ERRATUM FOR THE BOOK "NILPOTENT STRUCTURES IN ERGODIC THEORY" NOVEMBER 20, 2019

Part (i) of Proposition 19, Chapter 6 is erronious. To correct this, we modify the definitions of the space of cubes of a homogeneous space and of a nilmanifold. This necessitates small changes where these definitions are applied in other parts of Chapter 6 and in Chapter 12.

We thank Jiahao Qiu for bringing this error to our attention.

Modifications in Chapter 6: Cubes in an algebraic setting. There are no changes are needed though Corollary 18 and its proof.

Section 3. Starting with Part (i) of Proposition 19, we need to replace the use of the group $\mathsf{Q}^{[\![k]\!]}(\Gamma)$ by a different group (part (ii) of this proposition remains unchanged).

We introduce the following notation for the revised version of the cube group:

Notation. For a subgroup Γ of G and $k \geq 1$, let $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$ denote the subgroup of $G^{[\![k]\!]}$ spanned by elements of the form $z^{(\gamma)}$, where γ is a face of $[\![k]\!]$ of codimension j and $z \in G_j \cap \Gamma$.

Remark. More generally, we could define this subgroup using the notion of a filtered group (see Chapter 14 for the definition). We can define the facet group $\mathsf{Q}^{[\![k]\!]}(G^\bullet)$ associated to a filtration $G^\bullet$ of a group G. For a subgroup Γ of G, can define the induced filtration $\Gamma^\bullet$ on Γ and it can be proven that $\mathsf{Q}^{[\![k]\!]}(\Gamma^\bullet) = \mathsf{Q}^{[\![k]\!]}(G^\bullet) \cap \Gamma^{[\![k]\!]}$. If we take $G^\bullet$ to be the lower central series of G, then the group $\mathsf{Q}^{[\![k]\!]}(G^\bullet)$ coincides with $\mathsf{Q}^{[\![k]\!]}(G)$ and $\mathsf{Q}^{[\![k]\!]}(\Gamma^\bullet) = \mathsf{Q}^{[\![k]\!]}(\Gamma; G)$. However, we do not take this more general approach but only make the minimal changes to correct the error.

Using the notation introduced for $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$, Proposition 19 becomes:

Proposition 19 (Chapter 6)**.** *Let Γ be a subgroup of G. Then*

(i) $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$ *is the set of elements of the form $\psi(\mathbf{h})$ given by (19) where $h_j \in G_{\mathrm{codim}(\alpha_j)} \cap \Gamma$ for $j = 1, \ldots, 2^k$, and furthermore*

$$\mathsf{Q}^{[\![k]\!]}(\Gamma; G) = \mathsf{Q}^{[\![k]\!]}(G) \cap \Gamma^{[\![k]\!]}. \tag{1}$$

(ii) *If all the coordinates of a point of $\mathsf{Q}^{[\![k]\!]}(G)$ belong to Γ except possibly one of them, then this coordinate also belongs to ΓG_k.*

We note again that Part (ii) remains unchanged, and so we only give a corrected proof of Part (i).

Proof of Part (i). By definition, the set of elements of the form $\psi(\mathbf{h})$ given by (19), where $h_j \in G_{\text{codim}(\alpha_j)} \cap \Gamma$ for every j, is contained in $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$, and this group is clearly contained in $\mathsf{Q}^{[\![k]\!]}(G) \cap \Gamma^{[\![k]\!]}$. Thus we are left with checking the opposite inclusion, showing that $\mathsf{Q}^{[\![k]\!]}(G) \cap \Gamma^{[\![k]\!]}$ is contained in the set of elements of the form $\psi(\mathbf{h})$ given by (19), where $h_j \in G_{\text{codim}(\alpha_j)} \cap \Gamma$ for every j.

Write $\mathbf{g} \in \mathsf{Q}^{[\![k]\!]}(G) \cap \Gamma^{[\![k]\!]}$ as $\mathbf{g} = \Psi(\mathbf{h})$, as in (19). We proceed as in step four of the proof of Proposition 17. By (20) applied with $\ell = 1$, we obtain that $h_1 = g(\underline{0}) \in \Gamma$. By induction and using Equation (20) at each step, we obtain that $h_j \in \Gamma$ for every j. Thus $h_j \in G_{\text{codim}(\alpha_j)} \cap \Gamma$ and $\mathbf{g}$ has the desired form. □

We introduce a new result for use in Chapter 12:

Proposition 19a. *For every $k \geq 1$, $\mathsf{Q}^{[\![k]\!]}(G)_2 = \mathsf{Q}^{[\![k]\!]}(G_2; G)$.*

For consistency in notation, we should denote $\mathsf{Q}^{[\![k]\!]}(G)_2$ by $(\mathsf{Q}^{[\![k]\!]}(G))_2$, but we simplify this by removing the extra parentheses.

Proof. The inclusion $\mathsf{Q}^{[\![k]\!]}(G)_2 \subset \mathsf{Q}^{[\![k]\!]}(G) \cap G_2^{[\![k]\!]}$ is obvious, and by Part (i) of Proposition 19 this last group is equal to $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$. We prove the converse inclusion. By definition of $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$, it suffices to show that for every $j = 0, \ldots, k$, every face α of codimension j of $[\![k]\!]$, and every $z \in G_j \cap G_2$, we have $z^{(\alpha)} \in \mathsf{Q}^{[\![k]\!]}(G)_2$.

Note that $G_j \cap G_2 = G_{\max(j,2)}$. First assume that $j = 0$ or 1 and that α is a face of codimension j. For $g, h \in G$, we have $g^{[\![k]\!]} \in \mathsf{Q}^{[\![k]\!]}(G)$ and $h^{(\alpha)} \in \mathsf{Q}^{[\![k]\!]}(G)$, and so $[g,h]^{(\alpha)} = [g^{[\![k]\!]}, h^{(\alpha)}] \in \mathsf{Q}^{[\![k]\!]}(G)_2$. It follows that $z^{(\alpha)} \in \mathsf{Q}^{[\![k]\!]}(G)_2$ for every $z \in G_2$.

Assume now that $\text{codim}(\alpha) = j \geq 2$. Write $\alpha = \beta \cap \gamma$, where β is a facet and $\text{codim}(\gamma) = j-1$. For $g \in G$ and $h \in G_{j-1}$, we have that both $g^{(\beta)}$ and $h^{(\gamma)}$ belong to $\mathsf{Q}^{[\![k]\!]}(G)$ and thus $[g,h]^{(\alpha)} = [g^{(\beta)}, h^{(\gamma)}] \in \mathsf{Q}^{[\![k]\!]}(G)_2$. It follows that $z^{(\alpha)} \in \mathsf{Q}^{[\![k]\!]}(G)_2$ for every $z \in G_j$, proving the result. □

Section 4. No changes are needed until the definition of $\mathsf{Q}^{[\![k]\!]}(X)$ in Section 4.3. We change the definition of this group and define:

$$\mathsf{Q}^{[\![k]\!]}(X) = \mathsf{Q}^{[\![k]\!]}(G)/\mathsf{Q}^{[\![k]\!]}(\Gamma; G), \tag{2}$$

where $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$ is the group in (1).

For each use of $\mathsf{Q}^{[\![k]\!]}(X)$ throughout the remainder of Section 4, we use this new modified definition. Analogously, we replace each occurrence in the chapter of $\mathsf{Q}^{[\![k]\!]}(\Gamma)$ by $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$, and in the proof of Proposition 28, we replace $\mathsf{Q}^{[\![k-1]\!]}(\Gamma)$ by $\mathsf{Q}^{[\![k-1]\!]}(\Gamma; G_s\Gamma)$. The proofs, appealing to the modified Proposition 19 and modified definition, are unchanged.

Modifications in Chapter 12: Cubic structures in nilmanifolds. This chapter carries out the study of the cubic structure $\mathsf{Q}^{[\![k]\!]}(X)$, where $X = G/\Gamma$ is a nilmanifold. We continue using the modification in the definition given in (2):

$$\mathsf{Q}^{[\![k]\!]}(X) = \mathsf{Q}^{[\![k]\!]}(G)/\mathsf{Q}^{[\![k]\!]}(\Gamma; G),$$

where $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$ is the group in (1). This introduces numerous small notational changes in the chapter, with the replacement of each occurrence of $\mathsf{Q}^{[\![k]\!]}(\Gamma)$ by $\mathsf{Q}^{[\![k]\!]}(\Gamma; G)$.

The only significant changes required in the proofs occur in Theorem 3, and so for completeness we include the full statement and proof:

Theorem 3 (Chapter 12)**.** *Let $(X = G/\Gamma, m_X, T)$ be an ergodic nilsystem. Then for each $k \in \mathbb{N}$, the topological system $(\mathsf{Q}^{[\![k]\!]}(X), \mathsf{Q}^{[\![k]\!]}(T))$ is minimal, and hence is uniquely ergodic.*

Proof. Let $k \geq 1$ and let $\tau \in G$ be the element defining T. Since $\mathsf{Q}^{[\![k]\!]}(G)$ is spanned by $g^{(\alpha_i)}$ for $g \in G$ and $i = 1, \ldots, k+1$, it follows that $\mathsf{Q}^{[\![k]\!]}(G)$ is spanned by $\mathsf{Q}^{[\![k]\!]}(G^0)$ and $\tau^{(\alpha_i)}$, $i = 1, \ldots, k+1$. Thus $\mathsf{Q}^{[\![k]\!]}(G)$ is also spanned by $(\mathsf{Q}^{[\![k]\!]}(G))^0$ and the elements $\tau^{(\alpha_i)}$, $i = 1, \ldots, k+1$.

To show that $(\mathsf{Q}^{[\![k]\!]}(X), \mathsf{Q}^{[\![k]\!]}(T))$ is minimal, by Theorem 17 of Chapter 11 it suffices to show that the compact abelian group

$$W_k = \frac{\mathsf{Q}^{[\![k]\!]}(G)}{\mathsf{Q}^{[\![k]\!]}(G)_2 \cdot \mathsf{Q}^{[\![k]\!]}(\Gamma; G)},$$

endowed with the transformations induced by $T^{(\alpha_i)}$ for $1 \leq i \leq k+1$, is minimal.

Let $Z = G/(G_2\Gamma)$, let $p\colon G \to Z$ be the associated the quotient map, and let $\sigma = p(\tau)$. Then the transformation induced by T on Z is the translation by σ, which we also denote by T.

We claim that W_k can be identified with the compact abelian group $\mathsf{Q}^{[\![k]\!]}(Z)$, endowed with the transformations $T^{(\alpha_i)}$, $1 \leq i \leq k+1$.

To see this identification, let $q\colon G \to G/G_2$ be the quotient homomorphism. By Propositions 19a and 19 of Chapter 6, we have that

$$\mathsf{Q}^{[\![k]\!]}(G)_2 = \mathsf{Q}^{[\![k]\!]}(G_2; G) = \mathsf{Q}^{[\![k]\!]}(G) \cap G_2^{[\![k]\!]} \tag{3}$$

and this group is the kernel of the group homomorphism $q^{[\![k]\!]}$ from $\mathsf{Q}^{[\![k]\!]}(G)$ onto $\mathsf{Q}^{[\![k]\!]}(G/G_2)$ (see Proposition 11 of Chapter 6). Therefore,

W_k is naturally identified with the quotient

$$\frac{\mathsf{Q}^{\llbracket k \rrbracket}(G/G_2)}{q^{\llbracket k \rrbracket}(\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma;G))}.$$

We compute the image of $\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma;G)$ under the homomorphism $q^{\llbracket k \rrbracket}\colon \mathsf{Q}^{\llbracket k \rrbracket}(G) \to \mathsf{Q}^{\llbracket k \rrbracket}(G/G_2)$. By definition, $\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma;G)$ is the group spanned by elements of the form $\gamma^{(\alpha)}$ where α is a face of $\llbracket k \rrbracket$ and $\gamma \in \Gamma \cap G_{\mathrm{codim}(\alpha)}$. Thus $q^{\llbracket k \rrbracket}(\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma;G))$ is the group spanned by the elements $q^{\llbracket k \rrbracket}(\gamma^{(\alpha)}) = (q(\gamma))^{(\alpha)}$ for the same values of γ and α. If $\mathrm{codim}(\alpha) \geq 2$, then $\gamma \in G_2$ and $q(\gamma)$ is the identity. Therefore, $q^{\llbracket k \rrbracket}(\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma;G))$ is the subgroup of $\mathsf{Q}^{\llbracket k \rrbracket}(G/G_2)$ spanned by the elements $(q(\gamma))^{(\alpha)}$, where $\gamma \in \Gamma$ and α is either $\llbracket k \rrbracket$ or is a facet. By definition (see Section 2.3 of Chapter 6), these elements are the generators of $\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma G_2/G_2)$. Thus again using Proposition 19 of Chapter 6 and noting that G/G_2 and $\Gamma G_2/G_2$ are abelian, we have that

$$q^{\llbracket k \rrbracket}(\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma;G)) = \mathsf{Q}^{\llbracket k \rrbracket}(\Gamma G_2/G_2) = \mathsf{Q}^{\llbracket k \rrbracket}(G/G_2) \cap (\Gamma G_2/G_2)^{\llbracket k \rrbracket}.$$

Thus the group $q^{\llbracket k \rrbracket}(\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma;G))$ is the kernel of the quotient homomorphism from $\mathsf{Q}^{\llbracket k \rrbracket}(G/G_2)$ onto $\mathsf{Q}^{\llbracket k \rrbracket}(Z)$. Therefore, we have the natural identifications

$$W_k = \frac{\mathsf{Q}^{\llbracket k \rrbracket}(G/G_2)}{q^{\llbracket k \rrbracket}(\mathsf{Q}^{\llbracket k \rrbracket}(\Gamma;G))} = \mathsf{Q}^{\llbracket k \rrbracket}(Z),$$

and the isomorphism from W_k to $\mathsf{Q}^{\llbracket k \rrbracket}(Z)$ is associated to the quotient homomorphism $p^{\llbracket k \rrbracket}$ from $\mathsf{Q}^{\llbracket k \rrbracket}(G)$ onto $\mathsf{Q}^{\llbracket k \rrbracket}(Z)$. Finally we check that the dynamics can be identified. For $i = 1, \dots, k+1$, the transformation induced by $T^{(\alpha_i)}$ of $\mathsf{Q}^{\llbracket k \rrbracket}(X)$ is the translation by $\tau^{(\alpha_i)}$ and thus the transformation of $\mathsf{Q}^{\llbracket k \rrbracket}(Z)$ induced by $T^{(\alpha_i)}$ is the translation by $p^{\llbracket k \rrbracket}(\tau^{(\alpha_i)}) = \sigma^{(\alpha_i)}$. This completes the proof of the claim, showing the identification of ${}_k$ with the compact abelian group $\mathsf{Q}^{\llbracket k \rrbracket}(Z)$ with the transformations $T^{(\alpha_i)}$, $1 \leq i \leq k+1$.

We are left with showing that $\mathsf{Q}^{\llbracket k \rrbracket}(Z)$, endowed with same the transformations, is minimal. We make use of the parametrization of $\mathsf{Q}^{\llbracket k \rrbracket}(Z)$ as elements of the form $(g_{\underline{\epsilon}}\colon \underline{\epsilon} \in \llbracket k \rrbracket)$, where $g_{\underline{\epsilon}} = z + \underline{\epsilon} \cdot \underline{t}$ for $z \in Z$ and $\underline{t} = (t_1, \dots, t_k) \in Z^k$ (see Section 2.4 of Chapter 6). We define the map $F\colon Z^{k+1} \to Z^{\llbracket k \rrbracket}$ by setting

$$F(z, \underline{t}) = \bigl(z + \underline{\epsilon} \cdot \underline{t}\colon \underline{\epsilon} \in \llbracket k \rrbracket\bigr), \quad \text{for } z \in Z \text{ and } \underline{t} = (t_1, \dots, t_k) \in Z^k.$$

Thus F is an isomorphism from Z^{k+1} onto $\mathsf{Q}^{\llbracket k \rrbracket}(Z)$, and this isomorphism is continuous. For $i = 1, \dots, k+1$, let $S_i\colon Z^{k+1} \to Z^{k+1}$ be the transformation obtained by adding σ to the i^{th} coordinate. Then F is an isomorphism from the system $(Z, S_1, \dots, S_{k+1})$ to the system

$(\mathsf{Q}^{[\![k]\!]}(Z), T^{(\alpha_1)}, \ldots, T^{(\alpha_{k+1})})$. Since the first system is minimal, so is the second one. In particular, $\mathsf{Q}^{[\![k]\!]}(Z)$, endowed the transformations $T^{(\alpha_i)}$ for $1 \leq i \leq k+1$, is minimal. $\square$