Notes on classifying spaces

Nerve of a small category $\mathcal{C}$:

$$N_n\mathcal{C} = \{ i_0 \xrightarrow{f_1} i_1 \to \dots \xrightarrow{f_n} i_n \}$$

d_0: $[i_1 \to \dots \to i_n]$

d_k: $[i_0 \to \dots \to i_{k-1} \to i_{k+1} \to \dots \to i_n]$

d_n: $[i_0 \to \dots \to i_{n-1}]$

s_k: $[i_0 \to \dots \to i_k = i_k \to \dots \to i_n]$

$N[n] \simeq \Delta^n$ where $[n] = (0 \to 1 \to \dots \to n)$; $N(\mathcal{C}\times\mathcal{D}) \simeq N\mathcal{C} \times N\mathcal{D}$

$$|N\mathcal{C}| =: B\mathcal{C}$$

Functor $\mathcal{C} \xrightarrow{F} \mathcal{D} \rightsquigarrow N(F): N\mathcal{C} \to N\mathcal{D}$

Morphism of functors $F \overset{\varphi}{\Rightarrow} G \rightsquigarrow$

homotopy
$$\begin{tikzcd} N\mathcal{C} \arrow[r, bend left, "NF"] \arrow[r, bend right, "NG"'] \arrow[r, phantom, "\Downarrow"] & N\mathcal{D} \end{tikzcd}$$

(b/c φ defines $\mathcal{C} \times [1] \to \mathcal{D}$)

Cor. $\mathcal{C} \rightleftarrows \mathcal{D}$ adjoint functors $\Longrightarrow$ $N\mathcal{F}, N\mathcal{D}$ are homotopically inverse

If $\mathcal{C}$ has an initial or final object then $N\mathcal{C} \simeq N[0] \simeq pt$

(b/c there is an adjoint functor to $[0] \to \mathcal{C}$).

$F: \mathcal{C} \to \mathcal{D} \quad d \in ob(\mathcal{D})$

Categories

$$\begin{tikzcd} F/d \arrow[r] \arrow[rr, bend right, "\mathrm{const}_d"'] & \mathcal{C} \arrow[r, "F"] \arrow[d, phantom, "\Downarrow"] & \mathcal{D} \\ & {} & \end{tikzcd}$$

F/d: objs $c;\ F(c) \to d$

mors
$$\begin{tikzcd} F(c) \arrow[rr] \arrow[dr] & & F(c') \arrow[dl] \\ & d & \end{tikzcd}$$

$$\begin{tikzcd} d\backslash F \arrow[r] \arrow[rr, bend right, "\mathrm{const}_d"'] & \mathcal{C} \arrow[r] \arrow[d, phantom, "\Uparrow"] & \mathcal{D} \\ & {} & \end{tikzcd}$$

$d\backslash F$: objs $c;\ F(c) \leftarrow d$

mors
$$\begin{tikzcd} F(c) \arrow[rr] & & F(c') \\ & d \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$F/\mathcal{D}$: objs $c;\ d;\ F(c) \to d$

mors
$$\begin{tikzcd} F(c') \arrow[r, bend left] & F(c) \arrow[d] & \\ & d \arrow[r] & d' \end{tikzcd}$$

$\mathcal{D}\backslash F$: objs $c;\ d;\ F(c) \leftarrow d$

mors
$$\begin{tikzcd} F(c') & F(c) \arrow[l, bend right] & \\ & d \arrow[u] & d' \arrow[l] \end{tikzcd}$$

Quillen: For $C \xrightarrow{F} D$, $B(D \backslash F) \simeq BC$

Idea: bisimplicial set

$$X_{pq} = \left\{ \begin{array}{c} d_q \to \cdots \to d_0 \\ \downarrow \\ F(c_0) \to F(c_1) \to \cdots \to F(c_p) \end{array} \right\}$$

$$\downarrow$$

$\mathrm{diag}(X) = N(D\backslash F)$ $\quad Y_{pq} = N_p C \ni c_0 \to c_1 \to \cdots \to c_p$

pre-image of each of those: $N(D \backslash F(c_0)) \simeq *$

"fibration w/ contractible fibers"

Actual proof: later.

Also:

$$\begin{array}{ccc} & X_{pq} & \\ \swarrow & & \searrow \\ N_p C & & N_q D^{op} \end{array}$$

$N(d_0 \backslash F)$ $=$ preimage of $[d_q \to \cdots \to d_0]$

Quillen's Thm A | IF $N(d_0 \backslash F)$ contractible for every $d_0 \in ob(D)$, we get $BC \simeq BD^{op} \simeq BD$

a bit more precisely, to see that this is induced by f (but by what else?..):

$$\begin{array}{ccccc} NC & \xleftarrow{\sim} & N(D\backslash F) & \xrightarrow{\sim} & ND^{op} \\ {\scriptstyle NF}\downarrow & & \downarrow & & \| \\ ND & \xleftarrow{\sim} & N(D\backslash id_D) & \xrightarrow{\sim} & ND^{op} \end{array}$$

Thm (Quillen) 1) Let $f: X_{*,\bullet} \to Y_{*,\bullet}$ be a morphism of bisimplicial sets.

If $X_{*,p} \to Y_{*,p}$ is a homotopy equivalence for every p then $X \to Y$ is a hom. eq.

2) Let $X: \mathcal{C} \to \mathrm{Top}$ be a functor. Let

$$i \longmapsto X_i \text{ on objects}$$

$$X(p) := \coprod_{i_0 \to \dots \to i_p} X_{i_0} \,;\; Y(p) = \coprod_{i_0 \to \dots \to i_p} pt = N\mathcal{C}$$

$\overset{x_0}{[i_0]}$ $\quad$ $\overset{x_0}{[i_0 \xrightarrow{f} i_1]}$ $\quad$ $\overset{x_0}{[i_0 \to i_1 \to i_2]}$

$\overset{f(x_0)}{[i_1]}$

(In other words: $X(*) = pt \overset{h}{\underset{\mathcal{C}}{\times}} X_\bullet$.)

$$\overset{x_0}{[i_0 \to \dots \to i_p]} \longmapsto [i_0 \to \dots \to i_p]$$

Assume that for any $i_0 \xrightarrow{f} i_1$ in $\mathcal{C}$, $X_{i_0} \underset{h.e.}{\xrightarrow{\sim}} X_{i_1}$

Then for any $i \in \mathrm{Ob}(\mathcal{C})$

$$X_i \to X(*) \to Y(*)$$

is a homotopy fiber sequence.

Given $F: \mathcal{C} \to \mathcal{D}$

coCartesian arrow

$c_1 \to c_2$:

$$\begin{tikzcd}
c_1 \arrow[r, bend left] \arrow[d, mapsto] & c_2 \arrow[d, mapsto] \\
d_1 \arrow[r] \arrow[ur, phantom, "\mapsto"] & d_2
\end{tikzcd}$$

s.t.

$$\begin{tikzcd}
c_1 \arrow[r, bend left] \arrow[rr, bend left] \arrow[d, mapsto] & c_2 \arrow[r, "\exists !"] \arrow[d, mapsto] & c_2' \arrow[d, mapsto] \\
d_1 \arrow[r] \arrow[rr, bend right] & d_2 \arrow[r] & d_2'
\end{tikzcd}$$

for any $c_2' \mapsto d_2'$, , giving same

$$\begin{tikzcd}
\mathcal{C}(c_2, c_2') \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{C}(c_1, c_2') \arrow[d] \\
\mathcal{D}(d_2, d_2') \arrow[r] & \mathcal{D}(d_1, d_2')
\end{tikzcd}$$

F is a coCartesian opfibration if any

$$\begin{tikzcd}
c_1 \arrow[d, mapsto] & \\
d_1 \arrow[r] & d_2
\end{tikzcd}$$

$\exists$ Cartesian arrow

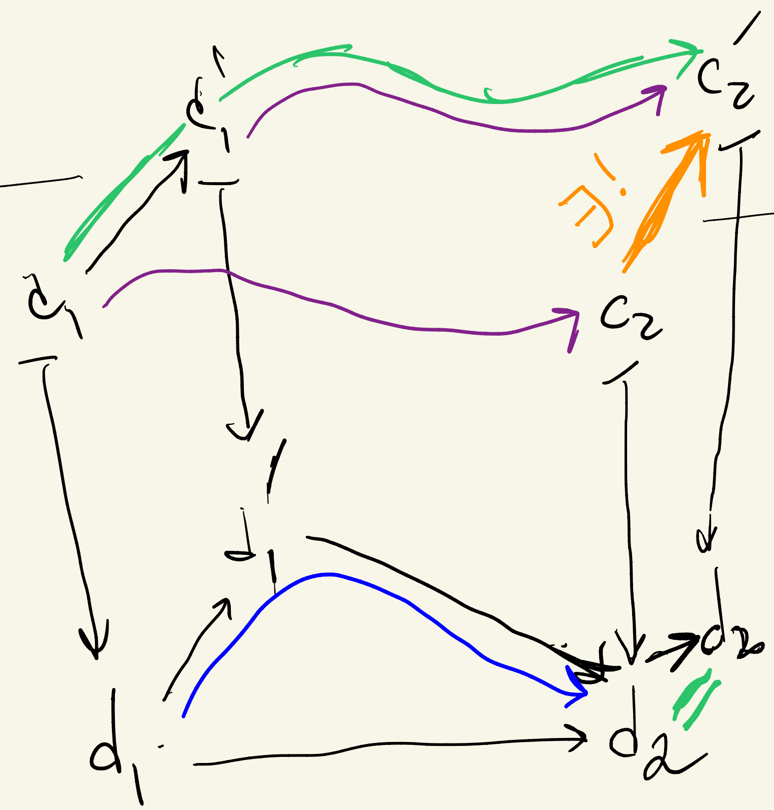
$$\begin{tikzcd}
c_1 \arrow[r, bend left] \arrow[d, mapsto] & c_2 \arrow[d, mapsto, "F"] \\
d_1 \arrow[r] \arrow[ur, phantom, "\mapsto"] & d_2
\end{tikzcd}$$

Note:

$$\begin{array}{ccc} \begin{tikzcd} c_1 \arrow[d] & \\ d_1 \arrow[r] & d_2 \end{tikzcd} & \rightsquigarrow & \begin{tikzcd} c_2 \arrow[d] \\ d_2 \end{tikzcd} \\ \cap & & \cap \\ Ob(F/d_2) & & Ob(F^{-1}(d_2)) \end{array}$$

Is this a functor? YES

$$P : F/d_2 \longrightarrow F^{-1}(d_2)$$



morphism in $F^{-1}(d_2)$

morphism in F/d_2

Fact: P is left adjoint to i

$$\begin{tikzcd} F/d_2 \arrow[r, bend left, "P"] & F^{-1}(d_2) \arrow[l, bend left, "i"] \end{tikzcd}$$

$$\begin{tikzcd}
 & & c_2' \arrow[d, mapsto] \\
c_1 \arrow[rru] \arrow[rr] \arrow[d, mapsto] & & c_2 \arrow[u, "\exists!"'] \arrow[d, mapsto] \\
d_1 \arrow[rr] & & d_2 = d_2
\end{tikzcd}$$

$$F^{-1}(d_2)\left(\begin{array}{c} c_2 \\ \downarrow \\ d_2 \end{array} , \begin{array}{c} c_2' \\ \downarrow \\ d_2 \end{array} \right)$$

$$\|$$

$$(F/d_2)\left(\begin{array}{c} c_1 \\ \downarrow \\ d_1 \to d_2 \end{array} , \begin{array}{c} c_2' \\ \downarrow \\ d_2 \end{array} \right)$$

CoCartesian fibrations vs pseudofunctors

1) Pseudofunctor $\mathcal{D} \to Cat$:

1) $(d \in Ob(\mathcal{D})) \longmapsto$ category $\mathcal{C}(d)$

2) $d_1 \xrightarrow{f} d_2$ in $\mathcal{D}$ $\longmapsto$ functor $\mathcal{C}(d_1) \xrightarrow{\phi(f)} \mathcal{C}(d_2)$

3) $d_1 \xrightarrow{f} d_2 \xrightarrow{g} d_3 \longmapsto$ natural transf.

$$
\begin{tikzcd}
\mathcal{C}(d_1) \arrow[r, "\phi(f)"'] \arrow[rr, bend left=50, "\phi(gf)"] \arrow[rr, phantom, bend left=25, "\Downarrow"] & \mathcal{C}(d_2) \arrow[r, "\phi(g)"'] & \mathcal{C}(d_3)
\end{tikzcd}
$$

<u>such that:</u>

$$
\begin{tikzcd}
\mathcal{C}(d_1) \arrow[r, "\phi(f)"'] \arrow[rrr, bend left=40, "\phi(hgf)"] \arrow[rrr, phantom, bend left=20, "\Downarrow = \Downarrow"] & \mathcal{C}(d_2) \arrow[r, "\phi(g)"'] & \mathcal{C}(d_3) \arrow[r, "\phi(h)"'] & \mathcal{C}(d_4)
\end{tikzcd}
$$

Given a coCartesian opfibration:

$$\mathcal{C} \xrightarrow{F} \mathcal{D}$$

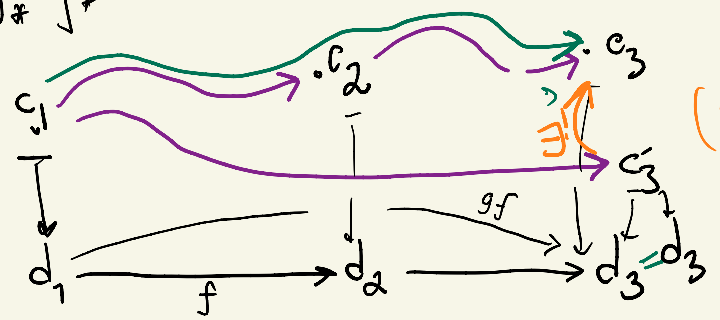
$\mathcal{C}(d) = F^{-1}(d)$

$f_* : \mathcal{C}(d_1) \to \mathcal{C}(d_2)$: $\quad \begin{bmatrix} c_1 & \\ \downarrow & \\ d_1 \longrightarrow d_2 \end{bmatrix} \longmapsto \begin{bmatrix} c_2 \\ \downarrow \\ d_2 \end{bmatrix}$

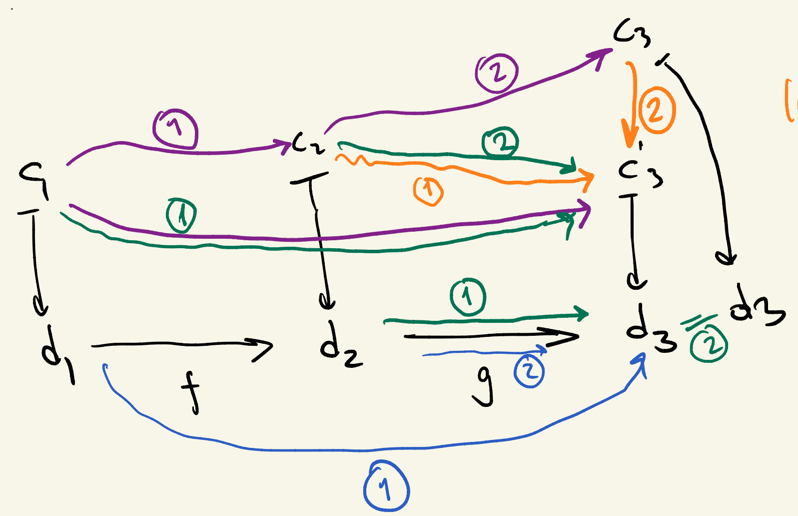
$$\begin{tikzcd} c_1 \arrow[r, bend left] \arrow[d, mapsto] & c_2 \arrow[d, mapsto] \\ d_1 \arrow[r] & d_2 \end{tikzcd}$$

$$f_* : \quad F^{-1}(d_1) \longleftrightarrow F/d_2 \xrightarrow{P} F^{-1}(d_2)$$

$(gf)_* \simeq g_* f_*$:



The inverse:
two steps ①, ②
using two
coCartesian
liftings



$(gf)_* \leftarrow g_* f_*$

Mutually inverse by uniqueness of ↑; by the same reason: $(hgf)_* \overset{\sim}{\underset{\sim}{\rightleftarrows}} h_* g_* f_*$

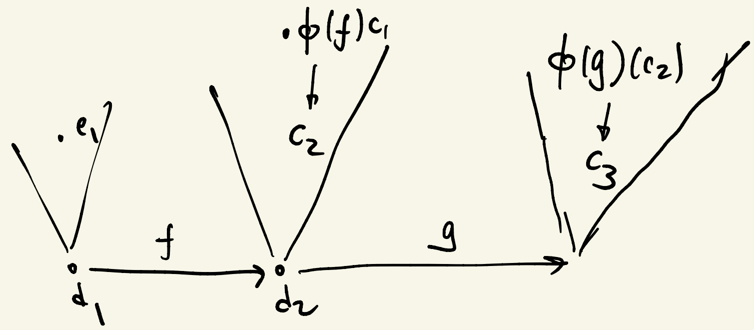
From pseudofunctors to co Cartesian fibrations:

$\phi : \mathcal{D} \to \mathcal{C}ats$

$ob(\mathbb{C}) = \{(d,c) \mid c \in Ob(\mathbb{C}(d))\}$

$(d_1, c_1) \to (d_2, c_2)$ in $\mathbb{C}$: $\quad d_1 \xrightarrow{f} d_2$ in $\mathcal{D}$; $\quad \phi(f)(c_1) \to c_2$ in $\mathbb{C}(d)$

Composition:

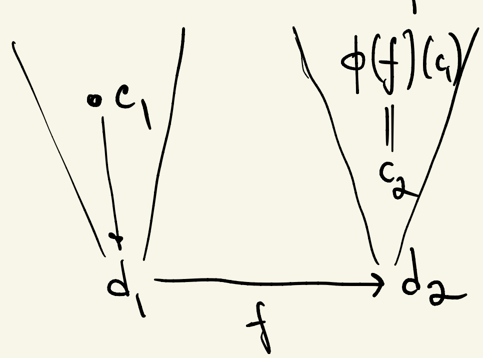


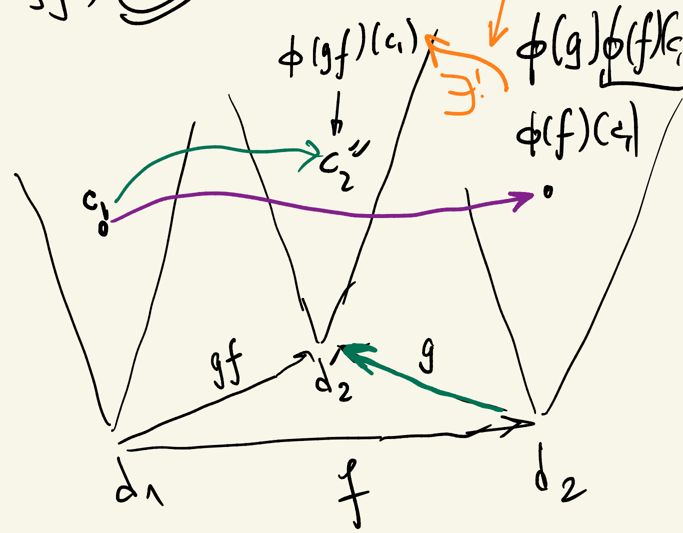
$$\begin{tikzcd} \phi(gf)(c_1) \arrow[r] \arrow[ddr] & \phi(g)\underbrace{\phi(f)(c_1)} \arrow[d] \\ & \underbrace{\phi(g)(c_2)} \arrow[d] \\ & c_3 \end{tikzcd}$$

need: $\phi(gf) \simeq \phi(g)\phi(f)$

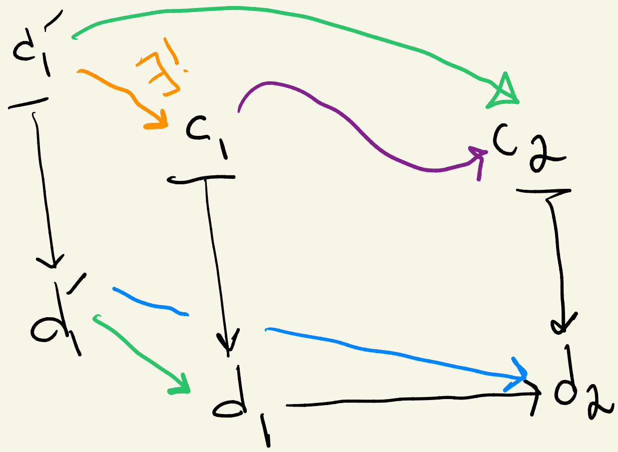
Associativity of composition:
from the condition $\quad \phi(hgf) \overset{=}{\rightrightarrows} \phi(h)\phi(g)\phi(f)$

The cocartesian lifting:





Cartesian arrow:

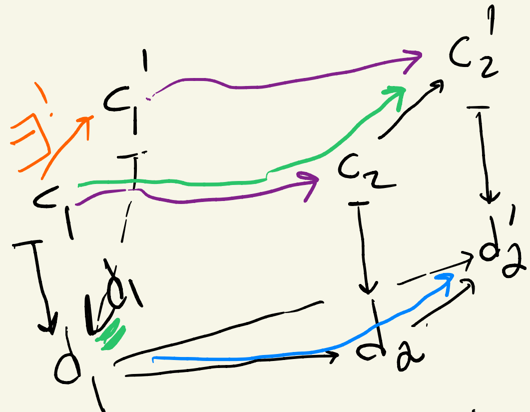


P: $\left[\begin{array}{ccc} & & c_2 \\ & & \downarrow F \\ d_1 & \to & d_2 \end{array}\right] \in \mathrm{Ob}(d_1 \backslash F)$

$\wr$

$\begin{array}{c} c_1 \\ \downarrow \\ d_1 \end{array} \in \mathrm{Ob}(F^{-1}(d_1))$

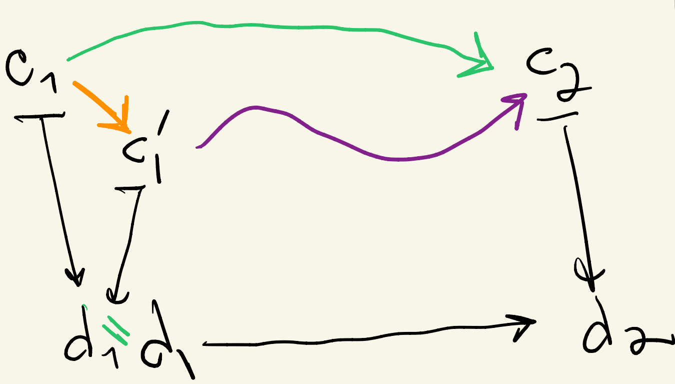
Functoriality:



$$\mathrm{Mor}_{F^{-1}(d_1)}\left(\begin{array}{c} c_1 \\ \downarrow \\ d_1 \end{array}, P\left[\begin{array}{ccc} & & c_2 \\ & & \downarrow F \\ d_1 & \longrightarrow & d_2 \end{array}\right]\right)$$

$$\wr\!\!\mid$$

$$\mathrm{Mor}_{d_1 \backslash F}\left(\begin{array}{c} c_1 \\ \downarrow \\ d_1 \end{array}, \left[\begin{array}{ccc} & & c_2 \\ & & \downarrow F \\ d_1 & \to & d_2 \end{array}\right]\right)$$



[P right adjoint to i]

Contravariant pseudofunctor $\mathcal{D} \to \mathrm{Cats}$:

$(d \text{ in } Ob(\mathcal{D})) \rightsquigarrow$ category $\mathcal{C}(d)$

$(f: d_1 \to d_2 \text{ in } \mathcal{D}) \rightsquigarrow$ functor $\phi(f): \mathcal{C}(d_1) \leftarrow \mathcal{C}(d_2)$

$(d_1 \xrightarrow{f} d_2 \xrightarrow{g} d_3) \rightsquigarrow$ isom of functors

$$\begin{tikzcd} \cdot & \cdot \arrow[l, "\phi(f)"'] & \cdot \arrow[l, "\phi(g)"'] \arrow[ll, bend right, "\phi(gf)"'] \arrow[l, phantom, shift right=4, "\Downarrow z"] \end{tikzcd}$$

such that

$$\begin{tikzcd} d_1 & d_2 \arrow[l] & d_3 \arrow[l] & d_4 \arrow[l] \arrow[lll, bend right] \arrow[ll, phantom, shift right=4, "\Downarrow = \Downarrow"] \end{tikzcd}$$

Quillen's Thm B Given $F: \mathcal{C} \to \mathcal{D}$, assume that for any $d_1 \xrightarrow{f} d_2$ in $\mathcal{D}$,

$$d_1 \backslash F \leftarrow d_2 \backslash F$$

induces h.e. on B.

Then $\forall d \in ob(\mathcal{D})$:

$$B(d \backslash F) \to B\mathcal{C} \to B\mathcal{D}$$

is a homotopy fibration sequence.

Pf Again, the bisimplicial set

$$X_{pq} = \left\{ \begin{tikzcd} d_q \arrow[r] & \cdots \arrow[r] & d_1 \arrow[r] & d_0 \arrow[d] & & & \\ & & & F(c_0) \arrow[r, bend right] & F(c_1) \arrow[r, bend right] & \cdots \arrow[r, bend right] & F(c_p) \end{tikzcd} \right\}$$

(Pre-image of $d_q \to \cdots \to d_0$) = Nerve$(d_0 \backslash F)$

By **Quillen's Thm** 1), 2): $D\backslash F \simeq X$

$$B\left(d\backslash F \longrightarrow X \longrightarrow D^{op} \right)$$

a homotopy fiber sequence.

$B(D\backslash F) \simeq BC$ — the usual

$B(d\backslash F)$

$$\begin{tikzcd}
d\backslash F \arrow[r] \arrow[d] & D\backslash F \arrow[r] \arrow[d] & D^{op} \arrow[d, phantom, "\wr"] \\
d\backslash D \arrow[r] \arrow[d, phantom, "\wr"] & D\backslash D \arrow[r, "\sim"] & D^{op} \\
*
\end{tikzcd}$$

Corollary Given a Cartesian fibration s.t.

$f^* : F^{-1}(d_1) \leftarrow F^{-1}(d_2)$

induce h.e. on B.

Then $\forall d$

$$B\left(F^{-1}(d) \to C \to D \right)$$

is a homotopy fiber sequence.

Example of a coCartesian fibration

Given $\mathcal{C}$.

$Ar(\mathcal{C}) = Tw(\mathcal{C})$:

objects: $X \xrightarrow{f} Y$ in $\mathcal{C}$

morphisms:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d] \\ X' \arrow[u] \arrow[r] & Y \end{tikzcd}$$

$$\begin{tikzcd} & (X \to Y) \arrow[dll, bend right] \arrow[drr, bend left] & \\ X^{op} & Tw(\mathcal{C}) \arrow[d] \arrow[dr] & Y \\ & \mathcal{C}^{op} & \mathcal{C} \end{tikzcd}$$

Composition:

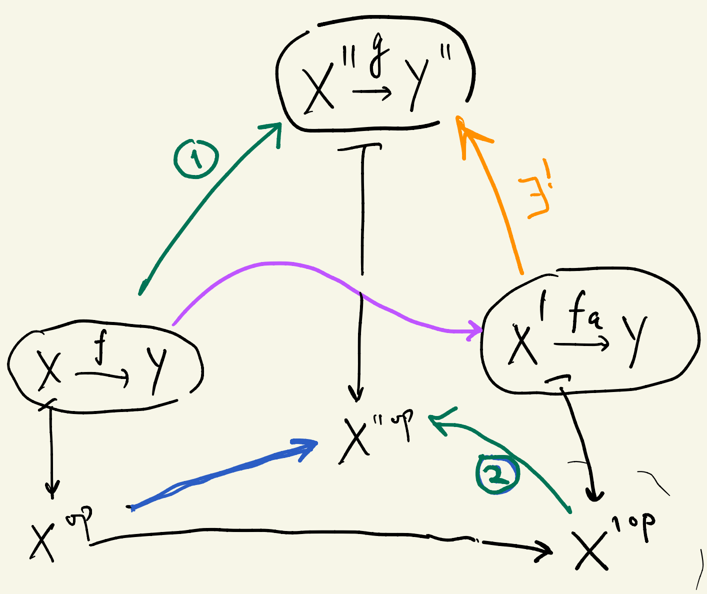
$$\begin{tikzcd} X \arrow[r] & Y \arrow[d] \arrow[dd, bend left] \\ X' \arrow[u] & Y' \arrow[d] \\ X'' \arrow[u] \arrow[uu, bend left] & Y'' \end{tikzcd}$$

Claim:

$$\begin{tikzcd} Tw(\mathcal{C}) \arrow[r] \arrow[d] \arrow[dr] & \mathcal{C}^{op} \\ \mathcal{C} & \mathcal{C}^{op} \times \mathcal{C} \end{tikzcd}$$

are cocartesian fibrations.

Pf



$$X \xleftarrow{a} X' \xleftarrow{b} X''$$

2

1

2

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d, "?"] \\ X'' \arrow[u, "ab"] \arrow[r, "g"'] & Y'' \end{tikzcd} \longleftrightarrow \begin{tikzcd} X' \arrow[r, "fa"] & Y \arrow[d, "?"] \\ X'' \arrow[u, "b"] \arrow[r, "g"'] & Y'' \end{tikzcd}$$

$\exists!$