IC-RIGIDITY AND GENERALIZATIONS OF A THEOREM OF MAHLER

BRYNA KRA AND SCOTT SCHMIEDING

Abstract. Mahler showed that for every irrational α and every base $b \geq 2$, there is a single positive integer n such that every digit in $\{0, \dots, b-1\}$ occurs infinitely often in the base-b expansion of $n\alpha$. Furstenberg later observed that dynamical methods can be used to give a different proof. Building on this perspective, we give generalizations of Mahler's theorem in two directions: the multipliers are a syndetic subset of certain multiplicative semigroups, and analogous versions hold in higher dimensions. Our main tool is IC-rigidity, a type of rigidity which is a hybrid of topological and measurable constraints.

1. Introduction

For an irrational $\alpha \in \mathbb{R}$, the orbit $\{n\alpha \mod 1 : n \geq 1\}$ is dense in the circle $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, and so in particular any prescribed finite block of digits occurs at the beginning of the expansion of some multiple $n\alpha$. Mahler [12] improved this, showing that for every irrational α and every base $b \geq 2$, there is a single positive integer n such that every digit in $\{0, \dots, b-1\}$ occurs infinitely often in the base-b expansion of $n\alpha$. Using an observation of Furstenberg, Alon and Peres [1, Corollary 7.2] used dynamical methods to give a simple proof of this theorem, and further showed that the set of n with this property can be restricted to certain subsets of $\mathbb{N}$. A quantitative improvement, along with a polynomial version, was given in Berend and Boshernitzan [2].

However, Mahler showed that more holds, meaning that the set of iterates with this property is large in a multiplicative sense. The set $J \subset \mathbb{N}$ is *multiplicatively syndetic* if there is a nonempty finite set $F \subset \mathbb{N}$ such that for all $n \in \mathbb{N}$, we have $nF \cap J \neq \emptyset$, where $nF = \{nf : f \in F\}$. Mahler showed that the set of n such that $n\alpha$ contains every digit in the base-b expansion is multiplicatively syndetic. Equivalently, the density of these iterates is positive with respect to every multiplicative Følner sequence Φ, and the lower bound can be chosen uniformly over all Følner sequences.

We use dynamical methods to give a new proof of Mahler's theorem, and further give a strengthening of conditions on the iterates, showing that they form a multiplicatively syndetic subset.

Theorem 1.1. *Let $1 \leq r \in \mathbb{N}$, let $b \geq 2$, and let $\alpha \in \mathbb{R}$ be irrational. Then*

$$\{n^r : \text{ every } 0 \leq i \leq b-1 \text{ appears infinitely often in the base } b\text{-expansion of } n^r\alpha\}$$

is a multiplicatively syndetic subset of the multiplicative semigroup $\{n^r : n \in \mathbb{N}\}$.

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It follows immediately that the set of n with this property is infinite. Mahler's theorem corresponds to the case $r = 1$.

This theorem is deduced as a corollary of a more general statement in Theorem 2.8: the result holds when $\{n^r : n \geq 1\}$ is replaced by any multiplicative semigroup having a largeness property that we call the *density Glasner property* (see Section 2.1).

The dynamical methods we use build on our previous work in [10]. For a compact metric space X, let $\mathcal{K}(X)$ denote the space of nonempty compact subsets of X, endowed with the Hausdorff metric, and we denote this metric by $d(\cdot, \cdot)$. Given a countable group (or semigroup) G acting continuously on X, an *invariant random compact* (IRC) is a Borel probability measure μ on $\mathcal{K}(X)$ which is invariant under the induced action of G on $\mathcal{K}(X)$. Adding in an assumption that each $g \in G$ acts surjectively on X, then the delta mass δ_X is always an IRC.

We say the action is *IC-rigid* if the only IRC μ such that μ-almost every compact is infinite is $\mu = \delta_X$. In [10] we showed that the *multiplicative action of* $\mathbb{N}$ *on* $\mathbb{T}$, given by $T_n \colon \mathbb{T} \to \mathbb{T}$ by $T_n(x) = nx \mod 1$ for all $n \geq 1$, has a weaker form of rigidity. We strengthen this to show that this action is IC-rigid, further showing that the same holds for sparser multiplicative semigroups in $\mathbb{N}$.

Theorem 1.2. *For every natural number $r \geq 1$, the action of the multiplicative semigroup $\{n^r : n \in \mathbb{N}\}$ by multiplication on $\mathbb{T}$ is IC-rigid.*

In particular, the multiplicative action of $\mathbb{N}$ on $\mathbb{T}$ is IC-rigid. As a corollary of this result, we show for every $r \geq 1$, $\varepsilon > 0$, and infinite compact set $Y \subset \mathbb{T}$ that is invariant under multiplication by p for some $p \geq 2$, there are many dilations of the form $T_{n^r}(Y)$ which are ε-dense (a precise statement is given in Corollary 2.7). This result, in turn, then allows us to deduce Theorem 1.1 in Section 2.

Studying the action of $\mathrm{SL}_d(\mathbb{Z})$ on $\mathbb{T}^d$, we obtain a higher dimensional version of Mahler's Theorem. Recall for a semigroup $\mathcal{S}$, a subset $J \subset \mathcal{S}$ is syndetic if there exists a finite set $F \subset \mathcal{S}$ such that $\mathcal{S} = \bigcup_{f \in F} f^{-1}J$ where $f^{-1}J = \{s \in \mathcal{S} : fs \in J\}$. To guarantee that $\delta_{\mathbb{T}^d}$ is an invariant IRC, we restrict our attention to the subset $\mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z})$ of $d \times d$ nonsingular integer matrices. For a $d \times d$ integer matrix A we define $T_A \colon \mathbb{T}^d \to \mathbb{T}^d$ by $T_A(x) = Ax \mod \mathbb{Z}^d$. Given a base $b \geq 2$ and $x \in \mathbb{T}^d$, we write

$$x = \sum_{r=1}^{\infty} \frac{c_r(x)}{b^r}$$

where each $c_r(x) = (x_r^{(1)}, \ldots, x_r^{(d)})$ satisfies $0 \leq x_r^{(i)} \leq b - 1$ for every $1 \leq i \leq d$.

Theorem 1.3. *Let $\alpha \in \mathbb{T}^d$ have at least one irrational coordinate. Then the set*

$$\{A \in \mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z}) : \text{ for every } w \in \{0, \ldots, b-1\}^d,\ c_r(T_A(\alpha)) = w \text{ for infinitely many } r\}$$

is syndetic in $\mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z})$.

Taking $d = 1$, we recover Mahler's theorem. Again IC-rigidity is the key ingredient; we prove this holds for certain subgroups of $\mathrm{SL}_d(\mathbb{Z})$.

Theorem 1.4. *Let Γ be a subgroup of $SL_d(\mathbb{Z})$ for $d \geq 2$ whose Zariski closure is Zariski-connected, and having the property that the action of Γ on $\mathbb{R}^d$ is irreducible. Then the action of Γ on $\mathbb{T}^d$ is IC-rigid.*

For example, the theorem applies to any subgroup of $\mathrm{SL}_d(\mathbb{Z})$ which is Zariski-dense in $\mathrm{SL}_d(\mathbb{R})$, such as any finite index subgroup of $\mathrm{SL}_d(\mathbb{Z})$.

Our approach is similar to that used in Theorem 1.2, but since the groups in question are not in general amenable we can not make use of Følner sequences. Instead, we make use of results on random walks for certain subgroups of $\mathrm{SL}_d(\mathbb{Z})$ in the work of He and de Saxce [9] and certain bounds from Bulinski and Fish [4].

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Methodologies. AI use was restricted to give a mathematical audit of the paper, resulting in the correction of small typos.

2. Multiplicatively syndetic Mahler Theorem

2.1. **IC-rigidity for certain multiplicative semigroups of $\mathbb{N}$.** We start by proving the action of certain multiplicative semigroups of $\mathbb{N}$ on $\mathbb{T}$ is IC-rigid. For $n \geq 1$, we define $T_n \colon \mathbb{T} \to \mathbb{T}$ by setting

$$T_n(x) = nx \bmod 1.$$

Given $\varepsilon > 0$ we say a set $Y \subset \mathbb{T}$ is ε-dense in $\mathbb{T}$ if for every $x \in \mathbb{T}$ there exists $y \in Y$ such that $d(x, y) < \varepsilon$. We define

$$D_\varepsilon = \{Y \in \mathcal{K}(\mathbb{T}) : Y \text{ is } \varepsilon\text{-dense in } \mathbb{T}\}.$$

Equivalently, we have that $D_\varepsilon = \{Y \in \mathcal{K}(\mathbb{T}) : d(Y, \mathbb{T}) < \varepsilon\}$. We let $\mathbf{1}_{D_\varepsilon}$ denote the indicator function of D_ε.

Theorem 2.1. *If $\mathcal{S} \subset \mathbb{N}$ is a multiplicative semigroup such that for every infinite $Y \subset \mathbb{T}$ and $\varepsilon > 0$ we have*

$$\lim_{N\to\infty} \frac{1}{|\mathcal{S} \cap [1, N]|} \sum_{n\in\mathcal{S}\cap[1,N]} \mathbf{1}_{D_\varepsilon}(T_n Y) = 1,$$

then the action of $\mathcal{S}$ on $\mathbb{T}$ by multiplication is IC-rigid.

Glasner [5] showed that for every infinite set $Y \subset \mathbb{T}$ and $\varepsilon > 0$, some integer dilation nY is ε dense. This motivates us to refer to a multiplicative semigroup $\mathcal{S} \subset \mathbb{N}$ satisfying the hypothesis of Theorem 2.1 as having the *density Glasner property.*

Proof. Suppose $\mathcal{S} \subset \mathbb{N}$ is a multiplicative semigroup having the density Glasner property. Let μ be an IRC for $\mathcal{S}$ acting on $\mathbb{T}$ such that μ-almost every compact set is infinite and let $\varepsilon > 0$. For $N \in \mathbb{N}$ and $\varepsilon > 0$, define $f_{N,\varepsilon} \colon \mathcal{K}(\mathbb{T}) \to \mathbb{R}$ by

$$f_{N,\varepsilon}(Y) = \frac{1}{|\mathcal{S} \cap [1, N]|} \sum_{n\in\mathcal{S}\cap[1,N]} \mathbf{1}_{D_\varepsilon}(T_n Y).$$

Since μ is T_n-invariant for every $n \in \mathcal{S}$, we have

$$\mu(D_\varepsilon) = \mu(T_n^{-1}(D_\varepsilon)) = \mu\left(\{Y \in \mathcal{K}(\mathbb{T}) : T_n Y \in D_\varepsilon\}\right).$$

Then for every $N \in \mathbb{N}$ we have

$$\int_{\mathcal{K}(\mathbb{T})} f_{N,\varepsilon} d\mu = \int_{\mathcal{K}(\mathbb{T})} \frac{1}{|\mathcal{S} \cap [1, N]|} \sum_{n\in\mathcal{S}\cap[1,N]} \mathbf{1}_{D_\varepsilon}(T_n Y) d\mu = \mu(D_\varepsilon).$$

By the assumption on $\mathcal{S}$ and since μ-almost every Y is infinite, the sequence $f_{N,\varepsilon}$ converges pointwise μ-almost everywhere to the constant function 1. By the dominated convergence theorem it follows that $\mu(D_\varepsilon) = 1$ and hence

$$\mu\left(\bigcap_{m=1}^{\infty} D_{\frac{1}{m}}\right) = 1.$$

Since $\bigcap_{m=1}^{\infty} D_{\frac{1}{m}} = \{\mathbb{T}\}$, it follows that $\mu = \delta_{\mathbb{T}}$. □

We show that some natural multiplicative semigroups in $\mathbb{N}$ have the density Glasner property. For $\mathcal{S} \subset \mathbb{N}$ and $N \geq 1$, write

$$A_N^{\mathcal{S}}(t) = \frac{1}{|\mathcal{S} \cap [1,N]|} \sum_{n \in \mathcal{S} \cap [1,N]} e^{2\pi i n t}.$$

Lemma 2.2. *Suppose $\mathcal{S} \subset \mathbb{N}$ is a multiplicative semigroup such that the associated $A_N^{\mathcal{S}}(t)$ satisfies the following:*

(1) If $t \in \mathbb{T}$ is irrational, then $A_N^{\mathcal{S}}(t) \to 0$ as $N \to \infty$.
(2) There exists a function $\kappa\colon \mathbb{N} \to \mathbb{R}^+$ such that $\kappa(R) \to 0$ as $R \to \infty$ and such that for every rational $\frac{p}{q} \in \mathbb{T}$ written in reduced form with $q > R$, we have $\limsup_{N\to\infty} |A_N^{\mathcal{S}}(\frac{p}{q})| \leq \kappa(R)$.

Then $\mathcal{S}$ has the density Glasner property.

We use the following standard Fourier detection result (see for instance [4, Lemma 2.3]).

Lemma 2.3. *Let $0 < \varepsilon < \frac{1}{2}$. There exists $M = M(\varepsilon) \geq 1$ and $C = C(\varepsilon) > 0$ such that for every $k \geq 1$, if $\{z_1, \ldots, z_k\}$ is not ε-dense in $\mathbb{T}$, then*

$$C \sum_{m=1}^{M} \Big|\frac{1}{k} \sum_{j=1}^{k} e^{2\pi i m z_j}\Big|^2 \geq 1.$$

Note both M and C depend on ε, but do not depend on either k or on the set $\{z_1, \ldots, z_k\}$.

Proof of Lemma 2.2. Suppose $\mathcal{S} \subset \mathbb{N}$ is a semigroup satisfying the hypotheses of Lemma 2.2. It suffices to show that for every infinite $Y \subset \mathbb{T}$ and every $\varepsilon > 0$, we have that

$$\lim_{N\to\infty} \frac{|\{n \in \mathcal{S} \cap [1,N] : T_n Y \text{ is not } \varepsilon\text{-dense in } \mathbb{T}\}|}{|\mathcal{S} \cap [1,N]|} = 0.$$

Suppose then that Y is infinite and fix R and k. Then we can choose $E_k = \{z_1, \ldots, z_k\} \subset Y$ such that $\operatorname{diam}(E_k) < \frac{1}{MR}$, and so $\max_{i,j}\{d(z_i, z_j)\} < \frac{1}{MR}$. Then for every $i \neq j$ and every $1 \leq m \leq M$, we have

$$0 < |m(z_i - z_j)| < \frac{1}{R}.$$

It follows that $m(z_i - z_j)$ is either irrational, or is rational with denominator greater than R (in reduced form). Note

$$\frac{1}{|\mathcal{S} \cap [1,N]|} \sum_{n \in \mathcal{S} \cap [1,N]} \Big|\frac{1}{k} \sum_{j=1}^{k} e^{2\pi i m n z_j}\Big|^2 = \frac{1}{k^2} \sum_{i,j=1}^{k} A_N^{\mathcal{S}}(m(z_i - z_j)).$$

If T_nY is not ε-dense, then T_nE_k is not ε-dense and so

$$\frac{|\{n \in \mathcal{S} \cap [1,N] : T_nY \text{ is not } \varepsilon\text{-dense in } \mathbb{T}\}|}{|\mathcal{S} \cap [1,N]|}$$
$$\leq \frac{C}{|\mathcal{S} \cap [1,N]|} \sum_{n \in \mathcal{S} \cap [1,N]} \sum_{m=1}^{M} \left|\frac{1}{k}\sum_{j=1}^{k} e^{2\pi i m n z_j}\right|^2 = \frac{C}{k^2}\sum_{m=1}^{M}\sum_{i,j=1}^{k} A_N^{\mathcal{S}}(m(z_i - z_j)).$$

Taking absolute values and lim sup as $N \to \infty$ and applying hypotheses (1) and (2) to each pair $z_i - z_j$, we have that

$$\limsup_{N\to\infty} \frac{|\{n \in \mathcal{S} \cap [1,N] : T_nY \text{ is not } \varepsilon\text{-dense in } \mathbb{T}\}|}{|\mathcal{S} \cap [1,N]|}$$
$$\leq CM\frac{1}{k^2}(k + k^2\kappa(R)) = CM\left(\frac{1}{k} + \kappa(R)\right)$$

(note that there are k cases where $i = j$). Taking $R \to \infty$ and $k \to \infty$ gives the statement. □

Berend and Peres [3] showed that for every infinite compact set $Y \subset \mathbb{T}$ and every $\varepsilon > 0$, the set of integers n such that T_nY is ε-dense has natural density one. Berend and Boshernitzan [2] extended Mahler's theorem to polynomial sequences. We extend these results to show that the density conclusion also holds for powers, and this result immediately implies Theorem 1.2.

Theorem 2.4. *Let $1 \leq r \in \mathbb{N}$ and let $\mathcal{S} = \{n^r : n \geq 1\}$. Then $\mathcal{S}$ has the density Glasner property, and hence the action of $\mathcal{S}$ by multiplication on $\mathbb{T}$ is IC-rigid.*

Proof. Fix $r \geq 1$ and let $\mathcal{S} = \{n^r : n \geq 1\}$. We check that $\mathcal{S}$ satisfies Lemma 2.2. For t irrational, by Weyl equidistribution (polynomial form) we have

$$\lim_{N\to\infty} A_N(t) = \lim_{N\to\infty} \frac{1}{\lfloor N^{\frac{1}{r}} \rfloor} \sum_{n \leq \lfloor N^{\frac{1}{r}} \rfloor} e^{2\pi i n^r t} = 0.$$

Suppose that $t = p/q$ in reduced form. Consider $F_r(p,q) = \frac{1}{q}\sum_{n=1}^{q} e^{2\pi i p n^r/q}$ and let $M_N = |\{n : n^r \leq N\}|$. Write $M_N = kq + s$ for some $0 \leq s < q$. Then

$$A_N(p/q) = \frac{kq}{M_N}F_r(p,q) + \frac{1}{M_N}\sum_{n=1}^{s} e^{2\pi i p n^r/q}$$

and so

$$\limsup_{N\to\infty} |A_N(p/q)| = |F_r(p,q)|.$$

By [7, 8] (also see the discussion in [11]), there exists a constant $C_r > 0$ (by [13], one can take $C_r = e^{C(\frac{r}{\varphi(r)})^2}$ for some constant C) such that $|qF_r(p,q)| \leq C_r q^{1-\frac{1}{r}}$ and so

$$|F_r(p,q)| \leq C_r q^{-\frac{1}{r}}.$$

Using $\kappa(R) = C_r R^{-\frac{1}{r}}$, we have $\kappa(R) \to 0$ as $R \to \infty$, and for $q > R$ we have

$$\limsup_{N\to\infty} |A_N(p/q)| \leq \kappa(R).$$

It follows by Lemma 2.2 that $\mathcal{S}$ is density Glasner, and hence by Theorem 2.1, the action of $\mathcal{S}$ is IC-rigid. □

Applying Theorem 2.4 with $r = 1$, we deduce the following.

Theorem 2.5. *Suppose $\mathcal{S} \subset \mathbb{N}$ is a multiplicative semigroup with positive lower density. Then the action of $\mathcal{S}$ on $\mathbb{T}$ by multiplication is IC-rigid.*

Proof. We follow the proof of Theorem 2.1, and so it suffices to show that for Y infinite, we have

$$\lim_{N\to\infty} \frac{|\{n \in \mathcal{S} \cap [1,N] : T_n Y \text{ is not } \varepsilon\text{-dense in } \mathbb{T}\}|}{|\mathcal{S} \cap [1,N]|} = 0.$$

Let Y be infinite and $\varepsilon > 0$. Let $B = \{n \in \mathbb{N} : T_n Y \text{ is not } \varepsilon\text{-dense in } \mathbb{T}\}$. Since $\mathcal{S}$ has positive lower density, there exists $\delta > 0$ such that $|\mathcal{S} \cap [1,N]| \geq \delta N$ for all sufficiently large N. Then

$$\limsup_{N\to\infty} \frac{|B \cap \mathcal{S} \cap [1,N]|}{|\mathcal{S} \cap [1,N]|} \leq \limsup_{N\to\infty} \frac{|B \cap [1,N]|}{|\mathcal{S} \cap [1,N]|} \leq \limsup_{N\to\infty} \frac{1}{\delta} \frac{|B \cap [1,N]|}{N} = 0,$$

by noting that the limit holds for $\mathcal{S} = \mathbb{N}$ by the $r = 1$ case of Theorem 2.4. □

2.2. **Further corollaries of IC-rigidity for the action of $\mathbb{N}$ on $\mathbb{T}$.** Replacing the use of weak IC-rigidity in [10] by the stronger IC-rigidity given in Theorems 2.4 and 2.5, we deduce several corollaries. Though the proofs are analogous, we include them for completeness.

Theorem 2.6. *Let $Y \subset \mathbb{T}$ be an infinite compact subset invariant under T_p for some $p \geq 2$ and let $\mathcal{S} \subset \mathbb{N}$ be a multiplicative semigroup with the density Glasner property. Then for every Følner sequence $(F_m)_{m\in\mathbb{N}}$ of the multiplicative semigroup $\mathcal{S}$ and every $\varepsilon > 0$, we have*

$$\lim_{m\to\infty} \frac{1}{|F_m|} \big|\{n \in F_m : T_n Y \text{ is } \varepsilon\text{-dense in } \mathbb{T}\}\big| = 1$$

Proof. Let $\mathcal{I}_p = \{Z \in \mathcal{K}(\mathbb{T}) : T_p Z = Z \text{ and } Z \text{ is infinite}\}$. Suppose $Y \in \mathcal{I}_p$ for some $p \geq 2$ and set $\mu_m = \frac{1}{|F_m|} \sum_{n\in F_m} \delta_{T_n Y}$. It suffices to show that the only weak* accumulation point of μ_m is $\delta_{\mathbb{T}}$. Every weak* accumulation point of μ_m is $\mathcal{S}$-invariant, and by Theorem 2.1 the action of $\mathcal{S}$ on $\mathbb{T}$ is IC-rigid, and so it suffices to show that for every weak* accumulation point μ of the measures $(\mu_m)_{m\in\mathbb{N}}$, we have $\mu(\mathcal{K}_{\text{fin}}(\mathbb{T})) = 0$.

We show that $\mu(\mathcal{K}_{\leq r}(\mathbb{T})) = 0$ for every $r \geq 1$. Since $\mathcal{K}_{\text{fin}}(\mathbb{T}) = \bigcup_{r\geq 1} \mathcal{K}_{\leq r}(\mathbb{T})$, this completes the proof.

Fix $r \geq 1$. We claim there exists $\delta_r > 0$ such that $d(\mathcal{I}_p, \mathcal{K}_{\leq r}(\mathbb{T})) \geq \delta_r$. If not, then since $\mathcal{K}_{\leq r}(\mathbb{T})$ is compact, there exists a finite set $E \in \mathcal{K}_{\leq r}(\mathbb{T})$ and sequence $Y_j \in \mathcal{I}_p$ such that $Y_j \to E$ in $\mathcal{K}(\mathbb{T})$ as $j \to \infty$. Since every $Y_j \in \mathcal{I}_p$ is T_p-invariant, it follows that E is as well. Choose $k \in \mathbb{N}$ such that $T_p^k(x) = x$ for every $x \in E$. Then there exists $\delta > 0$ such that $T_p^k(B(x,\delta)) \cap B(y,\delta) = \emptyset$ for every $x, y \in E$ with $x \neq y$, and T_p^k acts on $B(x,\delta)$ by a linear expansion by p^k for every $x \in E$. Then for some $j \in \mathbb{N}$, we have

$$Y_j \subset \bigcup_{x\in E} B(x,\delta).$$

Since Y_j is T_p-invariant, we have $T_p^k Y_j = Y_j$. Thus if $z \in Y_j \cap B(x,\delta)$ for some $x \in E$, then $T_p^k(z) \in Y_j \cap B(x,\delta)$, and hence $T_p^{\ell k}(z) \in Y_j \cap B(x,\delta)$ for every $\ell \geq 1$. However since T_p^k acts by a linear expansion by p^k on $B(x,\delta)$, this is only possible if $z = x$. Thus $Y_j \subset E$ contradicting that Y_j is infinite and proving the claim.

Since $T_nY \in \mathcal{I}_p$ for every $n \in \mathcal{S}$, it follows that $d(T_nY, K_{\leq r}(\mathbb{T})) \geq \delta_r$ for all such n. Letting $B_{\delta_r}(\mathcal{K}_{\leq r}(\mathbb{T})) = \{Y \in \mathcal{K}(T) : d(Y, \mathcal{K}_{\leq r}(\mathbb{T})) < \delta_r\}$, it follows that $\mu_m(B_{\delta_r}(\mathcal{K}_{\leq r}(\mathbb{T}))) = 0$ for every $m \geq 1$. Since $B_{\delta_r}(\mathcal{K}_{\leq r}(\mathbb{T}))$ is open, this means that $\mu(B_{\delta_r}(\mathcal{K}_{\leq r}(\mathbb{T}))) = 0$ and $\mu(\mathcal{K}_{\leq r}(\mathbb{T})) = 0$. □

Corollary 2.7. *Let $Y \subset \mathbb{T}$ be an infinite compact subset invariant under T_p for some $p \geq 2$, and let $\mathcal{S} \subset \mathbb{N}$ be a multiplicative semigroup with the density Glasner property. Then for every $\varepsilon > 0$, the set*

$$\{n \in \mathcal{S} : T_nY \text{ is } \varepsilon\text{-dense in } \mathbb{T}\}$$

is a multiplicatively syndetic subset of $\mathcal{S}$.

Proof. It suffices to show that if $J \subset \mathcal{S}$ satisfies

$$\lim_{k\to\infty} \frac{|J \cap F_k|}{|F_k|} = 1$$

for every Følner sequence for the multiplicative semigroup $\mathcal{S}$, then J is multiplicatively syndetic in $\mathcal{S}$. To see this, if J is not multiplicatively syndetic, then for every finite $F \subset \mathcal{S}$, there exists some $r \in \mathcal{S}$ such that $rF \subset \mathcal{S} \setminus J$. Let F_m be a Følner sequence for the multiplicative semigroup $\mathcal{S}$. For each m, pick $r_m \in \mathcal{S}$ such that $F_m r_m \subset \mathcal{S} \setminus J$ and set $\Psi_m = F_m r_m$. Then (Ψ_m) is also a Følner sequence for $\mathcal{S}$, and by construction lies entirely in $\mathbb{N} \setminus J$. □

2.3. **Generalizing Mahler's Theorem.**

Theorem 2.8. *Let $\mathcal{S} \subset \mathbb{N}$ be a multiplicative semigroup having the density Glasner property, let $1 \leq b \in \mathbb{N}$, and let $\alpha \in \mathbb{R}$ be irrational. Then*

$$\{n \in \mathcal{S} : \text{ every } 0 \leq i \leq b-1 \text{ appears infinitely often in the base } b\text{-expansion of } n\alpha\}$$

is a multiplicatively syndetic subset of $\mathcal{S}$.

Proof. Let $\alpha \in \mathbb{R}$ be irrational and $2 \leq b \in \mathbb{N}$. Let $\mathcal{B}$ be the set of limit points of the set $\{b^k\alpha \bmod 1 : k \geq 1\} \subset \mathbb{T}$. The set $\mathcal{B}$ is T_b-invariant. We claim $\mathcal{B}$ is infinite. Indeed, suppose instead that $\mathcal{B}$ were finite. Then since $\mathcal{B}$ is T_b-invariant, there exists $k \geq 1$ such that T_b^k pointwise fixes each element of $\mathcal{B}$, and every element of $\mathcal{B}$ is rational. Choosing small neighborhoods of each point in $\mathcal{B}$ and using the fact that T_b locally expands, it follows that $T_b^j(\alpha) \in \mathcal{B}$ for some j, which is a contradiction since α is irrational.

Set $\varepsilon = \frac{1}{2b}$ and define

$$J = \{n \in \mathcal{S} : T_n\mathcal{B} \text{ is } \varepsilon\text{-dense in } \mathbb{T}\}.$$

If $m \in J$ then $m\alpha$ in base b contains every $0 \leq i \leq b-1$ infinitely often. Indeed, if $m \in J$, then for every $0 \leq i \leq b-1$ we have $T_m\mathcal{B} \cap (\frac{i}{b}, \frac{i+1}{b}) \neq \emptyset$. Since each point in $T_m\mathcal{B}$ is an accumulation point of $\{b^n m\alpha \bmod 1 : n \geq 1\}$, it follows that infinitely many points from $\{b^n m\alpha \bmod 1 : n \geq 1\}$ belong to $(\frac{i}{b}, \frac{i+1}{b})$. Thus the result follows from Corollary 2.7. □

We refer to the proofs of [10, Theorem 5.7, Corollary 5.8] for the proofs of the next two.

Corollary 2.9. *Let $Y \subset \mathbb{T}$ be an infinite compact subset invariant under T_p for some $p \geq 2$, and suppose $\mathcal{S} \subset \mathbb{N}$ is a multiplicative semigroup with the density*

Glasner property. Then for every Følner sequence $(F_m)_{m\in\mathbb{N}}$ of $\mathcal{S}$, there exists a set $J \subset \mathcal{S}$ such that

$$\lim_{m\to\infty} \frac{|J \cap F_m|}{|F_m|} = 1$$

and

$$\lim_{n\to\infty, n\in J} T_n Y = \mathbb{T} \quad \text{in} \quad \mathcal{K}(\mathbb{T}).$$

Corollary 2.10. *Let $Y \subset \mathbb{T}$ be an infinite compact subset invariant under T_p for some $p \geq 2$ and suppose $\mathcal{S} \subset \mathbb{N}$ is a multiplicative semigroup with the density Glasner property. Let $W \subset \mathcal{S}$ be finite and fix $\varepsilon > 0$. Then for every Følner sequence $(F_m)_{m\in\mathbb{N}}$ of the multiplicative semigroup $\mathcal{S}$, we have*

$$\lim_{m\to\infty} \frac{1}{|F_m|} |\{n \in F_m : T_{wn}Y \text{ is } \varepsilon\text{-dense in } \mathbb{T} \text{ for all } w \in W\}| = 1.$$

Remark 2.11. Using Corollary 2.10 with $\mathcal{S} = \mathbb{N}$ we also have the following slightly stronger version of Theorem 2.8: Given $b, k \geq 2$ and $\alpha \in \mathbb{R}$ irrational, the set of $n \in \mathbb{N}$ such that every $0 \leq i \leq b-1$ appears infinitely often in the base b-expansions of each of $n\alpha, 2n\alpha, \ldots, kn\alpha$ is multiplicatively syndetic.

We also have the following.

Corollary 2.12. *Let $Y \subset \mathbb{T}$ be an infinite compact subset invariant under T_p for some $p \geq 2$ and let $\varepsilon > 0$. There exists a finite set $F \subset \mathbb{N}$ such that for every $n \geq 1$, there exists $f \in F$ such that the set*

$$J(n, \varepsilon) = \{k \in \mathbb{N} : T_{fn^k}Y \text{ is } \varepsilon\text{-dense in } \mathbb{T}\}$$

has positive upper density.

Proof. This is a consequence of Corollary 2.7 with $\mathcal{S} = \mathbb{N}$. Indeed, given such Y and $\varepsilon > 0$, by Corollary 2.7 the set

$$J = \{m \in \mathbb{N} : T_m Y \text{ is } \varepsilon\text{-dense in } \mathbb{T}\}$$

is multiplicatively syndetic. Choose $F = \{f_1, \ldots, f_k\}$ such that for every $m \in \mathbb{N}$ we have $mF \cap J \neq \emptyset$. Fix $n \geq 1$. Then for every $j \geq 1$, there exists $1 \leq i \leq k$ such that $f_i n^j \in J$, and so

$$\bigcup_{i=1}^{k} \{j : f_i n^j \in J\} = \mathbb{N}.$$

It follows that there exists some $f \in F$ such that

$$\overline{d}\left(\{j : fn^j \in J\}\right) \geq \frac{1}{|F|} > 0,$$

where $\overline{d}$ denotes the upper density. □

2.4. **Further IC-rigid actions.** We remark that there are other mechanisms for producing IC-rigid actions, and note one of them here. Recall an action of G on X is *extremely proximal* if for every $X \neq Y \in \mathcal{K}(X)$ there exists $x \in X$ such that $\{x\} \in \overline{\{gY : g \in G\}}$. See Glasner [6] for further background.

Proposition 2.13. *If (X, G) is extremely proximal and minimal with X infinite, then it is IC-rigid.*

Proof. Suppose (X, G) is extremely proximal and also minimal, and let $X \neq Y \in \mathcal{K}(X)$. Minimality together with extreme proximality implies for every $x \in X$ we have $\{x\} \in \overline{\{gY : g \in G\}}$. It follows that if $\emptyset \neq U \subset X$ is open then there exists $g \in G$ such that $gY \subset U$.

Fix $Z \in \mathcal{K}(X)$ with $Z \neq X$ and write $E(Z) = \{Y \in \mathcal{K}(X) : Y \subset Z\}$. Given $N \geq 1$, since X is infinite, we may choose nonempty disjoint open sets $U_1, \dots, U_N$ in X. By the above, for each $1 \leq i \leq N$ there exists $g_i \in G$ such that $g_i Z \subset U_i$, and hence the sets $g_i Z$ are pairwise disjoint. Suppose μ is an IRC for (X, G). Using invariance of μ we have

$$N\mu(E(Z)) = \sum_{i=1}^{N} \mu(E(Z)) \leq \sum_{i=1}^{N} \mu(g_i E(Z)) \leq 1$$

since each of the $g_i E(Z)$ are disjoint. Since this holds for all $N \geq 1$ we get that $\mu(E(Z)) = 0$. If V_i is a countable basis for X, then

$$\mathcal{K}(X) \setminus \{X\} \subset \bigcup_{i=1}^{\infty} E(X \setminus V_i)$$

and it follows that $\mu = \delta_X$. □

3. Subgroups of $\mathrm{SL}_d(\mathbb{Z})$

3.1. **IC-rigidity for certain subgroups of $SL_d(\mathbb{Z})$.** Throughout, we fix some $2 \leq d \in \mathbb{N}$. We slightly abuse notation and write

$$D_\varepsilon = \{Y \in \mathcal{K}(\mathbb{T}^d) : d(Y, \mathbb{T}^d) < \varepsilon\}.$$

Theorem 3.1. *Let Γ be a subgroup of $SL_d(\mathbb{Z})$ whose Zariski closure is Zariski-connected, and having the property that the action of Γ on $\mathbb{R}^d$ is irreducible. Then the action of Γ on $\mathbb{T}^d$ is IC-rigid.*

We make a slight abuse of notation and write the action of Γ on $\mathbb{T}^d$ as gE for $g \in \Gamma$ and $E \subset \mathbb{T}^d$.

For the remainder of this section, we fix a subgroup Γ of $\mathrm{SL}_d(\mathbb{Z})$ whose Zariski closure is Zariski-connected, and having the property that the action of Γ on $\mathbb{R}^d$ is irreducible. We let ρ be a fully-supported probability measure on Γ satisfying $\sum_{g \in \Gamma} \|g\|^\varepsilon \rho(g) < \infty$ for some $\varepsilon > 0$, and write ρ^{*n} for the n-fold convolution.

Before stating the main technical result, we recall a result that is implicit in the proof of [4, Theorem A]. We include the proof for convenience.

Lemma 3.2. *Let $\varepsilon > 0$. There exists $C_\varepsilon < \infty$ and $\eta > 0$ such that for every $E = \{x_1, \dots, x_k\} \subset \mathbb{T}^d$, we have*

$$\limsup_{n \to \infty} \rho^{*n} \left(\{g \in \Gamma : gE \text{ is not } \varepsilon\text{-dense in } \mathbb{T}^d\} \right) \leq C_\varepsilon (k^{-\eta} + k^{-1}),$$

and the constant C_ε does not depend on k.

Proof. Set $M = \lceil d/\varepsilon \rceil$ and define

$$\Phi_E(g) = \sum_{v \in B(M)} \sum_{i,j=1}^{k} e^{2\pi i (v \cdot g(x_i - x_j))}$$

where $B(M) = \{0 \neq v \in \mathbb{Z}^d : \|v\|_\infty \leq M\}$. Note $\Phi_E(g)$ is real and nonnegative, since it is also equal to $\sum_{v\in B(M)} |\sum_{i=1}^k e^{2\pi i v\cdot g(x_i)}|^2$. Using [4, Lemma 2.3], we have that if gE is not ε-dense, then

$$k^2 \leq C_d \varepsilon^{-d} \Phi_E(g) \tag{1}$$

for some constant C_d. We consider two cases. If $x_i - x_j$ is irrational, by the assumptions on Γ we can apply [9, Theorem 1.2] (see the reformulation in [4, Lemma 3.2]) and we have that

$$\lim_{n\to\infty} \int_\Gamma e^{2\pi i(v\cdot g(x_i - x_j))}\, d\rho^{*n} = 0.$$

On the other hand, if $x_i - x_j$ is rational, it has finite order in $\mathbb{T}^d$, say order q. Again using the assumptions on Γ we can apply [4, Lemma 3.2] to get $c > 0$ (which does not depend on $x_i - x_j$) such that for each v we have

$$\limsup_{n\to\infty} \int_\Gamma |e^{2\pi i(v\cdot g(x_i - x_j))}|\, d\rho^{*n} \leq 2\|v\| q^{-\frac{1}{c}}. \tag{2}$$

Without loss of generality, we assume $c \geq 1$. Let $h_q = |\{(i,j) : x_i - x_j \text{ has order } q\}|$. Then [4, Proposition 2.1] gives

$$\sum_{q\geq 2} h_q q^{-\beta} \leq C k^{2-\frac{\beta}{d+1}}. \tag{3}$$

Let $\beta = 1/c$ and set $\eta = \frac{\beta}{d+1}$. Each of the k-many $i = j$ cases in the definition of $\Phi_E(g)$ adds 1 to the sum for each v, in total contributing $|B(M)|k$. Since $|B(M)|$ is finite, applying the bounds in (2) and (3) we obtain

$$\limsup_{n\to\infty} \int_\Gamma \Phi_E(g)\, d\rho^{*n} \leq C'_\varepsilon \left(k^{2-\eta} + k\right) \tag{4}$$

for some $C'_\varepsilon > 0$. By Markov's inequality, it follows that

$$\rho^{*n}\left(\{g : \Phi_E(g) \geq C_d^{-1}\varepsilon^d k^2\}\right) \leq \frac{C_d\varepsilon^{-d}}{k^2} \int_\Gamma \Phi_E(g)\, d\rho^{*n}.$$

Combining this with (1) and (4), we have

$$\begin{aligned}
\limsup_{n\to\infty} \rho^{*n}\left(\{g : gE \text{ not } \varepsilon\text{-dense}\}\right) &\leq \limsup_{n\to\infty} \rho^{*n}\left(\{g : k^2 \leq C_d\varepsilon^{-d}\Phi_E(g)\}\right) \\
&\leq \limsup_{n\to\infty} \frac{C_d\varepsilon^{-d}}{k^2} \int_\Gamma \Phi_E(g)\, d\rho^{*n} \\
&\leq \frac{C_d\varepsilon^{-d}}{k^2} C'_\varepsilon \left(k^{2-\eta} + k\right) = C_\varepsilon\left(k^{-\eta} + k^{-1}\right).
\end{aligned}$$

□

This allows us to prove the main technical ingredient for showing IC-rigidity of the action of Γ on $\mathbb{T}^d$.

Proposition 3.3. *If $d \geq 2$, $Y \subset \mathbb{T}^d$ is compact and infinite, and $\varepsilon > 0$, then we have*

$$\lim_{n\to\infty} \rho^{*n}\left(\{g \in \Gamma : gY \text{ is } \varepsilon\text{-dense in } \mathbb{T}^d\}\right) = 1.$$

Proof. Given infinite compact $Y \subset \mathbb{T}^d$ and $\varepsilon > 0$, for each $k \geq 1$ we may select $F_k = \{x_1, \ldots, x_k\} \subset Y$. If gY is not ε-dense, then gF_k is not ε-dense. By Lemma 3.2, it follows that for each $k \geq 1$ we have that

$$\limsup_{n\to\infty} \rho^{*n}\left(\{g \in \Gamma : gY \text{ is not } \varepsilon\text{-dense in } \mathbb{T}^d\}\right) \leq C_\varepsilon(k^{-\eta} + k^{-1}).$$

Taking $k \to \infty$, we obtain

$$\limsup_{n\to\infty} \rho^{*n}\left(\{g \in \Gamma : gY \text{ is not } \varepsilon\text{-dense in } \mathbb{T}^d\}\right) = 0$$

and hence

$$\lim_{n\to\infty} \rho^{*n}\left(\{g \in \Gamma : gY \text{ is } \varepsilon\text{-dense in } \mathbb{T}^d\}\right) = 1. \qquad \square$$

We use this to complete the proof of Theorem 3.1.

Proof of Theorem 3.1. Suppose μ is an IRC for the action of Γ on $\mathbb{T}^d$ such that μ-almost every compact is infinite and let $\varepsilon > 0$. Recall $D_\varepsilon = \{Y \in \mathcal{K}(\mathbb{T}^d) : Y \text{ is } \varepsilon\text{-dense in } \mathbb{T}^d\}$. For $n \geq 1$, define $f_{n,\varepsilon} \colon \mathcal{K}(\mathbb{T}^d) \to \mathbb{R}$ by

$$f_{n,\varepsilon}(Y) = \rho^{*n}\left(\{g \in \Gamma : gY \in D_\varepsilon\}\right).$$

Then

$$\begin{aligned}\int_{\mathcal{K}(\mathbb{T}^d)} f_{n,\varepsilon}(Y)\, d\mu(Y) &= \int_{\mathcal{K}(\mathbb{T}^d)} \rho^{*n}\left(\{g : gY \in D_\varepsilon\}\right)\, d\mu(Y) \\ &= \int_{\mathcal{K}(\mathbb{T}^d)} \int_\Gamma \mathbf{1}_{D_\varepsilon}(gY)\, d\rho^{*n}\, d\mu(Y),\end{aligned}$$

and by Fubini's theorem this is equal to

$$\begin{aligned}\int_\Gamma \int_{\mathcal{K}(\mathbb{T}^d)} \mathbf{1}_{D_\varepsilon}(gY)\, d\mu\, d\rho^{*n} &= \int_\Gamma \mu\left(\{Y : gY \in D_\varepsilon\}\right)\, d\rho^{*n} \\ &= \int_\Gamma \mu(g^{-1}D_\varepsilon)\, d\rho^{*n}.\end{aligned}$$

Since μ is Γ-invariant, this last term is equal to

$$\int_\Gamma \mu(D_\varepsilon)\, d\rho^{*n} = \mu(D_\varepsilon).$$

By assumption, μ-almost every $Y \in \mathcal{K}(\mathbb{T}^d)$ is infinite, and so by Proposition 3.3, $f_{n,\varepsilon}(Y)$ converges to 1 for μ-almost every $Y \in \mathcal{K}(\mathbb{T}^d)$. It follows by the dominated convergence theorem that $\mu(D_\varepsilon) = 1$. Thus

$$\mu\left(\bigcap_{m=1}^{\infty} D_{\frac{1}{m}}\right) = 1.$$

Since $\bigcap_{m=1}^{\infty} D_{\frac{1}{m}} = \{\mathbb{T}^d\}$, we have $\mu = \delta_{\mathbb{T}^d}$. $\square$

3.2. **Corollaries and a higher-dimensional Mahler Theorem.** Fix $d \geq 2$. Given $p \geq 2$, we (in a minor abuse of notation) write $T_p \colon \mathbb{T}^d \to \mathbb{T}^d$ where $T_p(x) = px$ (since d is fixed throughout, there is no ambiguity as to the space on which T_p acts). Define

$$\mathcal{I}_p = \{Y \in \mathcal{K}(\mathbb{T}^d) : Y \text{ is infinite and } T_pY = Y\}.$$

Lemma 3.4. *Both of the following hold.*

(1) The set $\mathcal{I}_p$ is closed in $\mathcal{K}(\mathbb{T}^d)$ and hence compact.

(2) For every $r \geq 1$, there exists $\delta_r > 0$ such that $d(\mathcal{I}_p, \mathcal{K}_{\leq r}(\mathbb{T}^d)) \geq \delta_r$.

Proof. To check part (1), suppose there is a sequence $Y_j \in \mathcal{I}_p$ such that $Y_j \to Y$ and Y is finite. Then $T_p Y = Y$ and we may choose $k \geq 1$ such that T_p^k acts by the identity on Y, and choose $\delta > 0$ such that $T_p^k(B_\delta(x)) \cap B_\delta(y) = \emptyset$ for all $x \neq y \in Y$. It follows that $Y_j \subset \bigcup_{x \in Y} B_\delta(x)$ for j sufficiently large. But then by the invariance $T_p^k Y_j = Y_j$, if $z \in Y_j \cap B_\delta(x)$, then the choice of δ implies $T_p^{k\ell}(z) \in Y_j \cap B_\delta(x)$ for all $\ell \geq 0$. Then the local expansion of T_p forces $z = x$, and so $Y_j \subset Y$, a contradiction.

The proof of part (2) is the same as used in the proof of Theorem 2.6. □

We use this to obtain the analog of Theorem 2.8 in the higher-dimensional setting. To guarantee that $\delta_{\mathbb{T}^d}$ is an IRC, we restrict our attention to the subset $\mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z})$ of $d \times d$ nonsingular integer matrices.

Corollary 3.5. *Suppose $Y \in \mathcal{I}_p$. Then for every $\varepsilon > 0$, the set*

$$J_Y = \{A \in \mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z}) : T_A Y \text{ is } \varepsilon\text{-dense in } \mathbb{T}^d\}$$

is a syndetic subset of $\mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z})$.

Proof. Recall $D_\varepsilon = \{Y \in \mathcal{K}(\mathbb{T}^d) : d(Y, \mathbb{T}^d) < \varepsilon\}$, which is open in $\mathcal{K}(\mathbb{T}^d)$. Let $Y \in \mathcal{I}_p$. By Proposition 3.3, there exists $g \in \Gamma$ such that $gY \in D_\varepsilon$. Thus

$$\mathcal{I}_p \subset \bigcup_{g \in \Gamma} g^{-1}(D_\varepsilon).$$

Since $\mathcal{I}_p$ is compact, there exists a finite set $F \subset \Gamma$ such that

$$\mathcal{I}_p \subset \bigcup_{g \in F} g^{-1}(D_\varepsilon).$$

Let $A \in \mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z})$. Since T_A commutes with T_p and is finite to one (since A is nonsingular), we have $T_A Y \in \mathcal{I}_p$. Thus there exists $g \in F$ such that $g T_A Y \in D_\varepsilon$, and hence $gA \in J_Y$. It follows that J_Y is syndetic. □

We use this to prove the higher dimensional analog of Mahler's theorem.

Proof of Theorem 1.3. Let $\alpha \in \mathbb{T}^d$ have at least one irrational coordinate and let $\mathcal{M}(\alpha)$ be defined by

$$\{A \in \mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z}) : \text{ for every } w \in \{0, \dots, b-1\}^d,\ c_r(T_A(\alpha)) = w \text{ for infinitely many } r\}.$$

Let $\mathcal{B}$ be the set of accumulation points of the set

$$\{T_b^k(\alpha) : k \geq 1\}.$$

Then $\mathcal{B}$ is T_b-invariant and infinite, by an argument analogous to the one given in the first paragraph of the proof of Theorem 2.8.

Let $\varepsilon = \frac{1}{2b}$. Since $\mathcal{B}$ is also T_b-invariant, by Corollary 3.5 the set

$$J_{\mathcal{B}} = \{A \in \mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z}) : T_A \mathcal{B} \text{ is } \frac{1}{2b}\text{-dense in } \mathbb{T}^d\}$$

is syndetic in $\mathrm{M}_d^{\mathrm{NS}}(\mathbb{Z})$. It suffices then to show that $J_{\mathcal{B}} \subset \mathcal{M}(\alpha)$. To check this, suppose $A \in J_{\mathcal{B}}$, and so $T_A \mathcal{B}$ is $\frac{1}{2b}$-dense in $\mathbb{T}^d$. Since T_A commutes with T_b, the set $T_A \mathcal{B}$ is the set of accumulation points of $\{T_b^k T_A(\alpha) : k \geq 1\}$. It follows that for every $w = (a_1, \dots, a_d) \in \{0, \dots, b-1\}^d$, the T_b-orbit of $T_A(\alpha)$ enters the cube $Q_w = \prod_{i=1}^d (\frac{a_i}{b}, \frac{a_i+1}{b})$ infinitely many times. This implies $c_r(T_A \alpha) = w$ for infinitely many r, and so $A \in \mathcal{M}(\alpha)$, and hence $J_{\mathcal{B}} \subset \mathcal{M}(\alpha)$. □

References

[1] N. Alon and Y. Peres. Uniform dilations. *Geom. Funct. Anal.* **2** (1992), no. 1, 1–28.
[2] D. Berend and M. Boshernitzan. On a result of Mahler on the decimal expansions of $(n\alpha)$. *Acta Arith.* **66** (1994), no. 4, 315–322.
[3] D. Berend and Y. Peres. Asymptotically dense dilations of sets on the circle. *J. London Math. Soc. (2)* **47** (1993), no. 1, 1–17.
[4] K. Bulinski and A. Fish. Glasner property for linear group actions and their products. *Math. Z.* **303** (2023), no. 4, Paper No. 99, 12 pp.
[5] S. Glasner. Almost periodic sets and measures on the torus. *Israel J. Math.* **32** (1979), no. 2-3, 161–172.
[6] S. Glasner. Topological dynamics and group theory. *Trans. Amer. Math. Soc.* **187** (1974), 327–334.
[7] G. H. Hardy and J. E. Littlewood. Some problems of 'Partitio numerorum'; I: A new solution of Waring's problem. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1920* (1920), 33–54.
[8] G. H. Hardy and J. E. Littlewood. Some problems of 'Partitio numerorum' (VI): Further researches in Waring's Problem. *Math. Z.* **23** (1925), no. 1, 1–37.
[9] W. He and N. de Saxcé. Linear random walks on the torus. *Duke Math. J.* **171** (2022), no. 5, 1061–1133.
[10] B. Kra and S. Schmieding. Invariant Random Compacts. `arXiv:2605.03993`
[11] J. Loxton and R. Vaughan. The estimation of complete exponential sums. *Canad. Math. Bull.* **28** (1985), no. 4, 440–454.
[12] K. Mahler. Arithmetical properties of the digits of the multiples of an irrational number. *Bull. Austral. Math. Soc.* **8** (1973), 191–203.
[13] S. B. Stečkin. An estimate for Gaussian sums. (Russian) *Mat. Zametki* **17** (1975), no. 4, 579–588.

Department of Mathematics, Northwestern University, 2033 Sheridan Road, Evanston, IL 60208
Email address: `kra@math.northwestern.edu`

Department of Mathematics, 107 McAllister Building University Park, State College, PA 16802
Email address: `sks7247@psu.edu`