H-spaces

(X, x_0) $\quad \mu : X \times X \to X$ continuous

$$\mu(x_0, -) \text{ and } \mu(-, x_0) \simeq \mathrm{id}_X$$

L: Can change μ to have a strict unit x_0

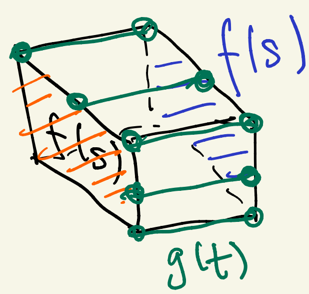
$$\mu(x_0, x) = \mu(x, x_0) = x$$

Lemma $\pi_1(X, x_0)$ is abelian, acting trivially on $\pi_n(X, x_0)$.

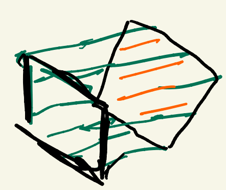
Pf $\quad f : (I^n, \partial I^n) \to (X, x_0) \qquad [f] \in \pi_n(X, x_0)$

$\quad g : (I, \partial I) \to (X, x_0) \qquad [g] \in \pi_1(X, x_0)$

$$I^{n+1} \to X \qquad (s,t) \mapsto \mu(f(s), g(t))$$

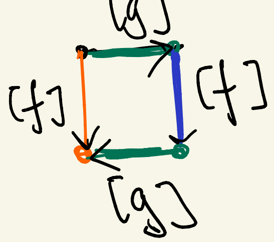


all •—• are $g(t)$



$\simeq \mathrm{Ad}_{[g]}([f])$

e.g.



$[f][g] = [g][f]$

conn. comp. of $(0,0) \in \mathrm{Ob}(S^{-1}S)$

$\quad\|$

Note: $+$ turns $B((S^{-1}S)_0)$ into an H-space.

(For us: $\mathcal{P} = \mathrm{Proj}_{\text{fin.gen.}}(A)$; $\mathcal{S} = \mathrm{Iso}(\mathcal{P})$).

Rmk:
How do we see the vanishing of commutators in $\pi_1((\mathcal{S}^{-1}\mathcal{S})_0)$?

Our loops:

$$\begin{tikzcd}
0+A \arrow[rr, "g"] & & 0+A \\
& A+0 \arrow[ul, "="] \arrow[ur, "="'] \arrow[u, phantom, "\lambda_g"] &
\end{tikzcd} \quad \text{in } \langle \mathcal{S},\mathcal{S}\rangle; \qquad
\begin{tikzcd}
(A,A) \arrow[rr] & & (A,A) \\
& (0,0) \arrow[ul] \arrow[ur] \arrow[u, phantom, "(\lambda_e,\lambda_g)"] &
\end{tikzcd} \quad \text{in } \mathcal{S}^{-1}\mathcal{S};$$

$$\wr \qquad\qquad\qquad\qquad \wr$$

$$\begin{tikzcd}
A+0 \arrow[r, bend left, "g"] \arrow[r, "="'] \arrow[r, phantom, "l_g"] & A
\end{tikzcd} \qquad
L_g := \begin{tikzcd}
(0,0) \arrow[r, bend left] \arrow[r] \arrow[r, phantom, "(l_e,l_g)"] & (A,A)
\end{tikzcd} \quad \text{in } \mathcal{S}^{-1}\mathcal{S}$$

A square in $\langle \mathcal{S},\mathcal{S}\rangle$:

$$P_{g,h}: \quad \begin{tikzcd}
A+A \arrow[rr, bend left, "h"] \arrow[rr, "g"'] & & A\oplus A \\
A+0 \arrow[u, "g"] \arrow[rr, "h"'] & & A+A \arrow[u, "g"] \arrow[u, bend right]
\end{tikzcd}$$

Compositions:

$$\Gamma: \quad A+A+0 \longrightarrow A\oplus A \quad (h,\ g)$$

$$\|$$

$$\llcorner: \quad A+A \longrightarrow A\oplus A \quad (h,\ g)$$

A square in $\mathcal{S}^{-1}\mathcal{S}$:

$$S_{g,h}: \quad \begin{tikzcd}
(A,A) \arrow[r] & (A\oplus A, A\oplus A) \\
(0,0) \arrow[u] \arrow[r] \arrow[ur, phantom, "P_{e,e}\times P_{g,h}"] & (A,A) \arrow[u]
\end{tikzcd}$$

Differ by the permutation automorphism $\overset{\sigma}{A+A}$

In $S^{-1}S$:

$$\begin{tikzcd}
(0,0) \arrow[r] \arrow[d] \arrow[rr, bend left, "L_h"] & (A,A) \arrow[d] & (0,0) \arrow[l] \arrow[d] \\
(A,A) \arrow[r] \arrow[ur, phantom, "S^{e,h}"] & (A^2,A^2) \arrow[ur, phantom, "{}^{e,e}S"] & (A,A) \arrow[l] \\
(0,0) \arrow[r] \arrow[u] \arrow[rr, bend right, "L_h"'] \arrow[uu, bend left, "L_g"] \arrow[ur, phantom, "S_{g,h}"] & (A,A) \arrow[u] \arrow[ur, phantom, "{}_{g,e}S"] & (0,0) \arrow[l] \arrow[u] \arrow[uu, bend right, "L_g"']
\end{tikzcd}$$

$L_g = L_h^{-1} L_g L_h$
in $\pi_1(B(S^{-1}S)_0)$

$BGL(A) \to B(S^{-1}S)_0$

S_n = Full subcat of S gen. by A^n

$$S_n \longrightarrow (S^{-1}S)_0 \qquad \begin{tikzcd} A^n \arrow[r, mapsto] \arrow[d, "g"'] & (A^n, A^n) \arrow[d, "1\ \ g"] \\ A^n \arrow[r, mapsto] & (A^n, A^n) \end{tikzcd}$$

$$\begin{tikzcd} A^n \arrow[d] \\ A+A^n \end{tikzcd} \quad \begin{tikzcd} S_n \arrow[d] \arrow[r] \arrow[dr, phantom, "\Leftarrow"] & (S^{-1}S)_0 \\ S_{n+1} \arrow[ur] \end{tikzcd} \qquad \begin{tikzcd} BGL_n \arrow[d] \arrow[r] \arrow[dr, phantom, "\text{homot.}"] & B(S^{-1}S)_0 \\ BGL_{n+1} \arrow[ur] \end{tikzcd}$$

easily promoted to: $\operatorname{hocolim}_n BGL_n(A) \to B(S^{-1}S)_0$

But $\bullet \to \bullet \to \bullet \to \dots$ is filtered; $\operatorname{hocolim}_n \xrightarrow{\sim} \operatorname{colim}_n = \varinjlim_n$

Get a morphism $BGL(A) \to B(S^{-1}S)_0$

R.H.S. is an H-space; isomorphism on H_*;
by the universal property:

$$\begin{tikzcd} BGL(A) \arrow[r] \arrow[d] & B(S^{-1}S)_0 \\ BGL(A)^+ \arrow[ur, "\sim"'] \end{tikzcd}$$

$BS^{-1}S \simeq \Omega BQP$

$P = \mathrm{Proj}_{f.g.}(A)$; $S = \mathrm{Isom}(P)$

Want: $(S^{-1}S)_0 \longrightarrow\ ? \longrightarrow QP$ fibration; $\qquad B?$ contractible

Construct $\boxed{?}$: 1) $\mathcal{E}$: objects: $M \overset{i}{\rightarrowtail} N$ in P

morphisms:

$$\begin{tikzcd} M_1 \arrow[r, tail, "i_1"] & N_1 \arrow[d, "i'"] \\ M_2 \arrow[u, "i"] \arrow[r, tail, "i_2"'] & N_2 \end{tikzcd} \qquad i_2 = i' i_1 i$$

1) $\mathcal{E} \longrightarrow (P, \text{mono})$: $\quad \boxed{M \rightarrowtail N} \mapsto N$

Claim: this is a coCartesian fibration.

$(\text{Fiber of } N) = \{M \rightarrowtail N\}$;

2) all fibers contractible: $\quad 0 \rightarrowtail N$ final obj

$$\begin{tikzcd} M \arrow[r, tail] & N \\ 0 \arrow[u] \arrow[ur, "1"'] & \end{tikzcd} \qquad \begin{array}{c} M \rightarrowtail N \\ \downarrow \\ M' \rightarrowtail N \end{array}$$

(P, mono) contractible: 0 is initial. $\Rightarrow B\mathcal{E} \simeq *$

3) S acts on $\mathcal{E}$: $\quad C : (M \overset{i}{\rightarrowtail} N) \longmapsto (C+M \xrightarrow{id_C + i} C+N)$

$B\mathcal{E}$ contractible $\Rightarrow$ the action is hom.inv. $\Rightarrow B\mathcal{E} \simeq BS^{-1}\mathcal{E} \Rightarrow BS^{-1}\mathcal{E} \simeq *$

We get $S^{-1}S \hookrightarrow S^{-1}\mathcal{E} \simeq *$

The functor $S^{-1}\mathcal{E} \to QP$

$$(M \overset{i}{\rightarrowtail} N) \xmapsto{\text{on objects}} \mathrm{coker}(i)$$

On morphisms:

$$\begin{tikzcd} M_1 \arrow[r, tail, "i_1"] & N_1 \arrow[d, tail, "i'"] \\ M_2 \arrow[u, tail, "i"] \arrow[r, tail, "i_2"'] & N_2 \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} \mathrm{coker}(i_1) \\ \mathrm{coker}(i_1 i) \arrow[u, two heads] \arrow[d, tail] \\ \mathrm{coker}(i' i_1 i) \end{tikzcd} \quad \left(\begin{tikzcd} \mathrm{im}(i_1) \\ \mathrm{im}(i_1 i) \arrow[u, hook] \arrow[d, two heads, "i'"] \\ \mathrm{im}(i' i_1 i) \end{tikzcd} \right)$$

Fact: this is a Cartesian fibration

(Fiber of $C \in ob(P) = ob(QP)$: $\{ M \overset{i}{\rightarrowtail} N$ w/ $\mathrm{coker}(i) \cong C \}$

$$\begin{tikzcd} M_1 \arrow[r, tail] & N_1 \arrow[r, two heads] \arrow[d, tail] & C \\ M_2 \arrow[u, tail] \arrow[r, tail] & N_2 \arrow[r, two heads] & C \arrow[u, "="] \end{tikzcd}$$

$\uparrow$, $\downarrow$ must be isom

Why Cartesian fibration?

In the language of contravariant pseudofunctors $QP \to Cat$:

$$QP^{op} \to Cat \; ; \; C \mapsto \{ M \overset{i}{\rightarrowtail} N \text{ w/ } \mathrm{coker}(i) \simeq C \} \cong S \text{ (since all exts. split)}$$

Define f^* for $f \in \mathrm{Mor}_{QP}$:

1)

$$
\begin{tikzcd}
\ker(u) \arrow[r, "="] \arrow[d, tail] & \ker(u) \arrow[d, tail] \\
M \arrow[r, tail] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & M' \arrow[d, "u'"] \\
C \arrow[r, tail, "i_!"'] & C'
\end{tikzcd}
\qquad
\left(
\begin{tikzcd}
\ker(u') \arrow[d, tail] \arrow[r, "="] & \ker u' \arrow[d, tail] \\
M \arrow[r, squiggly] & M'
\end{tikzcd}
\right)
\text{ in } \mathcal{E}
$$

$$
\begin{tikzcd}
\ker u \arrow[d, tail] \\
M
\end{tikzcd}
=
i_!^* \left(
\begin{tikzcd}
\ker(u') \arrow[d, tail] \\
M'
\end{tikzcd}
\right)
$$

2)

$$
\begin{tikzcd}
\ker(qu') \arrow[d, tail] & \ker(u') \arrow[l, tail] \arrow[d, tail] \\
M' \arrow[r, equal] \arrow[d, "u'"] & M' \arrow[d, "u'"] \\
C & C' \arrow[l, two heads, "q"]
\end{tikzcd}
\qquad
\begin{tikzcd}
\ker(qu') \arrow[d] \arrow[r, squiggly] & \ker u' \arrow[d, tail] \\
M' \arrow[r] & M'
\end{tikzcd}
\text{ in } \mathcal{E}
$$

$$
\begin{tikzcd}
\ker(qu') \arrow[d, tail] \\
M'
\end{tikzcd}
=
q^{!*} \left(
\begin{tikzcd}
\ker(u') \arrow[d, tail] \\
M'
\end{tikzcd}
\right)
$$

Have to check: 1) those are functors (clear from the description of $\mathcal{C}_c$ above);

2) $\phi(f,g) : (gf)^* \simeq f^* g^*$

key case: for a bicartesian square

$$\begin{tikzcd} A \arrow[r, tail, "i'"] \arrow[d, two heads, "q'"'] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, two heads, "q"] \\ C \arrow[r, tail, "i"] & D \end{tikzcd}$$

<u>recall</u>:

$$i'_! q'^! = q^! i_!$$

So have to check

$$(*) \qquad q'^{!*} i'^*_! \simeq i_!^* q^{!*} : \mathcal{C}_D \to \mathcal{C}_A$$

$$\begin{tikzcd} \ker(v') & \ker(u) \arrow[l, hook'] \arrow[r, "="] \arrow[d, tail] & \ker(u) \arrow[r, tail] \arrow[d, tail] & \ker(v) \\ & M' \arrow[r, tail] \arrow[d, two heads] \arrow[dd, bend right=60, "v'"'] \arrow[dr, phantom, "\ulcorner"] & M \arrow[d, two heads, "u"] \arrow[dd, bend left=60, "v"] & \\ & A \arrow[r, tail] \arrow[d, two heads] \arrow[dr, phantom, "\ulcorner"] & B \arrow[d, two heads] & \\ & C \arrow[r, tail] & D & \end{tikzcd}$$

equiv to:

$$(*): \quad \begin{tikzcd} M' \arrow[r, tail, two heads] \arrow[d, "v'"'] \arrow[dr, phantom, "\text{bicart}"] & M \arrow[d, "v"] \\ C \arrow[r, tail] & D \end{tikzcd}$$

So:

$$\begin{array}{c}\mathcal{E} \simeq * \\ \downarrow \\ QP\end{array}$$

(fiber of C) $\simeq \mathcal{S}$

For $f = i_! q^! = q'^! i'_!$ in QP:

f^* is $M \mapsto M + \ker q \; (\simeq M + \ker q')$

Now: $\mathcal{S}$ acts on $\mathcal{E}$; $\mathcal{S}^{-1}\mathcal{S}$ acts on $\mathcal{S}^{-1}\mathcal{E}$;

$$\begin{array}{c}\mathcal{S}^{-1}\mathcal{E} \sim * \\ \downarrow \\ QP\end{array}$$

Cartesian fibration;
fibers $\simeq \mathcal{S}^{-1}\mathcal{S}$;
f^* are homotopy equiv on $B(\mathcal{S}^{-1}\mathcal{S})$.

By Quillen Thm B:

$$B(\mathcal{S}^{-1}\mathcal{S}) \to \underset{\displaystyle *}{\underset{\wr\wr}{B(\mathcal{S}^{-1}\mathcal{E})}} \longrightarrow BQP$$

is a homotopy fiber sequence

$$B(\mathcal{S}^{-1}\mathcal{S}) \simeq \Omega BQP$$