Monoidal category $\mathcal{C}$

categ $\mathcal{M}$; functor $\otimes : \mathcal{M}\times\mathcal{M}\to\mathcal{M}$; . $\mathbb{1}\in \mathrm{Ob}(\mathcal{M})$

$$(X\otimes Y)\otimes Z \xrightarrow[\sim]{\Phi_{X,Y,Z}} X\otimes(Y\otimes Z)$$

$$\mathbb{1}\otimes X \underset{\lambda_X}{\xrightarrow{\sim}} X \underset{\rho_X}{\xleftarrow{\sim}} X\otimes\mathbb{1}$$

<u>such that:</u>

$$\begin{tikzcd}
(X\otimes\mathbb{1})\otimes Y \arrow[rr, "\Phi_{X,\mathbb{1},Y}", "\sim"'] \arrow[dr, "\sim", "\lambda_X\otimes \mathrm{id}_Y"'] & \arrow[d, phantom, "(\text{triangle})"] & X\otimes(\mathbb{1}\otimes Y) \arrow[dl, "\sim"', "\mathrm{id}_X\otimes\rho_Y"] \\
& {} &
\end{tikzcd}$$

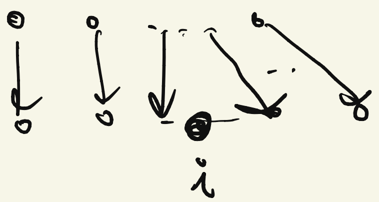
and

$$\begin{tikzcd}
((X\otimes Y)\otimes Z)\otimes U \arrow[rr, "\sim"] \arrow[d, "\sim"'] & & (X\otimes(Y\otimes Z))\otimes U \arrow[d, "\sim"] \\
(X\otimes Y)\otimes(Y\otimes Z) & (\text{pentagon}) & X\otimes((Y\otimes Z)\otimes U) \arrow[dl, "\sim"] \\
& X\otimes(Y\otimes(Z\otimes U)) \arrow[ul, "\sim"] &
\end{tikzcd}$$

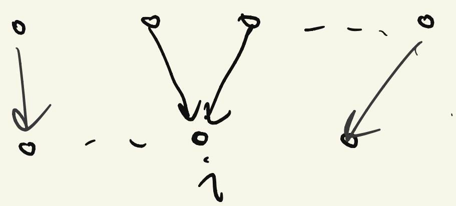
Monoidal categories as coCartesian fibrns:

Δ : $\quad ob(\Delta) = \{[n] \mid n \geq 0\}$ $\quad [n] = \{0, 1, \dots, n\}$

$\Delta([m], [n]) = \{\text{non-decreasing maps}\}$



$d_i : [n] \to [n+1]$ in Δ



$s_i : [n] \to [n-1]$ in Δ

Given $(\mathcal{M}, \otimes, \mathbb{1})$:

$\mathcal{M}^{\otimes n}$: objects: $[n]; M_1, \dots, M_n \in ob(\mathcal{M})$

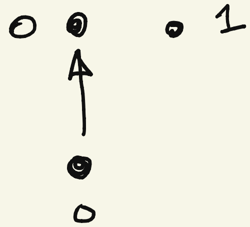
<u>Morphisms</u>

$$\begin{array}{c} ([n]; M_1, \dots, M_n) \\ \downarrow (\alpha; \{f_i\}) \\ ([m]; L_1, \dots, L_m) \end{array}$$

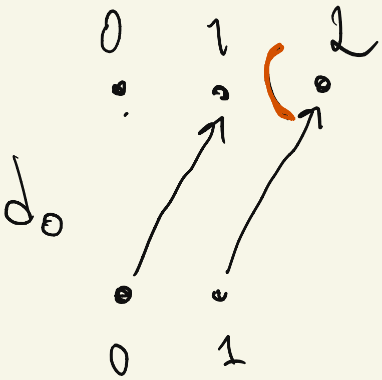
<u>where</u> $\alpha : [m] \to [n]$ <u>in Δ</u>

$f_i : M_{\alpha(i-1)+1} \otimes \dots \otimes M_{\alpha(i)} \to L_i$

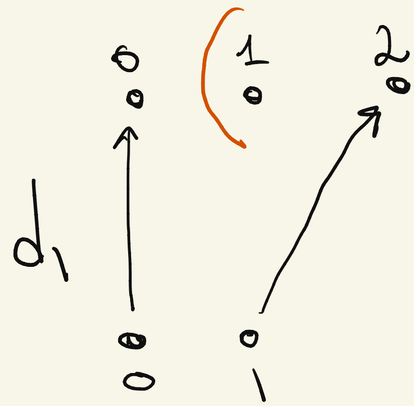
Examples



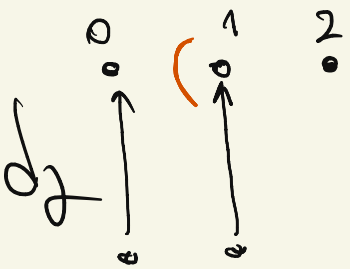
M_1 empty data



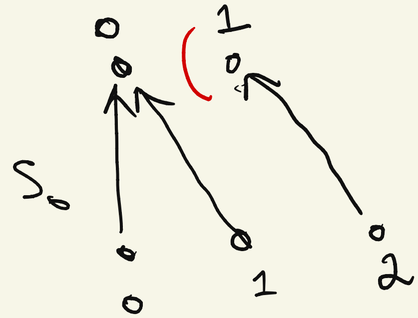
$M_2 \to L_1$



$M_1 \otimes M_2 \to L_1$



$M_1 \to L_1$



$M_1 \to L_2$

forgetting functor: $\mathcal{M}^{\otimes} \longrightarrow \Delta^{op}$

In the language of pseudofunctors:

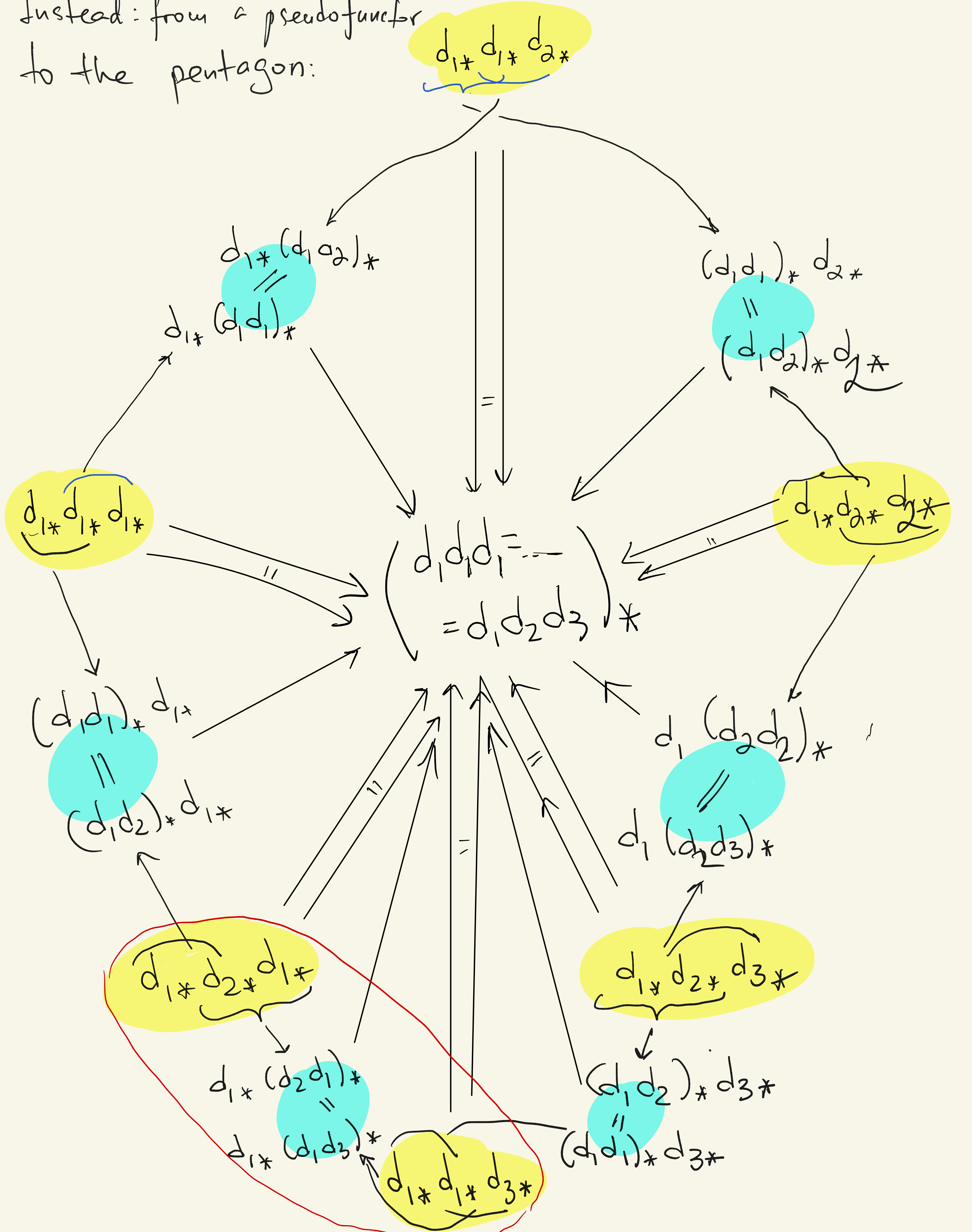
$\Delta^{op} \to Cat$; $[n] \mapsto \mathcal{M}^{\times n}$; for $\alpha \in \Delta([m],[n])$

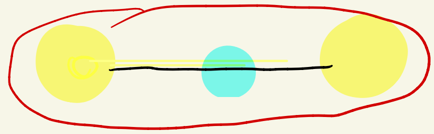
$\alpha_* : \mathcal{M}^{\times n} \longrightarrow \mathcal{M}^{\times m}$; $(M_1, \dots, M_n) \mapsto (L_1, \dots, L_m)$

$$L_i = M_{\alpha(i-1)+1} \otimes \dots \otimes M_{\alpha(i)}$$

(<u>not</u> done here: $c(\alpha, \beta) : \alpha_* \beta_* \Rightarrow (\alpha\beta)_*$)

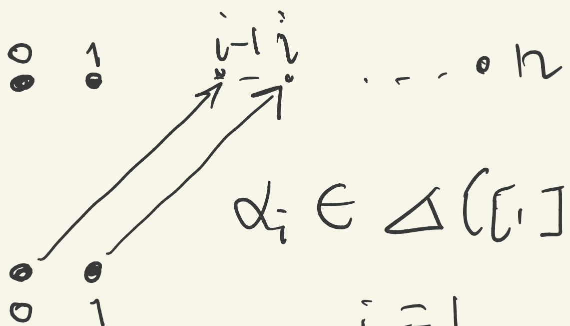
Instead: from a pseudofunctor to the pentagon:



What we get is a dodecagon (12). But one of the six  edges is identical:

$$d_{1*}d_{2*}d_{1*}(M_1,\dots,M_4) = (M_1\otimes M_2)\otimes(M_3\otimes M_4) = d_{1*}d_{3*}d_{1*}(M_1,\dots,M_4)$$

Characterization of monoidal categories in these terms:



$\alpha_i \in \Delta([1],[n])$ $\quad (\alpha_{1*},\dots,\alpha_{n*}) : M[n]\to M[1]^n$

$i=1,\dots,n$

Fact: Monoidal categories

$\updownarrow$

pseudofunctors $\Delta^{op}\to \mathcal{C}ats$ such that

$$(\alpha_{1*},\dots,\alpha_{n*}) : M[n] \xrightarrow{\sim} M[1]^{\times n}.$$

Symmetric monoidal categories

A monoidal category $(\mathcal{S}, \otimes, \phi, \lambda, \rho, \mathbb{1})$

together with

$$T_{X,Y} : X \otimes Y \xrightarrow{\sim} Y \otimes X$$

(natural transformation) such that:

$$T_{X,Y} \cdot T_{Y,X} = id_{Y \otimes X}$$

$$\begin{tikzcd}
(X \otimes Y) \otimes Z \arrow[r, "\phi"] \arrow[d, "T \otimes id_Z"] & X \otimes (Y \otimes Z) \arrow[r, "T"] & (Y \otimes Z) \otimes X \arrow[d, "\phi"] \\
(Y \otimes X) \otimes Z \arrow[r, "\phi"] & Y \otimes (X \otimes Z) \arrow[r, "id_Y \otimes T"] & Y \otimes (Z \otimes X)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{1} \otimes X \arrow[rr, "T_{1,X}"] \arrow[dr, "\sim", "\rho_X"'] & & X \otimes \mathbb{1} \arrow[dl, "\sim"', "\lambda_X"] \\
& X &
\end{tikzcd}$$

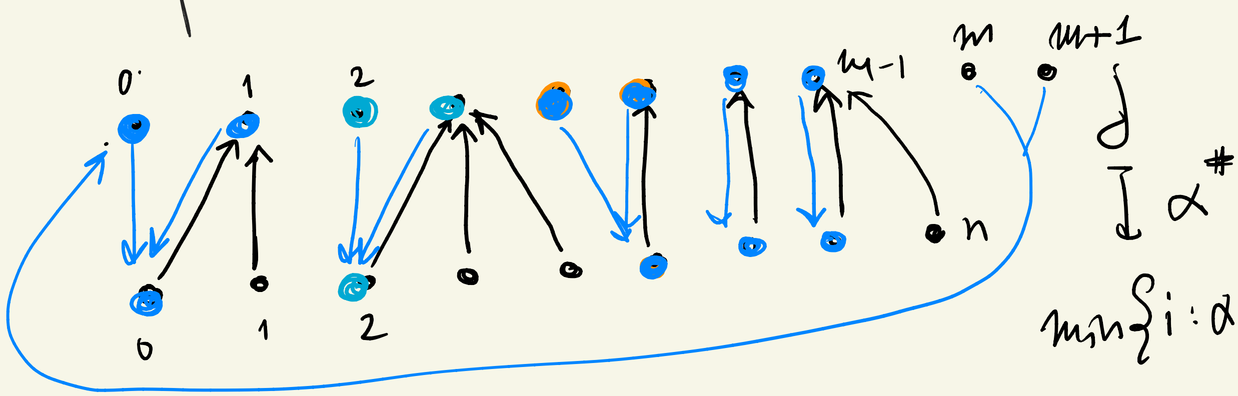
In terms of pseudofunctors: $\mathrm{Fin}_* \to \mathrm{Cats}$

Fin_*: objs: $[n] = \{0, 1, \ldots, n\}$

$\mathrm{Fin}_*([n],[m]) = \{\text{maps } [n] \to [m] \mid 0 \mapsto 0\}$

Functor $\Delta^{op} \to \mathrm{Fin}_*$: $[n] \mapsto [n]$

On morphisms: $\alpha \in \Delta([m],[n])$



$\min\{i : \alpha(i) \geq j\}$

(min of the empty set) $= 0$.

Rmk To see this better: periodize our maps.

$\Lambda([m],[n]) = \{$ monotonous maps $\mathbb{Z} \xrightarrow{f} \mathbb{Z}$:

$\alpha(i + m+1) = \alpha(i) + n+1 \}/\sim$

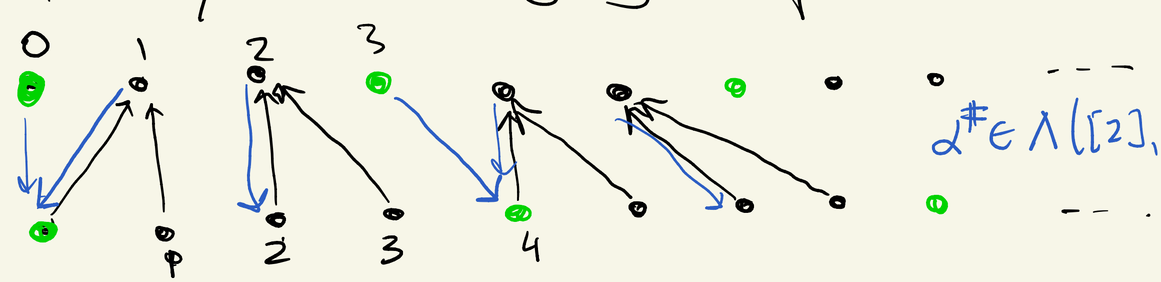
$\alpha(-) \sim \alpha(- + m+1)$

This is the cyclic category of Connes.

$\alpha \in \Delta([3],[2])$

$\cap$

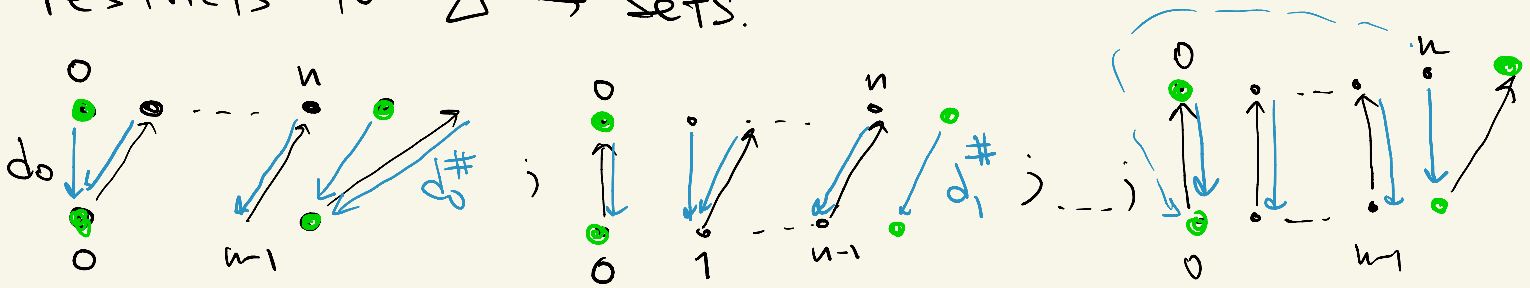
$\Lambda([3],[2])$



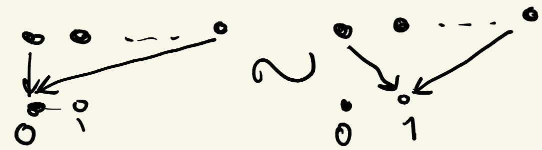
$\alpha^{\#} \in \Lambda([2],[3])$

$\Lambda \xrightarrow{\sim} \Lambda^{op}$; $\alpha \mapsto \alpha^{\#}$ as above.

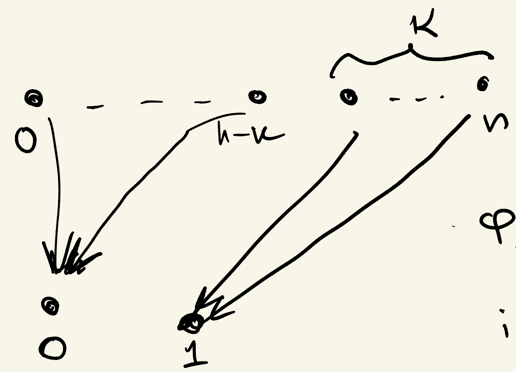
Rmk The tautological functor $Fin_* \to Sets$
$[n] \mapsto \{0,1,\dots,n\}$
restricts to $\Delta^{op} \to Sets$.



Compare: $\Delta^1[n] = \Delta([n],[1])$; $\mathbb{T}[n] = \Delta([n],[1])/\sim$



This is the standard simplicial circle.



$\varphi_k \longleftrightarrow k \in \{0,1,\dots,n\}$
identifies $\mathbb{T}$
with the tautological
Δ^{op}-set $[n] \mapsto \{0,\dots,n\}$.

Rmk For a commutative ring A:

$$A^{\otimes \mathbb{T}} : [n] \mapsto A^{\otimes \mathbb{T}[n]} \simeq A^{\otimes n+1}$$

is a simplicial ring

as we see from $d_\cdot^{\#}$ above: it is the standard simplicial model for Hochschild homology.

Action of an SMC on a category

$$\mathcal{S} \times \mathcal{X} \to \mathcal{X} \qquad s, x \longmapsto s \square x$$

$$\psi_{s_1, s_2, x} : (s_1 \otimes s_2) \square x \xrightarrow{\sim} s_1 \square (s_2 \square x)$$

$$\lambda_x : \quad \mathbb{1} \square x \xrightarrow{\sim} x$$

Subject to: pentagon for s_1, s_2, s_3, x;
triangle for $s, \mathbb{1}, x$

Category $\langle \mathcal{S}, \mathcal{X} \rangle$

Motivation:

$S \times X \to X$ monoid acting on a set (abelian)

$s, x \longmapsto s + x$

$S \backslash X = \{(s, x)\}/\sim \qquad X/\sim \qquad x \sim s + x, \exists s \in S$ (equiv rel gen by)

$S \backslash (\underbrace{S \times X}_{\text{diag action}}) : \qquad (s, x) \sim (u + s, u + x) \quad u \in S$

$\overset{\text{a.k.a.}}{=} -s + x$

In particular: $S \backslash (S \times S)$ (diagonal) $\qquad (s_1, s_2)$ a.k.a. $s_2 - s_1$

monoid structure:

$S \backslash (S \times S) = K_0(S) =$ the abelian grp envelope of S.

Now $\mathcal{S} \times \mathcal{X} \to \mathcal{X}$:

$\langle \mathcal{S}, \mathcal{X} \rangle$: $Ob(\langle \mathcal{S}, \mathcal{X} \rangle) = Ob(\mathcal{X})$

$\mathrm{Morp}_{\langle \mathcal{S}, \mathcal{X} \rangle}(x, y)$: (s, f) $s \in ob(\mathcal{S})$; $f : s + x \to y$

up to $\sim$

$$\begin{tikzcd}
s+x \arrow[d, "\wr"'] \arrow[d, shift left=4, "id_x"] \arrow[dr] & \\
t+x \arrow[r] & y
\end{tikzcd}$$

Composition:

$$x \xrightarrow{(s,f)} y \xrightarrow{(t,g)} z$$

$$\underbrace{(t+s)}_{\in} + x \cong t + \underset{z}{(s+x)} \longrightarrow t+y \longrightarrow z$$

$$\begin{tikzcd}
x \arrow[r, "{(s,f)}"] \arrow[rr, bend right] & y \arrow[r, "{(t,g)}"] & z \arrow[r, "{(u,h)}"] & w
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "{(s,f)}"'] \arrow[rr, bend right] & y \arrow[r, "{(t,g)}"'] & z \arrow[r, "{(u,h)}"'] & w
\end{tikzcd}$$

$$\begin{tikzcd}
(t+s);\ (t+s)+x \arrow[r] & t+(s+x) \arrow[d] \\
& t+y \arrow[d] \\
& z
\end{tikzcd}$$

$$\begin{tikzcd}
u+(t+s);\ (u+(t+s))+x \arrow[d] & \\
u+((t+s)+x) \arrow[r] & u+(t+(s+x)) \arrow[d] \\
& u+(t+y) \arrow[d] \\
& u+z \arrow[r] & w
\end{tikzcd}$$

$$\begin{tikzcd}
x \arrow[r, "{(s,f)}"'] & y \arrow[r, "{(t,g)}"'] \arrow[rr, bend right] & z \arrow[r, "{(u,h)}"'] & w
\end{tikzcd}$$

$$\begin{tikzcd}
((u+t);(u+t)+y \arrow[r, "\simeq"] & u+(t+y) \arrow[d] \\
& u+z \arrow[r] & w
\end{tikzcd}$$

$$\begin{tikzcd}
(u+t)+s;\ ((u+t)+s)+x \arrow[d] \\
(u+t)+(s+x) \arrow[d] \\
u+(t+(s+x)) \arrow[d]
\end{tikzcd}$$

By the pentagon for $\mathcal{S}, \mathcal{X}$:

$$\begin{tikzcd}
(u+(t+s); (u+(t+s))+x \arrow[d, "\phi\,2"'] \arrow[r, "\cdots"] & u) \arrow[d, equal] \\
((u+t)+s; ((u+t)+s)+x \arrow[r, "\cdots"] & u)
\end{tikzcd}$$

$2\,\phi$ $\quad$ id

commutes

More precisely:

$$\begin{tikzcd}
(u+(t+s)) \arrow[d, "\phi_{u,t,s}", "\sim"'] \arrow[rr] & & {} \arrow[d] \\
(u+t)+s \arrow[d] & \text{for } (\mathcal{S},\mathcal{X} & {} \\
{} \arrow[rr] & & u+(t+(s+x)) \arrow[dr] \\
& & & u
\end{tikzcd}$$

So: $\langle \mathcal{S}, \mathcal{X} \rangle$ a category

$$\mathcal{S}^{-1}\mathcal{X} := \langle \mathcal{S}, \mathcal{S}\times\mathcal{X} \rangle$$

Assumption: 1) All morphisms in $\mathcal{S}$ are isomorphisms

2) $\forall A$ in $\mathcal{S}$: $X \mapsto A+X$ is faithful

In which case: A) For $(s,f): X \to Y$ in $\langle \mathcal{S}, X \rangle$

s is defined up to unique isom

$$\begin{tikzcd} S+X \arrow[rr, "{(g, id_X)}", "\sim"'] \arrow[dr, "f"'] & & S+X \arrow[dl, "f"] \\ & & \end{tikzcd} \qquad (g, id_X) = id_{S+X} \overset{2)}{\Longrightarrow} g = id_S$$

B) $\langle \mathcal{S}, \mathcal{S} \rangle$ is contractible

unique morphism $0 \to s$ $\forall s$:

$$0+s \xrightarrow[id_s]{\sim} s$$

C) $\mathcal{S}^{-1}X = \langle \mathcal{S}, \mathcal{S} \times X \rangle$

$$\begin{tikzcd} \mathcal{S}^{-1}X \arrow[d, "\rho"'] \\ \langle \mathcal{S}, \mathcal{S} \rangle \end{tikzcd}$$

proj

is a coCartesian fibration

$\forall s \in ob(\mathcal{S})$: $\rho^{-1}(s) \simeq X$; for a morphism $(u,f): s_1 \to s_2$ in $\langle \mathcal{S}, \mathcal{S} \rangle$

$(u,f)_*: X \to X$ is $? \to ? + u$

D) $\mathcal{S}$ acts on $\underbrace{X \text{ invertibly}}$ $\Leftrightarrow$ $X \underset{h.e.}{\xrightarrow{\sim}} \mathcal{S}^{-1}X$

by def: $x \mapsto x+u$ induces h.e. on $B(X)$

Pf $\Leftarrow$: $\mathcal{S}$ acts on $\mathcal{S}^{-1}X$ invertibly.

$\Rightarrow$: from Quillen's Thm B (all $(u,f)_*$ are hom. eq.;

$\langle \mathcal{S}, \mathcal{S} \rangle$ contractible; $(u,f)_*^{-1}(c) \to \mathcal{S}^{-1}X \to *$

is a homotopy fibration). $\quad \overset{\wr}{X}$

Now: a bisimplicial set as before:

$$(p,q) \rightsquigarrow \left\{ \begin{tikzcd} p(F_0) \arrow[r, bend left] & p(F_1) & \cdots & p(F_{q-1}) \arrow[r, bend left] & p(F_q) \arrow[d] \\ & & & & B_0 \arrow[r] & B_1 \arrow[r] & \cdots \arrow[r] & B_p \end{tikzcd} \right\} = X_{p,q}$$

$\mathbb{Z} X_{*,\bullet}$: $\quad E^0_{pq} = \mathbb{Z} X_{p,q}$

(chains)

1) differential in p direction:

chains of $N(\underbrace{p(F_q)}_{\text{contractible}} / \langle S, S \rangle)$

$$E^2 = E^\infty = H_\bullet(S^{-1}X)$$

2) differential in q direction:

get (homology of $p^{-1}(B_0)$) $\simeq$ (homology of X)

$$E^2_{pq} = H_p(\langle S, S \rangle; H_q(X, \mathbb{Z})) \Rightarrow H^{p+q}(S^{-1}X)$$

To that, apply $\mathbb{Z}[(\pi_0 S)^{-1} \pi_0 S] \otimes_{\mathbb{Z}[\pi_0 S]} -$:

(which is exact)

$$E^2_{pq} = H_p\left(\langle S, S\rangle ; (\pi_0 S)^{-1} H_q(X)\right)$$

$$\Downarrow$$

$$(\pi_0 S)^{-1} H_{p+q}(S^{-1}X)$$

$$\shortparallel$$

$$H_{p+q}(S^{-1}X)$$

homology of a contractible space w/ coeffs in a local system.

Therefore

$$\pi_0(S)^{-1} H_\cdot(X, \mathbb{Z}) \simeq H_\cdot(S^{-1}X, \mathbb{Z})$$