–174.

[379] ———, *Hodge theoretic aspects of mirror symmetry*, in From Hodge theory to integrability and TQFT tt*-geometry, vol. 78 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2008, pp. 87–174.

[380] R. M. Kaufmann, *Moduli space actions on the Hochschild co-chains of a Frobenius algebra. I. Cell operads*, J. Noncommut. Geom., 1 (2007), pp. 333–384.

[381] ———, *Moduli space actions on the Hochschild co-chains of a Frobenius algebra. II. Correlators*, J. Noncommut. Geom., 2 (2008), pp. 283–332.

[382] ———, *A proof of a cyclic version of Deligne's conjecture via cacti*, Math. Res. Lett., 15 (2008), pp. 901–921.

[383] ———, *A detailed look on actions on Hochschild complexes especially the degree 1 coproduct and actions on loop spaces*, J. Noncommut. Geom., 16 (2022), pp. 677–716.

[384] R. M. Kaufmann, B. C. Ward, and J. J. Zúñiga, *The odd origin of Gerstenhaber brackets, Batalin-Vilkovisky operators, and master equations*, J. Math. Phys., 56 (2015), pp. 103504, 40.

[385] A. Kaygun and M. Khalkhali, *Excision in Hopf cyclic homology*, K-Theory, 37 (2006), pp. 105–128.

[386] D. Kazhdan, V. Nistor, and P. Schneider, *Hochschild and cyclic homology of finite type algebras*, Selecta Math. (N.S.), 4 (1998), pp. 321–359.

[387] B. Keller, *Derived categories and their uses*, in Handbook of algebra, Vol. 1, vol. 1 of Handb. Algebr., Elsevier/North-Holland, Amsterdam, 1996, pp. 671–701.

[388] ———, *Invariance of cyclic homology under derived equivalence*, in Representation theory of algebras (Cocoyoc, 1994), vol. 18 of CMS Conf. Proc., Amer. Math. Soc., Providence, RI, 1996, pp. 353–361.

[389] ———, *Basculement et homologie cyclique*, in Algèbre non commutative, groupes quantiques et invariants (Reims, 1995), vol. 2 of Sémin. Congr., Soc. Math. France, Paris, 1997, pp. 13–33.

[390] ———, *Invariance and localization for cyclic homology of DG algebras*, J. Pure Appl. Algebra, 123 (1998), pp. 223–273.

[391] ———, *On the cyclic homology of ringed spaces and schemes*, Doc. Math., 3 (1998), pp. 231–259.

[392] ———, *An overview of results on cyclic homology of exact categories*, in Algebras and modules, II (Geiranger, 1996), vol. 24 of CMS Conf. Proc., Amer. Math. Soc., Providence, RI, 1998, pp. 337–345.

[393] ———, *On the cyclic homology of exact categories*, J. Pure Appl. Algebra, 136 (1999), pp. 1–56.

[394] ———, *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl., 3 (2001), pp. 1–35.

[395] ———, *Introduction to A-infinity algebras and modules*, Homology Homotopy Appl., 3 (2001), pp. 1–35.

[396] ———, *Addendum to: "Introduction to A-infinity algebras and modules" [Homology Homotopy Appl.* **3** *(2001), no. 1, 1–35; MR1854636 (2004a:18008a)]*, Homology Homotopy Appl., 4 (2002), pp. 25–28.

[397] ———, *Hochschild cohomology and derived Picard groups*, J. Pure Appl. Algebra, 190 (2004), pp. 177–196.

[398] ———, *A-infinity algebras, modules and functor categories*, in Trends in representation theory of algebras and related topics, vol. 406 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2006, pp. 67–93.

[399] ———, *On differential graded categories*, in International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[400] ———, *On differential graded categories*, in International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zürich, 2006, pp. 151–190.

[401] ———, *Deformed Calabi-Yau completions*, J. Reine Angew. Math., 654 (2011), pp. 125–180. With an appendix by Michel Van den Bergh.

[402] B. Keller and W. Lowen, *On Hochschild cohomology and Morita deformations*, Int. Math. Res. Not. IMRN, (2009), pp. 3221–3235.

[403] ———, *On Hochschild cohomology and Morita deformations*, Int. Math. Res. Not. IMRN, (2009), pp. 3221–3235.

[404] M. Khalkhali, *An approach to operations on cyclic homology*, J. Pure Appl. Algebra, 107 (1996), pp. 47–59.

[405] ———, *On Cartan homotopy formulas in cyclic homology*, Manuscripta Math., 94 (1997), pp. 111–132.

[406] ———, *A survey of entire cyclic cohomology*, in Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), vol. 17 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1997, pp. 79–89.

[407] ———, *Operations on cyclic homology, the X complex, and a conjecture of Deligne*, Comm. Math. Phys., 202 (1999), pp. 309–323.

[408] ———, *Basic noncommutative geometry*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, 2009.

[409] ———, *A short survey of cyclic cohomology*, in Quanta of maths, vol. 11 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2010, pp. 283–311.

[410] ———, *Basic noncommutative geometry*, EMS Series of Lectures in Mathematics, European Mathematical Society (EMS), Zürich, second ed., 2013.

[411] M. Khalkhali and A. Moatadelro, *A Riemann-Roch theorem for the noncommutative two torus*, J. Geom. Phys., 86 (2014), pp. 19–30.

[412] M. Khalkhali and A. Pourkia, *Hopf cyclic cohomology in braided monoidal categories*, Homology Homotopy Appl., 12 (2010), pp. 111–155.

[413] M. Khalkhali and B. Rangipour, *Cyclic cohomology of (extended) Hopf algebras*, in Noncommutative geometry and quantum groups (Warsaw, 2001), vol. 61 of Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw, 2003, pp. 59–89.

[414] M. Khalkhali and B. Rangipour, *Invariant cyclic homology*, K-Theory, 28 (2003), pp. 183–205.

[415] ———, *On the cyclic homology of Hopf crossed products*, in Galois theory, Hopf algebras, and semiabelian categories, vol. 43 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 2004, pp. 341–351.

[416] ———, *On the generalized cyclic Eilenberg-Zilber theorem*, Canad. Math. Bull., 47 (2004), pp. 38–48.

[417] M. Khalkhali and B. Rangipour, *Cup products in Hopf-cyclic cohomology*, C. R. Math. Acad. Sci. Paris, 340 (2005), pp. 9–14.

[418] M. Khalkhali and B. Rangipour, *A note on cyclic duality and Hopf algebras*, Comm. Algebra, 33 (2005), pp. 763–773.

[419] M. Khalkhali and B. Rangipour, *Introduction to Hopf-cyclic cohomology*, in Noncommutative geometry and number theory, Aspects Math., E37, Friedr. Vieweg, Wiesbaden, 2006, pp. 155–178.

[420] M. Kontsevich, *Operads and motives in deformation quantization*, vol. 48, 1999, pp. 35–72. Moshé Flato (1937–1998).

[421] ———, *Deformation quantization of algebraic varieties*, Lett. Math. Phys., 56 (2001), pp. 271–294. EuroConférence Moshé Flato 2000, Part III (Dijon).

[422] ———, *Deformation quantization of Poisson manifolds*, Lett. Math. Phys., 66 (2003), pp. 157–216.

[423] ———, *XI Solomon Lefschetz Memorial Lecture series: Hodge structures in non-commutative geometry*, in Non-commutative geometry in mathematics and physics, vol. 462 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2008, pp. 1–21. Notes by Ernesto Lupercio.

[424] ———, *XI Solomon Lefschetz Memorial Lecture series: Hodge structures in non-commutative geometry*, in Non-commutative geometry in mathematics and physics, vol. 462 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2008, pp. 1–21. Notes by Ernesto Lupercio.

[425] ———, *Geometry in dg-categories*, in New spaces in mathematics—formal and conceptual reflections, Cambridge Univ. Press, Cambridge, 2021, pp. 554–592.

[426] M. Kontsevich and A. L. Rosenberg, *Noncommutative smooth spaces*, in The Gelfand Mathematical Seminars, 1996–1999, Gelfand Math. Sem., Birkhäuser Boston, Boston, MA, 2000, pp. 85–108.

[427] ———, *Noncommutative smooth spaces*, in The Gelfand Mathematical Seminars, 1996–1999, Gelfand Math. Sem., Birkhäuser Boston, Boston, MA, 2000, pp. 85–108.

[428] M. Kontsevich and Y. Soibelman, *Deformations of algebras over operads and the Deligne conjecture*, in Conférence Moshé Flato 1999, Vol. I (Dijon), vol. 21 of Math. Phys. Stud., Kluwer Acad. Publ., Dordrecht, 2000, pp. 255–307.

[429] M. Kontsevich and Y. Soibelman, *Notes on A_∞-algebras, A_∞-categories and non-commutative geometry*, in Homological mirror symmetry, vol. 757 of Lecture Notes in Phys., Springer, Berlin, 2009, pp. 153–219.

[430] M. Lesch, H. Moscovici, and M. J. Pflaum, *Relative pairing in cyclic cohomology and divisor flows*, J. K-Theory, 3 (2009), pp. 359–407.

[431] J.-L. Loday, *Cyclic homology, a survey*, in Geometric and algebraic topology, vol. 18 of Banach Center Publ., PWN, Warsaw, 1986, pp. 281–303.

[432] ———, *Homologies diédrale et quaternionique*, Adv. in Math., 66 (1987), pp. 119–148.

[433] ———, *Partition eulérienne et opérations en homologie cyclique*, C. R. Acad. Sci. Paris Sér. I Math., 307 (1988), pp. 283–286.

[434] ———, *Opérations sur l'homologie cyclique des algèbres commutatives*, Invent. Math., 96 (1989), pp. 205–230.

[435] ———, *Cyclic homology*, vol. 301 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, 1992. Appendix E by María O. Ronco.

[436] ———, *Cyclic homology*, vol. 301 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer-Verlag, Berlin, second ed., 1998. Appendix E by María O. Ronco, Chapter 13 by the author in collaboration with Teimuraz Pirashvili.

[437] ———, *From diffeomorphism groups to loop spaces via cyclic homology*, in Symétries quantiques (Les Houches, 1995), North-Holland, Amsterdam, 1998, pp. 727–755.

[438] ———, *Hochschild and cyclic homology: résumé and variations*, in Algebraic K-theory and its applications (Trieste, 1997), World Sci. Publ., River Edge, NJ, 1999, pp. 234–254.

[439] ———, *Algebraic K-theory and cyclic homology*, J. K-Theory, 11 (2013), pp. 553–557.

[440] ———, *Free loop space and homology*, in Free loop spaces in geometry and topology, vol. 24 of IRMA Lect. Math. Theor. Phys., Eur. Math. Soc., Zürich, 2015, pp. 137–156.

[441] J.-L. Loday and C. Procesi, *Cyclic homology and lambda operations*, in Algebraic K-theory: connections with geometry and topology (Lake Louise, AB, 1987), vol. 279 of NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., Kluwer Acad. Publ., Dordrecht, 1989, pp. 209–224.

[442] J.-L. Loday and D. Quillen, *Homologie cyclique et homologie de l'algèbre de Lie des matrices*, C. R. Acad. Sci. Paris Sér. I Math., 296 (1983), pp. 295–297.

[443] ———, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv., 59 (1984), pp. 569–591.

[444] ———, *Cyclic homology and the Lie algebra homology of matrices*, Comment. Math. Helv., 59 (1984), pp. 569–591.

[445] J.-L. Loday and B. Vallette, *Algebraic operads*, vol. 346 of Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Springer, Heidelberg, 2012.

[446] M. Marcolli and G. Tabuada, *Noncommutative motives and their applications*, in Commutative algebra and noncommutative algebraic geometry. Vol. I, vol. 67 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, New York, 2015, pp. 191–214.

[447] R. Meyer, *Excision in Hochschild and cyclic homology without continuous linear sections*, J. Homotopy Relat. Struct., 5 (2010), pp. 269–303.

[448] H. Moscovici, *Cyclic cohomology and invariants of multiply connected manifolds*, in Proceedings of the International Congress of Mathematicians, Vol. I, II (Kyoto, 1990), Math. Soc. Japan, Tokyo, 1991, pp. 675–688.

[449] ———, *Local index formula and twisted spectral triples*, in Quanta of maths, vol. 11 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2010, pp. 465–500.

[450] ———, *Equivariant Chern classes in Hopf cyclic cohomology*, Bull. Math. Soc. Sci. Math. Roumanie (N.S.), 58(106) (2015), pp. 317–330.

[451] ———, *Geometric construction of Hopf cyclic characteristic classes*, Adv. Math., 274 (2015), pp. 651–680.

[452] H. Moscovici and B. Rangipour, *Cyclic cohomology of Hopf algebras of transverse symmetries in codimension 1*, Adv. Math., 210 (2007), pp. 323–374.
[453] ———, *Hopf algebras of primitive Lie pseudogroups and Hopf cyclic cohomology*, Adv. Math., 220 (2009), pp. 706–790.
[454] ———, *Hopf cyclic cohomology and transverse characteristic classes*, Adv. Math., 227 (2011), pp. 654–729.
[455] H. Moscovici and F. Wu, *Straight Chern character for Witt spaces*, in Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), vol. 17 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1997, pp. 103–113.
[456] T. Moulinos, M. Robalo, and B. Toën, *A universal Hochschild-Kostant-Rosenberg theorem*, Geom. Topol., 26 (2022), pp. 777–874.
[457] F. Naef, *Poisson brackets in Kontsevich's "Lie world"*, J. Geom. Phys., 155 (2020), pp. 103741, 13.
[458] T. Natsume and R. Nest, *The cyclic cohomology of compact Lie groups and the direct sum formula*, J. Operator Theory, 23 (1990), pp. 43–50.
[459] T. Natsume and R. Nest, *The local structure of the cyclic cohomology of Heisenberg Lie groups*, J. Funct. Anal., 119 (1994), pp. 481–498.
[460] R. Nest, *Cyclic cohomology of crossed products with* **Z**, J. Funct. Anal., 80 (1988), pp. 235–283.
[461] R. Nest and B. Tsygan, *Algebraic index theorem*, Comm. Math. Phys., 172 (1995), pp. 223–262.
[462] ———, *Algebraic index theorem*, Comm. Math. Phys., 172 (1995), pp. 223–262.
[463] ———, *Algebraic index theorem for families*, Adv. Math., 113 (1995), pp. 151–205.
[464] ———, *Algebraic index theorem for families*, Adv. Math., 113 (1995), pp. 151–205.
[465] ———, *Formal versus analytic index theorems*, Internat. Math. Res. Notices, (1996), pp. 557–564.
[466] ———, *Formal versus analytic index theorems*, Internat. Math. Res. Notices, (1996), pp. 557–564.
[467] ———, *The Fukaya type categories for associative algebras*, in Deformation theory and symplectic geometry (Ascona, 1996), vol. 20 of Math. Phys. Stud., Kluwer Acad. Publ., Dordrecht, 1997, pp. 285–300.
[468] ———, *Product structures in (cyclic) homology and their applications*, in Operator algebras and quantum field theory (Rome, 1996), Int. Press, Cambridge, MA, 1997, pp. 416–439.
[469] ———, *On the cohomology ring of an algebra*, in Advances in geometry, vol. 172 of Progr. Math., Birkhäuser Boston, Boston, MA, 1999, pp. 337–370.
[470] V. Nistor, *Group cohomology and the cyclic cohomology of crossed products*, Invent. Math., 99 (1990), pp. 411–424.
[471] V. Nistor, *A bivariant Chern character for* p*-summable quasihomomorphisms*, K-Theory, 5 (1991), pp. 193–211.
[472] ———, *A bivariant Chern-Connes character*, Ann. of Math. (2), 138 (1993), pp. 555–590.
[473] ———, *Cyclic cohomology of crossed products by algebraic groups*, Invent. Math., 112 (1993), pp. 615–638.
[474] ———, *Higher McKean-Singer index formulae and noncommutative geometry*, in Representation theory of groups and algebras, vol. 145 of Contemp. Math., Amer. Math. Soc., Providence, RI, 1993, pp. 439–452.
[475] ———, *On the Cuntz-Quillen boundary map*, C. R. Math. Rep. Acad. Sci. Canada, 16 (1994), pp. 203–208.
[476] ———, *Higher index theorems and the boundary map in cyclic cohomology*, Doc. Math., 2 (1997), pp. 263–295.
[477] ———, *Super-connections and non-commutative geometry*, in Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), vol. 17 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1997, pp. 115–136.
[478] ———, *Higher orbital integrals, Shalika germs, and the Hochschild homology of Hecke algebras*, Int. J. Math. Math. Sci., 26 (2001), pp. 129–160.
[479] D. Perrot, *Pseudodifferential extension and Todd class*, Adv. Math., 246 (2013), pp. 265–302.
[480] A. Petrov, D. Vaintrob, and V. Vologodsky, *The Gauss-Manin connection on the periodic cyclic homology*, Selecta Math. (N.S.), 24 (2018), pp. 531–561.

[481] A. Polishchuk and L. Positselski, *Hochschild (co)homology of the second kind I*, Trans. Amer. Math. Soc., 364 (2012), pp. 5311–5368.

[482] A. Polishchuk and A. Vaintrob, *Matrix factorizations and cohomological field theories*, J. Reine Angew. Math., 714 (2016), pp. 1–122.

[483] A. Preygel, *Thom-Sebastiani and Duality for Matrix Factorizations, and Results on the Higher Structures of the Hochschild Invariants*, ProQuest LLC, Ann Arbor, MI, 2012. Thesis (Ph.D.)–Massachusetts Institute of Technology.

[484] M. Puschnigg, *Asymptotic cyclic cohomology*, Universität Heidelberg, Naturwiss.-Math. Gesamtfak., Heidelberg, 1993.

[485] ———, *Asymptotic cyclic cohomology*, vol. 1642 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1996.

[486] ———, *A survey of asymptotic cyclic cohomology*, in Cyclic cohomology and noncommutative geometry (Waterloo, ON, 1995), vol. 17 of Fields Inst. Commun., Amer. Math. Soc., Providence, RI, 1997, pp. 155–168.

[487] ———, *Explicit product structures in cyclic homology theories*, K-Theory, 15 (1998), pp. 323–345.

[488] ———, *Excision in cyclic homology theories*, Invent. Math., 143 (2001), pp. 249–323.

[489] ———, *Diffeotopy functors of ind-algebras and local cyclic cohomology*, Doc. Math., 8 (2003), pp. 143–245.

[490] ———, *Excision and the Hodge filtration in periodic cyclic homology: the case of splitting and invertible extensions*, J. Reine Angew. Math., 593 (2006), pp. 169–207.

[491] ———, *Characters of Fredholm modules and a problem of Connes*, Geom. Funct. Anal., 18 (2008), pp. 583–635.

[492] D. Quillen, *Algebra cochains and cyclic cohomology*, Inst. Hautes Études Sci. Publ. Math., (1988), pp. 139–174 (1989).

[493] ———, *Cyclic cohomology and algebra extensions*, K-Theory, 3 (1989), pp. 205–246.

[494] D. Quillen, *Chern-Simons forms and cyclic cohomology*, in The interface of mathematics and particle physics (Oxford, 1988), vol. 24 of Inst. Math. Appl. Conf. Ser. New Ser., Oxford Univ. Press, New York, 1990, pp. 117–134.

[495] D. Quillen, *Bivariant cyclic cohomology and models for cyclic homology types*, J. Pure Appl. Algebra, 101 (1995), pp. 1–33.

[496] B. Rangipour, *Constant and equivariant cyclic cohomology*, Lett. Math. Phys., 79 (2007), pp. 67–73.

[497] ———, *Cup products in Hopf cyclic cohomology via cyclic modules*, Homology Homotopy Appl., 10 (2008), pp. 273–286.

[498] ———, *Cyclic cohomology of corings*, J. K-Theory, 4 (2009), pp. 193–207.

[499] B. Rangipour and S. Sütlü, *Cyclic cohomology of Lie algebras*, Doc. Math., 17 (2012), pp. 483–515.

[500] M. Rivera, *On String Topology Operations and Algebraic Structures on Hochschild Complexes*, ProQuest LLC, Ann Arbor, MI, 2015. Thesis (Ph.D.)–City University of New York.

[501] M. Rivera and Z. Wang, *Singular Hochschild cohomology and algebraic string operations*, J. Noncommut. Geom., 13 (2019), pp. 297–361.

[502] ———, *Invariance of the Goresky-Hingston algebra on reduced Hochschild homology*, Proc. Lond. Math. Soc. (3), 125 (2022), pp. 219–257.

[503] P. Schapira and J.-P. Schneiders, *Elliptic pairs. I. Relative finiteness and duality*, no. 224, 1994, pp. 5–60. Index theorem for elliptic pairs.

[504] ———, *Elliptic pairs. II. Euler class and relative index theorem*, no. 224, 1994, pp. 61–98. Index theorem for elliptic pairs.

[505] T. Schedler, *Zeroth Hochschild homology of preprojective algebras over the integers*, Adv. Math., 299 (2016), pp. 451–542.

[506] D. Shklyarov, *Hirzebruch-Riemann-Roch theorem for differential graded algebras*, ProQuest LLC, Ann Arbor, MI, 2009. Thesis (Ph.D.)–Kansas State University.

[507] D. Shklyarov, *Hirzebruch-Riemann-Roch-type formula for DG algebras*, Proc. Lond. Math. Soc. (3), 106 (2013), pp. 1–32.

[508] ———, *Non-commutative Hodge structures: towards matching categorical and geometric examples*, Trans. Amer. Math. Soc., 366 (2014), pp. 2923–2974.

[509] D. Shklyarov, *Matrix factorizations and higher residue pairings*, Adv. Math., 292 (2016), pp. 181–209.

[510] ———, *On a Hodge theoretic property of the Künneth map in periodic cyclic homology*, J. Algebra, 446 (2016), pp. 132–153.

[511] ———, *On a Hodge theoretic property of the Künneth map in periodic cyclic homology*, J. Algebra, 446 (2016), pp. 132–153.

[512] ———, *On Hochschild invariants of Landau-Ginzburg orbifolds*, Adv. Theor. Math. Phys., 24 (2020), pp. 189–258. With an appendix by A. Basalaev and Shklyarov.

[513] B. Shoikhet, *Cohomology of the Lie algebras of differential operators: lifting formulas*, in Topics in quantum groups and finite-type invariants, vol. 185 of Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 1998, pp. 95–110.

[514] ———, *Lifting formulas. II*, Math. Res. Lett., 6 (1999), pp. 323–334.

[515] B. Shoikhet, *Integration of the lifting formulas and the cyclic homology of the algebras of differential operators*, Geom. Funct. Anal., 11 (2001), pp. 1096–1124.

[516] B. Shoikhet, *A proof of the Tsygan formality conjecture for chains*, Adv. Math., 179 (2003), pp. 7–37.

[517] ———, *A proof of the Tsygan formality conjecture for chains*, Adv. Math., 179 (2003), pp. 7–37.

[518] ———, *Tsygan formality and Duflo formula*, Math. Res. Lett., 10 (2003), pp. 763–775.

[519] ———, *Tsygan formality and Duflo formula*, Math. Res. Lett., 10 (2003), pp. 763–775.

[520] ———, *Tetramodules over a bialgebra form a 2-fold monoidal category*, Appl. Categ. Structures, 21 (2013), pp. 291–309.

[521] ———, *Differential graded categories and Deligne conjecture*, Adv. Math., 289 (2016), pp. 797–843.

[522] J. Simons and D. Sullivan, *The Atiyah Singer index theorem and Chern Weil forms*, Pure Appl. Math. Q., 6 (2010), pp. 643–645.

[523] A. Solotar and M. Vigué-Poirrier, *Dihedral homology of commutative algebras*, J. Pure Appl. Algebra, 109 (1996), pp. 97–106.

[524] ———, *Two classes of algebras with infinite Hochschild homology*, Proc. Amer. Math. Soc., 138 (2010), pp. 861–869.

[525] D. Sullivan, *Homotopy theory of the master equation package applied to algebra and geometry: a sketch of two interlocking programs*, in Algebraic topology—old and new, vol. 85 of Banach Center Publ., Polish Acad. Sci. Inst. Math., Warsaw, 2009, pp. 297–305.

[526] A. A. Suslin and M. Wodzicki, *Excision in algebraic* K*-theory and Karoubi's conjecture*, Proc. Nat. Acad. Sci. U.S.A., 87 (1990), pp. 9582–9584.

[527] ———, *Excision in algebraic* K*-theory*, Ann. of Math. (2), 136 (1992), pp. 51–122.

[528] ———, *Excision in algebraic* K*-theory*, Ann. of Math. (2), 136 (1992), pp. 51–122.

[529] G. Tabuada, *Invariants additifs de DG-catégories*, Int. Math. Res. Not., (2005), pp. 3309–3339.

[530] G. Tabuada, *Une structure de catégorie de modèles de Quillen sur la catégorie des dg-catégories*, C. R. Math. Acad. Sci. Paris, 340 (2005), pp. 15–19.

[531] G. Tabuada, *Differential graded versus simplicial categories*, Topology Appl., 157 (2010), pp. 563–593.

[532] ———, *Generalized spectral categories, topological Hochschild homology and trace maps*, Algebr. Geom. Topol., 10 (2010), pp. 137–213.

[533] ———, *Homotopy theory of dg categories via localizing pairs and Drinfeld's dg quotient*, Homology Homotopy Appl., 12 (2010), pp. 187–219.

[534] ———, *On Drinfeld's dg quotient*, J. Algebra, 323 (2010), pp. 1226–1240.

[535] ———, *A guided tour through the garden of noncommutative motives*, in Topics in noncommutative geometry, vol. 16 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2012, pp. 259–276.

[536] ———, *Products, multiplicative Chern characters, and finite coefficients via noncommutative motives*, J. Pure Appl. Algebra, 217 (2013), pp. 1279–1293.

[537] ———, *Bivariant cyclic cohomology and Connes' bilinear pairings in noncommutative motives*, J. Noncommut. Geom., 9 (2015), pp. 265–285.

[538] D. Tamarkin, *What do dg-categories form?*, Compos. Math., 143 (2007), pp. 1335–1358.

[539] ———, *What do dg-categories form?*, Compos. Math., 143 (2007), pp. 1335–1358.

[540] D. Tamarkin and B. Tsygan, *Cyclic formality and index theorems*, vol. 56, 2001, pp. 85–97. EuroConférence Moshé Flato 2000, Part II (Dijon).

[541] D. Tamarkin and B. Tsygan, *The ring of differential operators on forms in noncommutative calculus*, in Graphs and patterns in mathematics and theoretical physics, vol. 73 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2005, pp. 105–131.

[542] ———, *The ring of differential operators on forms in noncommutative calculus*, in Graphs and patterns in mathematics and theoretical physics, vol. 73 of Proc. Sympos. Pure Math., Amer. Math. Soc., Providence, RI, 2005, pp. 105–131.

[543] D. E. Tamarkin, *Operadic proof of M. Kontsevich's formality theorem*, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–The Pennsylvania State University.

[544] ———, *Operadic proof of M. Kontsevich's formality theorem*, ProQuest LLC, Ann Arbor, MI, 1999. Thesis (Ph.D.)–The Pennsylvania State University.

[545] J. Terilla and T. Tradler, *Deformations of associative algebras with inner products*, Homology Homotopy Appl., 8 (2006), pp. 115–131.

[546] R. W. Thomason and T. Trobaugh, *Higher algebraic* K*-theory of schemes and of derived categories*, in The Grothendieck Festschrift, Vol. III, vol. 88 of Progr. Math., Birkhäuser Boston, Boston, MA, 1990, pp. 247–435.

[547] B. Toën, *The homotopy theory of* dg*-categories and derived Morita theory*, Invent. Math., 167 (2007), pp. 615–667.

[548] ———, *Lectures on dg-categories*, in Topics in algebraic and topological K-theory, vol. 2008 of Lecture Notes in Math., Springer, Berlin, 2011, pp. 243–302.

[549] ———, *Lectures on dg-categories*, in Topics in algebraic and topological K-theory, vol. 2008 of Lecture Notes in Math., Springer, Berlin, 2011, pp. 243–302.

[550] ———, *Derived algebraic geometry*, EMS Surv. Math. Sci., 1 (2014), pp. 153–240.

[551] B. Toën and G. Vezzosi, *Brave new algebraic geometry and global derived moduli spaces of ring spectra*, in Elliptic cohomology, vol. 342 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2007, pp. 325–359.

[552] B. Toën and G. Vezzosi, *Chern character, loop spaces and derived algebraic geometry*, in Algebraic topology, vol. 4 of Abel Symp., Springer, Berlin, 2009, pp. 331–354.

[553] ———, *Algèbres simpliciales* S^1*-équivariantes, théorie de de Rham et théorèmes HKR multiplicatifs*, Compos. Math., 147 (2011), pp. 1979–2000.

[554] ———, *Caractères de Chern, traces équivariantes et géométrie algébrique dérivée*, Selecta Math. (N.S.), 21 (2015), pp. 449–554.

[555] T. Tradler, *Poincare duality induces a BV-structure on Hochschild cohomology*, ProQuest LLC, Ann Arbor, MI, 2002. Thesis (Ph.D.)–City University of New York.

[556] ———, *The Batalin-Vilkovisky algebra on Hochschild cohomology induced by infinity inner products*, Ann. Inst. Fourier (Grenoble), 58 (2008), pp. 2351–2379.

[557] T. Tradler, S. O. Wilson, and M. Zeinalian, *One more proof of the index formula for block Toeplitz operators*, J. Operator Theory, 76 (2016), pp. 171–174.

[558] T. Tradler and M. Zeinalian, *On the cyclic Deligne conjecture*, J. Pure Appl. Algebra, 204 (2006), pp. 280–299.

[559] ———, *On the cyclic Deligne conjecture*, J. Pure Appl. Algebra, 204 (2006), pp. 280–299.

[560] ———, *Algebraic string operations*, K-Theory, 38 (2007), pp. 59–82.

[561] ———, *Algebraic string operations*, K-Theory, 38 (2007), pp. 59–82.

[562] B. Tsygan, *Cyclic homology*, in Cyclic homology in non-commutative geometry, vol. 121 of Encyclopaedia Math. Sci., Springer, Berlin, 2004, pp. 73–113.

[563] ———, *On the Gauss-Manin connection in cyclic homology*, Methods Funct. Anal. Topology, 13 (2007), pp. 83–94.

[564] ———, *Noncommutative calculus and operads*, in Topics in noncommutative geometry, vol. 16 of Clay Math. Proc., Amer. Math. Soc., Providence, RI, 2012, pp. 19–66.

[565] B. L. Tsygan, *Homology of matrix Lie algebras over rings and the Hochschild homology*, Uspekhi Mat. Nauk, 38 (1983), pp. 217–218.

[566] V. G. Turaev, *Skein quantization of Poisson algebras of loops on surfaces*, Ann. Sci. École Norm. Sup. (4), 24 (1991), pp. 635–704.

[567] M. Van den Bergh, *Double Poisson algebras*, Trans. Amer. Math. Soc., 360 (2008), pp. 5711–5769.

[568] M. Vigué-Poirrier, *Cyclic homology and Quillen homology of a commutative algebra*, in Algebraic topology—rational homotopy (Louvain-la-Neuve, 1986), vol. 1318 of Lecture Notes in Math., Springer, Berlin, 1988, pp. 238–245.

[569] ———, *Sur l'algèbre de cohomologie cyclique d'un espace* 1*-connexe applications à la géometrie des variétés*, Illinois J. Math., 32 (1988), pp. 40–52.
[570] ———, *Homologie de Hochschild et homologie cyclique des algèbres différentielles graduées*, Astérisque, (1990), pp. 7, 255–267. International Conference on Homotopy Theory (Marseille-Luminy, 1988).
[571] ———, *Cyclic homology of algebraic hypersurfaces*, J. Pure Appl. Algebra, 72 (1991), pp. 95–108.
[572] ———, *Décompositions de l'homologie cyclique des algèbres différentielles graduées commutatives*, K-Theory, 4 (1991), pp. 399–410.
[573] ———, *Homologie cyclique des espaces formels*, J. Pure Appl. Algebra, 91 (1994), pp. 347–354.
[574] ———, *Homologie et* K*-théorie des algèbres commutatives: caractérisation des intersections complètes*, J. Algebra, 173 (1995), pp. 679–695.
[575] ———, *Finiteness conditions for the Hochschild homology algebra of a commutative algebra*, J. Algebra, 207 (1998), pp. 333–341.
[576] ———, *Hochschild homology criteria for trivial algebra structures*, Trans. Amer. Math. Soc., 354 (2002), pp. 3869–3882.
[577] ———, *Hochschild homology of finite dimensional algebras*, AMA Algebra Montp. Announc., (2003), pp. Paper 10, 5. Théories d'homologie, représentations et algèbres de Hopf.
[578] M. Vigué-Poirrier and D. Burghelea, *A model for cyclic homology and algebraic* K*-theory of* 1*-connected topological spaces*, J. Differential Geom., 22 (1985), pp. 243–253.
[579] C. Voigt, *Equivariant local cyclic homology and the equivariant Chern-Connes character*, Doc. Math., 12 (2007), pp. 313–359.
[580] ———, *Equivariant periodic cyclic homology*, J. Inst. Math. Jussieu, 6 (2007), pp. 689–763.
[581] ———, *Equivariant cyclic homology for quantum groups*, in K-theory and noncommutative geometry, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 151–179.
[582] ———, *A new description of equivariant cohomology for totally disconnected groups*, J. K-Theory, 1 (2008), pp. 431–472.
[583] ———, *Chern character for totally disconnected groups*, Math. Ann., 343 (2009), pp. 507–540.
[584] ———, *Cyclic cohomology and Baaj-Skandalis duality*, J. K-Theory, 13 (2014), pp. 115–145.
[585] A. A. Voronov and M. Gerstenkhaber, *Higher-order operations on the Hochschild complex*, Funktsional. Anal. i Prilozhen., 29 (1995), pp. 1–6, 96.
[586] C. Weibel, *Le caractère de Chern en homologie cyclique périodique*, C. R. Acad. Sci. Paris Sér. I Math., 317 (1993), pp. 867–871.
[587] ———, *Cyclic homology for schemes*, Proc. Amer. Math. Soc., 124 (1996), pp. 1655–1662.
[588] ———, *The Hodge filtration and cyclic homology*, K-Theory, 12 (1997), pp. 145–164.
[589] C. A. Weibel, *Nil* K*-theory maps to cyclic homology*, Trans. Amer. Math. Soc., 303 (1987), pp. 541–558.
[590] ———, *An introduction to homological algebra*, vol. 38 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 1994.
[591] C. A. Weibel and S. C. Geller, *étale descent for Hochschild and cyclic homology*, Comment. Math. Helv., 66 (1991), pp. 368–388.
[592] T. Willwacher, *Formality of cyclic chains*, Int. Math. Res. Not. IMRN, (2011), pp. 3939–3956.
[593] ———, *The homotopy braces formality morphism*, Duke Math. J., 165 (2016), pp. 1815–1964.
[594] M. Wodzicki, *Cyclic homology of differential operators*, Duke Math. J., 54 (1987), pp. 641–647.
[595] ———, *Cyclic homology of differential operators*, Duke Math. J., 54 (1987), pp. 641–647.
[596] ———, *Noncommutative residue. I. Fundamentals*, in K-theory, arithmetic and geometry (Moscow, 1984–1986), vol. 1289 of Lecture Notes in Math., Springer, Berlin, 1987, pp. 320–399.
[597] ———, *Noncommutative residue. I. Fundamentals*, in K-theory, arithmetic and geometry (Moscow, 1984–1986), vol. 1289 of Lecture Notes in Math., Springer, Berlin, 1987, pp. 320–399.
[598] ———, *Cyclic homology of differential operators in characteristic* $p > 0$, C. R. Acad. Sci. Paris Sér. I Math., 307 (1988), pp. 249–254.

[599] ———, *Cyclic homology of differential operators in characteristic* $p > 0$, C. R. Acad. Sci. Paris Sér. I Math., 307 (1988), pp. 249–254.

[600] ———, *Cyclic homology of pseudodifferential operators and noncommutative Euler class*, C. R. Acad. Sci. Paris Sér. I Math., 306 (1988), pp. 321–325.

[601] ———, *Cyclic homology of pseudodifferential operators and noncommutative Euler class*, C. R. Acad. Sci. Paris Sér. I Math., 306 (1988), pp. 321–325.

[602] ———, *The long exact sequence in cyclic homology associated with an extension of algebras*, C. R. Acad. Sci. Paris Sér. I Math., 306 (1988), pp. 399–403.

[603] ———, *The long exact sequence in cyclic homology associated with an extension of algebras*, C. R. Acad. Sci. Paris Sér. I Math., 306 (1988), pp. 399–403.

[604] ———, *Vanishing of cyclic homology of stable* C^**-algebras*, C. R. Acad. Sci. Paris Sér. I Math., 307 (1988), pp. 329–334.

[605] ———, *Excision in cyclic homology and in rational algebraic* K*-theory*, Ann. of Math. (2), 129 (1989), pp. 591–639.

[606] ———, *Excision in cyclic homology and in rational algebraic* K*-theory*, Ann. of Math. (2), 129 (1989), pp. 591–639.

[607] ———, *Homological properties of rings of functional-analytic type*, Proc. Nat. Acad. Sci. U.S.A., 87 (1990), pp. 4910–4911.

[608] ———, *Schematic cohomology of a topological space and the algebraic cyclic homology of* $C(X)$, C. R. Acad. Sci. Paris Sér. I Math., 310 (1990), pp. 129–134.

[609] ———, *Vestigia investiganda*, vol. 2, 2002, pp. 769–798, 806. Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[610] ———, *Vestigia investiganda*, Mosc. Math. J., 2 (2002), pp. 769–798, 806. Dedicated to Yuri I. Manin on the occasion of his 65th birthday.

[611] ———, *Algebras of* p*-symbols, noncommutative* p*-residue, and the Brauer group*, in Noncommutative geometry and global analysis, vol. 546 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2011, pp. 283–304.

[612] ———, *Algebras of* p*-symbols, noncommutative* p*-residue, and the Brauer group*, in Noncommutative geometry and global analysis, vol. 546 of Contemp. Math., Amer. Math. Soc., Providence, RI, 2011, pp. 283–304.