VOLODIN K-theory

$$\mathrm{Vol}(A) = \bigcup_{\sigma} BT^{\sigma}(A) \hookrightarrow BGL(A)$$

where:

σ are linear orders on $\{1,2,3,\dots\}$

$$T^{\sigma}(A) = \{(a_{ij}) \mid a_{ij} = \delta_{ij} \text{ \underline{unless} } i \underset{\sigma}{<} j\}$$

<u>Thm</u>

$$\mathrm{Vol}(A) \longrightarrow BGL(A) \to BGL(A)^{+}$$

is a homotopy fiber sequence.

<u>Follows from:</u>

① $\pi_1(\mathrm{Vol}(A)) = St(A)$

② $\pi_1(\mathrm{Vol}(A))$ acts on $\pi_n(\mathrm{Vol}(A))$ trivially for $n>1$.

③ $H_n(\mathrm{Vol}(A)) = 0, \quad n > 0.$

Main source: A. Suslin, On Equivalence of K-theories.

Start with ③.

Prop Let σ be a linear order on $\{1,2,\dots,n\}$. Let $T_n^\sigma \subset GL_n(A)$ be the subgroup $\{(a_{ij}) \mid a_{ij} = \delta_{ij} \text{ unless } i <_\sigma j\}$.

Then $H_*(BT_n^\sigma) \to H_*(\bigcup_{\sigma'} BT^{\sigma'})$ is $\overset{*>0}{\text{zero}}$,

Pf Example: $e_{12}^a = [e_{13}^1, e_{32}^a] = 0$ in $H_1(T_3^{1<3<2})$.

Example: same but for Lie algebras:

$$H_\cdot(t_n^\sigma) \longrightarrow H_\cdot(t_{n+1}^{\sigma'})$$

where

$1 <_\sigma 2 < \dots <_\sigma n$ and $1 <_{\sigma'} n+1 <_{\sigma'} 2 <_{\sigma'} 3 <\dots <_{\sigma'} n$

Indeed: $$E_{1i_1}^{a_1} \wedge \dots \wedge E_{1i_m}^{a_m} \wedge \bigwedge_{2 \le j_\ell, k_\ell \le n} E_{j_\ell k_\ell}^{b_\ell}$$

$(E_{ij}^a = (a \cdot \delta_{ij}))$

$$\Big\downarrow h_m$$

$$\sum_{k=1}^{m} (-1)^{k-1} E_{1i_1}^{a_1} \wedge \dots \wedge \underbrace{E_{1,n+1}^{1} \wedge E_{n+1,i_k}^{a_k}}_{} \wedge \dots \wedge E_{1,i_m}^{a_m} \wedge \bigwedge E_{j_\ell k_\ell}^{b_\ell}$$

$h = \frac{1}{m} h_m$ for $m > 0$; 0 for $m = 0$.

Then

$$[\partial, h] = i - i \circ \pi$$

where $\quad i : t_n^{\sigma} \subset t_{n+1}^{\sigma'}$

$$\begin{tikzcd} i \circ \pi : & t_n^{\sigma} \arrow[r] & t_{n-1}^{\sigma} \arrow[d] & = \{(a_{ij}) \mid 2 \le i,j \le n;\} \\ & & t_{n+1}^{\sigma'} & a_{ij} = 0,\ i \ge j \end{tikzcd}$$

Now use induction in n.

Now the proof:

$\sigma : \ 1 < 2 < \ldots < n$

$\varphi : \{1, 2, \ldots, n\} \to \mathbb{N}$

$$\varphi(1) < \underbrace{\ldots}_{} < \varphi(2) < \underbrace{\ldots}_{} < \varphi(n-1) < \underbrace{\ldots}_{} < \varphi(n)$$

enough in between

Also: by def,

$$\begin{array}{c}\varphi(1)\\ \shortparallel \\ \psi(1)\end{array} < \ldots < \underbrace{\psi(2) < \varphi(2)}_{\text{neighbors}} < \ldots < \underbrace{\psi(n) < \varphi(n)}_{\text{neighbors}} = N$$

We have two embeddings

$$i_\varphi, i_\psi : T_n \to T_N$$

Their images commute: T_n^φ and T_n^ψ.

Claim If at least k elements btwn $\varphi(i)$ and $\varphi(i+1)$, $\forall i$: then

$$H_k(T_n^\varphi) \xrightarrow{0} H_k(T_N).$$

Pf Induction in k (already saw for $k=1$). Also will use induction in n.

Key lemma: Let $2<i+1<j$. Put $u = e_{i,i+1}^{-1} \cdot e_{j,j+1}^{-1}$. Then

$$Ad_u(e_{ij}^a e_{i+1,j+1}^a) = e_{ij}^a e_{i+1,j+1}^a$$

$$Ad_u(e_{1i}^a) = e_{1i}^a e_{1,i+1}^a$$

$$i^{\varphi}\cdot i^{\psi} : T_n \xrightarrow{\Delta} T_n \times T_n \xrightarrow{i^{\varphi}\times i^{\psi}} T_N \times T_N \xrightarrow{\mu} T^N$$

$$(i^{\varphi'}\circ \pi)\cdot i^{\psi} : T_n \xrightarrow{\Delta} T_n \times T_n \xrightarrow{\pi \times id} T_{n-1} \times T_n \xrightarrow{i^{\varphi'}\times i^{\psi}} T_N \times T_N \xrightarrow{\mu} T_N$$

$$\pi : T_n \to T_{n-1} ; \qquad (a_{ij})_{1\le i,j\le n} \mapsto (a_{ij})_{2\le i,j \le n}$$

$$\varphi' = \varphi|\{2,\underset{k}{\ldots},n\} .$$

changing notation to

$$u = \prod_{i>1} e^{-1}_{\psi(i),\varphi(i)} = \prod_{i>1} e^{-1}_{\psi(i),\psi(i)+1} :$$

$$i^{\varphi}\cdot i^{\psi} = Ad_u((i^{\varphi'}\circ\pi)\cdot i^{\psi})$$

(from the comp. above

The action on homology:

$$\begin{tikzcd}
 & & H_K(T_N) \\
H_k(T_n) \arrow[r] & \bigoplus_{i=0}^{K} H_i(T_n)\times H_{k-i}(T_n) \arrow[r] \arrow[dr, dashed, "\text{by induction in } k"'] & \bigoplus_{i=0}^{k} H_i(T_N)\otimes H_i(T_N) \arrow[u, "\mu"] \\
 & & H_k(T_n)\oplus H_K(T_n) \arrow[u, shift right=3, "i^{\varphi}"'] \arrow[u, shift left=3, "i^{\psi}"]
\end{tikzcd}$$

Same as

$$\begin{tikzcd}
H_K(T_n) \arrow[r] & \bigoplus_{i=0}^{k} H_i(T_n)\times H_{k-i}(T_n) \arrow[r] & \bigoplus_{i=0}^{K} H_i(T_n)\otimes H_{k-i}(T_n) \arrow[d, shift right=3, "i^{\varphi}\circ\pi \text{ acts by zero by induction in } n"'] \arrow[d, shift left=3, "i^{\psi}"] \arrow[dl, dashed] \\
 & H_k(T_n) \arrow[r, "i^{\psi}"] & \bigoplus H_i(T_N)\otimes H_{k-i}(T_N) \arrow[d, "\mu"] \\
 & & H_K(T_N)
\end{tikzcd}$$

In other words:

$$H_K(T_n) \overset{i^{\psi}}{\underset{i^{\psi}+i^{\varphi}}{\rightrightarrows}} H_K(T_N) \text{ are SAME; } i^{\varphi} = 0$$

To finish the proof of ③: (a bit sketchily):

$C_*(\bigcup_{\sigma} BT^{\sigma})$ is quasi-isom to the Čech cplx

$$\begin{tikzcd}
\bigoplus_{\sigma_0} C_*(BT^{\sigma_0}) & \bigoplus_{\sigma_0,\sigma_1} C_*(BT^{\sigma_0}\cap BT^{\sigma_1}) \arrow[l, shift right, "\partial_0"'] \arrow[l, shift left, "\partial_1"'] & \cdots \arrow[l, shift right=2, "\partial_0"'] \arrow[l] \arrow[l, shift left=2, "\partial_2"']
\end{tikzcd}$$

$$\partial = \partial_0 - \partial_1 + \ldots \pm \partial_n$$

each $BT^{\sigma_0} \cap \ldots \cap BT^{\sigma_n}$ is BT^{σ} where σ is the partial order: $i <_{\sigma} j$ when $i <_{\sigma_k} j, \forall k$.

$BT^{\sigma} = \{(a_{ij}) \mid a_{ij} = \delta_{ij}$ unless $i <_{\sigma} j\}$.

By the above, all $C_*(BT^{\sigma_0} \cap \ldots \cap BT^{\sigma_n})$ are acyclic.

Now prove ①: $\pi_1(\mathrm{Vol}(A)) \simeq St(A)$.

But π_1 is generated by e_{ij}^{a} subject to relations that take place in one of the T^{σ}, therefore in $St(A)$ (since $T^{\sigma} \subset St$).

As for ②:

Define

$W(St(A); \{T^\sigma(A)\})$ to be the simplicial subset of $ESt(A)$:

$W_n = \{(g_0, \gamma_1, \gamma_2, \dots, \gamma_n) \mid g_0 \in St(A); \gamma_i \in T^\sigma(A)$ <u>all</u> for <u>some</u> $\sigma\}$

(<u>recall</u>: all $T^\sigma(A)$ are also subgrps of $St(A)$).

$$\begin{tikzcd}
 & (g_0, \gamma_1, \dots, \gamma_n) \arrow[dl, "d_0"'] \arrow[d, "d_1"] \arrow[drr] & & \\
(g_0\gamma_1, \dots) & (g_0, \gamma_1\gamma_2, \dots) & \dots & (g_0, \gamma_1, \dots, \gamma_{n-1})
\end{tikzcd}$$

$St(A)$ acts on W:

$$(g_0, \gamma_1, \dots, \gamma_n) \longmapsto (gg_0, \gamma_1, \dots, \gamma_n).$$

<u>Claim</u>: This action is homotopically trivial.

(Note: <u>of course</u> the action is homotopically trivial on $E(St(A))$ (which is contractible).

But why on the subset?

One way to prove it on $E(St(A))$:

$$\Delta^1 \times E(St) \longrightarrow E(St)$$

whose restriction to 1, resp. $0 \in \Delta^1$, is id, resp. g.

Pass to the homogeneous form of E:

$$\begin{tikzcd} E_n = \{(g_0, g_1, \dots, g_n)\} \arrow[d, "d_i"] & (g_0, \dots, g_n) \arrow[d, mapsto] \\ (g_0, \dots, \hat{g}_i, \dots, g_n) & (gg_0, \dots, gg_n) \end{tikzcd}$$

$0 \leq i \leq n$

$$(0, \dots, 0, 1, \dots, 1) \times (g_0, \dots, g_n) \mapsto (gg_0, \dots, gg_i, g_{i+1}, \dots, g_n)$$

Now: $W \subset E(St)$ is characterized by:

For some σ, $g_i^{-1} g_j \in T^\sigma$, $\forall i, j$

Problem: $g_i^{-1} g^{-1} g_j \notin T^\sigma$?

BUT: Let all g_i be in St_n; note that St_{n+1} is generated by $x^a_{n+1,i}$, $x^b_{i,n+1}$.

So: enough to consider 1) $g = x^a_{n+1,i}$ or 2) $x^b_{i,n+1}$.

But then $g_i^{-1} g^{-1} g_j \in T_{\sigma'}$: σ': 1) $n+1 < 1 < \dots < n$

2) $1 < 2 < \dots < n+1$

So: $W(St(A), \{T^\sigma(A)\}) \circlearrowleft St(A)$ freely;

$Vol(A) = W/St$; $\pi_1(Vol(A)) = St(A) \Longrightarrow$

$W = \widetilde{Vol}(A)$, and $\pi_n(W) = \pi_n(\widetilde{Vol}(A))$, $n>1$;

but π_1 acts on them trivially. ◀

From this:

$$\begin{tikzcd}
GL(A) \arrow[r] & W(GL(A);\{T^\sigma\}) \arrow[r, "GL(A)"] & Vol(A) \arrow[d, phantom, "/2"] & \\
\Omega BGL(A) \arrow[r] \arrow[u, "\wr"] & \Omega BGL(A)^+ \arrow[r] \arrow[u, "2"'] & \mathcal{F}(A) \arrow[r] & BGL(A) \arrow[r] & \dots
\end{tikzcd}$$

homotopy fiber sequences

$$\Omega BGL(A)^+ \simeq W(GL(A);\{T^\sigma\})$$