DIOPHANTINE NUMBERS, DIMENSION AND DENJOY MAPS

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ABSTRACT. We study the effect of the arithmetic properties of the rotation number on the minimal set of an aperiodic, orientation preserving diffeomorphism of the circle. The box dimension and the Hausdorff dimension may differ. The box dimension depends only on the smoothness of the map while the Hausdorff dimension depends on both the Diophantine class of the rotation number and on smoothness of the map. The dimension estimates give a generalization of Denjoy's Theorem on conjugacy that includes dependence on the Diophantine class of the rotation number.

1. INTRODUCTION

1.1. **History.** Poincaré [13] showed that to each orientation preserving homeomorphism of the circle $S^1 = \mathbb{R}/\mathbb{Z}$ is associated a unique real parameter $\alpha \in [0, 1)$, called the rotation number, so that the orbit structure of f is the same as that of the rigid rotation R_α, where $R_\alpha(t) = (t + \alpha) \bmod 1$, provided that α is irrational. More precisely, Poincaré showed that f and R_α are *semi-conjugate*, meaning there exists an orientation preserving self mapping h of the circle so that $h \circ f = R_\alpha \circ h$.

We say that f is *minimal* if for each $x \in S^1$, the orbit under f is dense. (In this setup, density of the orbit for a single $x \in S^1$ is equivalent to density for all x.) More generally, for a homeomorphism f the set Ω is minimal if Ω is non-empty, compact, invariant and minimal with resepct to these properties. This is equivalent to saying that Ω is non-empty, $f(\Omega) = \Omega$ and each forward f-orbit in Ω is dense in Ω. It follows from Poincaré's Theorem that if f is minimal, then f is *conjugate* to the rotation by α. That is, there exists an orientation preserving homeomorphism of the circle h so that $f = h^{-1} \circ R_\alpha \circ h$. We assume throughout that all rotation numbers are irrational, even when not explicitly stated.

Half a century later, Denjoy [3] showed that if the derivative of f is of bounded variation, then f is minimal and so conjugate to a rotation. Denjoy also constructed examples of C^1 diffeomorphisms that are not conjugate to rotations. Herman [6] constructed $C^{1+\delta}$ maps that are not conjugate to rotations for any $\delta < 1$. In these examples, the minimal set of f is necessarily a Cantor set Ω and is the set of accumulation points of the forward f orbit of every point $t \in S^1$. We refer to any orientation preserving map that is not conjugate to a rotation as a Denjoy map.

1.2. **Summary of Results.** The rotation number has a strong effect on properties of the diffeomorphism. One area that has been well understood is the relation between the differentiability of f, the differentiability of the conjugation and the arithmetic properties of the rotation number. (See, for example, Herman [6], Yoccoz [14] and Katznelson and Ornstein [8].) Without stating any precise theorem, we note that the results differ sharply for Diophantine and for Liouville rotation numbers.

Here we study the effect of the rotation number on a $C^{1+\delta}$ Denjoy example. In particular, we consider the relation between the arithmetic properties of the rotation number and the Hausdorff dimension of the minimal set Ω of a $C^{1+\delta}$ example. See Section 2 for the definitions.

We prove:

Theorem 1.1. *Assume that $0 < \delta < 1$ and that $\alpha \in (0,1)$ is of Diophantine class $\nu \in (0, \infty)$. Then an orientation preserving $C^{1+\delta}$ diffeomorphism of the circle with rotation number α and minimal set Ω_α^δ satisfies*

$$\dim_B \Omega_\alpha^\delta \geq \delta$$

and

$$\dim_H \Omega_\alpha^\delta \geq \frac{\delta}{\nu} \ .$$

Furthermore, these results are sharp.

Norton [11] proved a preliminary result along these lines, showing that the upper box dimension of the minimal set of a $C^{1+\delta}$ Denjoy map is bounded from below by δ. Katznelson and Ornstein [9] have shown that for some specific Denjoy maps, the Hausdorff dimension of the minimal set depends on the differentiability. Pinto [12] proved the special case when the rotation number is the golden mean. Our results include these and show that while the box dimension does not depend on the rotation number of the Denjoy example, the more subtle Hausdorff dimension does.

The estimates on dimension give a stronger version of Denjoy's original theorem about conjugacy that takes the arithmetic of the rotation number into account. Namely, we have the following immediate corollary to Theorem 1.1:

Corollary 1.2. *Assume that f is an orientation preserving homeomorphism of the circle with rotation number $\alpha \in (0,1)$ and assume that α satisfies a Diophantine condition of order $\nu \in (0, \infty)$. Let Ω denote the minimal set of f. If $\dim(\Omega) = \delta/\nu \in (0,1)$, then $f \notin C^{1+\beta}$ for any $\beta > \delta$.*

2. Preliminaries

2.1. **Construction of Denjoy maps.** We review the construction of the classic Denjoy map. (See [7] for details of the construction.) The classic Denjoy map is used to show that the estimates in Theorem 1.1 are sharp.

Definition 2.1. *A Denjoy sequence D^δ of class δ is an infinite collection of lengths $\{\ell_n\}_{n \in \mathbb{Z}}$ (non-negative numbers) so that*

$$\sum_{n \in \mathbb{Z}} \ell_n \leq 1$$

and

$$\varliminf_{n \to \pm\infty} \frac{\log |\ell_n - \ell_{n+1}|}{\log \ell_n} = 1 + \delta \ .$$

Definition 2.2. *The classic Denjoy sequence, denoted D_0^δ, consists of lengths*

$$\ell_n = c_\delta(|n| + 1)^{-1/\delta} \ ,$$

where $c_\delta^{-1} = \sum_{n \in \mathbb{Z}}(|n| + 1)^{-1/\delta}$.

We denote the set of all Denjoy sequences of class δ by $\mathcal{D}^\delta$.

Let $\alpha \in (0,1)$ be irrational and let $\delta \in (0,1)$. Let D^δ be a Denjoy sequence of class δ. In order to obtain a Denjoy map, we "blow up" each point $n\alpha$ of the orbit of 0 under the rotation by α to an open interval J_n of length $\ell_n \in D^\delta$. At each step we rescale the remaining part of the circle, maintaining the total length equal to 1.

A Denjoy map f is defined by mapping the interval J_n diffeomorphically to J_{n+1} with derivative of absolute value 1 at the endpoints. By continuity, f is defined on the entire circle and by choice of the sequence ℓ_n, f can be chosen to be $C^{1+\delta}$. By placement of the intervals J_n in the same order as the rotation by α, the resulting f has rotation number α.

We set

$$\Omega_\alpha^\delta = \Omega_\alpha^\delta(D^\delta) = S^1 \setminus \bigcup_{n \in \mathbb{Z}} J_n$$

For $x, y \in S^1$, we let (x,y) denote the shorter of the two arcs on the circle determined by x and y. (If this interval is not unique, we take the interval in the positive orientation.) For real x, let $\|x\|$ denote the distance to the nearest integer. For an interval $J \subset S^1$, $|J|$ denotes the length of the interval.

Let $h\colon \Omega_\alpha^\delta \to S^1$ be the semi-conjugacy determined by Poincaré's Theorem. For x, y not in the orbit of α (meaning $x, y \in S^1 \setminus \{n\alpha : n \in \mathbb{Z}\}$), h^{-1} is well defined and we have

$$(1) \qquad d_{\Omega_\alpha^\delta}(h^{-1}(x), h^{-1}(y)) = \sum_{\{n:n\alpha\in(x,y)\}} |J_n| + \left(1 - \sum_{n\in\mathbb{Z}} |J_n|\right) \|x - y\| .$$

If $x \in \{n\alpha : n \in \mathbb{Z}\}$ (similarly for y), $h^{-1}(x)$ consists of two points and we choose the appropriate pre-image. For example, if the interval (x, y) is positively oriented for $h^{-1}(x)$ we take the right pre-image (equivalently, the point closer to $h^{-1}(y)$).

We note that the total length $\sum \ell_n$ may be strictly less than 1. In this case, the invariant measure associated to Ω_α^δ has positive Lebesgue measure and the dimension properties are not relevant.

We also have some preliminary estimates on sizes of complementary intervals for general Denjoy maps, and not just for the classic Denjoy map. These bounds are summarized in the following lemma:

Lemma 2.3. *If $D^\delta \in \mathcal{D}^\delta$, then for all $0 < \theta < \delta < 1$ there exists n_θ such that for all $n > n_\theta$*

$$|J_n| > n^{-1/\theta} .$$

Proof. We note that if f is a $C^{1+\delta}$ diffeomorphism then f^{-1} is also a $C^{1+\delta}$ diffeomorphism.

Assume that $D^\delta \in \mathcal{D}^\delta$. Then for $0 < \theta < \delta < 1$ there exists n_θ such that for all $n \geq n_\theta$,

$$\left|\frac{|J_n|}{|J_{n+1}|} - D_{h^{-1}((n+1)\alpha)}f^{-1}\right| \leq |\max_{x\in J_{n+1}} D_x f^{-1} - D_{h^{-1}((n+1)\alpha)}f^{-1}| \leq |J_{n+1}|^\theta .$$

Rewriting, we have

$$(2) \qquad |J_n| \leq |J_{n+1}|(D_{h^{-1}((n+1)\alpha)}f^{-1} + |J_{n+1}|^\theta) .$$

Setting

$$a_n = |J_n|n^\beta ,$$

where $\beta = \theta^{-1} > 1$, it suffices to show that for n sufficiently large, a_n is bounded away from 0. Inequality (2) becomes

$$\frac{a_n}{n^\beta} \leq \frac{a_{n+1}}{(n+1)^\beta} \cdot \left(1 + \frac{a_{n+1}^\theta}{n+1}\right) .$$

Equivalently, for $n > n_\theta$ and bounded a_{n+1} we have

$$(3) \qquad \frac{a_n}{a_{n+1}} \leq \left(1 + \frac{a_{n+1}^\theta}{n+1}\right) \cdot \left(\frac{n}{n+1}\right)^\beta \leq \exp\left(\frac{a_{n+1}^\theta - \beta}{n+1}\right) .$$

For $a_{n+1}^{\theta} < \beta$ the right side of inequality (3) is less than 1, again for sufficiently large n. Hence a_n is bounded away from 0. $\qquad\square$

2.2. **Dimension.** Let Y be a bounded subset of a compact metric space X. Let $N(\varepsilon)$ denote the minimal number of ε balls needed to cover Y.

Definition 2.4. *For a subset Y of a compact metric space X, the upper box dimension of Y, denoted by $\dim_{UB} Y$ is given by*

$$\overline{\lim_{\varepsilon \to 0}} \frac{\log N(\varepsilon)}{\log 1/\varepsilon} \ .$$

The lower box dimension $\dim_{LB}$ is defined similarly, replacing the $\overline{\lim}$ by $\underline{\lim}$.

If $\dim_{UB} Y$ and $\dim_{LB} Y$ both exist and are equal, we define the box dimension of Y to be this value, and write $\dim_B Y = \dim_{UB} Y = \dim_{LB} Y$.

For a subset U of a metric space, we let $\mathrm{diam}(U)$ denote the diameter of the set U.

Definition 2.5. *Let $s \in [0, \infty]$. The s-dimensional Hausdorff measure $\mathcal{H}^s(Y)$ of a subset Y of a metric space X is defined by*

$$\mathcal{H}^s(Y) = \lim_{\varepsilon \to 0} \inf \left\{ \sum_{i=1}^{\infty} (\mathrm{diam}(U_i))^s : Y \subset \bigcup_{i=1}^{\infty} U_i \ \text{and} \ \sup_{i=1,\dots,\infty} \mathrm{diam}(U_i) \le \varepsilon \right\} \ .$$

It is easy to see that there exists a unique $s_0 = s_0(Y)$ such that

$$(4) \qquad \mathcal{H}^s(Y) = \begin{cases} \infty & \text{for } s < s_0 \\ 0 & \text{for } s > s_0 \ . \end{cases}$$

Definition 2.6. *The unique number s_0 given by (4) is defined to be the Hausdorff dimension of Y and is denoted by $\dim_H Y$.*

Standard arguments give that for a subset Y of a metric space X,

$$\dim_H Y \le \dim_{LB} Y \le \dim_{UB} Y$$

and that these inequalities may be strict.

The box dimension can also be defined in terms of covering sums, as is the Hausdorff dimension, with the only change being that the covering intervals all have equal length. We note that in order to estimate the box dimension, it suffices that the lengths of the covering intervals tend to 0 along a geometric sequence.

Lastly, we define the Hausdorff dimension of a measure:

Definition 2.7. *Let μ be a Borel probability measure on X. Then the Hausdorff dimension of the measure μ is defined by*

$$\dim_H \mu = \inf_Y \left\{ \dim_H Y : \mu(Y) = 1 \right\}.$$

A survey of the methods and results in dimension theory can be found in [4].

2.3. **Diophantine numbers.** Discriminating the Denjoy sets necessitates the following number theoretic definitions. See Cassels [2] for an overview of Diophantine numbers.

Definition 2.8. *An irrational number α is of Diophantine class $\nu = \nu(\alpha) \in \mathbb{R}^+$ if*

$$(5) \qquad\qquad \|q\alpha\| < \frac{1}{q^\mu}$$

has infinitely many solutions in integers q for $\mu < \nu$ and at most finitely many for $\mu > \nu$.

If α is not of a Diophantine class $\nu \in \mathbb{R}^+$ then α is said to be a Liouville number.

If α is of Diophantine class ν, we write $\alpha \in \mathrm{Dioph}(\nu)$. In a slight abuse of terminology, we say that a Liouville number $\alpha \in (0,1)$ has infinite Diophantine class and write $\alpha \in \mathrm{Dioph}(\infty)$.

We denote by $[a_1, a_2, \cdots]$ the (standard) continued fraction expansion of α. We use $\{q_n\}$ to denote the sequence of denominators of the corresponding finite approximants. Then

$$(6) \qquad\qquad \frac{1}{q_n(a_n + 2)} \leq \|q_n\alpha\| \leq \frac{1}{a_n q_n}$$

and

$$(7) \qquad\qquad q_{n+1} = a_n q_n + q_{n-1} .$$

3. STATEMENT OF RESULTS

We now have the notation for stating our results in detail.

Theorem 3.1. *Let $0 < \delta < 1$. Assume that the irrational $\alpha \in (0,1)$ and that $\{q_n\}$ is the sequence of the denominators for the continued fraction convergents. Then*

$$\min_{D^\delta \in \mathcal{D}^\delta} \dim_H \Omega_\alpha^\delta(D^\delta) = \delta \varliminf_{n \to \infty} \frac{\log q_n}{\log q_{n+1}} .$$

Furthermore, the minimum dimension is attained for the classic Denjoy sequence D_0^δ of class δ.

We postpone the proof of Theorem 3.1 until Section 5, and instead state and prove several corollaries of the Theorem.

Corollary 3.2. *Let* $0 < \delta < 1$. *If* $\alpha \in (0, 1)$ *is of Diophantine class* $\nu \in \mathbb{R}^+$, *then*

$$\min_{D^\delta \in \mathcal{D}^\delta} \dim_H \Omega_\alpha^\delta(D^\delta) = \frac{\delta}{\nu}$$

and

$$\dim_H \Omega_\alpha^\delta(D_0^\delta) = \frac{\delta}{\nu} \ .$$

Proof. If α is of Diophantine class ν, then

$$
\begin{aligned}
\varlimsup_{n \to \infty} \frac{\log q_n}{\log q_{n+1}} &= \varlimsup_{n \to \infty} \frac{\log q_n}{\log(a_n q_n + q_{n-1})} \\
&= \varlimsup_{n \to \infty} \frac{\log q_n}{\log(a_n q_n)} \cdot \frac{\log(a_n q_n)}{\log(a_n q_n + q_{n-1})} \\
&= \varlimsup_{n \to \infty} \frac{\log q_n}{\log q_n^\nu} = \frac{1}{\nu} \ .
\end{aligned}
$$

Hence by Theorem 3.1 $\min_{D^\delta \in \mathcal{D}^\delta} \dim_H \Omega_\alpha^\delta(D^\delta) = \delta/\nu$. $\qquad \square$

Corollary 3.2 proves the statements of Theorem 1.1 on Hausdorff dimension. The remaining part of Theorem 1.1 on the box dimension is proved in Section 4.

Corollary 3.3. *Let* $0 < \delta < 1$ *and let* $\alpha \in (0, 1)$ *be irrational. Then*

$$\min_{D^\delta \in \mathcal{D}^\delta} \dim_H \Omega_\alpha^\delta(D^\delta) = \delta$$

and

$$\dim_H \Omega_\alpha^\delta(D_0^\delta) = \delta$$

for Lebesgue almost all α.

Proof. This follows immediately from Corollary 3.2, since almost every number (with respect to Lebesgue measure) is of Diophantine class $1 + \varepsilon$ for any $\varepsilon > 0$. $\qquad \square$

Another immediate consequence of Corollary 3.2 is the following:

Corollary 3.4. *Let* $0 < \delta < 1$. *If* $\alpha \in (0, 1)$ *is a Liouville number then*

$$\min_{D^\delta \in \mathcal{D}^\delta} \dim_H \Omega_\alpha^\delta(D^\delta) = 0 \ .$$

4. The box dimension of $\Omega_\alpha^\delta(D_0^\delta)$

We start by computing the box dimension for the classic map.

For simplicity, we do not give the optimal result in the following estimate, as the constants do not affect our results. We use $[x]$ to denote the integer part of x.

Lemma 4.1. *Let $\{a_i\}$ be a sequence of positive numbers and let $n \in \mathbb{N}$. Then*

$$\sum_{k=1}^{n} \left(\left[\frac{na_k}{\sum_{i=1}^{n} a_i} \right] + 1 \right) \le 3n .$$

Proof. Without loss of generality, we assume that $a_1 \le a_2 \le \ldots \le a_n$. Let

$$m = \left| \left\{ 1 \le i \le n : a_i < \frac{\sum_{k=1}^{n} a_k}{n} \right\} \right| .$$

Clearly $m < n$. Then

$$\begin{aligned}
\sum_{k=1}^{n} \left(\left[\frac{na_k}{\sum_{i=1}^{n} a_i} \right] + 1 \right) &\le m + \sum_{k=m+1}^{n} \left(\left[\frac{na_k}{\sum_{i=1}^{n} a_i} \right] + 1 \right) \\
&\le m + 2 \sum_{k=m+1}^{n} \frac{na_k}{\sum_{i=1}^{n} a_i} \\
&\le m + 2n \le 3n .
\end{aligned}$$

$\square$

Theorem 4.2. *Let $0 < \delta < 1$ and let $\alpha \in (0,1)$ be irrational. Then*

$$\dim_B \Omega_\alpha^\delta(D_0^\delta) = \delta .$$

Proof. Throughout, we maintain the notation introduced in Section 2.1 in defining the classic Denjoy map. For simplicity of notation, we write $\Omega = \Omega_\alpha^\delta(D_0^\delta)$.

First we show that $\dim_B \Omega \le \delta$. For $n \in \mathbb{N}$, consider the set

$$\mathcal{J}_n = S^1 \setminus \left(\bigcup_{-n \le k \le n} J_k \right) .$$

It consists of $2n + 1$ disjoint intervals of total length

$$1 - \sum_{-n \le k \le n} |J_k| = c_n n^{(\delta-1)/\delta} ,$$

where the c_n are uniformly bounded. Letting $\{I_k\}_{k=1}^{2n+1}$ denote the $2n + 1$ complementary intervals of $\mathcal{J}_n$, $\Omega \subset \bigcup_{1 \le k \le 2n+1} I_k$. The $\delta + \varepsilon$ covering sum of

Ω can be estimated by covering the union by intervals of the average length, and so by intervals of length $\frac{n^{1-1/\delta}}{2n+1}$. Using Lemma 4.1, we have

$$\sum_{k=1}^{2n+1} \left(\left[\frac{|I_k|(2n+1)}{\sum_{i=1}^{2n+1} |I_i|} \right] + 1 \right) \left(\frac{n^{1-1/\delta}}{2n+1} \right)^{\delta+\varepsilon} \leq \frac{6n+3}{(2n+1)^{\delta+\varepsilon}} n^{\delta-1+\varepsilon-\varepsilon/\delta}$$

$$= Kn^{-\varepsilon/\delta}$$

for a uniformly bounded constant K.

By the observation that it suffices to consider a geometric sequence of lengths, we have $\dim_B \Omega \leq \delta$.

For the lower bound, let $\{U_k\}$ be a finite cover of Ω consisting of intervals of uniform length. Without loss of generality, we may assume that there exist $m, n \in \mathbb{N}$ such that no U_k intersects $\bigcup_{-m \leq \ell \leq n} J_\ell$. The orbit segment

$$\{r\alpha : -m - n - 1 \leq r \leq m + n + 1\}$$

contains at least one point in each of the $m + n + 1$ contiguous intervals I_k of $S^1 \setminus \bigcup_{-m \leq \ell \leq n} J_\ell$. This means that the length of any I_k is bounded from below by $c_\delta(2n + 2m + 3)^{-1/\delta}$. Moreover, the distance between any two intervals I_k and I_j with $k \neq j$ is at least $c_\delta \left(\max(n, m) \right)^{-1/\delta}$.

Thus in order to cover Ω by intervals of length $c_\delta(2n + 2m + 3)^{-1/\delta}$, we need at least $m + n + 1$ intervals. Hence, the covering sum is bounded from below by

$$0 < c_\delta^{\delta-\varepsilon} \leq (m + n + 1)c_\delta^{\delta-\varepsilon}(2n + 2m + 3)^{(\varepsilon-\delta)/\delta} \leq \sum_k |U_k|^{\delta-\varepsilon} .$$

This proves the other direction. $\qquad\square$

Combining the computation in the second half of the proof of Theorem 4.2 and Lemma 2.3, we obtain:

Theorem 4.3. *Let $0 < \delta < 1$ and let $\alpha \in (0,1)$ be irrational. Then*

$$\min_{D^\delta \in \mathcal{D}^\delta} \dim_{LB} \Omega_\alpha^\delta(D^\delta) \geq \delta .$$

Proof. We apply Lemma 2.3 for all $\beta < \delta$. $\qquad\square$

5. Proof of Theorem 3.1

5.1. Beatty sequences.

Definition 5.1. *Let $\alpha \in (0,1)$ be irrational and assume that $\{q_n\}$ is the sequence of denominators for the continued fraction expansion for α. For $n_0 \in \mathbb{N}$, the n_0-hitting sequence of α is defined to be the subsequence $\{r_k\}_{k \in \mathbb{Z}}$ of integers so that $r_k \alpha \in [0, q_{n_0}\alpha]$.*

Letting $[x]$ denote the greatest integer less than or equal to x, it is a classic result that:

Lemma 5.2 (Hedlund and Morse [5]). *Let $\alpha \in (0, 1)$ be irrational, let $n_0 \in \mathbb{N}$ and assume that $\{r_k\}$ is the n_0-hitting sequence of α. Then there exist $\lambda > 1$ and $\theta > 0$ so that $r_k = [k\lambda + \theta]$.*

In symbolic dynamics terminology, this means that the sequence $\{r_k\}$ is a *Beatty sequence*. It is easy to see that a Beatty sequence $\mathcal{B}$ is close to being arithmetic. More precisely, if $r_k, r_{k+1} \in \mathcal{B}$ are two consecutive elements in the Beatty sequence defined by the parameters $\lambda > 1$ and $\theta > 0$, then $r_{k+1} - r_k$ takes on only one of two values, either $[\lambda]$ or $[\lambda] + 1$.

Maintaining notation as before, we write

$$\Omega_\alpha^\delta(D^\delta) = S^1 \setminus \bigcup_{n \in \mathbb{Z}} J_n \ .$$

The semi-conjugacy h given by Poincaré's Theorem is two to one at the endpoints of the intervals J_n. We write $g_l(n\alpha)$ and $g_r(n\alpha)$ for the left and right preimages under h on the orbit of α, i.e. if $h(x) = h(y) = n\alpha$ and x, y are the standard representatives of the equivalence classes $(x), (y) \in \mathbb{R}/\mathbb{Z}$ with $x < y$ then $g_l(n\alpha) = x$ and $g_r(n\alpha) = y$. Formally, we should write the equivalence class instead of a representative of this class, but for simplicity we avoid this notation.

5.2. **Lower bound for the Hausdorff dimension.** We now prove the lower bound in Theorem 3.1.

Proof. (first part of Theorem 3.1) Let $\mathfrak{I} = \{I_k\}$ be a covering of $\Omega = \Omega_\alpha^\delta(D^\delta)$ by intervals and choose $t \in \mathbb{R}$ so that the covering sum

$$\mathfrak{S}^t(\mathfrak{I}) = \sum_k |I_k|^t > 1.$$

Without loss of generality, we may assume that each interval I_k has the form $I_k = [g_r(n\alpha), g_l(m\alpha)]$, where $n = n(k), m = m(k) \in \mathbb{Z}$ and $n < m$. (If not, the boundary points of the interval I_k fall into an interval $J_l \in \mathcal{D}^\delta$ and so we could shrink the interval while still covering Ω with a smaller covering sum.)

Given such a covering of Ω, its image under h covers S^1. Thus we estimate the distortion of the intervals in the covering of Ω with respect to the map h. If the distortion is not too large, the Hausdorff dimension of the cover cannot be too small, giving a lower bound on the dimension. The asymptotic distortion of a point by taking intervals containing the point and shrink to it, is nothing but the local Hölder exponent of the map h.

Therefore, we need a lower bound for the number

(8)
$$\frac{\log|(n\alpha, m\alpha)|}{\log|(g_r(n\alpha), g_l(m\alpha))|} .$$

In order to estimate the distances in (8), we use (1). To bound the first term on the right hand side of (1), we use the leading term in the sum and so need

$$\rho = \inf\{|r| \in \mathbb{N} : r\alpha \in [n\alpha, m\alpha]\} .$$

We have that $r\alpha \in [n\alpha, m\alpha]$ if and only if $(r - n)\alpha \in [0, (m - n)\alpha]$. Let $n_0 \in \mathbb{N}$ be the smallest natural number such that $q_{n_0}\alpha \in [0, (m - n)\alpha]$. Let $p \in \mathbb{N}$ be the largest number so that $p\|q_{n_0}\alpha\| \leq \|(m - n)\alpha\|$. We note that both p and q_{n_0} depend on the difference $n - m$ rather than on n. Thus for $0 \leq s \leq p - 1$ all the intervals $[sq_{n_0}\alpha, (s + 1)q_{n_0}\alpha]$ are contained in the interval $[0, (m-n)\alpha]$. Using that $[0, q_{n_0-2}\alpha] \supset [0, (m-n)\alpha]$ and applying (6) twice, we have for large n_0

$$1 \leq p \leq \frac{|[0, q_{n_0-2}\alpha]|}{|[0, q_{n_0}\alpha]|} \leq \frac{q_{n_0+1} - q_{n_0-1} + 2q_{n_0}}{q_{n_0-1} - q_{n_0-3}} \leq 2\frac{q_{n_0+1}}{q_{n_0-1}} .$$

By Lemma 5.2 the n_0-hitting sequence of α is of the form $r_k = [k\theta + \gamma]$ for some $\theta > 1$ and $\gamma > 0$. For this sequence, $r_0 = q_{n_0}$ and $r_1 = q_{n_0+1} + q_{n_0}$. Indeed, to find the value of r_1 we note first that $(q_{n_0+1} + q_{n_0})\alpha \in [0, q_{n_0}\alpha]$, as $\|q_{n_0+1}\alpha\| \leq \|q_{n_0}\alpha\|$ and this is the first such occurrence. On the other hand, any point $m\alpha \in [0, q_{n_0}\alpha]$ satisfies $m - q_{n_0} > q_{n_0+1}$.

More generally, since a Beatty sequence is close to being arithmetic, we have that for $k > 0$

(9) $$q_{n_0} + k(q_{n_0+1} - 1) < r_k < q_{n_0} + k(q_{n_0+1} + 1)$$

and a similar equation holds for $k < 0$. Furthermore, it follows that hitting sequences are defined for more general intervals. Namely, each interval $[sq_{n_0}\alpha, (s + 1)q_{n_0}\alpha]$ has the hitting sequence $r_k + sq_{n_0}$. Thus the hitting set for the interval $[0, p\|q_{n_0}\|] \subset [0, (n - m)\alpha]$ is the set

$$\bigcup_k \bigcup_{0 \leq s \leq p} \{r_k + sq_{n_0}\}$$

and for the interval $[0, (p + 1)\|q_{n_0}\|] \supset [0, (n - m)\alpha]$, the hitting sequence is the set

$$\bigcup_k \bigcup_{0 \leq s \leq p+1} \{r_k + sq_{n_0}\} .$$

Write $n = r_k + b$ for suitable k and some b with $0 \leq b \leq q_{n_0+1}$. Using the above description of the hitting sets for the intervals $[0, p\|q_{n_0}\|]$ and $[0, (p + 1)\|q_{n_0}\|]$, we estimate ρ so that

$$\min_{0 \leq s \leq p} (|b - sq_{n_0}|, |b - r_{k+1}|) \leq \rho \leq \min_{0 \leq s \leq p+1} (|b - sq_{n_0}|, |b - r_{k+1}|) .$$

To do so, we define

$$\rho(n,p) = \min_{0 \le s \le p+1} \left\{ r - n \in \{r_k + sq_{n_0}\}_{k \in \mathbb{Z}} \right\} .$$

We note that p actually depends on $n - m$, and so $\rho(n,p)$ is just a function of n and m.

By (1), the quantity in (8) is greater than or equal to:

$$(10) \qquad \frac{\log \|(m-n)\alpha\|}{\log \rho^{-1/\delta}} = \delta \frac{\log \|(m-n)\alpha\|}{\log \rho^{-1}} .$$

Thus a lower bound for (10) is obtained at the same value which maximizes the function

$$f(n,p) = \|(m-n)\alpha\| \rho(n,p) .$$

If n_0 is large enough and so $(n\alpha, m\alpha)$ is small, the definition of p gives that

$$\|(m-n)\alpha\| \le \|(p+1)q_{n_0}\alpha\| = (p+1)\|q_{n_0}\alpha\| \le (p+1)(q_{n_0+1} - q_{n_0-1})^{-1}$$

and

$$\|(m-n)\alpha\| \ge \|pq_{n_0}\alpha\| = p\|q_{n_0}\alpha\| \ge p(q_{n_0+1} - q_{n_0-1} + 2q_{n_0})^{-1}$$

by using (6) and (7). As well,

$$(11) \quad [(q_{n_0+1}-1-(p+1)q_{n_0})/2] \le \max_n \rho(n,p) \le [(q_{n_0+1}+1-(p+1)q_{n_0})/2] .$$

For n_0 sufficiently large, the maximum of $f(n,p)$ is attained at some

$$p_0 \in \left[q_{n_0+1}/(4q_{n_0}), 3q_{n_0+1}/(4q_{n_0}) \right] ,$$

and so $p_0 \le 2q_{n_0+1}/q_{n_0-1}$. Combining these estimates we have

$$\|(m-n)\alpha\| \le \left(\frac{3q_{n_0+1}}{4q_{n_0}} + 1 \right)(q_{n_0+1} - q_{n_0-1})^{-1} = Cq_{n_0}^{-1}$$

for some constant C. Thus by (11) there exist constants $c_1 > c_2$ so that

$$c_1 q_{n_0+1} \ge \rho(n,p_0) \ge c_2 q_{n_0+1} .$$

This implies that

$$\frac{\log \|n\alpha - m\alpha\|}{\log \|g_r(n\alpha) - g_l(m\alpha)\|} \ge \frac{\log \|(n-m)\alpha\|}{\log \rho(n,p_0)^{-1/\delta}} \ge \frac{\log(Cq_{n_0}^{-1})}{\log (c_1 q_{n_0+1})^{-1/\delta}} = \delta \frac{\log C^{-1}q_{n_0}}{\log c_1 q_{n_0+1}} .$$

This shows that for any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ so that

$$|h(I_k)| < C(\varepsilon)|I_k|^\kappa ,$$

where

$$\kappa = \delta \varliminf_{n \to \infty} \frac{\log q_{n_0}}{\log q_{n_0+1}} - \varepsilon .$$

Since $\mathcal{I}$ covers Ω_α^δ, the collection $\{h(I_k)\}_k$ covers S^1. Hence

$$(12) \qquad 1 = \dim_H S^1 \leq (\delta \varliminf_{n\to\infty} \frac{\log q_{n_0}}{\log q_{n_0+1}})^{-1} \dim_H \Omega \ .$$

Hence we have the lower bound for the Hausdorff dimension of the minimal set. $\qquad \square$

Let λ denote Lebesgue measure. With a slight modification, the same proof shows:

Corollary 5.3. *Let $0 < \delta < 1$. Assume that $\{q_n\}$ are the denominators in the continued fraction expansion of an irrational $\alpha \in (0,1)$. If $\lambda \circ h$ is the (unique) invariant measure on $\Omega_\alpha^\delta(D^\delta)$, then*

$$\dim_H \lambda \circ h \geq \delta \varliminf_{n\to\infty} \frac{\log q_n}{\log q_{n+1}} \ .$$

Proof. The only change needed is that in (12), we replace $1 = \dim_H S^1$ by $1 = \dim_H \lambda$. $\qquad \square$

5.3. **An upper bound for the dimension of the classic Denjoy map.**
We now prove that the bounds in Theorem 1.1 are sharp. For the lower bounds in the proof of Theorem 1.1, we used distortion estimates on the semi-conjugating map. We showed that for x in a C/q_m neighborhood of $cq_{m+1}\alpha$, the distortion is maximized. Unfortunately, this distortion estimate does not suffice for computing an upper bound for the Hausdorff dimension because these intervals only cover a set of full measure and not the whole circle. (See also Section 5.4 for an application.)

To find an upper bound for the Hausdorff dimension, we also need to consider points with less distortion and estimate the dimension for such points.

Proof. (second part of Theorem 1.1)
If $\alpha \in \mathrm{Dioph}(1)$, then by the previous sections

$$\delta = \dim_B \Omega_\alpha^\delta(\mathcal{D}_0^\delta) \geq \dim_H \Omega_\alpha^\delta(\mathcal{D}_0^\delta) \geq \delta \ ,$$

and so it suffices to consider $\alpha \in \mathrm{Dioph}(\nu)$ with $\nu > 1$.

Let $\{q_n\}$ denote the sequence of denominators in the continued fraction expansion of $\alpha \in \mathrm{Dioph}(\nu)$. We have

$$q_{n+1} - q_{n-1} \geq q_{n+1} - q_n$$

and by passing to a subsequence $\{n_k\}$, since $\nu > 1$ for $\varepsilon > 0$ such that $1/(\nu - \varepsilon) < 1$, we can assume that

$$q_{n_k+1} - q_{n_k-1} \geq q_{n_k+1}(1 - q_{n_k}^{1-\nu+\varepsilon}) \ .$$

By passing to a further subsequence, if necessary, we can assume that

$$q_{n_k}^{1-\nu+\varepsilon} < \frac{1}{k} \ .$$

The proof is divided into two parts. First we estimate the Hausdorff dimension of level sets with a fixed small Hölder exponent and then we estimate the dimension of the remaining set.

Fix a number s with $1/(\nu - \varepsilon) < s < 1$. Consider the positive integers

$$s_m^{(k)} \in (\varepsilon_k q_{n_k+1}, (1 - \varepsilon_k) q_{n_k+1}) \cap \mathbb{N} \ ,$$

where $\varepsilon_k = q_{n_k}^{(s-1)/s}$. Define intervals

(13) $$I_{s_m^{(k)}} = (s_m^{(k)}\alpha - 1/q_{n_k}, s_m^{(k)}\alpha + 1/q_{n_k}) \ ,$$

and

(14) $$\hat{I}_{s_m^{(k)}} = (s_m^{(k)}\alpha - 1/(2q_{n_k}), s_m^{(k)}\alpha + 1/(2q_{n_k})) \ .$$

Since $s > 1/(\nu - \varepsilon)$, we have that

$$\lim_{k\to\infty} \frac{\varepsilon_k q_{n_k+1}}{q_{n_k}} \to \infty \ .$$

By similar calculations to those in Section 5.2, since

$$\min\{|r| \in \mathbb{N} : r\alpha \in I_{s_m^{(k)}}\} \geq \varepsilon_k q_{n_k+1} \ ,$$

each of the intervals $I_{s_m^{(k)}}$ has a preimage $g(I_{s_m^{(k)}})$ with length bounded above by

(15) $$|g(I_{s_m^{(k)}})| \leq (\varepsilon_k q_{n_k+1})^{-1/\delta} \ .$$

Let

$$F_s = \bigcap_{K\in\mathbb{N}} \bigcup_{k\geq K} \hat{I}_{s_m^{(k)}} \ .$$

This set represents those points whose distortion is not too small and includes points where the distortion is not maximized.

For $x \in F_s$ there exist infinitely many intervals $I_{s_m^{(k)}}$ containing x with the distance of x to the endpoints larger that $\frac{1}{4}|I_{s_m^{(k)}}|$. By the Besicovitch covering lemma, there is a subcover of F_s of multiplicity at most 3 consisting

of intervals $I_{s_m^{(k)}}$. By (15) we can estimate the distortion:

$$
\begin{aligned}
\varliminf_{k\to\infty} \frac{\log |I_{s_m^{(k)}}|}{\log |g(I_{s_m^{(k)}})|} \;&\geq\; \varliminf_{k\to\infty} \frac{\log(2/q_{n_k})}{\log(\varepsilon_k q_{n_k+1})^{-1/\delta}} \\[2mm]
&=\; \varliminf_{k\to\infty} \frac{\delta \log q_{n_k}}{((s-1)/s)\log q_{n_k} + \log q_{n_k+1}} \\[2mm]
&=\; \varliminf_{k\to\infty} \frac{\delta \log q_{n_k}}{\log q_{n_k+1}} \left(\frac{1}{\nu}\cdot\frac{s-1}{s} + 1 \right)^{-1} \\[2mm]
&=\; \frac{s\delta}{s-1+s\nu}\; .
\end{aligned}
$$

(16)

Next we estimate the dimension of the complement $S^1 \setminus F_s$. The intervals $I_{s_m^{(k)}}$ cover a part of the circle avoiding the points $0, \alpha, \ldots, q_{n_k}\alpha$. For $l, r \in [0, q_{n_k})$, $d(l\alpha, r\alpha) \sim 1/q_{n_k}$, where by $\sim$ we mean that the ratio of the two quantities are bounded from above and from zero uniformly in all parameters. As well, $d(q_{n_k}\alpha, 0) \sim 1/q_{n_k+1}$.
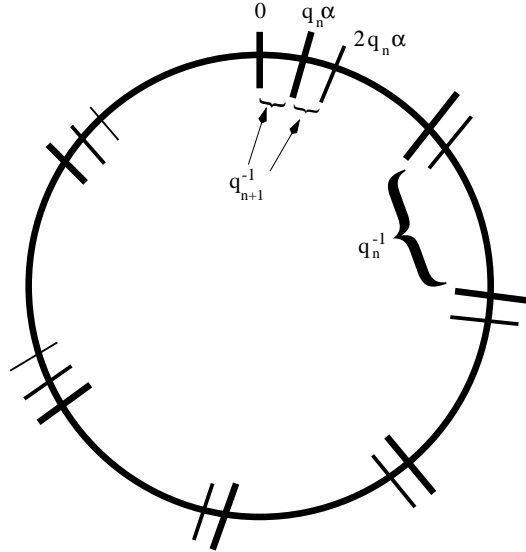

FIGURE 1

The gaps between these points are filled up from the left (or from the right, depending on which side of 0 the point $q_{n_k}\alpha$ lies) by points at distance $\sim 1/q_{n_k+1}$ for $l \in [q_{n_k}, q_{n_k+1})$. For $0 \leq l < q_{n_k}$, each point $l\alpha$ has a new (not necessarily immediate) "neighbor" at the q_{n_k}–th step, until the gaps are filled. This filling process stops before the last $\varepsilon_k q_{n_k+1}$ iterates, omitting the q_{n_k} gaps which are filled by the remaining iterates with frequencies differing

by at most one. This means that the set $\bigcup I_{s_m^{(k)}}$ omits q_{n_k} gaps of length at most

$$\left(\left[\frac{\varepsilon_k q_{n_k+1}}{q_{n_k}}\right] + 1\right) \cdot \frac{1}{q_{n_k+1}} \le \frac{2\varepsilon_k}{q_{n_k}} \ .$$

Hence, the upper box dimension of $S^1 \setminus F_s$ is bounded by

$$\dim_B(S^1 \setminus F_s) \ \le \ \lim_{k\to\infty} -\frac{\log q_{n_k}}{\log(2\varepsilon_k/q_{n_k})}$$

$$(17) \qquad\qquad = \ \lim_{k\to\infty} -\frac{\log q_{n_k}}{\log(2q_{n_k}^{(s-1)/s}/q_{n_k})} \ = \ s \ .$$

Therefore for τ sufficiently small, $\dim_B(F_s \setminus F_{s+\tau}) \le s$. This and (16) imply

$$(18) \qquad\qquad \dim_B g(F_s \setminus F_{s+\tau}) \le s \cdot \frac{s\delta}{s-1+s\nu} \le \frac{\delta}{\nu}.$$

We remark that this estimate holds for all $1/(\nu-\varepsilon) < s < 1$ and by the choice of sufficiently small ε, it holds for all $1/\nu \le s < 1$. Since $\dim_H F_1 = \delta/\nu$ by Section 5.2 it suffices to show that $\dim_B g(S^1 \setminus F_{1/\nu}) \le \delta/\nu$. But we know already from (17) that $\dim_B(S^1 \setminus F_{1/\nu}) \le 1/\nu$.

By Minkowski's Theorem ([2], p. 48), the intervals

$$\{(n\alpha, (n + q_m/4)\alpha) : 0 \le n \le q_m, m \in \mathbb{N}\}$$

cover all points in the circle that are not linearly independent of α and 1 over the rationals. Using the intervals $\{(n\alpha, (n + q_m)\alpha) : 0 \le n \le q_m, m \in \mathbb{N}\}$ in the distortion estimates, we have bounds on the distortion

$$\varliminf_{n\to\infty} \frac{\log q_{n+1}^{-1}}{\log(n + q_{n+1} + q_n)^{-1/\delta}} = \delta \ .$$

Therefore the dimension drops by a factor of at least δ and this implies that $\dim_B g(S^1 \setminus F_{1/\nu}) \le \delta/\nu$. $\qquad\square$

Corollary 5.4. *Let $0 < \delta < 1$ and let $\alpha \in \mathrm{Dioph}(\nu)$. Assume that h is the semi-conjugacy given by Poincaré's Theorem for a $C^{1+\delta}$ Denjoy map with rotation number α. Then*

$$\dim_H \lambda \circ h = \delta/\nu \ .$$

Proof. The lower bound is contained in Corollary 5.3. The measure cannot have larger dimension than its support and so we have equality. $\qquad\square$

5.4. **An application to discrepancy.** For a sequence $\{x_1, x_2, \ldots, x_N\}$ of N numbers in S^1 and a subset $Y \subset S^1$, let

$$A(Y; N) = \sum_{n=1}^{N} \chi_Y(x_n)$$

count the number of n with $x_n \in Y$.

Definition 5.5. *The discrepancy D_N of the N numbers $\{x_1, x_2, \ldots, x_N\}$ in S^1 is defined to be*

$$D_N = \sup_J \left| \frac{A(J; N)}{N} - |J| \right| ,$$

where the supremum is taken over all subintervals of S^1.

The discrepancy arises in many situations as an error estimate. Kuipers and Niederreiter [10] prove the following theorem, attributed to Behnke. (This is an exact bound on the discrepancy: the upper estimate on the discrepancy is attributed to Hecke and Ostrowski.)

Theorem 5.6 (Behnke [1]). *If $\alpha \in (0,1)$ is irrational of Diophantine class ν, then the discrepancy D_N of the fractional parts of the sequence $\{\alpha, 2\alpha, \ldots\}$ satisfies*

(19) $$D_N \geq cN^{-1/\nu+\tau}$$

for every $\tau > 0$ and some constant $c > 0$.

Proof. We maintain all the same notation as in Section 5.3.

Since the discrepancy D_N is always at least $1/N$, it suffices to consider $\nu > 1$.

We show that for sufficiently large N, the estimate in (19) holds with an arbitrary constant c. Accounting for the initial portion of the sequence D_N, we then obtain the estimate of (19) for some $c > 0$.

We proceed by contradiction. Assume instead that for some $\tau > 0$ and some $c > 0$, we have

(20) $$D_N \leq cN^{-1/\nu+\tau} .$$

Assume $\varepsilon/(\nu - \varepsilon) < \tau\nu$. We procced using a similar method to that used in the proof of Theorem 1.1. Choose the sequence $\{q_{n_k}\}$ (as in the proof of the second part of Theorem 1.1) so that $q_{n_k} \leq q_{n_k+1}^{1/(\nu-\varepsilon)}$.

Consider the positive integers

$$s_m^{(k)} \in \left(\frac{1}{4}q_{n_k+1}, \frac{3}{4}q_{n_k+1} \right) \cap \mathbb{N} .$$

Define intervals

(21) $$I_{s_m^{(k)}} = \left(s_m^{(k)}\alpha - 1/(2q_{n_k}), s_m^{(k)}\alpha + 1/(2q_{n_k}) \right) .$$

By similar calculations to those in Section 5.2, we have that for some $c_3 > 0$ independent of k,

$$c_3 q_{n_k+1} \geq \min\{|r| \in \mathbb{N} : r\alpha \in I_{s_m^{(k)}}\} \geq c_4 q_{n_k+1} .$$

Therefore each of the intervals $I_{s_m^{(k)}}$ has a preimage $g(I_{s_m^{(k)}})$ with length bounded by

$$(22) \qquad \left(\frac{1}{c_3 q_{n_k+1}}\right)^{-1/\delta} \leq |g(I_{s_m^{(k)}})| \leq \left(\frac{1}{c_4 q_{n_k+1}}\right)^{-1/\delta} .$$

Let

$$F_s^{(k)} = \bigcup_m I_{s_m^{(k)}} .$$

We claim that (20) implies that for sufficiently large k, the set $F_s^{(k)}$ covers the whole circle and hence $g(F_s^{(k)}) \supset \Omega$.

Indeed, (20) implies that for any interval J of length $1/q_{n_k}$

$$\begin{aligned} A(J, \frac{1}{2}q_{n_k+1}) &\geq \frac{q_{n_k+1}}{2q_{n_k}} - c2^{1/\nu+\tau-1}q_{n_k+1}^{1-1/\nu+\tau} \\ &\geq \frac{1}{2}q_{n_k+1}^{1-1/(\nu-\varepsilon)} - c2^{1/\nu+\tau-1}q_{n_k+1}^{1-1/\nu+\tau} \\ &\geq \tilde{c}q_{n_k+1}^{1-1/(\nu-\varepsilon)} \end{aligned}$$

for some constant $\tilde{c} > 0$. Thus for sufficiently large k, the sequence $\{n\alpha\}_{0 \leq n \leq 1/2q_{n_k+1}}$ hits any interval of length $1/q_{n_k}$ and so does the sequence $\{1/2q_{n_k+1}\alpha + n\alpha\}_{0 \leq n \leq 1/2q_{n_k+1}}$. This proves the claim.

Now we estimate the distortion by using (22):

$$\varliminf_{k \to \infty} \frac{\log |I_{s_m^{(k)}}|}{\log |g(I_{s_m^{(k)}})|} = \varliminf_{k \to \infty} \frac{-\log q_{n_k}}{\log q_{n_k+1}^{-1/\delta}} = \frac{\delta}{\nu}.$$

Since by (22) the intervals $g(I_{s_m^{(k)}})$ have approximately the same length for fixed k, the above distortion estimate implies that using the coverings $\{g(I_{s_m^{(k)}})\}$ we have

$$1 = \dim_B S^1 \geq \frac{\nu}{\delta} \dim_B \Omega ,$$

a contradiction with Theorem 4.2 for $\nu > 1$.

$\square$

References

1. H. Behnke, *Zur Theorie der diophantischen Approximationen, I*, Abh. Math. Sem. Hamburg **3** (1924), 261-318; II. ibid. **4** (1926), 33-46.
2. J. W. S. Cassels, *An introduction to Diophantine approximation*, Cambridge University Press, Cambridge, 1957.
3. A. Denjoy, *Sur les courbes défines par les équations différentielles á la surface du tore*, Journal de Mathématiques Pures et Appliquées **2** (1932), 333-375.
4. K. Falconer. *The geometry of fractal sets.* Cambridge University Press, Cambridge, 1985.

5. G. A. Hedlund and M. Morse, *Symbolic dynamics II: Sturmian trajectories*, American Journal of Mathematics **62** (1940), 1-42.

6. M. R. Herman, *Sur la conjugaison différentiable des difféomorphismes du cercle á des rotations*, Publications de l'Institute de Mathématiques des Hautes Études Scientifiques **49** (1979), 5-234.

7. A. Katok and B. Hasselblatt, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge Univ. Press, Cambridge 1995.

8. Y. Katznelson and D. Ornstein. *The differentiability of the conjugation of certain diffeomorphisms of the circle*, Ergodic Theory and Dynamical Systems **9** (1989), 643-680.

9. Y. Katznelson and D. Ornstein. *Private communication* 1999.

10. L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Wiley, New York, 1974.

11. A. Norton, *Denjoy's Theorem with exponents*, Proceedings of the American Mathematics Society **127** (1999), 3111-3118.

12. A. Pinto. *Poster session, Edinburgh and private communication* 2000.

13. H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, I. Journal de Mathématiques Pures et Appliquées, 3. série, **7** (1881), 375-422; II. **8** (1882), 251-286; III. 4. série, **1** (1885), 167-244; IV. **2** (1886), 151-217.

14. J. -C. Yoccoz, *Conjugaison différentiable des difféomorphismes du cercle dont le nombre de rotation vérifie une condition diophantienne*, Annales Scientifiques de l'École Normale Superieure **17** (1984), 333-361.

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