Exact categories

$\mathcal{C} \subset \mathcal{A}$ full additive subcategory $\mathcal{C}$ of an Abelian category $\mathcal{A}$, closed under extensions

Admissible mono/epi in $\mathcal{C}$:

$X' \rightarrowtail X$
$X \twoheadrightarrow X''$
part of a short exact sequence

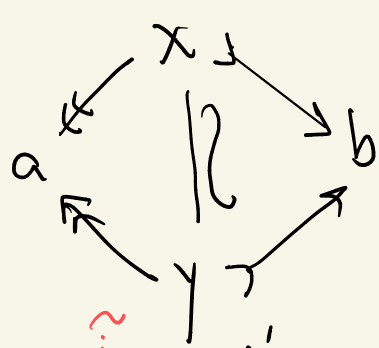
$0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$

Category $Q\mathcal{C}$

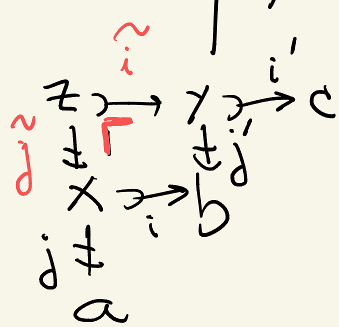
$\mathrm{Ob}\, Q\mathcal{C} = \mathrm{Ob}\,\mathcal{C}$; $\mathrm{Mor}_{Q\mathcal{C}}(a,b) =$

$= \left\{ \text{iso classes of } a \overset{j}{\twoheadleftarrow} X \overset{i}{\rightarrowtail} b \;\; (\text{admissible } j, i) \right\}$

isomorphism:



Composition:



$Z \xrightarrow{j\tilde{j}} a$ admissible: $Z \in \mathcal{C}$ since $\ker(\tilde{j}) = \ker(j')$; $\ker(j\tilde{j})$ is an extension of $\ker j$ by $\ker \tilde{j}$, both in $\mathcal{C}$.

Two subcategories:

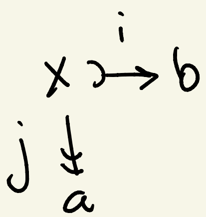
$\{\rightarrowtail\}$ and $\{\twoheadrightarrow\}^{op}$

$i: a \rightarrowtail b$ $\quad i_! : a \to b$ in $\mathcal{QC}$

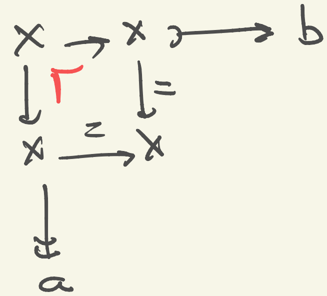
$\qquad\qquad\qquad\quad \| \quad a \xleftarrow{=} a \rightarrowtail b$

$j: a \twoheadleftarrow b$ $\quad j^! : a \to b$ in $\mathcal{QC}$

$\qquad\qquad\qquad\quad a \twoheadleftarrow b \xrightarrow{=} b$

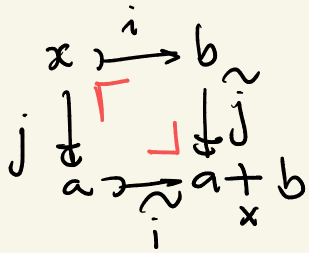


is the composition $a \xrightarrow{j^!} x \xrightarrow{i_!} b$

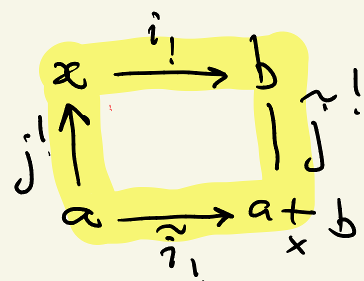


✓

But also:



(bicartesian)

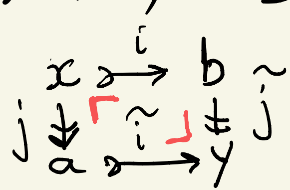


Commutes in $\mathcal{QC}$

Get a characterization of $\mathcal{QC}$:

- Subcat of $i_!$, $\xrightarrow{i}$; • Subcat of $j^!$, $\overset{j}{\twoheadrightarrow}$;

Rel: $\tilde{j}^! i_! = \tilde{i}_! j^!$, $\forall$



bicartesian square in $\mathcal{C}$

$K_0(\mathcal{C}) \to \pi_1 Q\mathcal{C}$

For $x_1 \overset{i}{\rightarrowtail} x$ in $\mathcal{C}$: $i_! : x_1 \to x$ in $Q\mathcal{C}$

$$\begin{array}{ccc} x_1 & \rightarrowtail & x \\ \big\downarrow{\scriptstyle =} & & \\ x_1 & & \end{array}$$

For $x_1 \overset{q}{\twoheadleftarrow} x$ in $\mathcal{C}$: $q^! : x_1 \to x$ in $Q\mathcal{C}$:

$$\begin{array}{ccc} x & \overset{=}{\rightarrowtail} & x \\ \big\downarrow & & \\ x_1 & & \end{array}$$

For M in $Ob(\mathcal{C})$:

$(M) =$ 

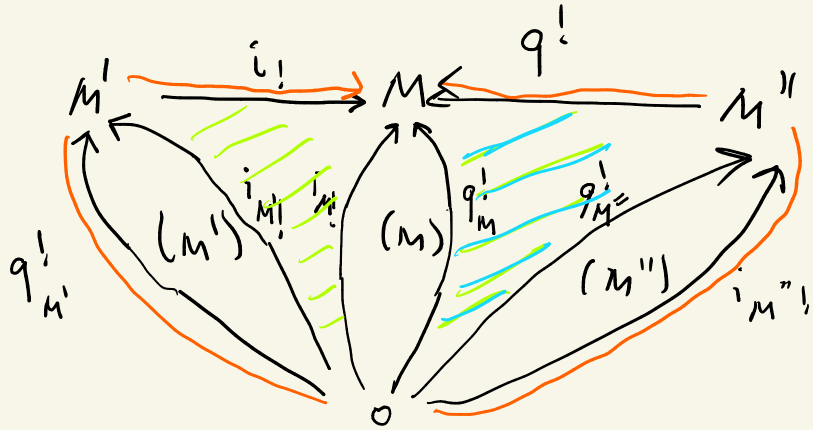
where $i_M : 0 \rightarrowtail M$

$q_M : M \twoheadrightarrow 0$

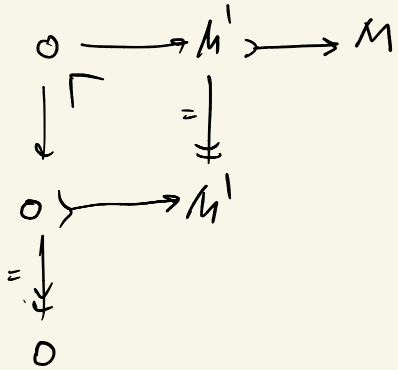
Claim $[M] \mapsto (M)$ well-defined

Pf

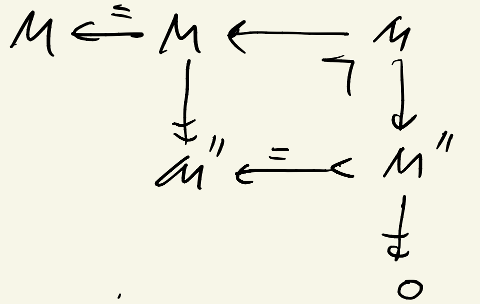
$$M' \overset{i}{\rightarrowtail} M \overset{q}{\twoheadrightarrow} M''$$



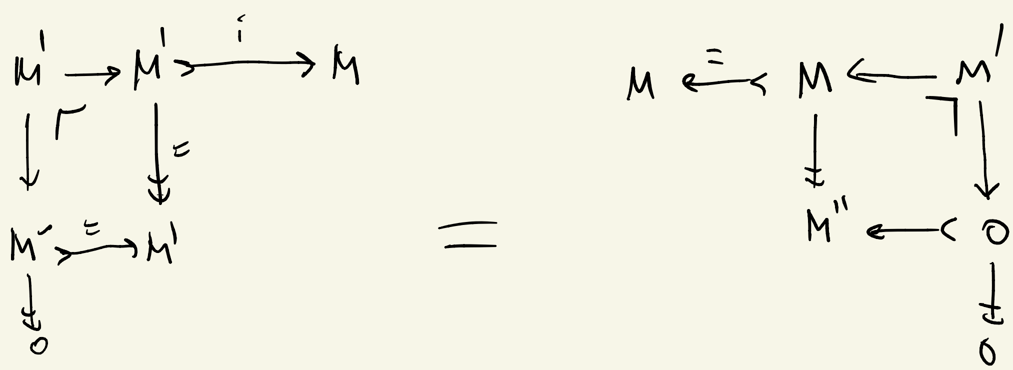
1) $i_! i_{M'!} = i_{M!}$



2) $q^! q^!_{M''} = q^!_M$

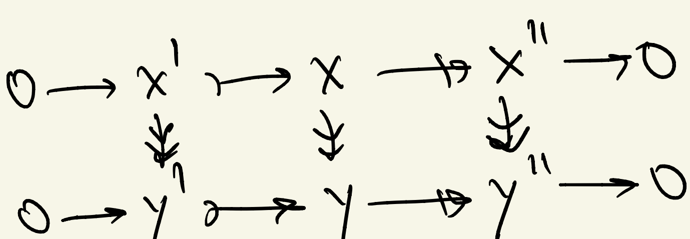


3) $i_! q^!_{M'} = q^! i_{M''!}$



"The splitting principle"

Let $\mathcal{E}$ be the category of short exact sequences in $\mathcal{C}$. Morphisms – comm. diagrams; admissible mono/epi – just



(or $\twoheadrightarrow$)

Theorem The functor

$$(s,t) : [0 \to X' \rightarrowtail X \twoheadrightarrow X'' \to 0] \mapsto (X', X'')$$

induces a homotopy equivalence

$$B\,Q\mathcal{E} \xrightarrow{\simeq} B(QC \times QC)$$

Sketch of the proof

Claim: For any $(M', M'') = d$ in $\mathrm{Ob}(QC \times QC)$:

$(s,t)/(M', M'') = F/d$ is contractible.

Pf $(s,t)/(M', M'')$: objects are

$$\begin{array}{ccccccccc}
0 \to & N' & \rightarrowtail & N & \twoheadrightarrow & N'' & \to 0 \\
 & \uparrow\!\!\uparrow & & & & \uparrow\!\!\uparrow & \\
 & X' & & & & X'' & \\
 & \downarrow & & & & \downarrow & \\
 & M' & & & & M'' &
\end{array}$$

$$(s,t)/(M_1,M_2) \supset \mathcal{C}_2 \supset \mathcal{C}_3 \sim *$$

$\mathcal{C}_2$:

$$\begin{array}{ccccccccc}
0 & \to & N' & \rightarrowtail & N & \twoheadrightarrow & N'' & \to & 0 \\
 & & \downarrow{\scriptstyle j^!} & & & & \uparrow\!\uparrow & & \\
 & & M' & & & & X'' & & \\
 & & & & & & \downarrow\!\!\!\!\rightarrowtail & & \\
 & & & & & & M'' & &
\end{array}$$

(i.e. $X' \overset{=}{\rightarrowtail} M'$)

$\mathcal{C}_3$:

$$\begin{array}{ccccccccc}
0 & \to & N' & \rightarrowtail & N & \twoheadrightarrow & N'' & \to & 0 \\
 & & \downarrow{\scriptstyle j^!} & & & & \downarrow{\scriptstyle i_!} & & \\
 & & M' & & & & M'' & &
\end{array}$$

(i.e. $\begin{array}{c} N'' \\ \uparrow = \\ X'' \end{array}$)

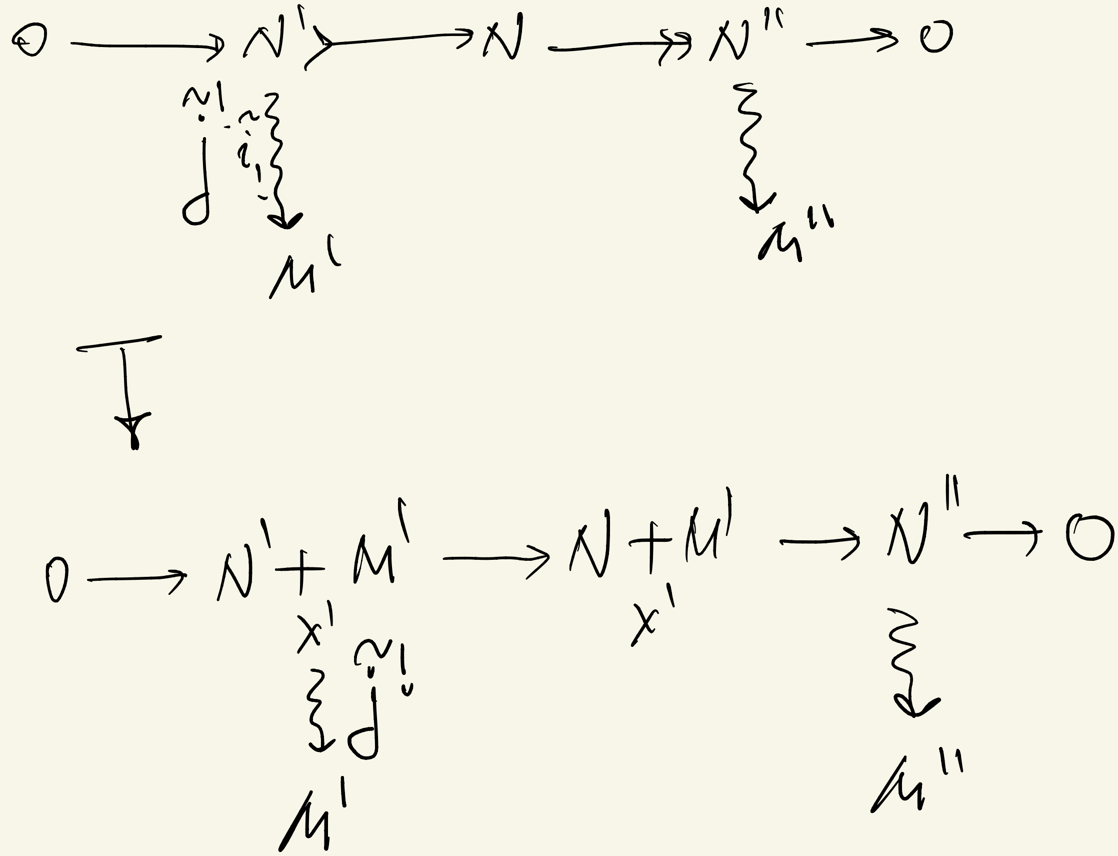
$$(s,t)/(M_1,M_2) \underset{i_2}{\overset{F_2 \text{ left adjoint}}{\rightleftarrows}} \mathcal{C}_2$$

<u>Recall</u>:

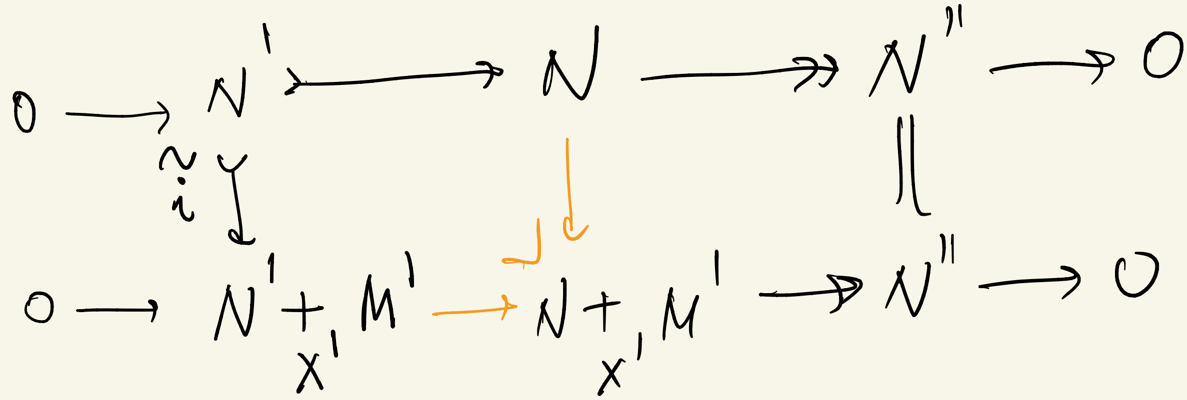
$$\begin{array}{ccccc}
 & & N' & & \\
 & {\scriptstyle j}\nearrow & & \searrow{\scriptstyle \tilde{i}} & \\
X' & < & & > & N' +_{X'} M' \\
 & {\scriptstyle i}\searrow & & \nearrow{\scriptstyle \tilde{j}} & \\
 & & M' & &
\end{array}$$

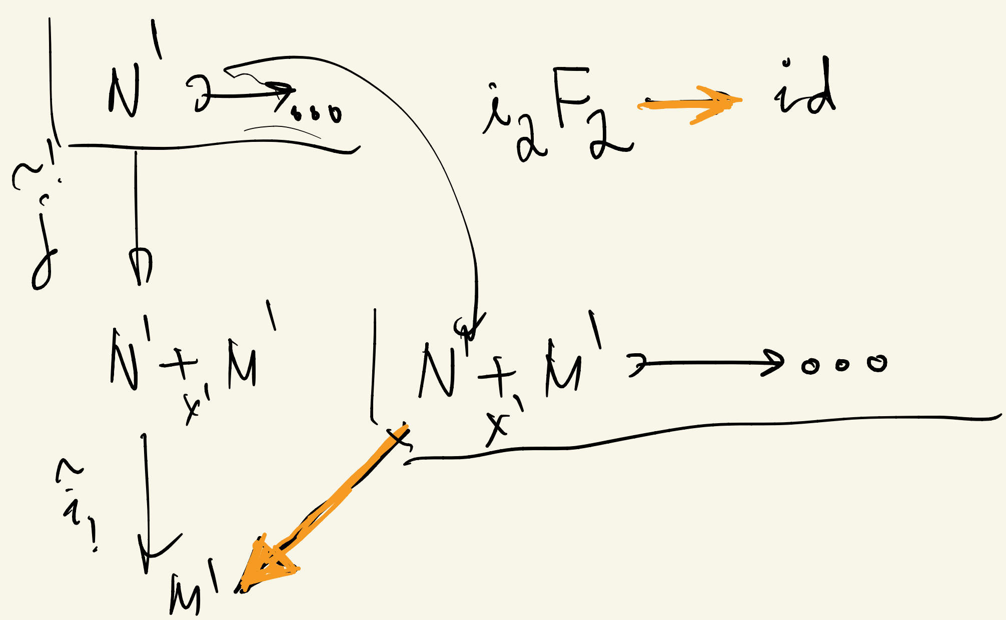
$$i_! j^! = \tilde{j}^! \tilde{i}_!$$

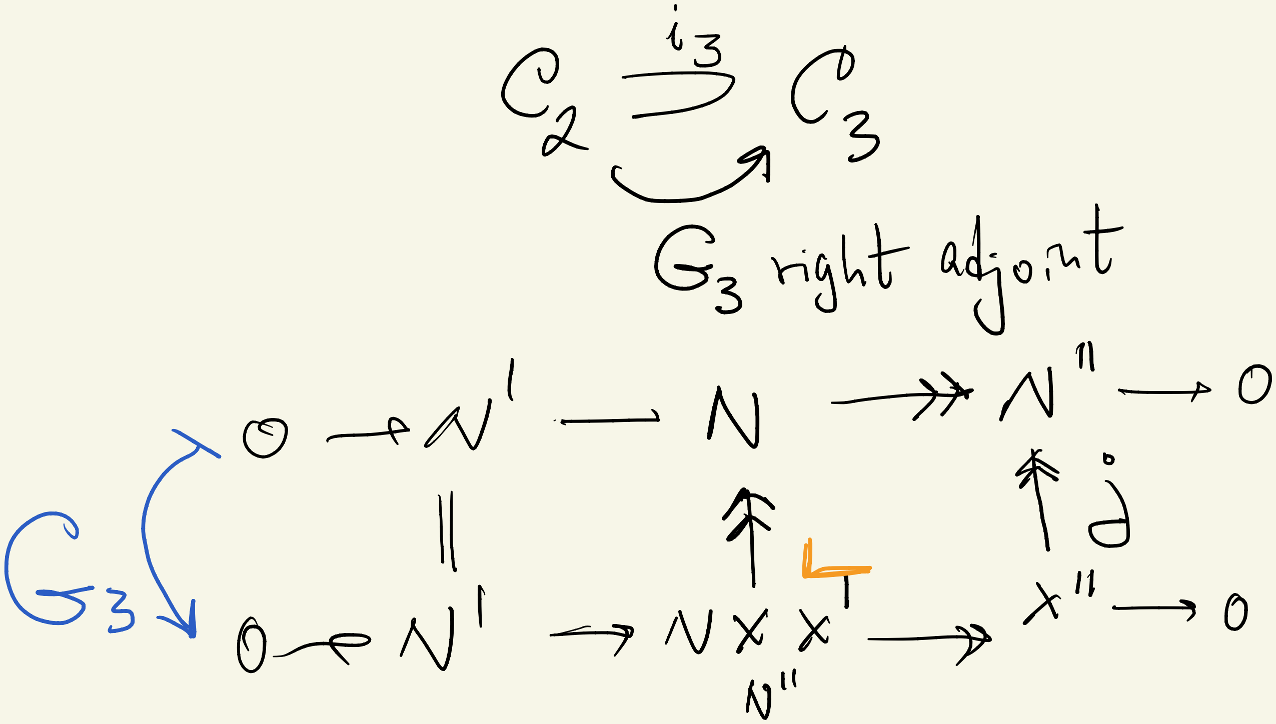
F_2 :

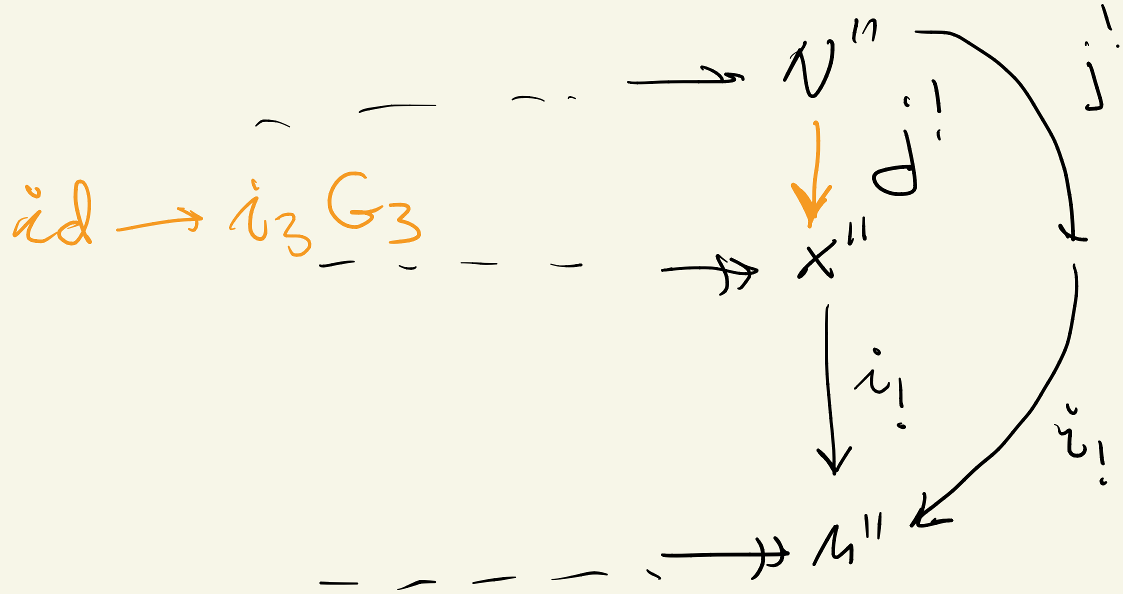


where

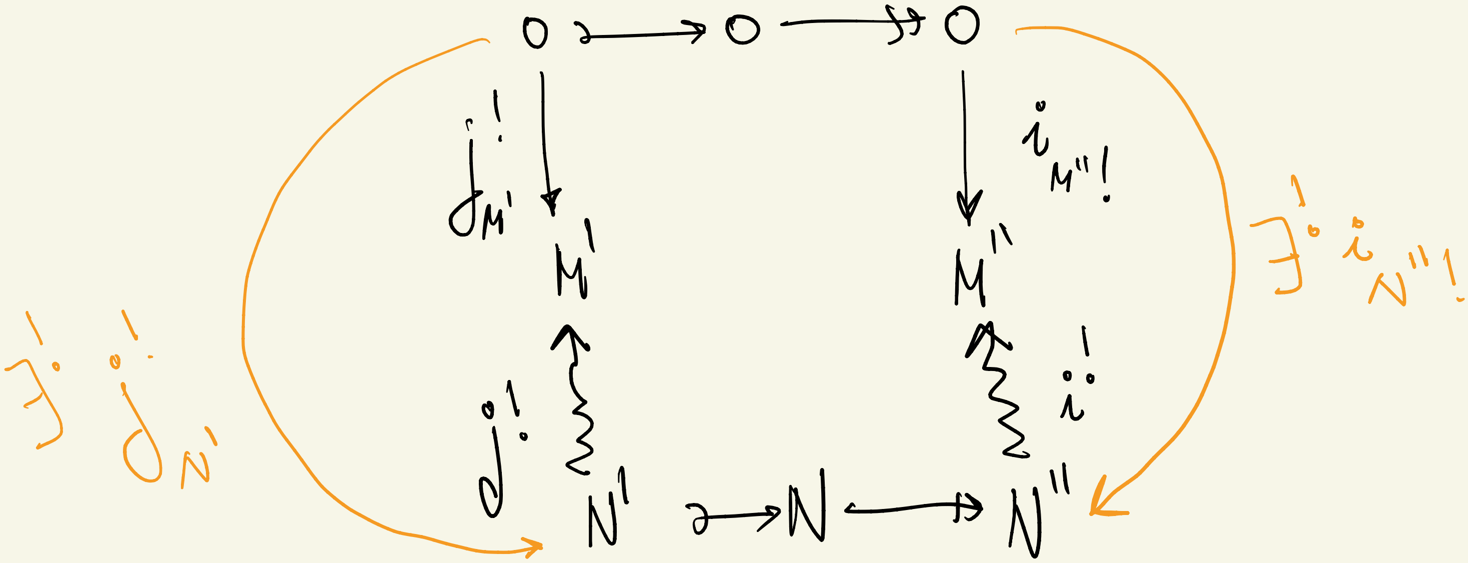


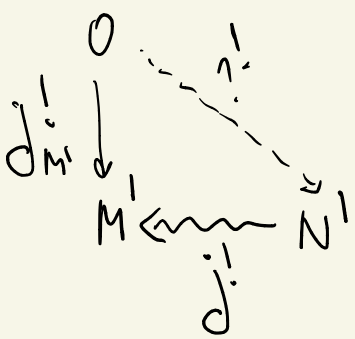



$\mathcal{C}_2 \xrightarrow{i_3} \mathcal{C}_3$
G_3 right adjoint
$0 \to N' \to N \to N'' \to 0$
$0 \to N' \to N \times_{N''} X'' \to X'' \to 0$
G_3
j

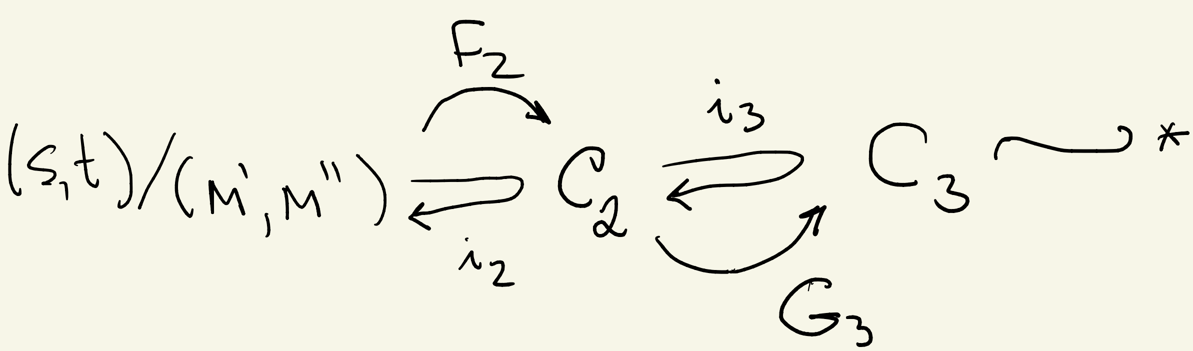

N''
X''
M''
$j^!$
$j^!$
$i_!$
$i_!$
$id \to i_3 G_3$

Initial object of $\mathcal{C}_3$:





$j_{M'} \overset{?}{=} j$



Conclusion: $B((s,t)/(M',M'')) \simeq *$, $\forall (M',M'')$

By Quillen's Theorem A: $BQ\mathcal{E} \xrightarrow{\simeq} BQ\mathcal{C} \times BQ\mathcal{C}$

From the splitting principle:

Thm Given exact functors $F', F, F'': \mathcal{C} \to \mathcal{D}$ and natural transformations $F' \overset{\varphi}{\Rightarrow} F \overset{\psi}{\Rightarrow} F''$ such that $0 \to F'(c) \to F(c) \to F''(c) \to 0$ is a short exact sequence in $\mathcal{D}$ for any c in $Ob(\mathcal{C})$. Then

$$F = F' + F'' : K_\cdot^Q(\mathcal{C}) \to K_\cdot^Q(\mathcal{D})$$

(Pf: pass through $\mathcal{E}$, ...)

Also:

Thm $K_0(\mathcal{C}) \xrightarrow{\sim} \pi_1 BQ(\mathcal{C})$

(this justifies the definition

$$K_\cdot^Q(\mathcal{C}) = \pi_{\cdot+1} BQ(\mathcal{C}))$$

The resolution theorem

Let $\mathcal{M}$ be an exact category, $\mathcal{P} \subset \mathcal{M}$ a full add. subcategory closed under extensions and such that:

(i) $0 \to M' \to P \to P'' \to 0$ s.e.s. in $\mathcal{M}$,

P, P'' in $\mathcal{P} \Rightarrow M'$ in $\mathcal{P}$.

(ii) For any M in $\mathcal{M}$ exists resolution

$$0 \to P_n \to \cdots \to P_0 \to M \to 0$$

Then $K_\cdot^Q(\mathcal{P}) \simeq K_\cdot^Q(\mathcal{M})$

The dévissage theorem

Let $\mathcal{B} \subset \mathcal{A}$ a full Abelian subcategory of an abelian category, closed under sub and quot objs and under finite products.

Assume every object of $\mathcal{A}$ has a finite filtration with quotients in $\mathcal{B}$. Then

$$K_\cdot^Q(\mathcal{B}) \xrightarrow{\sim} K_\cdot^Q(\mathcal{A})$$