ANALYSIS OF TWO STEP NILSEQUENCES

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Abstract. Nilsequences arose in the study of the multiple ergodic averages associated to Furstenberg's proof of Szemerédi's Theorem and have since played a role in problems in additive combinatorics. Nilsequences are a generalization of almost periodic sequences and we study which portions of the classical theory for almost periodic sequences can be generalized for two step nilsequences. We state and prove basic properties for 2-step nilsequences and give a classification scheme for them.

1. Introduction

Traditional Fourier analysis has been used with great success to study problems that are linear in nature. For example, Roth used Fourier techniques (the circle method) to show that a set of integers with positive upper density contains infinitely many arithmetic progressions of length 3 and van der Corput used such methods to show that the primes contain infinitely many arithmetic progressions of length 3. These questions can be reformulated into solving a certain linear equation in a subset of the integers, making them tractable to Fourier analysis; the linear nature of the equation is a key ingredient in the use of Fourier techniques.

Finer analysis is needed to study longer arithmetic progressions or more complicated patterns. This analysis is necessarily quadratic (or of higher order) in nature. The basic objects in such an analysis are the *nilsequences* (the precise definition is given in Section 2). Nilsequences were introduced in [2] for studying some problems in ergodic theory and additive combinatorics that arose from the multiple ergodic averages introduced by Furstenberg [5] in his ergodic theoretic proof of Szemerédi's Theorem. Two-step nilsequences play a role in the recent work of Green and Tao (see [6] and [7]) on arithmetic progressions of length 4 in the primes; higher order nilsequences are conjectured to play an analogous role for longer arithmetic progressions.

In spite of the recent uses of nilsequences in a variety of contexts, as of yet there is no systematic study of them. Even some of the basic properties are neither stated nor proved. In this paper, we carry out such an analysis for 2-step nilsequences and hope that this clarifies their role and properties. In a forthcoming paper [8], we give applications of the results in this paper.

Almost periodic sequences and their classically understood properties are a model we would like to emulate for nilsequences. Almost periodic sequences (which are exactly 1-step nilsequences) arise naturally in understanding the correlation sequence

$$\Big(\int f\cdot f\circ T^n\,d\mu\colon n\in\mathbb{Z}\Big)\ ,$$

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where $(X, \mathcal{X}, \mu, T)$ is a measure preserving system and $f \in L^\infty(\mu)$. In a similar way, 2-step nilsequences arise in [2] for understanding the double correlation sequence

$$\Big(\int f \cdot f\circ T^n \cdot f \circ T^{2n}\, d\mu \colon n \in \mathbb{Z}\Big)\ .$$

However, there are fundamental differences between the 1-step (almost periodic) and 2-step nilsequences that make it impossible to carry over the classical results in a straightforward manner. Namely, we no longer have functions such as the trigonometric functions that form a basis.

On the other hand, there is a class of nilsequences called *elementary nilsequences* (see Definition 3) that are the building blocks for all nilsequences. Using these, we obtain a density result (Theorem 5) analogous to the density of trigonometric polynomials in L^2. Restricting to the elementary nilsequences, we can classify them such that each elementary nilsequence belongs to a class with a particularly simple representative of this class.

Throughout this article, we restrict ourselves to 2-step nilsequences. Some of the analysis here can be carried out for higher levels, but most of the results are relative to the previous level of almost periodic sequences. The detailed analysis of higher order nilsequences begins with a firm understanding of the 2-step ones.

2. Definitions and examples

2.1. **Almost periodic sequences.**

Notation. In general, we denote a sequence by $\mathbf{a} = (a_n \colon n \in \mathbb{Z})$. Given $\mathbf{a}$, we use $\sigma\mathbf{a}$ to denote the *shifted* sequence. Thus for $k \in \mathbb{Z}$, by $\sigma^k\mathbf{a}$, we mean the shifted sequence $\sigma^k\mathbf{a} = (a_{n+k} \colon n \in \mathbb{Z})$.

For $t \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, we use the standard notation $e(t) = \exp(2\pi i t)$. The *exponential linear sequence* $\mathbf{e}(t)$ is defined by $\mathbf{e}(t) = (e(nt) \colon n \in \mathbb{Z})$.

Proposition 1 (and definition)**.** *For a bounded sequence $\mathbf{a} = (a_n \colon n \in \mathbb{Z})$, the following properties are equivalent:*

(i) *There exist a compact abelian group G, an element τ of G, and a continuous function f on G such that $a_n = f(\tau^n)$ for every $n \in \mathbb{Z}$.*
(ii) *The sequence $\mathbf{a}$ is a uniform limit of linear combinations of exponential sequences.*
(iii) *The family of translated sequences $\{\sigma^k\mathbf{a} \colon k \in \mathbb{Z}\}$ is relatively compact under the ℓ^∞-norm.*

A sequence satisfying these properties is called almost periodic*.*

The family of almost periodic sequences forms a sub-algebra of ℓ^∞, closed under complex conjugation, the shift and under uniform limits. We denote this subalgebra by $\mathcal{AP}$.

2.2. **2-step nilsequences.**

Definition 1. When G is a group, G_2 denotes its commutator subgroup, that is the subgroup spanned by elements of the form $ghg^{-1}h^{-1}$ for $g, h \in G$. G is 2-*step nilpotent* if G_2 is included in the center of G.

If G is a 2-step nilpotent Lie group and $\Gamma \subset G$ is a discrete and cocompact subgroup, then the compact manifold $X = G/\Gamma$ is called a 2-*step nilmanifold*. The action of G on X by left translation is written $(g, x) \mapsto g \cdot x$ for $g \in G$ and $x \in X$.

A 2-step nilmanifold $X = G/\Gamma$, endowed with the translation $T\colon x \mapsto \tau \cdot x$ by some fixed element τ of G is called a 2*-step nilsystem*.

If $f\colon X \to \mathbb{C}$ is a continuous function, $\tau \in G$ and $x_0 \in X$, then $(f(\tau^n \cdot x_0)\colon n \in \mathbb{Z})$ is a *basic* 2*-step nilsequence*. A 2*-step nilsequence* is a uniform limit of basic 2-step nilsequences.

With small modifications, this generalizes condition (i) in Proposition 1. There is a technical difficulty that explains the need to take uniform limits of basic nilsequences, rather than just nilsequences. Namely, $\mathcal{AP}$ is closed under the uniform norm, while the family of basic 2-step nilsequences is not. An inverse limit of rotations on compact abelian Lie groups is also a rotation on a compact abelian group, but the same result does not hold for nilsystems: the inverse limit of a sequence of 2-step nilmanifolds is not, in general, the homogeneous space of some locally compact abelian group (see [16]). The other conditions in Proposition 1 do not have straightforward modifications that generalize for 2-step nilsequences. After introducing some particular classes of 2-step nilsequences, we develop analogs of these other conditions.

There is an alternate characterization of 2-step nilsequences:

Proposition (see [9])**.** *A sequence* $\mathbf{a} = (a_n\colon n \in \mathbb{Z})$ *is a* 2*-step nilsequence if there exist an inverse limit* (X, T) *of* 2*-step nilsystems, a continuous function* f *on* X*, and a point* $x_0 \in X$ *such that* $a_n = f(T^n x_0)$ *for every* $n \in \mathbb{Z}$*.*

Definition 1, with the obvious changes, defines a k-step nilsequence. However, in this article we restrict ourselves to studying 2-step nilsequences.

Convention. We often omit the 2-step from our terminology and just refer to a 2-step nilsequence as a nilsequence. Similarly, a nilsystem refers to a 2-step nilsystem.

We introduce some particular 2-step nilsequences that play a particular role in the sequel.

2.3. Affine systems and quadratic sequences.

Notation. For $t \in \mathbb{T}$, let $\mathbf{q}(t) = (q_n(t)\colon n \in \mathbb{Z})$ denote the sequence given by

$$q_n(t) = e\Big(\frac{n(n-1)}{2}t\Big) \quad \text{for } n \in \mathbb{Z}\ .$$

This sequence is called a *quadratic exponential sequence.*

We show that any sequence $\mathbf{q}(t)$ is a nilsequence. We first remark that this sequence is almost periodic if and only if t is rational and that in this case it is periodic.

Let $G = \mathbb{Z} \times \mathbb{T} \times \mathbb{T}$, $\Gamma = \mathbb{Z} \times \{0\} \times \{0\}$ with multiplication defined by

$$(m, x, y) \cdot (m', x', y') = (m + m', x + x', y + y' + mx')\ .$$

Then G is a 2-step nilpotent Lie group, with $G_2 = \{0\} \times \{0\} \times \mathbb{T}$, and Γ is a discrete cocompact subgroup. Set $X = G/\Gamma$. Fix some $\alpha, \beta \in \mathbb{T}$, and set $\tau = (1, \alpha, \beta) \in G$ and endow X with the translation T by τ.

We can give another description of this system. The map $(m, x, y) \mapsto (x, y)$ from $G \to \mathbb{T}^2$ induces a homeomorphism from X to $\mathbb{T}^2$. Identifying these spaces, T becomes the familiar skew transformation $(x, y) \mapsto (x + \alpha, y + x + \beta)$ of $\mathbb{T}^2$.

We describe some nilsequences arising from this system. Set $x_0 = (0, 0) \in X$. Let f be the continuous function on X given by $f(x, y) = e(y)$. For every $n \in \mathbb{Z}$,

$$f(T^n x_0) = e\big(\frac{n(n-1)}{2}\alpha + n\beta\big) = q_n(\alpha)e(n\beta)\ .$$

Thus every product of a quadratic exponential sequence and an exponential sequence is a basic nilsequence (in particular, an exponential sequence is a basic nilsequence).

2.4. **Heisenberg nilsystems.**

2.4.1. *Heisenberg groups.* Let $d \geq 1$ be an integer. The *Heisenberg group of dimension* $2d + 1$ is $G = \mathbb{R}^{2d+1}$, identified with $\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$ and endowed with multiplication given by

$$(x, y, z)\cdot(x', y', z') = (x+x', y+y', z+z'+\langle x \mid y'\rangle) \quad \text{where } x, y, x', y' \in \mathbb{R}^d,\ z, z' \in \mathbb{R}$$

and $\langle \cdot \mid \cdot\rangle$ is the usual inner product on $\mathbb{R}^d$. The commutator subgroup G_2 of G is $\{0\}^d \times \{0\}^d \times \mathbb{R}$ and G is a 2-step nilpotent Lie group. We set $\Gamma = \mathbb{Z}^{2d+1}$. Then Γ is a discrete cocompact subgroup of G. The nilmanifold $N_d = G/\Gamma$ is called the *Heisenberg nilmanifold of dimension* $2d + 1$.

We give another representation of this nilmanifold, which is more useful in the sequel. The subgroup $K = \{0\}^d \times \{0\}^d \times \mathbb{Z}$ is normal in G, and is included in Γ. We thus have that $N_d = H_d/\Lambda_d$, where $N_d = G/K$, $\Lambda_d = \Gamma/K$. That is,

$$H_d = \mathbb{R}^d \times \mathbb{R}^d \times \mathrm{S}^1,\ \Lambda_d = \mathbb{Z}^d \times \mathbb{Z}^d \times \{1\}\ ,$$

where the multiplication is given by

(1)
$$(x, y, z) \cdot (x', y', z') = \big(x + x', y + y', zz'e(\langle x \mid y'\rangle)\big) \quad (x, y, x', y' \in \mathbb{R}^d,\ z, z' \in \mathrm{S}^1)\ .$$

(Note that S^1 denotes the circle while $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ is called the torus.) When N_d is endowed with the translation by some element τ of H_d, (N_d, T) is called a *Heisenberg nilsystem.* The commutator subgroup of H_d is $\{0\}^d \times \{0\}^d \times \mathrm{S}^1$.

2.4.2. *Nilsequences arising from Heisenberg nilsystems.*

Notation. For $s, t \in \mathbb{T}$, we set

$$\kappa(s, t) := \sum_{k\in\mathbb{Z}} \exp(-\pi(t + k)^2)e(ks)\ . \tag{2}$$

We write $\boldsymbol{\omega}(\alpha, \beta) = (\omega_n(\alpha, \beta) \colon n \in \mathbb{Z})$ for the sequence defined by

$$\omega_n(\alpha, \beta) = \kappa(n\alpha, n\beta)e(\frac{n(n-1)}{2}\alpha\beta) \quad \text{for } n \in \mathbb{Z}\ .$$

We show that products of sequences of this type are 2-step nilsequences.

Remark. The function κ is linked to the classical theta function

$$\theta(u, z) = \sum_{n\in\mathbb{Z}} \exp(\pi izn^2 + 2\pi inu)$$

by

$$\kappa(s, t) = \exp(-\pi t^2)\theta(s + it, i)\ . \tag{3}$$

This is a particular case of well known relations between Heisenberg groups and theta functions.

We start by explaining how to build explicit continuous functions on N_d. It is not as easy to do this as it is for affine systems, because the nilmanifold H_d is not homeomorphic to a torus.

Let ϕ be a continuous function on $\mathbb{R}^d$, tending to zero sufficiently fast at infinity. Define a function $\tilde{f}$ on H_d by

$$\tilde{f}(x,y,z) = z \sum_{k\in\mathbb{Z}^d} \phi(y+k) e(\langle k \mid x\rangle) \quad (x,y\in\mathbb{R}^d,\ z\in \mathrm{S}^1)\ .$$

We note that for all $k,\ell\in\mathbb{Z}^d$, we have $\tilde{f}\big((x,y,z)\cdot(k,\ell,1)\big) = \tilde{f}(x,y,z)$. In other words, the function $\tilde{f}$ on G is invariant under right translations by elements of Λ_d, and thus it induces a continuous function f on N_d.

The most interesting case is when ϕ is the Gaussian function $\phi(x) = \exp(-\pi\|x\|^2)$, where $\|\cdot\|$ is the usual euclidean norm in $\mathbb{R}^d$.

In this case we have:

$$\tilde{f}(x,y,z) = z\prod_{j=1}^{d} \kappa(x_j,y_j)\ . \tag{4}$$

Set $\tau = (\alpha,\beta,e(\gamma)) \in H_d$, where $\alpha = (\alpha_1,\dots,\alpha_d)$ and $\beta = (\beta_1,\dots,\beta_d)$ belong to $\mathbb{R}^d$ and $\gamma\in\mathbb{T}$ and define T to be the translation by τ on N_d. For every $n\in\mathbb{Z}$, we have

$$\tau^n = (n\alpha, n\beta, e(n\gamma)e\big(\frac{n(n-1)}{2}\langle\alpha\mid\beta\rangle\big))\ .$$

Let x_0 be the image in N_d of the element $(0,0,1)$ of H_d and f the function on N_d associated to the function $\tilde{f}$ defined by (4). Then for every $n\in\mathbb{Z}$, we have

$$\begin{aligned} f(T^n x_0) &= e(n\gamma)e\big(\frac{n(n-1)}{2}\langle\alpha\mid\beta\rangle\big)\prod_{j=1}^{d}\kappa(n\alpha_j,n\beta_j) \\ &= e_n(\gamma)\omega_n(\alpha_1,\beta_1)\cdot\ldots\cdot\omega_n(\alpha_d,\beta_d)\ . \end{aligned} \tag{5}$$

2.5. **Some counterexamples.** There are other natural quadratic-type sequences that are particularly simple to construct, and so it is natural to ask if these are (basic) nilsequences. For example, consider the sequence $(e(\lfloor n\alpha\rfloor n\beta)\colon n\in\mathbb{Z})$, where $\lfloor x\rfloor$ denotes the greatest integer less than or equal to x. This is not a nilsequence, as it can not be obtained as a uniform limit of sequences associated to continuous functions on nilmanifolds. On the other hand, this sequence is a "Besicovitch" basic nilsequence, meaning that it is a limit of basic nilsequences in quadratic averages. This can be seen by considering the simpler sequence $(e(\lfloor n\alpha\rfloor\beta)\colon n\in\mathbb{Z})$. Although at first glance it looks like an almost periodic sequence, it is not: under the uniform norm, the sequence $(e(\lfloor n\alpha+\gamma\rfloor\beta)\colon n\in\mathbb{Z})$ does not depend continuously on γ. The family of shifted sequences $(a_{n+k}\colon n\in\mathbb{Z})$ is not relatively compact under the uniform norm, but only in quadratic average. A similar problem occurs with the sequence $(e([n\alpha]n\beta)\colon n\in\mathbb{Z})$. A more complete explanation is given in Appendix C.

Objects similar to our nilsequences arise in several other contexts. For example, Green and Tao [6] use nilsequences to give asymptotics for the number of progressions of length 4 in the primes. Their definition is slightly different: the underlying group G is assumed to be connected and simply connected. Moreover, in all their uses of nilsequences the function defining it is taken to be Riemann integrable. This weaker hypothesis on the function (we assume continuity) accounts for the difference in sequences that arise. In particular, sequences with integer parts can be obtained from Riemann integrable functions, but not from continuous functions. Perhaps more important is the difference in point of view. In Green and Tao, they consider finite (albeit long) sequences taken from a nilsequence. The importance is that this point of view is local and in this context, quadratic sequences are the natural local model for 2-step nilsequences. In giving a global model for 2-step nilsequences, this program no longer can be carried through.

In [3], Bergelson and Leibman show that all generalized polynomials arise by evaluating a piecewise polynomial mapping on a nilmanifold. However, in order to obtain all generalized polynomials, they necessarily must allow functions with discontinuities. Our nilsequences are recurrent, while some generalized polynomials take on non-recurring values.

3. Results

3.1. **First properties.**

Notation. We denote the family by of (2-step) nilsequences by $\mathcal{N}^2$.

Some classical properties of 2-step nilsystems are recalled in Section 4.

Proposition 2.

- *The family of nilsequences is a sub-algebra of $\ell^\infty(\mathbb{Z})$ that contains $\mathcal{AP}$ and is invariant under complex conjugation, shift and uniform limits.*
- *If $\mathbf{a}$ is a nilsequence and $(I_j)_{j\in\mathbb{N}}$ is a sequence of intervals of $\mathbb{Z}$ whose lengths tend to infinity, then the limit*

$$\mathrm{Av}(\mathbf{a}) := \lim_{j\to\infty} \frac{1}{|I_j|} \sum_{n\in I_j} a_n$$

exists and does not depend on choice of the sequence (I_j) of intervals.

In the second statement of this proposition, the averages over the intervals (I_j) can be replaced by the averages over any Følner sequence.

If $\mathbf{a}$ and $\mathbf{b}$ are nilsequences, then the product sequence $\mathbf{a}\,\overline{\mathbf{b}} := (a_n\overline{b_n} \colon n \in \mathbb{Z}\}$ is a nilsequence. This, together with the second part of Proposition 2 justifies the following definition:

Definition 2. The *inner product* of two nilsequences $\mathbf{a}$ and $\mathbf{b}$ is defined to be

$$\langle \mathbf{a} \mid \mathbf{b}\rangle := \mathrm{Av}(\mathbf{a}\,\overline{\mathbf{b}}) = \lim_{N\to+\infty} \frac{1}{N} \sum_{n=0}^{N-1} a_n\overline{b_n}\ .$$

The *quadratic norm* of the nilsequence $\mathbf{a}$ is

$$\|\mathbf{a}\|_2 := \mathrm{Av}(|\mathbf{a}|^2)^{1/2} = \lim_{N\to+\infty} \frac{1}{N} \Big(\sum_{n=0}^{N-1} |a_n|^2\Big)^{1/2}\ .$$

In Section 4.2 we check that $\|\cdot\|_2$ is a norm on the space $\mathcal{N}^2$ of nilsequences.

3.2. **Elementary 2-step nilsequences.** There is no direct generalization of the second property of Proposition 1 for 2-step nilsequences, as there is no strict analog of exponential sequences. However, some particular nilsequences play a similar role, and these are the building blocks for all nilsequences.

Definition 3. A 2-step nilmanifold X is an *elementary (2-step) nilmanifold* if it can be written as $X = G/\Gamma$ as in Definition 1, where G_2 is the circle group S^1. An *elementary nilsystem* is a nilsystem $(X = G/\Gamma, T)$ such that X is an elementary nilmanifold.

When $X = G/\Gamma$ is an elementary nilmanifold (written such that $G_2 = \mathrm{S}^1$), we write $\mathcal{C}_1(X)$ for the family of continuous functions f on X such that

$$f(u \cdot x) = uf(x) \text{ for all } u \in G_2 = \mathrm{S}^1 \text{ and all } x \in X\ . \tag{6}$$

If (X, T) is an elementary nilsystem, $f \in \mathcal{C}_1(X)$ and $x_0 \in X$, the sequence $\big(f(T^n x_0)\colon n \in \mathbb{Z}\big)$ is a *basic elementary nilsequence*. A uniform limit of basic elementary nilsequences is an *elementary nilsequence*.

By convention, every almost periodic sequence is also considered to be an elementary nilsequence.

There are several things left imprecise in the definition of an elementary nilmanifold. First, the same nilmanifold X can be written as G/Γ in different ways, and the term "elementary nilmanifold" refers in fact to a given presentation of X and not only to the nilmanifold. Moreover, when we say that "$G_2 = \mathrm{S}^1$" we do not mean only that these groups are isomorphic, but also that we make the choice of a particular isomorphism that defines the identification.

Definition 4. Let $(X = G/\Gamma, T)$ be an elementary nilsystem. The same system but with the opposite identification of G_2 with S^1 is called the *conjugate system* of X and is written $(\bar{X}, T)$. We note that for $f \in \mathcal{C}_1(X)$, we have that $\bar{f} \in \mathcal{C}_1(\bar{X})$.

Under the usual identification of $\mathbb{T}$ with S^1, the affine system (X, T) introduced in Section 2.3 is an elementary nilsystem, the function f belongs to $\mathcal{C}_1(X)$ and the sequence $(q_n(\alpha)e_n(\beta)\colon n \in \mathbb{Z})$ is a basic elementary nilsequence.

The Heisenberg system (N_d, T) introduced in Section 2.4.1 is an elementary nilsystem, and the sequence defined in (5) is a basic elementary nilsequence.

Proposition 3. *The family of elementary nilsequences is closed under complex conjugation, the shift, taking products and uniform limits.*

All these properties follow immediately from the definitions, other than the closure under products. This is proved in Section 6 (Proposition 6).

The next Proposition (proved in Section 4.5) is a nilsequence version of the characterization of almost periodic sequences given by Part (ii) of Proposition 1:

Proposition 4. *Every nilsequence is a uniform limit of finite sums of elementary nilsequences.*

In this statement, we can obviously substitute "basic elementary nilsequences" for elementary nilsequences.

The family of 2-step nilsequences has no characterization similar to property (iii) of Proposition 1. A characterization is given in [9], but it has a completely different nature, relying on recurrence (or combinatorial) properties of the sequence more than on topological ones. However, the smaller class of elementary nilsequences does (see Section 9 for the proof):

Theorem 1. *Let* $\mathbf{a} = (a_n : n \in \mathbb{Z})$ *be a bounded sequence. The following are equivalent:*

(i) $\mathbf{a}$ *is an elementary nilsequence.*
(ii) *There exists a compact (in the norm topology) subset* $K \subset \ell^\infty(\mathbb{Z})$ *such that for all* $k \in \mathbb{Z}$*, there exists* $t \in \mathbb{T}$ *such that the sequence* $\mathbf{e}(t)\sigma^k\mathbf{a} = \big(e(nt)a_{n+k} : n \in \mathbb{Z}\big)$ *belongs to* K.

3.3. **The class of an elementary nilsequence.** Theorem 1 makes it clear that the fundamental objects are not elementary nilsequences, but rather elementary nilsequences considered relative to almost periodic sequences. This motivates the next definition.

Definition 5. Let $\mathbf{a} = (a_n : n \in \mathbb{Z})$ be a bounded sequence. The *class* $\mathcal{S}(\mathbf{a})$ of $\mathbf{a}$ is the norm closure in ℓ^∞ of the vector space spanned by sequences of the form $\mathbf{e}(t)\sigma^k\mathbf{a} = \big(e(nt)a_{n+k} : n \in \mathbb{Z}\big)$ for $t \in \mathbb{T}$ and $k \in \mathbb{Z}$.

The class of an elementary nilsequence contains only elementary nilsequences. If $\mathbf{a}$ is a non-identically zero almost periodic sequence, then $\mathcal{S}(\mathbf{a}) = \mathcal{AP}$.

Remark. The reason for the introduction of the shift in the definition of the class $\mathcal{S}(\mathbf{a})$ is not obvious at this point but is clarified later (proof of Corollary 3). If we use the quadratic norm the situation is different and the shift is not needed (see the proof of Theorem 5 in Section 8.6).

In Section 6.3 we show:

Theorem 2.

(i) *The classes of two elementary nilsequences are either identical or are orthogonal. This means that if* $\mathbf{a}$ *and* $\mathbf{b}$ *are non-identically zero elementary nilsequences, then:*
 - *If* $\mathbf{b} \in \mathcal{S}(\mathbf{a})$*, then* $\mathcal{S}(\mathbf{b}) = \mathcal{S}(\mathbf{a})$.
 - *If* $\mathbf{b} \notin \mathcal{S}(\mathbf{a})$*, then* $\mathrm{Av}(\mathbf{a}\overline{\mathbf{b}}) = 0$*, that is*

$$\frac{1}{N}\sum_{n=0}^{N-1} a_n\overline{b_n} \to 0 \text{ as } N \to \infty\ .$$

(ii) *The classes form a partition of the set of elementary nilsequences into open subsets.*
(iii) *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are elementary nilsequences, then there exists an elementary nilsequence* $\mathbf{c}$ *such that* $\mathcal{S}(\mathbf{a}) \cdot \mathcal{S}(\mathbf{b}) := \{\mathbf{a}'\mathbf{b}' : \mathbf{a}' \in \mathcal{S}(\mathbf{a}),\ \mathbf{b}' \in \mathcal{S}(\mathbf{b})\}$ *is included in* $\mathcal{S}(\mathbf{c})$.
(iv) *Endowed with this multiplication, the family of classes is an abelian group.*

3.4. **Classification.** In light of the last two parts of Theorem 2, it would be nice to classify elementary nilsequences. Optimally, one would like to classify the elementary nilsequences by giving an explicit representative in every class. Unfortunately, this is not possible as there is no reasonable (i.e. Borel) way to choose this representative. However, we can partially succeed in giving a classification, by reducing to some particular nilsequences.

Our classification of elementary nilsequences is related to, but not identical to, a classification of elementary nilsystems that we do not carry out here. In this classification, non-isomorphic nilsystems may give rise to the same class of nilsequences.

Notation. Let $d \geq 1$ be an integer. We write J_{2d} for the $2d \times 2d$ matrix

$$J_{2d} = \begin{pmatrix} 0 & \mathrm{Id}_d \\ -\mathrm{Id}_d & 0 \end{pmatrix}$$

where the 0 represents the $d \times d$ zero matrix and Id_d the $d \times d$ identity matrix.

Let $\mathrm{Sp}_{2d}(\mathbb{Q})$ denote the group of $2d \times 2d$ matrices M with rational entries which preserve the antisymmetric bilinear form associated to J_{2d}, that is, with

$$M^t J_{2d} M = J_{2d} \ .$$

In an equivalent way, M can be written in $d \times d$ blocks as

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A^tC and D^tB are symmetric and $D^tA - B^tC = \mathrm{Id}_d$.

Convention. In the next theorem and in the sequel, the empty product is by convention set to be 1.

Theorem 3. *Let $t \in \mathbb{T}$ be either equal to 0 or irrational, let $d \geq 0$ be an integer and let $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d$ be $2d$ reals, rationally independent modulo 1. Then the sequence $\mathbf{q}(t)\,\boldsymbol{\omega}(\alpha_1, \beta_1) \cdot \ldots \cdot \boldsymbol{\omega}(\alpha_d, \beta_d)$ is an elementary nilsequence.*

Every elementary nilsequence belongs to the class of a sequence of this type.

Let $t, t' \in \mathbb{T}$, $d, d' \geq 0$ and $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d$ and $\alpha'_1, \dots, \alpha'_{d'}, \beta'_1, \dots, \beta'_{d'}$ be two familes, each of which is a family of real numbers that are rationally independent modulo 1. Then the sequences $\mathbf{q}(t)\,\boldsymbol{\omega}(\alpha_1, \beta_1) \cdot \ldots \cdot \boldsymbol{\omega}(\alpha_d, \beta_d)$ and $\mathbf{q}(t')\,\boldsymbol{\omega}(\alpha'_1, \beta'_1) \cdot \ldots \cdot \boldsymbol{\omega}(\alpha'_{d'}, \beta'_{d'})$ belong to the same class if and only if $d = d'$ and there exist a $2d \times 2d$ matrix $Q \in \mathrm{Sp}_{2d}(\mathbb{Q})$ and integers $m, k_1, \dots, k_d, \ell_1, \dots, \ell_d$ with $m \geq 1$ such that

$$Q \begin{pmatrix} \alpha_1 + k_1/m \\ \vdots \\ \alpha_d + k_d/m \\ \beta_1 + \ell_1/m \\ \vdots \\ \beta_d + \ell_d/m \end{pmatrix} = \begin{pmatrix} \alpha'_1 \\ \vdots \\ \alpha'_d \\ \beta'_1 \\ \vdots \\ \beta'_d \end{pmatrix} \quad \text{and } m(t - t') = \sum_{i=1}^{d} (k_i \beta_i - \ell_i \alpha_i) \bmod 1 \ .$$

If $d = 0$ in the first part of the theorem, then the sequence is $\mathbf{q}(t)$ and the hypothesis of independence is vacuous. If $d = d' = 0$ in the second part of the theorem, then the sequences are $\mathbf{q}(t)$ and $\mathbf{q}(t')$ and the condition means simply that $t - t'$ is rational.

Remark. We show (Lemmas 6 and 8) that the sequence appearing in the first part of the theorem arises from an elementary nilsystem associated to a connected Lie group if and only if t belongs to the subgroup of $\mathbb{T}$ spanned by $\alpha_1, \dots, \beta_d$. Therefore, non-connected groups can not be avoided in the definition of nilsequences.

3.5. **Density results.** Together, the examples of nilsequences introduced in Sections 2.3 ad 2.4.2 span the 2-step nilsequences and so we give them a name:

Notation. We write $\mathcal{M}$ for the family of sequences of the form

$$\mathbf{e}(s)\mathbf{q}(t)\boldsymbol{\omega}(\alpha_1, \beta_1) \cdot \ldots \cdot \boldsymbol{\omega}(\alpha_d, \beta_d)$$

where $s, t \in \mathbb{T}$, $d \geq 0$ is an integer and $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d$ are reals that are rationally independent modulo 1.

Theorem 4. *The space of* 2*-step nilsequences is the closed shift invariant linear space spanned by the family* $\mathcal{M}$ *of sequences.*

For the quadratic norm, we have a density result analog of Theorem 4, without needing to include shifts of the sequences:

Theorem 5. *The linear span of the family* $\mathcal{M}$ *of sequences in dense in the space* $\mathcal{N}^2$ *of* 2*-step nilsequences under the quadratic norm.*

However, Theorem 3 shows that $\mathcal{M}$ is not a basis for $\mathcal{N}^2$.

3.6. **Higher order nilsequences.** Some of the analysis in this paper for 2-step nilsequences can be carried out for higher levels. We can define elementary k-step nilsequences by simply replacing G_2 by G_k. Then arbitrary k-step nilsequences can be approximated by elementary ones. However, the problem then becomes finding the analog of the span of (elementary) k-step nilsequences and giving a characterization of (elementary) k-step nilsequences.

One of the advantages of working with 2-step nilsequences is that we have the family of theta functions and associated theta sequences that simplify the classification. Representations of k-step nilpotent groups for $k \geq 3$ are significantly more complicated, and so the related classification of k-step nilsequences is a difficult problem.

4. Basic properties of nilsystems and nilsequences

4.1. **Nilmanifolds and nilsystems.** We recall some classical properties of nilmanifolds and nilsystems. For nilmanifolds, we refer to the seminal paper by Malcev [12]. For nilsystems, we refer to [1], [14], [15] and [11]; these papers deal only with the case of connected nilpotent Lie groups and their results were extended to the non-connected case in [10].

In section 2.3, we saw that every exponential sequence is a nilsequence. Therefore, $\mathcal{N}^2 \supset \mathcal{AP}$. The family of 2-step nilsystems is closed under Cartesian products. This, together with Definition 1, immediately implies the first statement of Proposition 2.

In the rest of this section, $(X = G/\Gamma, T)$ is assumed to be a 2-step nilsystem, where T is the left translation by $\tau \in G$ and x_0 is a point of X.

Let Y be the closed orbit of x_0, that is the closure in X of $\{T^n x_0 \colon n \in \mathbb{Z}\}$. Then (Y, T) can be given the structure of a 2-step nilsystem or of a rotation on a compact abelian group. More precisely, there exists a closed subgroup H of G containing τ such that $\Lambda := \Gamma \cap H$ is cocompact in H and Y can be identified with H/Λ. H is a Lie group, either abelian or 2-step nilpotent.

By substituting H for G and Λ for Γ in the definitions of basic 2-step nilsequences and of elementary nilsequences, we can restrict to the case that the orbit $\{T^n x_0 \colon n \in \mathbb{Z}\}$ of x_0 is dense in X. It is classical that this hypothesis implies:

(i) (X, T) is minimal (that is, every orbit is dense) and uniquely ergodic: the unique T-invariant probability measure is the *Haar measure* μ of X, which is invariant under the action of G on X.

Henceforth we assume that our nilsystem satisfies this assumption.

This immediately implies the second statement of Proposition 2: if a basic nilsequence $\mathbf{a}$ is defined as in Definition 1, then

$$\mathrm{Av}(\mathbf{a}) = \int f \, d\mu$$

and the existence of the average in the case of a general nilsequence follows by density.

4.2. **The quadratic norm.** The quadratic norm $\|\mathbf{a}\|_2$ of a 2-step nilsequence was introduced in Section 3.1 (Definition 2); we check here that it is a norm.

We only need to show that if $\mathbf{a}$ is a non-identically zero nilsequence, then $\|\mathbf{a}\|_2 \neq 0$. Let $\mathbf{a}$ be a non-identically zero basic nilsequence. Write it as $\big(f(T^n x_0) \colon n \in \mathbb{Z}\big)$, where (X, T) is a minimal nilsystem, $f \in \mathcal{C}(X)$ and $x_0 \in X$. By minimality, we have that $\|\mathbf{a}\|_\infty = \|f\|_\infty$ and that

for every δ with $0 < \delta < \|\mathbf{a}\|_\infty$, the set $\{n \in \mathbb{Z} \colon |a_n| \geq \delta\}$ is syndetic.

This property is stable under uniform limits and thus holds for all nilsequences. The announced result follows. $\square$

We remark that $\|\mathbf{a}\|_2 \leq \|\mathbf{a}\|_\infty$ for every nilsequence $\mathbf{a}$.

4.3. **Reduced form of a 2-step nilsystem.** We can make some further simplifying assumptions about the nilsystem, as in [2]. We only sketch the justification of these reductions.

Let G_0 be the connected component of the identity in G and let G_1 be the subgroup of G spanned by G_0 and τ. Since G_1 is open and the projection $G \to G/\Gamma = X$ is an open map, the image of G_1 under this projection is an open subset of X. This subset is invariant under T and thus is equal to X by minimality. Therefore, substituting G_1 for G and $\Gamma \cap G_1$ for Γ, we can reduce to the case that

(ii) G is spanned by the connected component G_0 of the identity and τ.

Let $\mathcal{Z}(G)$ be the center of G. We can substitute $G/(\Gamma \cap \mathcal{Z}(G))$ for G, and $\Gamma/(\Gamma \cap \mathcal{Z}(G))$ for Γ. Therefore we can reduce to the case that

(iii) The intersection of Γ with the center of G is trivial.

This implies that Γ does not contain any nontrivial normal subgroup of G. Therefore the action of G on X is faithful (the action of $g \in G$ on X is the identity only if g is the unit element), and G can be viewed as a group of transformations of X.

These properties imply that

(iv) Γ is abelian.
(v) G_2 is a finite dimensional torus.

Moreover, by minimality of (X, T), the subgroup spanned by Γ and τ is dense in G. Therefore, if an element γ of Γ commutes with τ then it belongs to the center of G and is trivial. So we have:

(vi) The map $\gamma \mapsto [\gamma, \tau] \colon \Gamma \to G_2$ is one to one.

Definition 6. Let $(X = G/\Gamma, T)$ be a (2-step) nilsystem (X, T), where T is the translation by $\tau \in G$. We say that (X, T) is *in reduced form* if G, Γ, τ satisfy the hypotheses (ii) and (iii).

By the preceding reductions, we see that in the definition of 2-step basic nilsequences we can restrict to the case that the nilsystem is minimal and in reduced form. We claim that in the definition of an elementary nilsequences we can restrict ourselves to minimal elementary nilsystems written in reduced form.

Indeed, let $(X = G/\Gamma, T)$ be an elementary nilsystem, $x_0 \in X$, $f \in \mathcal{C}_1(X)$ and let $\mathbf{a} = (f(T^n x)\colon n \in \mathbb{Z})$. We can assume without loss that this sequence is not almost periodic.

Let Y be the closed orbit of x_0 under T. Then (Y, T) is a not a rotation on a compact abelian group and thus is a 2-step nilsystem. We write $Y = G'/\Gamma'$, where G' is a closed subgroup of G and $\Gamma' = \Gamma \cap G'$. As this system is minimal, we can make the second reduction and assume that property (ii) holds. Then G'_2 is a nontrivial closed connected subgroup of $G_2 = \mathrm{S}^1$, and thus $G'_2 = G_2$ and (Y, T) is an elementary nilsystem. Moreover, the restriction of f to Y belongs to $\mathcal{C}_1(Y)$.

Now we make the last reduction, and write $Y = G''/\Gamma''$ by taking the quotient of G' and Γ' by $\mathcal{Z}(G') \cap \Gamma'$. The existence of the non-identically zero function f in $\mathcal{C}_1(Y)$ implies that $G'_2 \cap \Gamma'$ is trivial, and thus the natural projection $G'_2 \to G''_2$ is an isomorphism. We thus have that G''_2 is the circle group and that f belongs to $\mathcal{C}_1(Y = G''/\Gamma'')$. Our claim is proved.

Convention. Henceforth, all nilsystems are implicitly assumed to be minimal and in reduced form.

The *Kronecker factor* of an ergodic system (X, T, μ) is the factor whose σ-algebra is generated by all the eigenfunctions of T. We note the following result for later use (see for example [10]).

Proposition 5. *Let $X = G/\Gamma$ where G is a 2-step nilpotent Lie group, Γ is a closed cocompact subgroup and let T be translation by some $\tau \in G$. Let X be endowed with its Haar measure μ. Assume that property* (ii) *is satisfied. Then the following properties are equivalent:*

(1) *(X, T) is minimal.*
(2) *The subgroup of G spanned by τ and Γ is dense in G.*
(3) *The subgroup of G spanned by τ and ΓG_2 is dense in G.*

If these properties are satisfied, then the Kronecker factor of the ergodic system (X, μ, T) is the compact abelian group $Z = G/G_2\Gamma$ endowed with translation by the image of τ.

4.4. **The examples of Section 2.3 and 2.4, revisited.** By the preceding criteria, the affine system defined in Section 2.3 is minimal if and only if α is irrational. We remark that this system is written in reduced form.

The first presentation of the Heisenberg system $N_d = G/\Gamma$ given in Section 2.4 was not reduced. The reduction to the presentation $N_d = H_d/\Lambda_d$ we carried out in that Section is exactly the reduction explained above to the case that hypothesis (iii) holds. In this presentation, N_d is written in reduced form. By the criterion of Proposition 5, the nilsystem (N_d, T) is minimal if and only if the reals $\alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d$ are rationally independent modulo 1.

4.5. **Decomposition into elementary nilsequences.** In this Section, we prove Proposition 4 and introduce some notation used in the sequel.

Let $(X = G/\Gamma, T)$ be a minimal 2-step nilsystem in reduced form.

The restriction of the left translation $G \times X \to X$ to G_2 defines a continuous action of G_2 on X, and this action commutes with T. The quotient of X under this action is the compact abelian group $Z = G/G_2\Gamma$.

For every character χ of G_2, let $\mathcal{C}_\chi(X)$ denote the space of continuous functions f on X satisfying

$$f(u \cdot x) = \chi(u)\, f(x) \quad (u \in G_2,\ x \in X)\ . \tag{7}$$

If X is an elementary nilsystem, that is, if $G_2 = \mathrm{S}^1$, we identify the dual group $\widehat{G_2}$ with $\mathbb{Z}$ and the notation in (7) coincides with the notation in (6) of Definition 3.

Note that $\mathcal{C}_0(X)$ is the space of functions on X which factorize through Z. Every space $\mathcal{C}_\chi(X)$ is invariant under T and under multiplication by functions belonging to $\mathcal{C}_0(X)$. The spaces $\mathcal{C}_\chi(X)$ corresponding to different characters χ are orthogonal with respect to the inner product of $L^2(\mu)$.

Let $f \in \mathcal{C}(X)$. For every $\chi \in \widehat{G_2}$, we define a function f_χ on X by

$$f_\chi(x) = \int_{G_2} f(u \cdot x)\, \overline{\chi(u)}\, dm_{G_2}(u)\ ,$$

where m_{G_2} denotes the Haar measure of G_2. Clearly, for every $\chi \in \widehat{G_2}$. the function f_χ belongs to the space $\mathcal{C}_\chi(X)$. Moreover,

$$f = \sum_{\chi \in \widehat{G_2}} f_\chi$$

where the series converges in $L^2(X)$, and in uniform norm whenever f is sufficiently smooth. By density, every continuous function f on X can be approximated uniformly by finite sums of the form

$$\sum_{i=1}^{k} f_i\ ,\ \text{ where } f_i \in \mathcal{C}_{\chi_i}(X) \text{ and } \chi_i \in \widehat{G_2} \text{ for every } i\ .$$

Therefore, every basic nilsequence $\mathbf{a}$ arising from X as in Definition 1 can be approximated uniformly by finite sums of nilsequences of the form $\mathbf{b} = (f(T^n x_0) \colon n \in \mathbb{Z})$, where f belongs to $\mathcal{C}_\chi(X)$ for some $\chi \in \widehat{G_2}$.

We are left with describing sequences $\mathbf{b}$ of this form. If χ is trivial, then this sequence is almost periodic. Assume that χ is not trivial. Let Y be the quotient of X under the action of the subgroup $\ker(\chi)$ of G_2, $p \colon X \to Y$ be the natural projection and S be the transformation induced by T on Y. Then it is immediate that (Y, S) is an elementary nilsystem. Moreover, every $f \in \mathcal{C}_\chi(X)$ can be written as $f = \phi \circ p$ for some continuous function ϕ on Y belonging to $\mathcal{C}_1(Y)$. Therefore, the sequence $\mathbf{b}$ is equal to $(\phi(S^n y_0))$, where $y_0 = p(x_0)$. This completes the proof of Proposition 4. □

5. The Bohr extension of an elementary nilsystem

5.1. **The space of nilsequences associated to an elementary nilsystem.**

Definition 7. Let $(X = G/\Gamma, T)$ be an elementary nilsystem (minimal and written in reduced form). We write $\mathcal{N}^2(X)$ for the closed linear subspace of $\ell^\infty(\mathbb{Z})$ spanned by sequences of the form $\big(f(T^n x_0) \colon n \in \mathbb{Z}\big)$ where $f \in \mathcal{C}_1(X)$ and $x_0 \in X$.

Lemma 1. *Let $(X = G/\Gamma, T)$ be an elementary nilsystem.*

(i) *$\mathcal{N}^2(X)$ is invariant under the shift and multiplication by almost periodic sequences.*

(ii) *For every $x_1 \in X$, $\mathcal{N}^2(X)$ is the closed linear subspace of $\ell^\infty(\mathbb{Z})$ spanned by sequences of the form $\big(e(nt) f(T^n x_1) \colon n \in \mathbb{Z}\big)$ where $t \in \mathbb{T}$ and $f \in \mathcal{C}_1(X)$.*

Proof. (i) The invariance under the shift is obvious. It remains to show that every sequence $\mathbf{a}$ of the form $(e(nt)f(T^n x_0) \colon n \in \mathbb{Z})$ for some $f \in \mathcal{C}_1(X)$ and $t \in \mathbb{T}$ belongs to $\mathcal{N}^2(X)$.

Set $K = \{[\tau, g] \colon g \in G\}$ and first we show that $K = \mathrm{S}^1$. We have that K is a subgroup of $\mathbb{T}$. Since G is spanned by G_0 and τ, we have that K is connected and so is either equal to $\mathbb{T}$ or is trivial. If K is trivial, then τ belongs to the center $\mathcal{Z}(G)$ of G. Given $x \in X$, its closed orbit is thus included in $\mathcal{Z}(G) \cdot x$, and, since it is equal to X by minimality, we have $\mathcal{Z}(G) \cdot x = X$. This means that the system (X, T) is a rotation on a compact abelian group, a contradiction. Thus $K = \mathrm{S}^1$.

Now let f, x_0, t and $\mathbf{a}$ be as above. Pick $g \in G$ such that $e(t) = [g, \tau]$ and set $x_1 = g \cdot x_0$ and $h(x) = f(g^{-1} \cdot x)$ for every $x \in X$. For every $n \in \mathbb{Z}$,

$$h(T^n x_1) = h(\tau^n g \cdot x_0) = h([\tau, g]^n g \tau^n \cdot x_0) = e(nt) h(g \tau^n \cdot x_0) = e(nt) f(\tau^n \cdot x_0) = a_n$$

and we are done.

(ii) Fix $x_1 \in X$. Let M denote the closed span of sequences of the form $\big(e(nt)f(T^n x_1) \colon n \in \mathbb{Z}\big)$ where $t \in \mathbb{T}$ and $f \in \mathcal{C}_1(X)$. We need to show that for every $x_0 \in X$ and every $h \in \mathcal{C}_1(X)$, we have that the sequence $\big(h(T^n x_0) \colon n \in \mathbb{Z}\big)$ belongs to M.

Choose $g \in G$ such that $g \cdot x_1 = x_0$. Let $t \in \mathbb{T}$ be such that $e(t) = [\tau, g] \in G_2$ and define $f(x) = h(g \cdot x)$. Then for $n \in \mathbb{Z}$, by the same computation as above, we have that $h(T^n x_0) = e(nt) f(\tau^n \cdot x_1)$ and we are done. □

Corollary 1. *Let (X, T) be an elementary nilsystem. Then $\mathcal{N}^2(X) \perp \mathcal{AP}$, meaning that for every $\mathbf{a} \in \mathcal{N}^2(X)$ and every $\mathbf{b} \in \mathcal{AP}$, we have $\mathrm{Av}(\mathbf{a}\,\mathbf{b}) = 0$.*

If $\mathbf{a}$ is an elementary nilsequence and is not almost periodic, then $\mathbf{a} \perp \mathcal{AP}$.

Proof. Every function in $\mathcal{C}_1(X)$ obviously has zero integral with respect to the Haar measure of X. Thus by definition and unique ergodicity, every sequence in $\mathcal{N}^2(X)$ has zero average. The first statement follows immediately from Part (i) of Lemma 1. The second statement follows by density. □

5.2. **Construction of the Bohr extension.** Let $(X = G/\Gamma, T)$ be a minimal elementary 2-step nilsystem in reduced form $X = G/\Gamma$. We use following notation.

T denotes the translation by $\tau \in G$, $\pi \colon X \to Z = G/G_2\Gamma$ and $p \colon G \to Z$ are the natural projections, $\sigma = p(\tau)$ and S is the translation by σ on Z. (So $\pi \colon (X, T) \to (Z, S)$ is a factor map.) Z is endowed with its Haar measure m_Z. Let $\mathrm{B}(\mathbb{Z})$ denote the Bohr compactification of $\mathbb{Z}$: it is the dual of the circle group endowed with the discrete topology and it contains $\mathbb{Z}$ as a dense subgroup. Let $\mathrm{B}(\mathbb{Z})$ be endowed with its Haar measure $m_{\mathrm{B}(\mathbb{Z})}$. We write R for the (minimal) translation by 1 in $\mathrm{B}(\mathbb{Z})$. There exists a (well defined) continuous group homomorphism $r \colon \mathrm{B}(\mathbb{Z}) \to Z$ such that $r(1) = \sigma$. This homomorphism is onto, and it is a factor map from $(\mathrm{B}(\mathbb{Z}), R)$ to (Z, S).

Define

$$\tilde{X} = \{(x, z) \in X \times \mathrm{B}(\mathbb{Z}) \colon \pi(x) = r(z)\}$$

and let $\tilde{T}$ be the restriction of $T \times R$ to $\tilde{X}$. We write $q_1 \colon \tilde{X} \to X$ and $q_2 \colon \tilde{X} \to \mathrm{B}(\mathbb{Z})$ for the natural projections.

The system $(\tilde{X}, \tilde{T})$ is called the *Bohr extension* of (X, T).

We give a second presentation of this system. Define:

$$\tilde{G} = \big\{(g, z) \in G \times \mathrm{B}(\mathbb{Z}) \colon p(g) = r(z)\big\} \ ; \ \tilde{\Gamma} = \big\{(\gamma, 0) \colon \gamma \in \Gamma\big\}$$

and define $\tilde{\tau} = (\tau, 1) \in G \times \mathrm{B}(\mathbb{Z})$. Then $\tilde{G}$ is a closed subgroup of $G \times \mathrm{B}(\mathbb{Z})$, $\tilde{\tau} \in \tilde{G}$ and $\tilde{\Gamma}$ is a discrete subgroup of $\tilde{G}$. The map $(g, z) \mapsto (p(g), z)$ from $G \times \mathrm{B}(\mathbb{Z})$ to $X \times \mathrm{B}(\mathbb{Z})$ induces a homeomorphism of $\tilde{G}/\tilde{\Gamma}$ onto $\tilde{X}$ and we identify these spaces. Under this identification, $\tilde{T}$ is the translation by $\tilde{\tau}$ on $\tilde{X}$.

We have that $\tilde{G}_2 = \{(u, 0) \colon u \in G_2\}$ and thus can be identified with the circle group S^1. The action of this group on $\tilde{X}$ is given by

$$u \cdot (x, z) = (u \cdot x, z) \quad (u \in \mathrm{S}^1 = G_2,\ (x, z) \in \tilde{X}) .$$

This action of G_2 commutes with $\tilde{T}$ and the quotient of $\tilde{X}$ by this action can be identified with $\mathrm{B}(\mathbb{Z})$ in a natural way, the identification being given by the projection q_2.

The next result follows classically from the fact that all eigenfunctions of X factorize through Z and that $(\mathrm{B}(\mathbb{Z}), R)$ is a rotation. For completeness, we give a proof in a appendix A.

Lemma 2. $(\tilde{X}, \tilde{T})$ *is uniquely ergodic and the topological support of its invariant measure is equal to* $\tilde{X}$*.*

Corollary 2. $(\tilde{X}, \tilde{T})$ *is minimal.*

5.3. **Applications to elementary nilsequences.** We keep the notation of the previous sections. As for elementary nilsystems, we define $\mathcal{C}_1(\tilde{X})$ to be the space of continuous functions f on $\tilde{X}$ such that

$$f(u \cdot x, w) = u f(x, w) \quad (u \in \mathrm{S}^1,\ (x, w) \in \tilde{X}) .$$

Lemma 3. *Let* $\tilde{x}_0 \in \tilde{X}$*. The map*

$$f \mapsto \big(f(\tilde{T}^n \tilde{x}_0 \colon n \in \mathbb{Z})\big) \tag{8}$$

is an isometry of $\mathcal{C}_1(\tilde{X})$ *onto the space* $\mathcal{N}^2(X)$*.*

Proof. Let $x_0 = q_1(\tilde{x}_0)$. We identify $\widehat{\mathrm{B}(\mathbb{Z})}$ with $\mathbb{T}$ (with the discrete topology).

Clearly, $\mathcal{C}_1(\tilde{X})$ is the closed (under the uniform norm) linear span of the family of functions $(x, w) \mapsto \phi(x)\chi(w)$ for $\phi \in \mathcal{C}_1(X)$ and $\chi \in \widehat{\mathrm{B}(\mathbb{Z})}$. If f is a function of this type, then for every $n \in \mathbb{Z}$, we have

$$f(\tilde{T}^n \tilde{x}_0) = \phi(T^n x_0) e(n\chi) .$$

Thus the sequence $(f(\tilde{T}^n \tilde{x}_0))$ belongs to $\mathcal{N}^2(X)$ by part (i) of Lemma 1. By density, the same property remains valid for every $f \in \mathcal{C}_1(\tilde{X})$ and thus formula (8) defines a map $\jmath \colon \mathcal{C}_1(\tilde{X}) \to \mathcal{N}^2(X)$.

If $f \in \mathcal{C}_1(\tilde{X})$ and $\mathbf{a} = \jmath(f)$, by minimality of $(\tilde{X}, \tilde{T})$ we have that

$$\|\mathbf{a}\|_\infty = \sup_{n \in \mathbb{Z}} |f(\tilde{T}^n \tilde{x}_0)| = \|f\|_\infty$$

and the map $\jmath$ is an isometry.

By Part (ii) of Lemma 1, $\mathcal{N}^2(X)$ is the closed linear span of the family of sequences of the type $(e(nt)h(T^n x_0))$, where $t \in \mathbb{T}$ and $h \in \mathcal{C}_1(X)$. We are left with showing that every sequence of this type belongs to the range of $\jmath$. Let χ be the character of $\mathrm{B}(\mathbb{Z})$ corresponding to t and let $f(x, w) = h(x)\chi(w)$. This function belongs to $\mathcal{C}_1(\tilde{X})$, and its image under the isometry $\jmath$ is the given sequence. $\square$

Corollary 3. *Let (X,T) be an elementary nilsystem. Then $\mathcal{N}^2(X)$ is irreducible, meaning that it does not contain any closed proper nontrivial subspace that is invariant under the shift and multiplication by linear exponential sequences.*

In particular, for every non-zero $\mathbf{a}\in\mathcal{N}^2(X)$, we have that $\mathcal{S}(\mathbf{a})=\mathcal{N}^2(X)$.

Proof. Let $\mathbf{a}$ a non-identically zero elementary nilsequence belonging to $\mathcal{N}^2(X)$. We need to show that the class $\mathcal{S}(\mathbf{a})$ of this sequence is equal $\mathcal{N}^2(X)$.

Through the isometry defined in Lemma 3, the sequence $\mathbf{a}$ corresponds to a non-identically zero function f on $\tilde{X}$ that belongs to $\mathcal{C}_1(\tilde{X})$. Since for every $t\in\mathbb{T}$ and every $k\in\mathbb{Z}$ the sequence $\mathbf{e}(t)\sigma^k\mathbf{a}$ corresponds to the function $\chi\circ p_2\cdot T^kf$, where χ is the character of $\mathrm{B}(\mathbb{Z})$ associated to t, the space $\mathcal{S}(\mathbf{a})$ corresponds through this isometry to the closed linear subspace $\mathcal{F}$ of $\mathcal{C}_1(\tilde{X})$ spanned by the functions $\chi\circ q_2\cdot f\circ T^n$, where $\chi\in\widehat{\mathrm{B}(\mathbb{Z})}$ and $n\in\mathbb{Z}$. It remains to show that this space is equal to $\mathcal{C}_1(X)$.

Let U be the non-empty open subset $\{x\in\tilde{X}\colon f(x)\neq 0\}$ of $\tilde{X}$. As $f\in\mathcal{C}_1(\tilde{X})$ and the quotient of $\tilde{X}$ under the action of S^1 is equal to $\mathrm{B}(\mathbb{Z})$, we have that $U=p_2^{-1}(V)$ where V is some open subset of $\mathrm{B}(\mathbb{Z})$.

Since the closed linear span of the functions $\chi\circ p_2$ for $\chi\in\widehat{\mathrm{B}(\mathbb{Z})}$ is equal to $\{h\circ p_2\colon h\in\mathcal{C}(\mathrm{B}(\mathbb{Z}))\}$, we have that the closed linear span $\mathcal{F}$ of the family $\chi\circ q_2\cdot f$ for $\chi\in\widehat{\mathrm{B}(\mathbb{Z})}$ is equal to

$$\{g\in\mathcal{C}_1(\tilde{X})\colon g=0 \text{ outside of } U\}\ .$$

By minimality, there exists an integer $m\geq 1$ such that the sets R^jV, $0\leq j\leq m$, cover $\mathrm{B}(\mathbb{Z})$. We chose a partition of the unity $\{\phi_j\colon 0\leq j\leq m\}$ in $\mathcal{C}(\mathrm{B}(\mathbb{Z}))$, subordinated to this cover. For any $h\in\mathcal{C}_1(\mathrm{B}(\mathbb{Z}))$, each of the functions $h\cdot\phi_j\circ p_2$ belongs to $\mathcal{F}$, and thus h belongs to $\mathcal{F}$. □

In this proof, the shift was only needed because the function f on $\tilde{X}$ may vanish at some point.

Corollary 4. *Let $(X=G/\Gamma,T)$ and $(X'=G'/\Gamma',T')$ be two elementary nilsystems and assume that there exists a factor map $q\colon(X,T)\to(X',T')$ commuting with the action of $\mathrm{S}^1=G_2=G_2'$ on these systems. Then $\mathcal{N}^2(X)=\mathcal{N}^2(X')$.*

Proof. The factor map induces an inclusion of $\mathcal{N}^2(X)$ in $\mathcal{N}^2(X')$ and equality follows from the irreducibility of $\mathcal{N}^2(X')$. □

Remark. One can check that in this case $\widetilde{X'}=\tilde{X}$ and this leads to another proof of the same result.

Corollary 5. *Let $\mathbf{a}$ be a non-zero basic elementary nilsequence. Then there exists an elementary nilsystem X such that $\mathcal{S}(\mathbf{a})=\mathcal{N}^2(X)$.*

6. Taking products

We know that the product of two nilsequences is a nilsequence. We consider here the case of two elementary nilsequences. If $\mathcal{A}$ and $\mathcal{B}$ are two families of sequences, we write

$$\mathcal{A}\cdot\mathcal{B}:=\{\mathbf{a}\,\mathbf{b}\colon\mathbf{a}\in\mathcal{A},\ \mathbf{b}\in\mathcal{B}\}\ .$$

Let (X, T) be an elementary nilsystem. Then for every $\phi, \psi \in \mathcal{C}_1(X)$, the function $\phi\bar{\psi}$ belongs to $\mathcal{C}_0(X)$ and for every every $x_0 \in X$, the sequence $(\phi(T^n x_0)\overline{\psi(T^n x_0)} : n \in \mathbb{Z})$ is almost periodic. Thus

$$\mathcal{N}^2(X) \cdot \overline{\mathcal{N}^2(X)} \subset \mathcal{AP} \ .$$

6.1. **Orthogonality.**

Proposition 6. *If $(X = G/\Gamma, T)$ and $(X' = G'/\Gamma', T')$ are two elementary nilsystems, then exactly one of the following statements holds:*

- *$\mathcal{N}^2(X) = \mathcal{N}^2(X')$, and in this case $\mathcal{N}^2(X) \cdot \overline{\mathcal{N}^2(X')} \subset \mathcal{AP}$.*
- *$\mathcal{N}^2(X) \perp \mathcal{N}^2(X')$, and in this case there exists and elementary nilsystem (Y, S) such that $\mathcal{N}^2(X) \cdot \overline{\mathcal{N}^2(X')} \subset \mathcal{N}^2(Y)$.*

More precisely, $\mathcal{N}^2(X) = \mathcal{N}^2(X')$ if and only if there exists a closed subgroup H of $G \times G'$ such that

(i) *$(\tau, \tau') \in H$.*
(ii) *The commutator subgroup H_2 of H is the diagonal subgroup of $\mathbb{T}^2 = G_2 \times G'_2$.*
(iii) *The subgroup $\Lambda = (\Gamma \times \Gamma') \cap H$ of H is cocompact in H.*

The invariance under products of the family of elementary nilsequences follows immediately. Indeed, the product of an elementary nilsequence by an almost periodic sequence is a nilsequence; Proposition 6 shows that the product of two basic elementary nilsequences is an elementary nilsequence, and the general case follows by density.

Proof.

6.1.1. First we assume that there exists no group H satisfying properties (i), (ii) and (iii) above.

Let $x_0 \in X$ be the image of the unit element of G and $x'_0 \in X'$ the image of the unit element of G'. Then $\Gamma \times \Gamma'$ is the stabilizer of (x_0, x'_0) under the action of $G \times G'$ on $X \times X'$.

Let W be the closed orbit under $T \times T'$ of the point (x_0, x'_0) in the nilsystem $(X \times X', T \times T')$. It is classical that $(W, T \times T')$ is a two step nilsystem. Write $W = H/\Lambda$, where H, Λ satisfy properties (i) and (iii) above. By hypothesis, property (ii) is not satisfied and H_2 is not equal to the diagonal subgroup of $\mathbb{T}^2$.

By minimality, the natural projection $W \to X$ is onto. It follows that the image of the natural projection $H \to G$ has countable index in G. Therefore this image is open in G and thus contains G_0. As it contains τ, it is equal to G. For the same reason, the natural projection $H \to G'$ is onto.

Therefore H is not abelian. Its commutator subgroup is therefore not the trivial subgroup of $\mathbb{T}^2$. Let χ be the restriction to H_2 of the character $(u, u') \mapsto u - u'$ of $\mathbb{T}^2$. Thus χ is not the trivial character of H_2.

Let (Y, S) be the elementary nilsystem obtained by taking the quotient of W by the subgroup $\ker(\chi)$ of G_2 as in Section 4.5. Let $p \colon W \to Y$ be the natural projection and set $y_0 = p(x_0, x'_0)$.

Let $\phi \in \mathcal{C}_1(X)$ and $\phi' \in \mathcal{C}_1(X')$. Let Φ be the restriction to Y of the function $(x, x') \mapsto \phi(x)\overline{\phi'(x')}$. This function belongs to $\mathcal{C}_\chi(W)$ and thus can be written as $\Psi \circ p$ for some function $\Psi \in \mathcal{C}_1(Y)$. For every $n \in \mathbb{Z}$, we have

$$\phi(T^n x_0)\overline{\phi'((T')^n x'_0)} = \Phi((T \times T')^n (x_0, x'_0)) = \Psi(S^n y_0) \ .$$

Since $\mathcal{N}^2(X)$ is the closed linear span of the family of sequences of the form $(\phi(T^n x_0)e(nt))$, and similarly for $\mathcal{N}^2(X')$, we have that $\mathcal{N}^2(X)\cdot\overline{\mathcal{N}^2(X')} \subset \mathcal{N}^2(Y)$. By irreducibility we have that

$$\text{the closed linear span of } \mathcal{N}^2(X)\cdot\overline{\mathcal{N}^2(X')} \text{ is equal to } \mathcal{N}^2(Y)\ .$$

Since every sequence in $\mathcal{N}^2(Y)$ has zero average, $\mathcal{N}^2(X) \perp \mathcal{N}^2(X')$.

6.1.2. We now assume that there exists a subgroup H of $G\times G'$ with properties (i), (ii) and (iii).

Let L be the closed subgroup of $G\times G'$ spanned by Λ and $\sigma=(\tau,\tau')$.

Set $W=L/\Lambda$ and let $S\colon W\to W$ be the translation by $\sigma=(\tau,\tau')$. Then (W,S) is a 2-step nilsystem and the natural projections $W\to X$ and $W\to X'$ are factor maps. By the same argument as in Section 6.1.1, the natural projections $L\to G$ and $L\to G'$ are onto and L is not abelian. The commutator subgroup of L is thus a nontrivial subgroup of H_2, and by hypothesis, it is the diagonal subgroup of $\mathbb{T}^2$. Since L_2 is connected, $L_2=H_2$.

Substituting L for H, we are reduced to the case that

(iv) H is the closed subgroup of $G\times G'$ spanned by Λ and σ.

It follows that (W,S) is an elementary nilsystem. The natural factor maps $W\to X$ and $W\to X'$ commute with the actions of $\mathbb{T}=G_2=G'_2=H_2$ on X,X' and W, respectively. By Corollary 4, we have that $\mathcal{N}^2(X)=\mathcal{N}^2(W)=\mathcal{N}^2(X')$. □

6.2. For later use, we show:

Lemma 4. *Let X,X' be as in Proposition 6 and let H be a closed subgroup of $G\times G'$ satisfying the conditions* (i), (ii) *and* (iii) *of Proposition 6. Then there exist a subgroup Γ_1 of Γ with finite index, a subgroup Γ'_1 of Γ' with finite index and an isomorphism $\Phi\colon\Gamma_1\to\Gamma'_1$ such that*

$$\text{for every } \gamma\in\Gamma_1,\quad [\Phi(\gamma),\tau']=[\gamma,\tau]\ ; \tag{9}$$

$$\Lambda=\big\{\big(\gamma,\Phi(\gamma)\big)\colon\gamma\in\Gamma_1\big\}\ . \tag{10}$$

Proof. Let Γ_1 and Γ'_1 be the images of Λ under the natural projections $H\to G$ and $H\to G'$, respectively.

By (iii), Λ is cocompact in H. We have shown above that the natural projection $H\to G$ is onto. It follows that Γ_1 is cocompact in G and thus it has finite index in Γ. By the same proof, Γ'_1 has finite index in Γ'.

For every $\gamma\in\Gamma_1$, by definition there exists $\gamma'\in\Gamma'_1$ with $(\gamma,\gamma')\in\Lambda$. Properties (i) and (ii) imply that $[\gamma',\tau']=[\gamma,\tau]$. Since the map $\gamma'\mapsto[\gamma',\tau']$ is one to one (see Section 4.3), this condition completely determines γ' for a given γ. Thus there exists a group homomorphism $\Phi\colon\Gamma_1\to\Gamma'$ satisfying (9) and (10). By exchanging the role played by the two coordinates, we have that Φ is a group isomorphism from Γ_1 onto Γ'_1. □

6.3. **Proof of Theorem 2.** We start with a lemma.

Lemma 5. *For every non-almost periodic elementary nilsequence* $\mathbf{a}$*, there exist a basic elementary nilsequence* $\mathbf{b}$ *with* $\mathcal{S}(\mathbf{a})=\mathcal{S}(\mathbf{b})$ *and an elementary nilsystem* (X,T) *with* $\mathcal{S}(\mathbf{a})=\mathcal{N}^2(X)$.

Proof. Let $\mathbf{a}$ be a non-almost periodic elementary nilsequence; in particular, $\mathbf{a}$ is not identically zero. There exists a sequence $(\mathbf{b}(j) \colon j \geq 1)$ of basic elementary nilsequences converging uniformly to $\mathbf{a}$. For every $j \geq 1$, let (X_j, T_j) be an elementary nilsystem such that $\mathbf{b}(j)$ belongs to $\mathcal{N}^2(X_j)$.

As the quadratic norm is a norm (Section 4.2), $\|\mathbf{a}\|_2 > 0$ and $\|\mathbf{a} - \mathbf{b}(j)\|_2 \to 0$ as $j \to +\infty$. Therefore, there exists $i \geq 1$ such that $\|\mathbf{b}(i) - \mathbf{b}(j)\|_2 < \|\mathbf{b}(i)\|_2$ for every $j > i$.

This implies that for $j > i$ the sequence $\mathbf{b}(j)$ is not orthogonal to the sequence $\mathbf{b}(i)$. The spaces $\mathcal{N}^2(X_j)$ and $\mathcal{N}^2(X_i)$ are not orthogonal and by Proposition 6, these spaces are equal. Therefore $\mathbf{a}(j)$ belongs to $\mathcal{N}^2(X_i)$ for every $j > i$ and so $\mathbf{a} \in \mathcal{N}^2(X_i)$. By Corollary 5, this space is equal to $\mathcal{S}(\mathbf{b}(j))$ □

We now prove Theorem 2 by collecting the results of the preceding Sections.

(i) Let $\mathbf{a}$ and $\mathbf{b}$ be two non-identically zero elementary nilsequences. We want to show that the spaces $\mathcal{S}(\mathbf{a})$ and $\mathcal{S}(\mathbf{b})$ are equal or orthogonal. If one of these sequences, say $\mathbf{a}$, is almost periodic then $\mathcal{S}(\mathbf{a}) = \mathcal{AP}$ and the result follows from Corollary 1.

Assume now that $\mathbf{a}$ and $\mathbf{b}$ are not almost periodic. By Lemma 5 and Corollary 3, there exist two elementary nilsystems (X, T) and (X', T') with $\mathcal{S}(\mathbf{a}) = \mathcal{N}^2(X)$ and $\mathcal{S}(\mathbf{b}) = \mathcal{N}^2(X')$ and the result follows from Proposition 6.

(ii) The same proof used for Lemma 5 shows that every class is open.

(iii) Follows directly from Lemma 5 and Proposition 6.

(iv) The unit element for the multiplication is $\mathcal{AP}$. Each class different from $\mathcal{AP}$ is equal to $\mathcal{N}(X)$ for some elementary nilsystem X, and by Proposition 6, the inverse of this class is $\mathcal{N}(\bar{X})$. □

7. Classification: reduction to the connected case

Here we begin the proof of Theorem 3, considering first the case of an elementary system arising from a non-connected group.

Proposition 7. *Let $(X = G/\Gamma, T)$ be a (minimal) elementary nilsystem. Then there exist $t \in \mathbb{T}$ and an elementary nilsystem $(X' = G'/\Gamma', T')$ with G' connected such that $\mathcal{N}^2(X) = \mathbf{q}(t) \cdot \mathcal{N}^2(X')$.*

Proof. Assume that $(X = G/\Gamma, T)$ is an elementary nilsystem and assume that T, as usual, is rotation by the element $\tau \in G$. Let G_0 denote the connected component of the identity in G and set $\Gamma_0 = \Gamma \cap G_0$ and $X_0 = G_0\Gamma/\Gamma \cong G_0/\Gamma_0$. We have that X_0 is a connected component of X and is open in X.

First step. We first build a particular element $\tau_0 \in G_0\Gamma$ such that the system (X_0, T_0), where T_0 is translation by τ_0, is minimal.

Since X is minimal, there exists $k \in \mathbb{N}$ such that $T^k X_0 \cap X_0$ is nonempty. Choose k to the be least integer such that this intersection is nonempty. (If X is connected, then $X = X_0$ and $k = 1$.) By definition of X_0, K is also the smallest integer with $\tau^k \in G_0\Gamma$ and $T^k X_0 = X_0$. Since G is spanned by G_0 and τ, it follows that $G_0\Gamma$ is spanned by G_0 and τ^k. Since (X, T) is minimal, the set

$$\{\tau^n\gamma \colon n \in \mathbb{Z}, \gamma \in \Gamma\} \cap G_0\Gamma = \{\tau^{kn}\gamma \colon n \in \mathbb{Z}, \gamma \in \Gamma\}$$

is dense in $G_0\Gamma$. It follows that (X_0, T^k) is a minimal system.

Since $\tau^k \in G_0\Gamma$ and G_0 is connected, there exists $\theta \in \Gamma$ and $\tau_1 \in G_0$ such that

$$\tau^k = \tau_1^k \theta \ . \tag{11}$$

We now consider two cases, depending on whether G_0 is abelian or not.

The abelian case. First assume that G_0 is abelian. Then X_0 is a compact connected abelian group of finite dimension and so is a torus. Furthermore, X_0 contains the one dimensional torus $G_2 = \mathrm{S}^1$ as a closed subgroup, and so X_0 can be identified with $Z_0 \times G_2$, where

$$Z_0 = G_0\Gamma/G_2\Gamma \cong G_0/G_2\Gamma_0 \ .$$

Here Z_0 is an open and connected subgroup of $Z = G/G_2\Gamma$ and so Z_0 is isomorphic to S^d for some $d \in \mathbb{N}$. Let $\pi\colon X \to Z$ denote the natural projection and so $\pi(\tau)$ denotes the image of τ in Z. Let α denote the image of the element $\tau_1 \in G_0\Gamma$ in Z_0. By Equation (11), $\alpha^k = \pi(\tau)^k$. Since (X_0, T^k) is minimal, the translation by α^k on Z_0 is minimal. Therefore, there exists $v \in G_2$ such that the translation by (α^k, v) on $Z_0 \times G_2$ is minimal. Since G_2 is connected, there exists $u \in G_2$ such that $v = u^k$.

Let τ_0 be a lift of (α, u) in G_0 and let T_0 be the translation by τ_0 on X_0. Then (X_0, T_0) is a rotation system and (X_0, T_0^k) is minimal. It follows that (X_0, T_0) is minimal. Since τ_0^k and τ^k both project to $\pi(\tau)^k$ on Z, there exists $\eta \in \Gamma$ and $w \in G_2$ such that $\tau^k = \tau_0^k \eta w$. Thus we have constructed an element τ_0 of G_0 such that:

(i) $\tau^k = \tau_0^k \eta w$ for some $\eta \in \Gamma$ and $w \in G_2$;
(ii) The system (X_0, T_0), where T_0 is the translation by τ_0, is minimal;
(iii) The system (X_0, T_0) is either a rotation on a compact abelian group or is an elementary nilsystem.

The nonabelian case. We now turn to the case that G_0 is not abelian. We take $\tau_0 = \tau_1$, $\eta = 1$ and $w = 1$ and show that the properties (i)–(iii) are satisfied.

Since G_0 is now assumed to be nonabelian, we have that $(G_0)_2$ is a nontrivial torus. It is a subgroup of $G_2 = \mathrm{S}^1$ and $(G_0)_2 = G_2$.

Again, G is spanned by G_0 and τ. Let $\gamma \in \Gamma$ and write $\gamma = g_0\tau^j$ for some $g_0 \in G_0$ and $j \in \mathbb{Z}$. By definition, Γ is the stabilizer of some point of X and so $j = km$ for some $m \in \mathbb{Z}$. Thus

$$\gamma = g_0\tau^{km} = g_0\tau_0^{km}[\tau_0, \theta]^{km(m-1)/2}\theta^m = g_1\theta^m$$

for some $g_1 \in G_0$. Therefore $g_1 \in G_0 \cap \Gamma = \Gamma_0$ and we conclude that Γ is spanned by Γ_0 and θ. Recall that the set $\{\tau^{kn}\gamma\colon n \in \mathbb{Z}, \gamma \in \Gamma\}$ is dense in $G_0\Gamma$. By Equation (11), this set is included in

$$\{\tau_0^{kn}u\gamma\colon n \in \mathbb{Z}, u \in G_2, \gamma \in \Gamma\}$$

and so this set is also dense in $G_0\Gamma$. Since G_0 is open, the set

$$\{\tau_0^{kn}u\gamma_0\colon n \in \mathbb{Z}, u \in G_2, \gamma \in \Gamma\} \cap G_0 = \{\tau_0^{kn}u\gamma_0\colon n \in \mathbb{Z}, u \in G_2, \gamma \in \Gamma_0\}$$

is dense in G_0. Since G_0 is connected and $(G_0)_2 = G_0$, we have that $(X_0 = G_0/\Gamma_0, T_0^k)$ is minimal. Thus (X_0, T_0) is an elementary nilsystem satisfying properties (i)–(iii).

Second step. Finally we compare the spaces $\mathcal{N}^2(X)$ and $\mathcal{N}^2(X_0)$. Let x_0 be the image in X of the unit element of G. The space $\mathcal{N}^2(X_0)$ is spanned by translates of sequences of the form

$$\mathbf{a} = (\phi(T^n x_0) \colon n \in \mathbb{Z}) \ ,$$

where $\phi \in \mathbb{C}_1(X)$ vanishes outside of X_0. By definition of T_0, we have $a_n = 0$ if $n \notin k\mathbb{Z}$. We also use ϕ to denote the restriction of ϕ to X_0 and define the sequence $\mathbf{b}$ by

$$b_n = \frac{1}{k}\sum_{j=0}^{k-1} e(jn)\phi(T_0^n x_0) = \begin{cases} \phi(T_0^n x_0) & \text{if } n \in k\mathbb{Z} \\ 0 & \text{otherwise} \end{cases} .$$

By construction, $\tau^k = \tau_0^k \eta w$ for some $\eta \in \Gamma$ and $w \in G_2$. Let $u \in G_2$ be defined by $\theta \cdot x_0 = u \cdot x_0$. If $n = km$ for some $m \in \mathbb{Z}$, we have that

$$\begin{aligned} a_n = \phi((\tau_0^k \theta w)^m \cdot x_0) &= \phi(\tau_0^{km}[\tau_0, \eta]^{km(m-1)/2} w^m \theta^m \cdot x_0) \\ &= [\tau_0, \eta]^{km(m-1)/2}\, w^m u^n\, \phi(\tau_0^{km} \cdot x_0) = e_n(s) q_n(t) b_n \end{aligned}$$

for some $s, t \in \mathbb{T}$ which do not depend on choice of ϕ.

For $n \notin k\mathbb{Z}$, we have that $a_n = 0 = e_n(s) q_n(t) b_n$ and so $\mathbf{a} = \mathbf{q}(t)\mathbf{e}(s)\mathbf{b}$. Thus $\mathbf{a} \in \mathbf{q}(t)\mathcal{N}^2(X_0)$. Since $\mathcal{N}^2(X)$ and $\mathbf{q}(t)\mathcal{N}^2(X_0)$ are irreducible, we have that $\mathcal{N}^2(X) = \mathbf{q}(t)\mathcal{N}^2(X_0)$. In conclusion, we have constructed an elementary nilsystem associated to a connected group such that $\mathcal{N}^2(X) = \mathbf{q}(t)\mathcal{N}^2(X')$. □

8. Classification: The connected case

8.1. **Reduction to Heisenberg systems.** For Heisenberg systems we use the definitions and notation of Section 2.4.1. The condition for minimality of Heisenberg nilsystems was given in Section 4.4.

Throughout this section, we consider $\mathcal{N}^2(X)$ for various spaces and various transformations on X. To minimize confusion, we write $\mathcal{N}^2(X, T)$ instead of the usual $\mathcal{N}^2(X)$.

Lemma 6. *Let $(X = G/\Gamma, T)$ be an elementary nilsystem (minimal, written in reduced form) where G is connected. Then there exists an Heisenberg minimal system (N_d, T') with $\mathcal{N}^2(X, T) = \mathcal{N}^2(H_d, T')$.*

Proof. By Lemma 12 in Appendix B, (X, T) is the product of of an elementary system (X', T'') of the form given by the lemma and a rotation. Since $\mathcal{N}^2(X')$ is invariant under taking products with almost periodic sequences, it is easy to check that $\mathcal{N}^2(X, T) = \mathcal{N}^2(X', T'')$.

We can therefore reduce to the case that $G = \mathbb{R}^{2d} \times \mathrm{S}^1$ and $\Gamma = \mathbb{Z}^{2d} \times \{1\}$ for some $d \geq 1$, with multiplication given by

$$(x, z) \cdot (x', z') = (x + x', zz' e(\langle Ax \mid x' \rangle) \quad (x, x' \in \mathbb{R}^{2d},\ z, z' \in \mathrm{S}^1)$$

where A is a $2d \times 2d$ matrix with integer entries such that the matrix $B = A - A^t$ is nonsingular. We build a Heisenberg nilsystem (N_d, T') with $\mathcal{N}^2(X, T) = \mathcal{N}^2(N_d, T')$.

Let J_{2d} be as in Section 3.4. Choose a $2d \times 2d$ matrix Φ with rational entries such that

$$\Phi^t J_{2d} \Phi = B \ .$$

(Existence of such a matrix follows using an antisymmetric version of Gauss decomposition of a quadratic form.) Let $\tau = (\delta, \gamma) \in \mathbb{R}^{2d} \times \mathrm{S}^1 = G$ be the element defining T, $\tau' = (\Phi(\delta), 1) \in \mathbb{R}^{2d} \times \mathrm{S}^1 = H_d$ and let T' be the translation by τ' on $N_d = H_d/(\mathbb{Z}_d \times \mathbb{Z}_d \times \{1\})$.

Since (X, T) is minimal, the coordinates of δ are rationally independent modulo 1 by Proposition 5. Since Φ has rational entries, the coordinates of $\Phi(\delta)$ are rationally independent modulo 1 and (N_d, T') is minimal. Define

$$H = \big\{\big((x, z), (x', z')\big) \in G \times H_d \colon x' = \Phi(x),\ z' = z\big\}\ .$$

Then H is a subgroup of $G \times N_d$ satisfying the conditions of Proposition 6 and thus $\mathcal{N}^2(X, T) = \mathcal{N}^2(N_d, T')$. □

8.2. The case of Heisenberg systems.

Lemma 7. *Let $(N_d = H_d/\Lambda_d, T)$ and $(N_{d'} = H_{d'}/\Lambda_{d'}, T')$ be two minimal Heisenberg systems, where the transformations T and T' are respectively the translation by $\tau = (\alpha, \beta, \gamma) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathrm{S}^1 = H_d$ and translation by $\tau' = (\alpha', \beta', \gamma') \in \mathbb{R}^{d'} \times \mathbb{R}^{d'} \times \mathrm{S}^1 = H_{d'}$.*

Then $\mathcal{N}^2(N_d, T) = \mathcal{N}^2(N_{d'}, T')$ if and only if $d = d'$ and there exists a matrix $Q \in \mathrm{Sp}_{2d}(\mathbb{Q})$ mapping (α, β) to (α', β').

(We consider Q as a linear map from $\mathbb{R}^d \times \mathbb{R}^d$ to itself.)

Proof. First assume that $\mathcal{N}^2(N_d, T) = \mathcal{N}^2(N_{d'}, T')$. Let H, Λ be as in Proposition 6 and $\Gamma_1, \Gamma_1', \Phi$ as in Lemma 4. Since Γ_1 is of finite index in the group Λ_d which is isomorphic to $\mathbb{Z}^d$, it is itself isomorphic to $\mathbb{Z}^d$. In the same way, Γ_1' is isomorphic to $\mathbb{Z}^{d'}$. Since $\Phi\colon \Gamma_1 \to \Gamma_1'$ is an isomorphism, $d = d'$. Moreover, Φ is given by a $2d \times 2d$ matrix Q with rational entries. Relation (9) of Lemma 4 means exactly that Q belongs to $\mathrm{Sp}_{2d}(\mathbb{Q})$, and relation (i) of Proposition 6 that Q maps (α, β) to (α', β').

Conversely, assume that $d = d'$ and that the matrix $Q \in \mathrm{Sp}_{2d}(\mathbb{Q})$ maps (α, β) to (α', β'). We set

$$H = \big\{\big((x, y, z), (x', y', z')\big) \in N_d \times N_d \colon (x', y') = Q(x, y),\ z' = z\big\}$$

where $x, y, x', y' \in \mathbb{R}^d$ and $z, z' \in \mathbb{R}$. Then H satisfies all the conditions of Proposition 6 and thus $\mathcal{N}^2(N_d, T) = \mathcal{N}^2(N_{d'}, T')$. □

8.3. Heisenberg systems and quadratic exponential sequences.

Lemma 8. *Let (N_d, T) be a minimal Heisenberg nilsystem and let $t \in \mathbb{T}$ be such that for any integer $m > 0$, mt does not belong to the eigenvalue group of (N_d, T). Then the sequence $\mathbf{q}(t)$ does not belong to $\mathcal{N}^2(N_d, T)$ and there exists an elementary nilsystem (X, S) with $\mathbf{q}(t) \cdot \mathcal{N}^2(N_d, T) = \mathcal{N}^2(X, S)$.*

Moreover, for every Heisenberg system $(N_{d'}, T')$, $\mathbf{q}(t) \cdot \mathcal{N}^2(N_d, T) \neq \mathcal{N}^2(N_{d'}, T')$.

Proof. Let $G = H_d \times \mathbb{Z} \times \mathrm{S}^1$, endowed with multiplication given by

$$(g, m, z) \cdot (g', m', z') = (gg'z'^m, m + m', zz')\ ,$$

where we consider z'^m as an element of $\mathrm{S}^1 = (N_d)_2$. Then G is a 2-step nilpotent Lie group, with $G_2 = (H_d)_2 \times \{0\} \times \{1\}$. The subgroup $\Gamma = \Lambda_d \times \mathbb{Z} \times \{1\}$ is a discrete cocompact subgroup of G. Write $X = G/\Gamma$, $\sigma = (\tau, 1, e(t))$, and let S be the translation by σ on X.

Note that G is spanned by $G_0 = H_d \times \{0\} \times \mathrm{S}^1$ and σ, and that the rotation induced by σ on $G/\Gamma G_2 = \mathbb{T}^{2d} \times \mathbb{T}$ is minimal by the hypothesis on t. Therefore (X, S) is minimal. It is an elementary nilsystem, written in reduced form.

The map $(g, m, z) \mapsto (g, z)$ from G to $H_d \times \mathrm{S}^1$ induces a homeomorphism from X onto $N_d \times \mathrm{S}^1$. We identify these spaces. The transformation S has the form $(x, z) \mapsto (z \cdot Tx, ze(t))$. If h is a function belonging to $\mathcal{C}_1(N_d, T)$, then the function $h\colon (x, z) \mapsto f(x)$ belongs to $\mathcal{C}_1(X, S)$ and satisfies $h(S^n(x, 0)) = q_n(t)f(T^n x)$ for every $x \in X$ and every $n \in \mathbb{Z}$. We deduce that $\mathcal{N}^2(X, S)$ is included in $\mathbf{q}(t) \cdot \mathcal{N}^2(N_d, T)$ and these spaces are equal by irreducibility (Corollary 3). In particular, $\mathbf{q}(t) \cdot \mathcal{N}^2(N_d, T)$ does not contain the constant sequence 1; substituting $-t$ for t we have that $\mathbf{q}(t) \notin \mathcal{N}^2(N_d, T)$.

We are left with showing that when $(N_{d'}, T')$ is a minimal Heisenberg system, we have $\mathcal{N}^2(N_{d'}, T') \neq \mathbf{q}(t) \cdot \mathcal{N}^2(N_d, T)$. Assume that these spaces are equal, that is, that $\mathcal{N}^2(X, S) = \mathcal{N}^2(N_{d'}, T')$. Let the subgroups Λ_1' of $\Lambda_{d'}$ and Γ_1 of Γ and the isomorphism $\Phi\colon \Lambda_1' \to \Gamma_1$ be given by Proposition 6 and Lemma 4. Since Γ_1 is of finite index in $\Gamma = \Lambda_d \times \mathbb{Z} \times \{1\}$, the restriction to Γ_1 of the natural projection of Γ onto Λ_d can not be one to one. Therefore, there exists $0 \neq \gamma \in \Lambda_1'$ such that $\Phi(\gamma) = (0, m, 1)$ for some non-zero $m \in \mathbb{Z}$. By property (9) of Lemma 4, we have that

$$[\gamma, \tau'] = [\Phi(\gamma), \sigma] = e(mt) \ .$$

But $[\gamma, \tau']$ is an eigenvalue of $(N_{d'}, T')$ and thus of (N_d, T), a contradiction of the hypothesis. □

Lemma 9. *Let (N_d, T) and $(N_{d'}, T')$ be two minimal Heisenberg systems of dimensions $2d + 1$ and $2d' + 1$ respectively, with translations given by*

$$\tau = (\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d, e(\gamma)) \text{ and } \tau' = (\alpha_1', \ldots, \alpha_{d'}', \beta_1', \ldots, \beta_{d'}', e(\gamma'))$$

respectively. Let $t \in \mathbb{T}$.

Then $\mathbf{q}(t)\,\mathcal{N}^2(N_d, T) = \mathcal{N}^2(N_{d'}, T')$ if and only if $d = d'$ and there exist a $2d \times 2d$ matrix $Q \in \mathrm{Sp}_{2d}(\mathbb{Q})$ and integers $m, k_1, \ldots, k_d, \ell_1, \ldots, \ell_d$ with $m \geq 1$ such that

$$Q \begin{pmatrix} \alpha_1 + k_1/m \\ \vdots \\ \alpha_d + k_d/m \\ \beta_1 + \ell_1/m \\ \vdots \\ \beta_d + \ell_d/m \end{pmatrix} = \begin{pmatrix} \alpha_1' \\ \vdots \\ \alpha_d' \\ \beta_1' \\ \vdots \\ \beta_d' \end{pmatrix} \tag{12}$$

$$\text{and} \quad mt = \sum_{i=1}^{d} (k_i\beta_i - \ell_i\alpha_i) \bmod 1 \ . \tag{13}$$

Proof.

a) Let (N_d, T) be a minimal Heisenberg system of dimension $2d + 1$, where T is the translation by $\tau = (\alpha_1, \ldots, \alpha_d, \beta_1, \ldots, \beta_d, e(\gamma))$. Let $m, k_1, \ldots, k_d, \ell_1, \ldots, \ell_d$ be integers with $m \geq 1$ and assume that $t \in \mathbb{T}$ satisfies (13). Let

$$\tau' = (\alpha_1 + k_1/m, \ldots, \alpha_d + k_d/m, \beta_1 + \ell_1/m, \ldots, \beta_d + k_d/m, e(\gamma'))$$

where $\gamma' \in \mathbb{T}$ is arbitrary and T' the translation by τ' on N_d. The first $2d$ coordinates of τ' are rationally independent modulo 1 and thus (N_d, T') is minimal. We compare $\mathcal{N}^2(N_d, T)$ and $\mathcal{N}^2(N_d, T')$.

We have that $\tau'^m = \tau^m u\gamma$, where u is some element of $(H_d)_2 = \mathrm{S}^1$ and

$$\gamma = (k_1, \ldots, k_d, \ell_1, \ldots, \ell_d, 1) \in \Gamma \ .$$

An immediate computation gives that for every integer p,

$$\tau'^{mp} = \tau^{mp} v^{mp} e\big(\frac{mp(mp-1)}{2} t\big)\gamma^p$$

for some $v \in (H_d)_2 = \mathrm{S}^1$.

Let x_0 be the image in N_d of the unit element of H_d and let f be a function on N_d, belonging to $\mathcal{C}_1(N_d)$, with $f(x_0) \neq 0$. Let $\mathbf{a}$ and $\mathbf{b}$ be the sequences given by $a_n = \mathbf{1}_{m\mathbb{Z}}(n) f(T^n x_0)$ and $b_n = \mathbf{1}_{m\mathbb{Z}}(n) f(T'^n x_0)$. These sequence are not identically zero, the sequence $\mathbf{a}$ belongs to $\mathcal{N}^2(N_d, T)$ and the sequence $\mathbf{b}$ belongs to $\mathcal{N}^2(N_d, T')$ by definition. But for every integer n we have $b_n = v^n q_n(t) a_n$ by the above computation and thus the sequence $\mathbf{b}$ belongs to $\mathbf{q}(t)\,\mathcal{N}^2(N_d, T)$. Since the spaces $\mathbf{q}(t)\,\mathcal{N}^2(N_d, T)$ and $\mathcal{N}^2(N_d, T')$ are irreducible (Corollary 3), they are equal: $\mathbf{q}(t)\,\mathcal{N}^2(N_d, T) = \mathcal{N}^2(N_d, T')$.

b) Now assume that $d = d'$ and that $t \in \mathbb{T}$, the matrix $Q \in \mathrm{Sp}_{2d}(\mathbb{Q})$, and the integers $m, k_1, \dots, k_d, \ell_1, \dots, \ell_d$ satisfy the conditions (12) and (13). We need to show that $\mathbf{q}(t) \cdot \mathcal{N}^2(H_d, T) = \mathcal{N}^2(H_d, T')$. By using Lemma 7, we immediately reduce to the case that Q is the identity matrix. The result then follows from part a) above.

c) Assume now that $\mathbf{q}(t) \cdot \mathcal{N}^2(H_d, T) = \mathcal{N}^2(H_d, T')$. We show that there exist $m, k_1, \dots, k_d, \ell_1, \dots, \ell_d$ and Q satisfying (12) and (13).

By Lemma 8, there exists a non-zero integer m such that mt is an eigenvalue of X and thus there exist integers $k_1, \dots, k_d, \ell_1, \dots, \ell_d$ such that relation (13) holds. For $1 \leq j \leq d$ we set $\alpha''_j = \alpha_j + k_j/m$ and $\beta''_j = \beta_j + \ell_j/m$. Let T'' be the translation by $\tau'' = (\alpha''_1, \dots, \alpha''_d, \beta''_1, \dots, \beta''_d, \gamma)$ on N_d. By Part a), (N_d, T'') is minimal and $\mathcal{N}^2(N_d, T'') = \mathbf{q}(t) \cdot \mathcal{N}^2(N_d, T) = \mathcal{N}^2(N_{d'}, T')$. The conclusion follows from Lemma 7. $\square$

8.4. **Proof of Theorem 3.** a) We check that if $t, d, \alpha_1, \dots, \alpha_d, \beta_1, \dots, \beta_d$ are as in the statement of the theorem, then the sequence $\mathbf{a} = \mathbf{q}(t)\boldsymbol{\omega}(\alpha_1, \beta_1). \dots .\boldsymbol{\omega}(\alpha_d, \beta_d)$ is an elementary nilsequence.

If $d = 0$ and $t = 0$, this sequence is constant. If $d = 0$ and t is irrational, then $\mathbf{a} = \mathbf{q}(t)$ and this sequence belongs to $\mathcal{N}^2(Y)$ for some affine elementary nilsystem (Section 2.3).

Assume now that $d > 0$. By the discussion of Section 2.4.2, the sequence $\mathbf{b} = \boldsymbol{\omega}(\alpha_1, \beta_1) \cdot \dots \cdot \boldsymbol{\omega}(\alpha_d, \beta_d)$ belongs to $\mathcal{N}^2(N_d, T)$ for some Heisenberg system N_d. Let $t \in \mathbb{T}$ be an irrational and let (Y, T') be an affine elementary nilsystem with $\mathbf{q}(t) \in \mathcal{N}^2(Y, T')$. If for every integer $m \neq 0$, mt does not belong to the eigenvalue group of N_d, then by Lemma 8 the sequence $\mathbf{q}(-t)$ does not belong to $\mathcal{N}^2(N_d, T)$. Thus $\mathcal{N}^2(Y, T') \neq \mathcal{N}^2(N_d, T)$. By Lemma 8, there exists an elementary nilsystem (X, S) with $\mathbf{q}(t) \cdot \mathcal{N}^2(N_d, T) = \mathcal{N}^2(X, S)$. Therefore $\mathbf{q}(t)\mathbf{b}$ belongs to $\mathcal{N}^2(X, S)$ and is an elementary nilsequence.

We are left with the case that mt belongs to the group of eigenvalues of (N_d, T) for some integer $m \neq 0$. Let $k_1, \dots, k_d, \ell_1, \dots, \ell_d$ be given by (13), Q be the identity $2d \times 2d$ matrix and set $\alpha'_j = \alpha_j + k_j/m$ and $\beta'_j = \beta_j + \ell_j/m$ for $1 \leq j \leq d$. Then $(\alpha'_1, \dots, \alpha'_d, \beta'_1, \dots, \beta'_d)$ are rationally independent modulo 1 and N_d endowed with the translation T' by $\tau' = (\alpha'_1, \dots, \alpha'_d, \beta'_1, \dots, \beta'_d)$ is a minimal Heisenberg system. By Lemma 9, $\mathbf{a} = \mathbf{q}(t)\mathbf{b}$ belongs to $\mathcal{N}^2(N_d, T')$ and thus is an elementary nilsequence.

b) Let $\mathbf{a}$ be a non-identically zero elementary nilsequence. We show that its class contains some sequence of the announced type.

If it is almost periodic, then its class is $\mathcal{AP}$ and thus contains the constant sequence 1. This is the exactly the sequence $\mathbf{q}(0)$. Now assume that $\mathbf{a}$ is not almost periodic. By Lemma 5, there exists a minimal nilsystem X with $\mathcal{S}(\mathbf{a}) = \mathcal{N}^2(X)$. By Proposition 7, there exist $t \in \mathbb{T}$ and an elementary nilsystem $(X' = G'/\Gamma', T')$ with G' connected and $\mathcal{N}^2(X,T) = \mathbf{q}(t) \cdot \mathcal{N}^2(X',T')$. If t is rational, we have that $\mathbf{q}(t) \cdot \mathcal{N}^2(X',T') = \mathcal{N}^2(X',T')$ and thus we can assume that t is equal to 0 or is irrational. By Lemma 6, we can assume that (X',T') is a Heisenberg system and $\mathcal{N}^2(X',T')$ contains a sequence of the type $\boldsymbol{\omega}(\alpha_1,\beta_1) \cdot \ldots \cdot \boldsymbol{\omega}(\alpha_d,\beta_d)$ satisfying the independence condition. We conclude that $\mathcal{S}(\mathbf{a})$ contains a sequence of the announced form.

c) Let $t,t' \in \mathbb{T}$ be equal to 0 or be irrationals, $d,d' \geq 0$ and let $\alpha_1,\ldots,\alpha_d,\beta_1,\ldots,\beta_d$ and $\alpha'_1,\ldots,\alpha'_{d'},\beta'_1,\ldots,\beta'_{d'}$ satisfy the independence condition. Assume that the sequences $\mathbf{q}(t)\,\boldsymbol{\omega}(\alpha_1,\beta_1)\cdot\ldots\cdot\boldsymbol{\omega}(\alpha_d,\beta_d)$ and $\mathbf{q}(t')\,\boldsymbol{\omega}(\alpha'_1,\beta'_1)\cdot\ldots\cdot\boldsymbol{\omega}(\alpha'_{d'},\beta'_{d'})$ belong to the same class. Then the sequences $\mathbf{q}(t-t')\,\boldsymbol{\omega}(\alpha_1,\beta_1)\cdot\ldots\cdot\boldsymbol{\omega}(\alpha_d,\beta_d)$ and $\boldsymbol{\omega}(\alpha'_1,\beta'_1)\cdot\ldots\cdot\boldsymbol{\omega}(\alpha'_{d'},\beta'_{d'})$ belong to the same class. By the first part of Lemma 8, d and d' are either both zero or both non-zero. If $d = d' = 0$, then $\mathbf{q}(t-t')$ is almost periodic by Proposition 6 and thus $t-t'$ is rational. If d and d' are non-zero, then Lemma 9 gives the announced relations between the parameters $t,t',d,d',\alpha_1,\ldots,\beta'_d$.

The converse implication also follows from the same lemmas. □

8.5. **Proof of Theorem 4.** The theorem follows immediately from Theorem 3 and Proposition 4. □

8.6. **Proof of Theorem 5.**

Lemma 10. *Let $\mathbf{a}$ be a sequence belonging to the family $\mathcal{M}$. Then the linear span of $\{\mathbf{e}(u)\mathbf{a} \colon u \in \mathbb{T}\}$ is dense in $\mathcal{S}(\mathbf{a})$ for the quadratic norm.*

Proof of Lemma 10. If $\mathbf{a} = \mathbf{e}(s)\mathbf{q}(t)$ for some $s,t \in \mathbb{T}$, then one can easily check that $\mathcal{S}(\mathbf{a}) = \mathcal{AP}\cdot\mathbf{a}$. Thus the linear span of $\{\mathbf{e}(u)\mathbf{a} \colon u \in \mathbb{T}\}$ is dense in $\mathcal{S}(\mathbf{a})$ for the uniform norm. Thus it is dense for the quadratic norm.

Now consider the case that $\mathbf{a} = \mathbf{e}(s)\mathbf{q}(t)\mathbf{b}$ where $\mathbf{b} = \boldsymbol{\omega}(\alpha_1,\beta_1)\cdot\ldots\cdot\boldsymbol{\omega}(\alpha_d,\beta_d)$ with $d \geq 1$ and the independence condition satisfied. Since $\mathcal{S}(\mathbf{a}) = \mathbf{q}(t)\mathcal{S}(\mathbf{b})$, we can reduce to the case that $s = t = 0$, meaning that $\mathbf{a} = \mathbf{b}$.

In Section 2.4.2, we built an explicit minimal Heisenberg system (N_d,T), $x_0 \in N_d$ and an explicit function $f \in \mathcal{C}_1(N_d)$ such that $b_n = f(T^n x_0)$ for every $n \in \mathbb{Z}$.

Let $K = \{w \in N_d \colon f(w) = 0\}$. Since $f \in \mathcal{C}_1(N_d)$, $K = \pi^{-1}(L)$ where π is the natural projection from N_d onto $Z = H_d/(H_d)_2\Lambda_d$ and L is a closed subset of Z. Recall that Z can be identified with $\mathbb{T}^{2d}$.

The relation (3) between the function κ defined in (2) and the theta function shows that $\tilde{f}(x,y,z)$ defined by (4) vanishes only when one of the coordinates x_j or y_j of x or y is equal to $1/2$ modulo 1. Therefore, the subset L of $Z = \mathbb{T}^{2d}$ has zero Haar measure. It follows (as in the proof of Corollary 3) that the linear span of the family of functions $\{f\cdot\chi\circ\pi \colon \chi \in \widehat{Z}\}$ is dense in $\mathcal{C}_1(N_d)$ for the $L^2(N_d)$ norm.

For $\chi \in \widehat{Z}$ and for every $n \in \mathbb{Z}$, we have

$$(f\cdot\chi\circ\pi)(T^n x_0) = e(ns)f(T^n x_0) = e(ns)b_n\ ,$$

where $s \in \mathbb{T}$ is the eigenvalue associated to χ. Let E be the group of eigenvalues of T. We deduce that the closure in quadratic norm of the linear span of the family $\{\mathbf{e}(s)\mathbf{b}\colon s \in E\}$ of sequences contains all sequences of the form $\big(h(T^n x_0)\colon n \in \mathbb{Z}\big)$ for $h \in \mathcal{C}_1(N_d)$. Therefore, the closure in quadratic norm of the family $\{\mathbf{e}(s)\mathbf{b}\colon s \in \mathbb{T}\}$ of sequences contains all sequences of the form $\big(e(nt)h(T^n x_0)\colon n \in \mathbb{Z}\big)$ for $h \in \mathcal{C}_1(N_d)$ and $t \in \mathbb{T}$. By Lemma 1, it therefore contains $\mathcal{S}(\mathbf{b})$. The announced result follows. □

We note that this explains the difference in the definition of the class of a bounded sequence, if we use the quadratic norm instead of the supremum norm. If an elementary sequence $\mathbf{a}$ is such that the sequence $(|a_n|\colon n \in \mathbb{Z})$ is bounded from below, then the shift can be removed from the definition: $\mathcal{S}(\mathbf{a})$ is spanned by sequences of the form $\mathbf{e}(t)\mathbf{a}$. However, there are very few elementary nilsequences of this type: they all are products of an almost periodic sequence and a quadratic sequence $\mathbf{q}(t)$, as defined in Section 2.3.

Proof of Theorem 5. The result follows immediately from Proposition 4, Theorem 3 and Lemma 10. □

9. Characterization of elementary nilsequences

In this Section we show Theorem 1.

9.1. **The "only if" part.**

First we show the easy part of Theorem 1. We restate it here for convenience.

Proposition. *Let $\mathbf{a}$ be an elementary nilsequence. There exists a compact (in the norm topology) subset $K \subset \ell^\infty(\mathbb{Z})$ such that for all $k \in \mathbb{Z}$, there exists $t \in \mathbb{T}$ such that the sequence $\mathbf{e}(t)\sigma^k\mathbf{a} = \big(e(nt)a_{n+k}\colon n \in \mathbb{Z}\big)$ belongs to K.*

Proof. If $\mathbf{a}$ is almost periodic, then the result follows immediately from the characterization (iii) of almost periodic sequences in Proposition 1. Thus we assume that $\mathbf{a}$ is not almost periodic.

By Lemma 5, there exists an elementary nilsystem (X, T) with $\mathbf{a} \in \mathcal{N}^2(X)$. Let $(\tilde{X}, \tilde{T})$ be the Bohr extension of (X, T) as defined in Section 5.2 and let $\tilde{x}_0 \in \tilde{X}$. By Lemma 3, there exists a continuous function f on $\tilde{X}$ with $a_n = f(\tilde{T}^n\tilde{x}_0)$ for all $n \in \mathbb{Z}$.

Recall that $\tilde{X}$ can be written as $\tilde{G}/\tilde{\Gamma}$, where $\tilde{G}$ is a locally compact 2-step nilpotent group, $\tilde{\Gamma}$ is a closed cocompact subgroup, and $\tilde{T}$ is translation by some $\tilde{\tau} \in \tilde{G}$. The natural projection $\tilde{G} \to \tilde{X}$ is an open map and by compactness, we deduce that there exists a compact subset $\tilde{H}$ of $\tilde{G}$ such that $\tilde{G} = \tilde{H}\tilde{\Gamma}$. Let K be the subset of $\ell^\infty(\mathbb{Z})$ consisting of sequences of the form

$$\big(f(\tilde{g}\tilde{\tau}^n \cdot \tilde{x}_0)\colon n \in \mathbb{Z}\big)$$

for $\tilde{g} \in \tilde{H}$. Since the map $(\tilde{g}, \tilde{x}) \mapsto \tilde{g} \cdot \tilde{x}$ is continuous, the function $(\tilde{g}, \tilde{x}) \mapsto f(\tilde{g} \cdot \tilde{x})$ is uniformly continuous on $\tilde{H} \times \tilde{X}$ and K is a compact subset of $\ell^\infty(\mathbb{Z})$.

Let k be an integer. Pick $\tilde{g} \in \tilde{H}$ and $\tilde{\gamma} \in \tilde{\Gamma}$ with $\tilde{\tau}^k = \tilde{g}\tilde{\gamma}$. The sequence $\mathbf{b} = \big(f(\tilde{g}\tilde{\tau}^n \cdot \tilde{x}_0)\colon n \in \mathbb{Z}\big)$ belongs to K. Taking $t \in \mathbb{T}$ with $e(t) = [\tilde{\gamma}, \tilde{\tau}] \in \tilde{G}_2 = \mathrm{S}^1$, we have that for all $n \in \mathbb{Z}$,

$$(\sigma^k\mathbf{a})_n = a_{n+k} = f(\tilde{\tau}^k\tilde{\tau}^n \cdot \tilde{x}_0) = f(\tilde{g}\tilde{\gamma}\tilde{\tau}^n \cdot \tilde{x}_0) = e(nt)f(\tilde{g}\tilde{\tau}^n \cdot \tilde{x}_0) = e(nt)b_n\ .$$

Thus $\sigma^k\mathbf{a} = \mathbf{e}(t)\mathbf{b}$ and this completes the proof. □

9.2. **The "if" part.** In the next Subsections we show the implication (ii) $\Longrightarrow$ (i) of Theorem 1. We restate it for convenience.

Proposition. *Let* $\mathbf{a} = (a_n \colon n \in \mathbb{Z})$ *be a bounded sequence and assume that there exists a compact (in the norm topology) subset* K *of* $\ell^\infty(\mathbb{Z})$ *such that for all* $k \in \mathbb{Z}$*, there exists* $t \in T$ *such that the sequence* $\mathbf{e}(t)\sigma^k\mathbf{a} = \big(e(nt)a_{n+k} \colon n \in \mathbb{Z}\big)$ *belongs to* K*. Then* $\mathbf{a}$ *is an elementary nilsequence.*

We use the following notation. Recall that $\|\cdot\|_\infty$ denotes the uniform norm on $\ell^\infty(\mathbb{Z})$. Let B be a subset of $\ell^\infty(\mathbb{Z})$. For $t \in \mathbb{T}$, $\mathbf{e}(t) \cdot B := \{\mathbf{e}(t)\mathbf{b} \colon \mathbf{b} \in B\}$. If $k \in \mathbb{Z}$, $\sigma^k B := \{\sigma^k \mathbf{b} \colon \mathbf{b} \in B\}$. If z is a number, $zB = \{z\mathbf{b} \colon \mathbf{b} \in B\}$.

Before the proof, we need a simple lemma:

Lemma 11. *Let* E *be a syndetic subset of* $\mathbb{Z}$ *of syndetic constant* L*, meaning that*

$$\bigcup_{j=0}^{L-1} (j + E) = \mathbb{Z} \ .$$

If $t \in \mathbb{T}$ *is such that*

$$|t| < 1/3L \text{ and } |e(nt) - 1| < 1/3L \text{ for every } n \in E \ ,$$

then $t = 0$.

Proof of the lemma. Assume that t satisfies the above properties. For every $n \in \mathbb{Z}$ there exists j with $0 \leq j < L$ such that $n - j \in E$ and by hypothesis the point nt belongs to the interval $(jt - 1/3L, jt + 1/3L)$. Therefore, $\mathbb{Z}t$ is included in the union U of the intervals $(jt - 1/3L, jt + 1/3L)$ for $0 \leq j < L$. The complement of U in $\mathbb{T}$ contains a closed interval J of length $1/3L$. As the sequence $(nt \colon n \in \mathbb{Z})$ avoids J and $|t| < 1/3L$, we have $t = 0$. □

9.3. **First reductions.** We now turn to the proof of the Proposition. We can obviously assume that the sequence $\mathbf{a}$ is not almost periodic and in particular not identically zero. We can also assume that

$$\|\mathbf{a}\|_\infty = 1 \ .$$

Assume that $\mathbf{a}$ belongs to K. Substituting the compact set

$$\mathrm{S}^1 \cdot K := \{u\mathbf{b} \colon u \in \mathrm{S}^1, \ \mathbf{b} \in K\}$$

for K, we can assume that K is invariant under multiplication by constants of modulus 1. Write

$$\Omega = \{u\mathbf{e}(t) \cdot \sigma^n \mathbf{a} \colon u \in \mathrm{S}^1, \ t \in \mathbb{T}, \ n \in \mathbb{Z}\}$$

and let $\bar{\Omega}$ denote the norm closure of Ω in $\ell^\infty(\mathbb{Z})$. We remark that $\bar{\Omega}$ is invariant under the shift σ, under multiplication by constants of modulus 1 and under the operators of multiplication by exponential sequences. By hypothesis, we have

$$\Omega \subset \bigcup_{t \in \mathbb{T}} \mathbf{e}(t) \cdot K \ .$$

Substituting $K \cap \bar{\Omega}$ for K we can assume that

$$K \subset \bar{\Omega} \ .$$

9.4. **First step.** We obviously have that

$$\|\mathbf{b}\|_\infty = 1 \text{ for every } \mathbf{b} \in \bar{\Omega}\ .$$

Claim 1. $\bar{\Omega}$ is closed under pointwise convergence.

Proof. Let $(\mathbf{b}_j : j \in \mathbb{N})$ be a sequence in Ω, converging pointwise to a sequence $\mathbf{b}$. For every $j \in \mathbb{N}$, we write

$$\mathbf{b}_j = \mathbf{e}(t_j)\mathbf{c}_j \text{ where } t_j \in \mathbb{T} \text{ and } \mathbf{c}_j \in K \text{ for every } j\ .$$

By passing to a subsequence, we can assume that the sequence $(t_j : j \in \mathbb{N})$ converges to some t in $\mathbb{T}$ and that the sequence $(\mathbf{c}_j : j \in \mathbb{N})$ converges uniformly to some $\mathbf{c} \in K$. Thus $\mathbf{b} = \mathbf{e}(t)\mathbf{c}$. Since $\bar{\Omega}$ is invariant under multiplication by $\mathbf{e}(t)$ and contains K, we have that $\mathbf{b} \in \bar{\Omega}$. □

Claim 2. The set $S = \{n \in \mathbb{Z} \colon |a_n| \geq 2/3\}$ is syndetic.

Proof. Assume that S is not syndetic. Then there exists a sequence $(k_j : j \in \mathbb{N})$ of integers such that $(\sigma^{k_j}\mathbf{a})$ converges pointwise to a sequence $\mathbf{b}$ with $\|\mathbf{b}\|_\infty \leq 2/3$. By Claim 1, $\mathbf{b} \in \bar{\Omega}$, and thus $\|\mathbf{b}\|_\infty = 1$, a contradiction. □

In the sequel, we write L for the syndetic constant of the set S.

¿From the last claim we immediately deduce:

Claim 3. For every $\mathbf{b} \in \bar{\Omega}$, the set $\{n \in \mathbb{Z} \colon |b_n| \geq 1/2\}$ is syndetic of syndetic constant L.

Claim 4. For every compact subset Q of $\bar{\Omega}$, the family of sets $\big\{\mathbf{e}(t) \cdot Q \colon t \in \mathbb{T}\big\}$ is locally finite. More precisely, every ball B of radius $r < 1/25\,L$ intersects $\mathbf{e}(t) \cdot Q$ for only finitely many values of t.

Proof. Let B be a ball of radius $r < 1/25\,L$ and let $(t_j : j \in \mathbb{N})$ be a sequence of distinct elements of $\mathbb{T}$ such that $B \cap \mathbf{e}(t_j) \cdot Q \neq \emptyset$ for every $j \in \mathbb{N}$.

For every j, we chose $\mathbf{c}_j \in Q$ with $\mathbf{e}(t_j)\mathbf{c}_j \in B$. Passing to a subsequence, we can assume that the sequence $(t_j : j \in \mathbb{N})$ converges to some t in $\mathbb{T}$ and that the sequence $(\mathbf{c}_j : j \in \mathbb{N})$ converges uniformly to some sequence $\mathbf{c} \in Q$. We note that $\mathbf{c} \in \bar{\Omega}$.

For all $i, j \in \mathbb{N}$, by the triangle inequality,

$$\begin{aligned} \|\mathbf{e}(t_j - t_i)\mathbf{c}\|_\infty \leq \|\mathbf{e}(t_j)\mathbf{c} - \mathbf{e}(t_j)\mathbf{c}_j\|_\infty + \|\mathbf{e}(t_j)\mathbf{c}_j - \mathbf{e}(t_i)\mathbf{c}_i\|_\infty + \|\mathbf{e}(t_i)\mathbf{c} - \mathbf{e}(t_i)\mathbf{c}_i\|_\infty \\ \leq \|\mathbf{c} - \mathbf{c}_j\|_\infty + \|\mathbf{c} - \mathbf{c}_i\|_\infty + 2r \end{aligned}$$

because $\mathbf{e}(t_i)\mathbf{c}_i$ and $\mathbf{e}(t_j)\mathbf{c}_j$ belong to B.

Let i and j be sufficiently large. The last quantity is $< 4r$ and thus

$$|e((t_j - t_i)n) - 1| < 8r < 1/3L$$

for n in the set $F = \{n \in \mathbb{Z} \colon |c_n| \geq 1/2\}$. As we also have that $|t_i - t_j| < 1/3L$, Claim 3 and Lemma 11 give that $t_i = t_j$, a contradiction. □

We immediately deduce:

Claim 5. For every compact subset Q of $\bar{\Omega}$, the union of the sets $\mathbf{e}(t) \cdot Q$ for $t \in \mathbb{T}$ is closed.

Claim 6.
$$\bar{\Omega} = \bigcup_{t\in\mathbb{T}} \mathbf{e}(t)\cdot K\ .$$

Proof. The union on the right hand side is closed by Claim 5, it contains Ω and is included in $\bar{\Omega}$ by construction. □

Claim 7. $\bar{\Omega}$ is locally compact.

Proof. Every closed ball of radius $r < 1/25L$ is covered by sets of the form $\mathbf{e}(t)K$ by Claim 6 and intersects only finitely many of them by Claim 4. It is thus included in a finite union of sets of this form and so is compact. □

9.5. **A group of transformations.** Let $\mathrm{Isom}(\bar{\Omega})$ denote the group of isometries of $\bar{\Omega}$, endowed with the topology of pointwise convergence. This topology coincides with the compact-open topology, as well as with the topology of pointwise convergence on the dense subset Ω of $\bar{\Omega}$. We write the action of this group on Ω as $(g,\mathbf{b})\mapsto g\cdot\mathbf{b}$ for $g\in\mathrm{Isom}(\bar{\Omega})$ and $\mathbf{b}\in\bar{\Omega}$. Let G be the closure in $\mathrm{Isom}(\bar{\Omega})$ of the group spanned by the shift and the operators of multiplication by $\mathbf{e}(t)$ for $t\in\mathbb{T}$. This operator is written simply $\mathbf{e}(t)$.

We remark that for $t\in\mathbb{T}$, the commutator of σ and $\mathbf{e}(t)$ is multiplication by the constant $e(-t)$. Therefore, the commutator subgroup of G is the group of multiplication by constants of modulus 1, identified with S^1, and G is 2-step nilpotent.

Claim 8. Let $(u_i\colon i\in I)$ be a generalized sequence (a filter) in S^1, $(t_i\colon i\in I)$ a generalized sequence in $\mathbb{T}$, and $(k_i\colon i\in I)$ a generalized sequence in $\mathbb{Z}$. For every $i\in I$, let $g_i = u_i\mathbf{e}(t_i)\sigma^{k_i}\in G$. Then the generalized sequence $(g_i\colon i\in I)$ converges in G if and only if

(i) The generalized sequence $(g_i\cdot\mathbf{a}\colon i\in I)$ converges in $\bar{\Omega}$;
(ii) The generalized sequence $(u_i\colon i\in I)$ converges in S^1;
(iii) The generalized sequence $(k_i\colon i\in I)$ converges in the Bohr group $\mathrm{B}(\mathbb{Z})$, meaning that for every $t\in\mathbb{T}$ the generalized sequence $(k_it\colon i\in I)$ converges in $\mathbb{T}$.

Proof. Let $\mathbf{b} = u\mathbf{e}(t)\sigma^k\mathbf{a}\in\Omega$, with $u\in\mathcal{S}^1$, $t\in\mathbb{T}$, $k\in\mathbb{Z}$. For every $i\in I$ we have
$$g_i\cdot\mathbf{b} = e(k_it-kt_i)\,u\mathbf{e}(t)\sigma^k\,g_i\cdot\mathbf{a}\ .$$
The three conditions of the claim are equivalent to the property that $g_i\cdot\mathbf{b}$ converges for every $\mathbf{b}\in\Omega$. □

Claim 9. G acts transitively on $\bar{\Omega}$.

Proof. Let $\mathbf{b}\in\bar{\Omega}$. Since Ω is dense in $\bar{\Omega}$, there exist sequences $(s_i\colon i\in I)$, $(t_i\colon i\in I)$ and $(k_i\colon i\in I)$ as in Claim 8 such that that $e(s_i)\mathbf{e}(t_i)\sigma^{k_i}\cdot\mathbf{a}\to\mathbf{b}$. Substituting subsequences for these sequences we can assume that properties (ii) and (iii) are satisfied. The limit g of the sequence $(e(s_i)\mathbf{e}(t_i)\sigma^{k_i})$ satisfies $g\cdot\mathbf{a}=\mathbf{b}$. □

Claim 10. For every compact subset L of $\bar{\Omega}$, the subset $\tilde{L} = \{g\in G\colon g\cdot\mathbf{a}\in L\}$ of G is compact. Moreover, G is locally compact.

Proof. Let $\mathbf{b}\in\Omega$. Choose $h\in G$ such that $h\cdot\mathbf{a}=\mathbf{b}$. For every $g\in\tilde{L}$, we have $g\cdot\mathbf{b} = ghg^{-1}h^{-1}hg\cdot\mathbf{a}$. Thus $\tilde{L}\cdot\mathbf{b}$ is included in the compact set $\mathrm{S}^1\,h\cdot L$ and so is relatively compact. The first part of the Claim follows from Ascoli's Theorem, and the second part from the fact that $\bar{\Omega}$ is locally compact (Claim 7). □

9.6. **The space X.** Let Γ be the subgroup $\{\mathbf{e}(t) \colon t \in \mathbb{T}\}$ of G. It follows immediately from Claim 4 that Γ is discrete and closed in G.

Write $X = G/\Gamma$ endowed with the quotient topology. For $h \in G$, the map $g \mapsto hg$ from G to G induces a homeomorphism of X that we write $x \mapsto h \cdot x$. We remark that the map $G \times X \to X$ defined by $(h, x) \mapsto h \cdot x$ is continuous.

By Claim 10, $\tilde{K} = \{g \in G \colon g \cdot \mathbf{a} \in K\}$ is a compact subset of G and by Claim 6 we have that G is the union $\Gamma\tilde{K}$ of the sets $\mathbf{e}(t) \cdot \tilde{K}$ for $t \in \mathbb{T}$. Since K is invariant under multiplication by constants of modulus one, $\tilde{K}$ is invariant under multiplication by S^1 and $\Gamma\tilde{K} = \tilde{K}\Gamma$. Therefore the restriction to $\tilde{K}$ of the natural projection $G \to X = G/\Gamma$ is onto and we have:

Claim 11. X is compact.

Let the function $\tilde{f}$ on G be defined by

$$\tilde{f}(g) = \big(g^{-1} \cdot \mathbf{a}\big)_0 \ ,$$

where the subscript 0 denotes the coordinate corresponding to 0. This function is continuous and for every $t \in \mathbb{T}$,

$$\tilde{f}(g\mathbf{e}(t)) = \big(\mathbf{e}(-t)g^{-1} \cdot \mathbf{a}\big)_0 = \big(g^{-1} \cdot \mathbf{a}\big)_0 = \tilde{f}(g) \ .$$

(Recall that $\mathbf{e}(-t)\mathbf{b}$ is the product of the sequence $\mathbf{b}$ and the sequence $\mathbf{e}(-t) = (e(-nt) \colon n \in \mathbb{Z})$. Therefore we have $(\mathbf{e}(-t)\mathbf{b})_0 = b_0$.) Thus the function $\tilde{f}$ induces a continuous function f on X. We remark that for every $x \in X$ and every $u \in \mathrm{S}^1$, we have $f(u \cdot x) = uf(x)$. Using terminology similar to that in Definition 3 and Section 5.3, we say that the function f belongs to $\mathcal{C}_1(X)$.

Let $T \colon X \to X$ be the map $x \mapsto \sigma^{-1} \cdot x$ and let x_0 be the image in X of the unit element of G. For every $n \in \mathbb{Z}$, we have

$$f(T^n x_0) = \tilde{f}(\sigma^{-n}) = \big(\sigma^n \cdot \mathbf{a})_0 = a_n \ .$$

9.7. **Conclusion.** This situation was studied in [9] in the case that G is metrizable, but the same proof extends to the present case. We outline the method.

It is classical [13] that the locally compact 2-step nilpotent group G is an inverse limit of 2-step nilpotent Lie groups. This means that there exists a decreasing family (K_i) of compacts subgroups of the center of G, with trivial intersection, such that G/K_i is a 2-step nilpotent Lie group for every i. Therefore the system (X, T) is the inverse limit of the 2-step nilsystems $(X_i = G/K_i\Gamma, T_i)$ where T_i is the transformation of X_i induced by T. For every i, the commutator subgroup of G/K_i is the circle group, and thus (X_i, T_i) is an elementary nilsystem. For every i, let $p_i \colon X \to X_i$ be the natural projection and set $x_i = p_i(x_0)$. The function f on X is the uniform limit of functions of the form $f_i \circ p_i$, where $f_i \in \mathcal{C}_1(X_i)$ for every i. This gives us that the sequence $\mathbf{a}$ is the uniform limit of the sequences $\mathbf{a}(i) := (f_i(T_i^n x_i) \colon n \in \mathbb{Z})$, and so is an elementary nilsequence. □

Appendix A. Proof of Lemma 2

We recall the notation introduced in Section 5.2. We have that $(X = G/\Gamma, T)$ is an elementary nilsystem, where T is the translation by $\tau \in G$ and μ is its Haar measure. The Kronecker factor of this system is (Z, m_Z, S) and $\pi \colon X \to Z$ is the factor map. As before, $p \colon G \to Z$ is the natural homomorphism, so that S is the translation by $\sigma = p(\tau)$ on Z. Finally, $\mathrm{B}(\mathbb{Z})$ is the Bohr compactification of $\mathbb{Z}$, endowed with the translation R by 1 and with its Haar measure m, and $r\colon \mathrm{B}(\mathbb{Z}) \to Z$ is the continuous homomorphism characterized by $r(1) = \sigma$.

The system $(\tilde{X}, \tilde{T})$ denotes the Bohr extension of (X, T): $\tilde{X} = \{(x, z) \in X \times \mathrm{B}(\mathbb{Z})\colon \pi(x) = r(z)\}$ endowed with the translation $\tilde{T} = T \times R$; $q_1\colon \tilde{X} \to X$ and $q_2\colon \tilde{X} \to \mathrm{B}(\mathbb{Z})$ are the natural factor maps.

We first show that the system $(\tilde{X}, \tilde{T})$ is uniquely ergodic and that the invariant measure is the conditional independent joining (we recall the definition below).

A.1. **Notation.** Let ν be a $\tilde{T}$-invariant measure on $\tilde{X}$.

By unique ergodicity of $(\mathrm{B}(\mathbb{Z}), R)$ and (X, T), the images of ν under q_1 and q_2 are equal to m and μ, respectively (In other words, ν is a joining). We write $\mathbb{E}_r\colon L^2(m) \to L^2(m_Z)$ and $\mathbb{E}_\pi\colon L^2(\mu) \to L^2(m_Z)$ for the operators of conditional expectation, defined as usual by:

for every $\phi \in L^2(m)$, $\psi \in L^2(\mu)$ and $h \in L^2(m_Z)$,

$$\int_Z \mathbb{E}_r\phi \cdot h \, dm_Z = \int \phi \cdot h \circ r \, dm \text{ and } \int \mathbb{E}_\pi\psi \cdot h \, dm_Z = \int \psi \cdot h \circ \pi \, d\mu \ .$$

We claim that ν is the conditional independent joining $\mu \times_Z m$ of μ and m over Z, meaning it is the measure on $X \times \mathrm{B}(\mathbb{Z})$ defined by:

for every $\phi \in L^2(\mu)$ and $\psi \in L^2(m)$,

$$\int \phi(x)\psi(w) d\mu \times_Z m(x, w) = \int \mathbb{E}_p\phi(z) \cdot \mathbb{E}_r\psi(z) \, dm_Z(z) \ .$$

A.2. **Proof of the Claim.** The measure ν defines a continuous operator $\Phi\colon L^2(m) \to L^2(\mu)$ by:

for every $\phi \in L^2(\mu)$ and every $\psi \in L^2(m)$,

$$\int \phi(x)\psi(w) \, d\nu(x, w) = \int \Phi\psi(x) \, \phi(x) \, d\mu(x) \ .$$

Since ν is invariant under $\tilde{T} = T \times R$, we have that $\Phi \circ R = T \circ \Phi$. If ψ is a character of $\mathrm{B}(\mathbb{Z})$, it is an eigenfunction of $(\mathrm{B}(\mathbb{Z}), R)$. Thus $\Phi\phi$ is an eigenfunction of X and by hypothesis $\Phi\psi$ factorizes through Z. Since the characters of $\mathrm{B}(\mathbb{Z})$ span $L^2(m)$, the same property holds for every $\psi \in L^2(m)$.

Therefore the last integral is equal to

$$\int \Phi\psi(x) \, \mathbb{E}_p\phi \circ p(x) \, d\mu(x) = \int \psi(w) \, \mathbb{E}_p\phi \circ p(x) \, d\nu(x, w) \ .$$

Since ν is concentrated on $\tilde{X}$, we have that $p(x) = r(w)$ for ν-almost every (x, w) and the last integral can be rewritten as

$$\int \psi(w) \, \mathbb{E}_p\phi \circ r(w) \, d\nu(x, w) = \int \psi(w) \, \mathbb{E}_p\phi \circ r(w) \, dm(w) \ .$$

The projection of ν on Z is equal to m and using the definition of the operator $\mathbb{E}_p$, this is equal to

$$\int \mathbb{E}_p\phi(z) \, \mathbb{E}_r\psi(z) \, dm_Z(z) = \int \phi(x) \, \psi(w) \, d\mu \times_Z m(x, w) \ .$$

By definition of this measure, and our claim is proved.

A.3. In particular, $(\tilde{X},\tilde{T})$ is uniquely ergodic with invariant measure $\mu\times_Z m$. The topological support of this measure is clearly equal to $\tilde{X}$. $\square$

Appendix B. Elementary nilsystems arising from connected groups

We explain how elementary nilsystems with a connected group have a simple form. Similar results appear in [10]. Simply connected, connected nilpotent Lie groups admitting a discrete cocompact subgroup have been well understood since Malcev [12] and we refer to Chapter 5 of [4] for their properties. The proof reduces to this case and we only give a sketch of the reduction.

Lemma 12. *Let $(X=G/\Gamma,T)$ be an elementary nilsystem (minimal, written in reduced form) with connected group G. Then (X,T) is the product of a rotation on a finite dimensional torus with an elementary nilsystem $(X'=G'/\Gamma',T')$ of the following type:*

$d\geq 1$ is an integer, A is a $2d\times 2d$ matrix with integer entries such that the matrix $B=A-A^t$ is nonsingular; $G=\mathbb{R}^{2d}\times \mathrm{S}^1$ with multiplication given by

$$(x,z)\cdot(x',z')=\big(x+x',zz'e(\langle Ax\mid x'\rangle)\big)$$

and $\Gamma=\mathbb{Z}^d\times\{1\}$.

Proof. Let $\tilde{G}$ be the simply connected covering of G, $p\colon\tilde{G}\to G$ be the natural projection and let $\tilde{\Gamma}=p^{-1}(\Gamma)$. Then $\tilde{G}$ is a simply connected, connected 2-step nilpotent Lie group, the kernel of p is a discrete subgroup of the center of $\tilde{G}$, and $\tilde{\Gamma}$ is a discrete cocompact subgroup of $\tilde{G}$. Since G_2 is the circle group, $\tilde{G}_2$ is of dimension 1.

Let $W_1,\dots,W_m,Z$ be a Malcev base of the second kind of the Lie algebra of $\tilde{G}$. This means that every $g\in\tilde{G}$ can be written in a unique way as

$$g=\exp(w_1W_1)\dots\exp(w_mW_m)\exp(zZ)\text{ with }w_1,\dots,w_m,z\in\mathbb{R}\ ,$$

that $\tilde{G}_2=\{\exp(sZ)\colon s\in\mathbb{R}\}$, and that

$$\tilde{\Gamma}=\{\exp(p_1W_1)\dots\exp(p_mW_m)\exp(qZ)\colon p_1,\dots,p_m,q\in\mathbb{Z}\}\ .$$

We use the coordinates of G associated to this base and identify $\tilde{G}$ with $\mathbb{R}^m\times\mathbb{R}$, $\tilde{G}_2$ with $\{0\}\times\mathbb{R}$ and $\tilde{\Gamma}$ with $\mathbb{Z}^m\times\mathbb{Z}$. Then the multiplication in $\tilde{G}$ is given by

$$(w_1,\dots,w_m,z)\cdot(w'_1,\dots,w'_m,z')=(w_1+w'_1,\dots,w_m+w'_m,z+z'+\langle Aw\mid w'\rangle)$$

where $w=(w_1,\dots,w_m)$, $w'=(w'_1,\dots,w'_m)$ and A is an $m\times m$ matrix with integer entries. The commutator map is given by

$$\big[(w_1,\dots,w_m,z)\cdot(w'_1,\dots,w'_m,z')\big]=(0,\dots,0,\langle Bw\mid w'\rangle)\ ,$$

where $B=A^t-A$.

We can write $m=2d+q$ and find a base of $\mathbb{Z}^{2d}\times\mathbb{Z}^q$ of the form $(e_1,\dots,e_{2d},f_1,\dots,f_q)$ such that $(f_1,\dots,f_q)$ is a base of the kernel of the matrix B. The center of $\tilde{G}$ is thus the linear span of the vectors $f_1,\dots,f_q$ and $\exp(Z)=(0,\dots,0,1)$.

By hypothesis, the intersection of the center of G and Γ is trivial. It follows that $f_1,\dots,f_q$ and $\exp(Z)$ belong to the kernel of the projection p.

Let Λ be the subgroup spanned by these elements, $G'=\tilde{G}/\Lambda$ and let Γ' be the image of $\tilde{\Gamma}$ in G'. Then $X=G'/\Gamma'$ and G',Γ' have the announced form. $\square$

Appendix C. Non-examples of nilsequences

C.1. **A non-almost periodic example.** First we show that the sequence $\mathbf{b}$ defined by $b_n = e(\lfloor n\alpha\rfloor\beta\colon n\geq 1)$, where α,β are independent over the rationals, is not almost periodic.

Let $(k_j\colon j\geq 1)$ be a sequence of integers such that the sequence $(k_j\alpha\colon j\geq 1)$ converges to 0 modulo 1 and $(\lfloor k_j\alpha\rfloor\beta\colon j\geq 1)$ converges to some γ modulo 1.

If $\mathbf{b}$ is almost periodic, then by passing to a subsequence, which we also denote by $(k_j\colon j\geq 1)$, we can assume that the sequence $(\sigma^{k_j}\mathbf{b}\colon j\geq 1)$ converges uniformly to some sequence $\mathbf{c}$. On the other hand, it is easy to check that $(\sigma^{k_j}\mathbf{b}\colon j\geq 1)$ converges to $e(\gamma)\mathbf{b}$ in quadratic norm, but not uniformly.

We have that $\mathbf{c}=e(\gamma)\mathbf{b}$, a contradiction.

C.2. **A 2-step nonexample.** We show that the sequence $e(\lfloor n\beta\rfloor n\alpha)$, where α,β are independent over the rationals, is not a 2-step nilsequence.

We proceed by contradiction. Assume that this sequence is a 2-step nilsequence. Then using Proposition 2, the sequence $\mathbf{a}$ given by

$$a_n = e(\lfloor n\beta\rfloor n\alpha - \frac{n(n-1)}{2}\alpha\beta)$$

also is a 2-step nilsequence.

Consider the Heisenberg nilmanifold $N_1 = G/\Gamma$, endowed with translation by $\tau=(\alpha,\beta,0)$. Let e denote the image in N_1 of the unit element of G. The subset $F=[0,1)\times[0,1)\times\mathbb{T}$ of $H_1=\mathbb{R}\times\mathbb{R}\times\mathbb{T}$ is a fundamental domain for the projection $p\colon H_1\to N_1$. Thus there exists a unique function h on N_1 such that $h(p(x,y,z)) = e(-z)$ for every $(x,y,z)\in F$. Note that h is Riemann integrable, but is not continuous, and that for every n we have $a_n = h(T^n e)$. (In fact a general statement along these lines holds. Let (X,T) be a minimal k-step nilsystem, $x_0\in X$, and f be a Riemann integrable function on X. If the sequence $(f(T^n x_0)\colon n\geq 1)$ is a k-step nilsequence, then f is continuous.)

Let $(h_j\colon j\geq 1)$ be a sequence of continuous functions on N_1 converging to h in $L^2(N_1)$ and for every j, let $\mathbf{a}(j)$ denote the sequence $(h_j(T^n e)\colon n\in\mathbb{Z})$. By Proposition 3 and Lemma 1, each of these sequences is an elementary nilsequence belonging to $\mathcal{N}^2(N_1)$. As $j\to+\infty$, $\mathbf{a}(j)$ converges to $\mathbf{a}$ in quadratic norm (see Section 3.1).

By Proposition 4 and Theorem 2, proceeding as in the proof of Lemma 5, it is easy to deduce that if a nilsequence is a limit in the quadratic norm of elementary nilsequences belonging to $\mathcal{N}^2(X)$, then it also belongs to $\mathcal{N}^2(X)$. In particular, it is an elementary nilsequence. Therefore $\mathbf{a}$ is an elementary nilsequence.

Let $(k_j\colon j\geq 1)$ be a sequence of integers such that $k_j\alpha$ and $k_j\beta$ tend to 0 modulo 1 and $\frac{k_j(k_j-1)}{2}\alpha\beta$ converges modulo 1 to some γ.

By Theorem 1, passing to a subsequence of $(k_j\colon j\geq 1)$ if necessary, we can assume that there exists a sequence $(t_j\colon j\geq 1)$ in $\mathbb{T}$ such that the sequence $(\mathbf{e}(t_j)\sigma^{k_j}\mathbf{a}\colon j\geq 1)$ converges uniformly as $j\to\infty$ to some sequence.

On the other hand, it is easy to check that the sequence $(\mathbf{e}(k_j\alpha\beta)\sigma^{k_j}\mathbf{a}\colon j\geq 1)$ converges as $j\to\infty$ in quadratic norm, but not uniformly. Therefore the sequence $\big(\mathbf{e}(t_j-k_j\alpha\beta)\colon j\geq 1\big)$ converges in quadratic norm but not uniformly, and this is impossible.

References

[1] L. Auslander, L. Green and F. Hahn. Flows on homogeneous spaces. *Ann. Math. Studies* **53**, Princeton Univ. Press (1963).

[2] V. Bergelson, B. Host and B. Kra, with an Appendix by I. Ruzsa. Multiple recurrence and nilsequences. *Inventiones Math.* **160** (2005), 261-303.

[3] V. Bergelson and A. Leibman. Distribution of values of bounded generalized polynomials. *Acta Math.* **198** (2007) 155-230.

[4] L. Corwin and F.P. Greenleaf, *Representations of nilpotent Lie groups and their applications*, Cambridge University Press (1990).

[5] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. d'Analyse Math.* **31** (1977), 204–256.

[6] B. Green and T. Tao. Linear equations in the primes. To appear, *Annals of Math.*

[7] B. Green and T. Tao. Quadratic uniformity of the Möbius function. To appear, *Annales de l'Institut Fourier*.

[8] B. Host and B. Kra. Uniformity seminorms on $\ell^\infty(\mathbb{Z})$ and an inverse theorem. In preparation.

[9] B. Host and A. Maass. Nilsystèmes d'ordre deux et parallélépipèdes. To appear, *Bull. Math. Soc. France*.

[10] A. Leibman. Pointwise convergence of ergodic averages for polynomial sequences of rotations of a nilmanifold. *Ergod. Th. & Dynam. Sys.* **25** (2005), 201-113.

[11] E. Lesigne. Sur une nil-variété, les parties minimales associées à une translation sont uniquement ergodiques. *Ergod. Th. & Dynam. Sys.* **11** (1991), 379–391.

[12] A. Malcev. On a class of homogeneous spaces. *Amer. Math. Soc. Transl.* **9** (1962), 276-307.

[13] D. Montgomery and L. Zippin. *Topological Transformation Groups.* Interscience Publishers (1955).

[14] W. Parry. Ergodic properties of affine transformations and flows on nilmanifolds. *Amer. J. Math. 91* (1969), 757–771.

[15] W. Parry. Dynamical systems on nilmanifolds. *Bull. London Math. Soc.* **2** (1970), 37–40.

[16] D. J. Rudolph. Eigenfunctions of $T \times S$ and the Conze-Lesigne algebra. *Ergodic Theory and its Connections with Harmonic Analysis*, Eds. K. Petersen and I. Salama. Cambridge University Press, New York (1995), 369–432.

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