A POINT OF VIEW ON GOWERS UNIFORMITY NORMS

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Abstract. Gowers norms have been studied extensively both in the direct sense, starting with a function and understanding the associated norm, and in the inverse sense, starting with the norm and deducing properties of the function. Instead of focusing on the norms themselves, we study associated dual norms and dual functions. Combining this study with a variant of the Szemerédi Regularity Lemma, we give a decomposition theorem for dual functions, linking the dual norms to classical norms and indicating that the dual norm is easier to understand than the norm itself. Using the dual functions, we introduce higher order algebras that are analogs of the classical Fourier algebra, which in turn can be used to further characterize the dual functions.

1. Introduction

In his seminal work on Szemerédi's Theorem, Gowers [1] introduced uniformity norms $U(d)$ for each integer $d \geq 1$, now referred to as Gowers norms or Gowers uniformity norms, that have played an important role in the developments in additive combinatorics over the past ten years. In particular, Green and Tao [3] used Gowers norms as a tool in their proof that the primes contain arbitrarily long arithmetic progressions; shortly thereafter, they made a conjecture [5], the Inverse Conjecture for the Gowers norms, on the algebraic structures underlying these norms. Related seminorms were introduced by the authors [8] in the setting of ergodic theory, and the ergodic structure theorem provided a source of motivation in the formulation of the Inverse Conjecture. For each integer $d \geq 1$ and $\delta > 0$, Green and Tao introduce a class $\mathcal{F}(d, \delta)$ of "$(d-1)$-step nilsequences of bounded complexity," which we do not define here, and then showed:

Inverse Theorem for Gowers Norms (Green, Tao, and Ziegler [7]). *For each integer $d \geq 1$ and $\delta > 0$, there exists a constant $C = C(d, \delta) >$*

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0 such that for every function f on $\mathbb{Z}/N\mathbb{Z}$ with $|f| \leq 1$ and $\|f\|_{U(d)} \geq \delta$, there exists $g \in \mathcal{F}(d, \delta)$ with $\langle g; f\rangle \geq C$.

Another approach was later developed by Szegedy, outlined in the announcement [12] for the article [11].

We are motivated by the work of Gowers in [2]. Several ideas come out of this work, in particular the motivation that there are advantages to working with algebra norms. The Gowers norms $U(d)$ are classically defined in $\mathbb{Z}/N\mathbb{Z}$, but we choose to work in a general compact abelian group. For most of the results presented here, we take care to distinguish between the group $\mathbb{Z}/N\mathbb{Z}$ and the interval $[1, \ldots, N]$, of the natural numbers $\mathbb{N}$. On the other hand, for applications in additive combinatorics, the results are often more directly proved without this separation. This is a conscious choice that allows us to separate what about Gowers norms is particular to the combinatorics of $\mathbb{Z}/N\mathbb{Z}$ and what is more general. In summary, our point of view is that of harmonic analysis, rather than combinatorial.

More generally, the Gowers norms can be defined on a nilmanifold. This is particularly important in the ergodic setting, where analogous seminorms were defined by the authors in [8] in an arbitrary measure preserving system; these seminorms are exactly norms when the space is a nilmanifold. While we restrict ourselves to abelian groups in this article, most of the results can be carried out in the more general setting of a nilmanifold without significant changes.

Instead of focusing on the Gowers norms themselves, we study the associated dual norms that fit within this framework , as well as the associated dual functions. Moreover, in the statement of the Inverse Theorem, and more generally in uses of the Gowers norms, one typically assumes that the functions are bounded by 1. In the duality point of view, instead we study functions in the dual space itself, allowing us to consider functions that are within a small L^1 error from functions in this space. This allows us to restrict ourselves to dual functions of functions in a certain L^p class (Theorem 3.8). In particular, the further development of the material on dual functions leads us to a new decomposition result for anti-uniform functions of the type favored in additive combinatorics.

This leads to a rephrasing of the Inverse Theorem in terms of dual functions (see Section 2.2 for precise meanings of the term) in certain L^p classes, and in this form the Gowers norms do not appear explicitly (Section 3.3). This reformulates the Inverse Theorem more in a classical analysis context.

The dual functions allow us to introduce algebras of functions on the compact abelian group Z. For $d = 2$, this corresponds to the classical Fourier algebra. Finding an interpretation for the higher order uniformity norms is hard and no analogs of Fourier analysis with simple formulas, such as Parseval, exist. For $d > 2$, the higher order Fourier algebras are analogs of the classical case of the Fourier algebra. These algebras allow us to further describe the dual functions. Starting with a dual function of level d, we find that it lies in the Fourier algebra of order d, giving us information on its dual norm $U(d)^*$, and by an approximation result, we gain further insight into the original function.

The main result of the paper is a theorem on compactness (Theorem 5.2) of dual functions, obtained by applying a variation of the Szemerédi Regularity Lemma. Roughly speaking, the theorem states that within a given error, all shifts of a dual function of degree d are well approximated by elements of a module of bounded rank on the ring of dual functions of degree $d-1$. This property is reminiscent of the uniform almost periodicity norms introduced by Tao [13] and close to some combinatorial properties used, in particular, in [7].

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2. Gowers norms: definition and elementary bounds

2.1. **Notation.** Throughout, we assume that Z is a compact abelian group and let μ denote Haar measure on Z. If Z is finite, then μ is the uniform measure; the classical case to keep in mind is when $Z = \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ and the measure of each element is $1/N$.

All functions are implicitly assumed to be real valued. When Z is infinite, we also implicitly assume that all functions and sets are measurable. For $1 \leq p \leq \infty$, $\|\cdot\|_p$ denotes the $L^p(\mu)$ norm; if there is a need to specify a different measure ν, we write $\|\cdot\|_{L^p(\nu)}$.

We fix an integer $d \geq 1$ throughout and the dependence on d is implicit in all statements.

We have various spaces of various dimensions: 1, d, 2^d. Ordinary letters t are reserved for spaces of one dimension, vector notation $\vec{t}$ for dimension d, and bold face characters $\mathbf{t}$ for dimension 2^d.

If f is a function on Z and $t \in Z$, we write f_t for the function on Z defined by

$$f_t(x) = f(x+t),$$

where $x \in Z$. If f is a μ-integrable function on Z, we write

$$\mathbb{E}_{x\in Z} f(x) = \int f(x)\, d\mu(x).$$

We use similar notation for multiple integrals. If f and g are functions on Z, we write

$$\langle f; g\rangle = \mathbb{E}_{x\in Z} f(x)g(x),$$

assuming that the integral on the right hand side is defined.

If d is a positive integer, we set

$$V_d = \{0,1\}^d.$$

Elements of V_d are written as $\vec{\epsilon} = \epsilon_1\epsilon_2\cdots\epsilon_d$, without commas or parentheses. Writing $\vec{0} = 00\cdots 0 \in V_d$, we set

$$\widetilde{V}_d = V_d \setminus \{\vec{0}\}.$$

For $\mathbf{x} \in Z^{2^d}$, we write $\mathbf{x} = (x_{\vec{\epsilon}} \colon \vec{\epsilon} \in V_d)$.

For $\vec{\epsilon} \in V_d$ and $\vec{t} = (t_1, t_2, \ldots, t_d) \in Z^d$ we write

$$\vec{\epsilon}\cdot\vec{t} = \epsilon_1 t_1 + \epsilon_2 t_2 + \cdots + \epsilon_d t_d\ .$$

2.2. **The uniformity norms and the dual functions: definitions.** The *uniformity norms*, or *Gowers norms*, $\|f\|_{U(d)}$, $d \geq 2$, of a function $f \in L^\infty(\mu)$ are defined inductively by

$$\|f\|_{U(1)} = \big|\mathbb{E}_x f(x)\big|$$

and for $d \geq 2$,

$$\|f\|_{U(d)} = \Big(\mathbb{E}_t \|f.f_t\|_{U(d-1)}^{2^{d-1}}\Big)^{1/2^d}.$$

Note that $\|\cdot\|_{U(1)}$ is not actually a norm. (See [1] for more on these norms and [8] for a related seminorm in ergodic theory.) If there is ambiguity as to the underlying group Z, we write $\|\cdot\|_{U(d,Z)}$.

These norms can also be defined by closed formulas:

$$\|f\|_{U(d)}^{2^d} = \mathbb{E}_{x\in Z,\,\vec{t}\in Z^d} \prod_{\vec{\epsilon}\in V_d} f(x+\vec{\epsilon}\cdot\vec{t}). \tag{1}$$

We rewrite this formula. Let Z_d be the subset of Z^{2^d} defined by

$$Z_d = \Big\{(x+\vec{\epsilon}\cdot\vec{t}\colon \vec{\epsilon}\in V_d)\colon x\in Z,\ \vec{t}\in\mathbb{Z}^d\Big\}. \tag{2}$$

This set can be viewed as the "set of cubes of dimension d" (see, for example, [1] or [8]). It is easy to check that Z_d is a closed subgroup of Z^{2^d}. Let μ_d denote its Haar measure. Then Z_d is the image of $Z^{d+1} = Z\times Z^d$ under the map $(x,\vec{t}) \mapsto (x+\vec{\epsilon}\cdot\vec{t}\colon \vec{\epsilon}\in V_d)$. Furthermore, μ_d is the image of $\mu\times\mu\times\ldots\times\mu$ (taken $d+1$ times) under the same map. If $f_{\vec{\epsilon}}$, $\vec{\epsilon}\in V_d$, are functions in $L^\infty(\mu)$, then

$$\mathbb{E}_{x\in Z,\,\vec{t}\in Z^d} \prod_{\vec{\epsilon}\in V_d} f_{\vec{\epsilon}}(x+\vec{\epsilon}\cdot\vec{t}) = \int_{Z_d} \prod_{\vec{\epsilon}\in V_d} f_{\vec{\epsilon}}(x_{\vec{\epsilon}})\, d\mu_d(\mathbf{x}).$$

In particular, for $f \in L^\infty(\mu)$,

$$\|f\|_{U(d)}^{2^d} = \int_{Z_d} \prod_{\vec{\epsilon} \in V_d} f(x_{\vec{\epsilon}})\, d\mu_d(\mathbf{x}). \tag{3}$$

Associating the coordinates of the set V_d with the coordinates of the Euclidean cube, we have that the measure μ_d is invariant under permutations that are associated to the isometries of the Euclidean cube. These permutations act transitively on V_d.

For $d = 2$, by Parseval's identity we have that

$$\|f\|_{U(2)} = \|\widehat{f}\|_{\ell^4(\widehat{Z})}, \tag{4}$$

where $\widehat{Z}$ is the dual group of Z and $\widehat{f}$ is the Fourier transform of f. For $d \geq 3$, no analogous simple formula is known and the interpretation of the Gowers uniformity norms is more difficult. A deeper understanding of the higher order norms is, in part, motivation for the current work.

We make use of the "Cauchy-Schwarz-Gowers Inequality" (CSG) (see [1]) used in the proof of the subadditivity of Gowers norms:

Cauchy-Schwarz-Gowers Inequality. *Let $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in V_d$, be 2^d functions belonging to $L^\infty(\mu)$. Then*

$$\text{(CSG)} \quad \left| \mathbb{E}_{x \in Z,\, \vec{t} \in Z^d} f_{\vec{\epsilon}}(x + \vec{\epsilon} \cdot \vec{t}) \right| = \left| \int_{Z_d} \prod_{\vec{\epsilon} \in V_d} f_{\vec{\epsilon}}(x_{\vec{\epsilon}})\, d\mu_d(\mathbf{x}) \right| \leq \prod_{\vec{\epsilon} \in \{0,1\}^d} \|f_{\vec{\epsilon}}\|_{U(d)}.$$

Applying the Cauchy-Schwarz-Gowers Inequality with half of the functions equal to f and the other half equal to the constant 1, we deduce that

$$\|f\|_{U(d+1)} \geq \|f\|_{U(d)}$$

for every $f \in L^\infty(\mu)$.

Definition 2.1. For $f \in L^\infty(\mu)$, define the *dual function* $\mathcal{D}_d f$ on Z by

$$\mathcal{D}_d f(x) = \mathbb{E}_{\vec{t} \in Z^d} \prod_{\vec{\epsilon} \in \widetilde{V}_d} f(x + \vec{\epsilon} \cdot \vec{t}). \tag{5}$$

It follows from the definition that

$$\|f\|_{U(d)}^{2^d} = \langle \mathcal{D}_d f; f \rangle. \tag{6}$$

More generally, we define:

Definition 2.2. If $f_{\vec{\epsilon}} \in L^\infty$ for $\epsilon \in \widetilde{V}_d$, we write

$$\mathcal{D}_d(f_{\vec{\epsilon}} \colon \vec{\epsilon} \in \widetilde{V}_d)(x) = \mathbb{E}_{\vec{t} \in Z^d} \prod_{\vec{\epsilon} \in \widetilde{V}_d} f_{\vec{\epsilon}}(x + \vec{\epsilon} \cdot \vec{t}). \tag{7}$$

We call such a function the *cubic convolution product of the functions* $f_{\vec{\epsilon}}$.

There is a formal similarity between the cubic convolution product and the classic convolution product; for example,

$$\mathcal{D}_2(f_{01}, f_{10}, f_{11})(x) = \mathbb{E}_{t_1 t_2 \in Z} f_{01}(x+t_1) f_{10}(x+t_2) f_{11}(x+t_1+t_2).$$

2.3. **Elementary bounds.** For $\vec{\epsilon} \in V_d$ and $\alpha \in \{0,1\}$, we write $\vec{\epsilon}\alpha = \epsilon_1 \dots \epsilon_d \alpha \in V_{d+1}$, maintaining the convention that such elements are written without commas or parentheses. Thus

$$V_{d+1} = \{\vec{\epsilon}0 \colon \vec{\epsilon} \in V_d\} \cup \{\vec{\epsilon}1 \colon \vec{\epsilon} \in V_d\}.$$

The image of Z_{d+1} under each of the two natural projections on Z^{2^d} is Z_d, and the image of the measure μ_{d+1} under these projections is μ_d.

Lemma 2.3. *Let $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in \widetilde{V}_d$, be $2^d - 1$ functions in $L^\infty(\mu)$. Then for all $x \in Z$,*

$$\Big| \mathcal{D}_d(f_{\vec{\epsilon}} \colon \vec{\epsilon} \in \widetilde{V}_d)(x) \Big| \leq \prod_{\vec{\epsilon} \in \widetilde{V}_d} \|f_{\vec{\epsilon}}\|_{2^{d-1}}. \tag{8}$$

In particular, for every $f \in L^\infty(\mu)$,

$$\|\mathcal{D}_d f\|_\infty \leq \|f\|_{2^{d-1}}^{2^d - 1}. \tag{9}$$

Proof. Without loss, we can assume that all functions are nonnegative. We proceed by induction on $d \geq 2$.

For nonnegative f_{01}, f_{10} and $f_{11} \in L^\infty(\mu)$,

$$\begin{aligned} \mathcal{D}_2(f_{01}, f_{10}, f_{11})(x) &= \mathbb{E}_{t_1 \in Z} f_{01}(x+t_1) \mathbb{E}_{t_2 \in Z} f_{10}(x+t_2) f_{11}(x+t_1+t_2) \\ &\leq \mathbb{E}_{t_1 \in Z} f_{01}(x+t_1) \|f_{10}\|_2 \|f_{11}\|_2 \\ &\leq \|f_{01}\|_2 \|f_{10}\|_2 \|f_{11}\|_2. \end{aligned}$$

This proves the case $d = 2$. Assume that the result holds for some $d \geq 2$. Let $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in \widetilde{V}_{d+1}$, be nonnegative and belong to $L^{2^d(\mu)}$. Then

$$\begin{aligned} &\mathcal{D}_{d+1}(f_{\vec{\epsilon}} \colon \vec{\epsilon} \in \widetilde{V}_{d+1})(x) \\ &\qquad = \mathbb{E}_{\vec{s} \in Z^d} \Big(\prod_{\vec{\eta} \in \widetilde{V}_d} f_{\vec{\eta}0}(x + \vec{\eta} \cdot \vec{s}) \mathbb{E}_{u \in Z} \prod_{\vec{\theta} \in V_d} f_{\vec{\theta}1}(x + \vec{\theta} \cdot \vec{s} + u) \Big). \end{aligned}$$

For every $\vec{s} \in Z^d$ and every $x \in Z$, by the Hölder Inequality,

$$\mathbb{E}_{u\in Z} \prod_{\vec{\theta}\in V_d} f_{\vec{\theta}1}(x + \vec{\theta}\cdot\vec{s} + u) \leq \prod_{\vec{\theta}\in V_d} \|f_{\vec{\theta}1}\|_{2^d}.$$

On the other hand, by the induction hypothesis, for every $x \in Z$,

$$\mathbb{E}_{\vec{s}\in Z^d} \prod_{\vec{\eta}\in \widetilde{V}_d} f_{\vec{\eta}0}(x + \vec{\eta}\cdot\vec{s}) \leq \prod_{\vec{\eta}\in \widetilde{V}_d} \|f_{\vec{\eta}0}\|_{2^{d-1}} \leq \prod_{\vec{\eta}\in \widetilde{V}_d} \|f_{\vec{\eta}0}\|_{2^d}$$

and (8) holds for $d + 1$. $\square$

Corollary 2.4. *Let $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in V_d$, be 2^d functions belonging to $L^\infty(\mu)$. Then*

$$\left|\mathbb{E}_{x\in Z,\vec{t}\in Z^d} \prod_{\vec{\epsilon}\in V_d} f_{\vec{\epsilon}}(x + \vec{\epsilon}\cdot\vec{t})\right| \leq \prod_{\vec{\epsilon}\in V_d} \|f_{\vec{\epsilon}}\|_{2^{d-1}}. \tag{10}$$

In particular, for $f \in L^\infty(\mu)$,

$$\|f\|_{U(d)} \leq \|f\|_{2^{d-1}}. \tag{11}$$

By the corollary, the definition (1) of the Gowers norm $U(d)$ can be extended by continuity to the space $L^{2^{d-1}}(\mu)$, and if $f \in L^{2^{d-1}}(\mu)$, then the integrals defining $\|f\|_{U(d)}$ in Equation (1) exist and (11) holds. Using similar reasoning, if $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in V_d$, are 2^d functions belonging to $L^{2^{d-1}}(\mu)$, then the integral on the left hand side of (10) exists, the Cauchy-Schwarz-Gowers (CSG) remains valid, and (10) holds. If we have 2^{d-1} functions in $L^{2^{d-1}}(\mu)$, then Inequality (8) remains valid. Similarly, the definitions and results extend to $\mathcal{D}_d f$ and to cubic convolution products for functions belonging to $L^{2^{d-1}}(\mu)$.

The bounds given here (such as (11)) can be improved and made sharp. In particular, one can show that

$$\|f\|_{U(d)} \leq \|f\|_{2^d/(d+1)}$$

and

$$\|\mathcal{D}f\|_\infty \leq \|f\|^{2^d-1}_{(2^d-1)/d}.$$

We omit the proofs, as they are not used in the sequel.

When Z is infinite, we define the *uniform space of level d* to be the completion of $L^\infty(\mu)$ under the norm $U(d)$. As d increases, the corresponding uniform spaces shrink. A difficulty is that the uniform space may contain more than just functions. For example, if $Z = \mathbb{T} := \mathbb{R}/\mathbb{Z}$, the uniform space of level 2 consists of distributions T on $\mathbb{T}$ whose Fourier transform $\widehat{T}$ satisfies $\sum_{n\in\mathbb{Z}} |\widehat{T}(n)|^4 < +\infty$.

Corollary 2.5. *Let $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in V_d$, be 2^d functions on Z and let $\vec{\alpha} \in V_d$. Assume that $f_{\vec{\alpha}} \in L^1(\mu)$ and $f_{\vec{\epsilon}} \in L^{2^{d-1}}(\mu)$ for $\vec{\epsilon} \neq \vec{\alpha}$. Then*

$$\Big|\mathbb{E}_{x\in Z,\, \vec{t}\in Z^d} \prod_{\vec{\epsilon}\in V_d} f_{\vec{\epsilon}}(x+\vec{\epsilon}\cdot\vec{t})\Big| \le \|f_\alpha\|_1 \prod_{\substack{\vec{\epsilon}\in V_d\\ \vec{\epsilon}\neq\vec{\alpha}}} \|f_{\vec{\epsilon}}\|_{2^{d-1}}.$$

Proof. The left hand side is equal to

$$\Big|\int_{Z_d} f_{\vec{\alpha}}(x_{\vec{\alpha}}) \prod_{\substack{\vec{\epsilon}\in V_d\\ \vec{\epsilon}\neq\vec{\alpha}}} f_{\vec{\epsilon}}(x_{\vec{\epsilon}})\, d\mu_d(\mathbf{x})\Big|$$

Using the symmetries of the measure μ_d, we can reduce to the case that $\vec{\alpha} = \vec{0}$, and then the result follows from Lemma 2.3. □

We note for later use:

Lemma 2.6. *For every $f \in L^{2^{d-1}}(\mu)$, $\mathcal{D}_d f(x)$ is a continuous function on Z.*

More generally, if $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in \widetilde{V}_d$ are $2^d - 1$ functions belonging to $L^{2^{d-1}}(\mu)$, then the cubic convolution product $\mathcal{D}_d(f_{\vec{\epsilon}} \colon \vec{\epsilon} \in \widetilde{V}_d)(x)$ is a continuous function on Z.

Proof. By density and (8), it suffices to prove the result when $f_{\vec{\epsilon}} \in L^\infty(\mu)$ for every $\vec{\epsilon} \in \widetilde{V}_d$. Furthermore, we can assume that $|f_{\vec{\epsilon}}| \le 1$ for every $\vec{\epsilon} \in \widetilde{V}_d$. Let g be the function on Z defined in the statement. For $x, y \in Z$, we have that

$$|g(x) - g(y)| \le \sum_{\vec{\epsilon}\in\widetilde{V}_d} \|f_{\vec{\epsilon},x} - f_{\vec{\epsilon},y}\|_1$$

and the result follows. □

3. Duality

3.1. **Anti-uniform spaces.** Consider the space $L^{2^{d-1}}(\mu)$ endowed with the norm $U(d)$. By (11), the dual of this normed space can be viewed as a subspace of $L^{2^{d-1}/(2^{d-1}-1)}(\mu)$, with the duality given by the pairing $\langle\cdot;\cdot\rangle$. Following Green and Tao [3], we define

Definition 3.1. The *anti-uniform space of level d* is defined to be the dual space of $L^{2^{d-1}}(\mu)$ endowed with the norm $U(d)$. Functions belonging to this space are called *anti-uniform functions of level d*. The norm on the anti-uniform space given by duality is called the *anti-uniform norm of level d* and is denoted by $\|\cdot\|^*_{U(d)}$.

Obviously, when Z is finite, then every function on Z is an anti-uniform function.

It follows from the definitions that

$$\|f\|^*_{U(d+1)} \leq \|f\|^*_{U(d)}$$

for every $f \in L^\infty(\mu)$. Thus, as d increases, the corresponding anti-uniform spaces increase.

More explicitly, a function $g \in L^{2^{d-1}/(2^{d-1}-1)}(\mu)$ is an anti-uniform function of level d if

$$\sup\bigl\{|\langle g; f\rangle| \colon f \in L^{2^{d-1}}(\mu),\ \|f\|_{U(d)} \leq 1\bigr\} < +\infty.$$

In this case, $\|g\|^*_{U(d)}$ is defined to be equal to this supremum. Again, in case there may be ambiguity as to the underlying space Z, we write $\|\cdot\|^*_{U(d,Z)}$.

We conclude:

Corollary 3.2. *For every anti-uniform function g of level d, $\|g\|^*_{U(d)} \geq \|g\|_{2^{d-1}/(2^{d-1}-1)}$.*

For $d = 2$, the anti-uniform space consists of functions $g \in L^2(\mu)$ with $\|\widehat{g}\|_{\ell^{4/3}(\widehat{Z})}$ finite, and for these functions,

$$\|g\|^*_{U(2)} = \|\widehat{g}\|_{\ell^{4/3}(\widehat{Z})}. \tag{12}$$

This example illustrates that the $\|\cdot\|^*_{U(d)}$ can not be bounded by the norm $\|\cdot\|_{2^{d-1}/(2^{d-1}-1)}$, meaning that there is no bound for the converse direction of Corollary 3.2.

The dual spaces allow us to give an equivalent reformulation of the Inverse Theorem in terms of dual norms. By a dual formulation, we mean the determination of a family $\mathcal{F}'(d,\delta)$, for each integer $d \geq 1$ and each $\delta > 0$, of "$(d-1)$-step nilsequences of bounded complexity" that satisfies the following statement:

Inverse Theorem, Dual Form. *For each integer $d \geq 1$ and each $\delta > 0$, every function g on $\mathbb{Z}_N$ with $\|g\|^*_{U(d)} \leq 1$ can be written as $g = h + \psi$ with $h \in \mathcal{F}'(d,\delta)$ and $\|\psi\|_1 \leq \delta$.*

Remark 3.3. Note that the the function g is not assumed to be bounded.

We show that this formulation is equivalent to the Inverse Theorem, with simple and explicit relations between the classes $\mathcal{F}(d,\delta)$ and $\mathcal{F}'(d,\delta)$, but with different relations in each direction.

Proof. First assume the Inverse Theorem and let $\mathcal{F} = \mathcal{F}(d,\delta)$ be the class of nilsequences and $C = C(d,\delta)$ be as in its formulation. Let

$$K = \widetilde{\mathcal{F}} + B_{L^1(\mu)}(C),$$

where $\widetilde{\mathcal{F}}$ denotes the convex hull of $\mathcal{F}$ and the second term is the ball of radius C in $L^1(\mu)$. Let g be a function with $|\langle g; h\rangle| \leq C$ for all $h \in K$. In particular, $|g| \leq 1$ and $|\langle g; h\rangle| \leq C$ for all $h \in K$. By the Inverse Theorem, we have that $\|g\|_{U(d)} < \delta$. By the Hahn-Banach Theorem, $K \supset B_{U(d)^*}(C/\delta)$, the ball of radius C/δ in the anti-uniform norm of level d. Thus

$$B_{U(d)^*}(1) \subset (\delta/C)\widetilde{\mathcal{F}} + B_{L^1(\mu)}(\delta).$$

Taking $\mathcal{F}'(d,\delta)$ to be $(\delta/C)\widetilde{\mathcal{F}}$, we have the statement.

Conversely, assume the Dual Form. Say that $\mathcal{F}' = \mathcal{F}'(d,\delta/2)$ is the convex hull of $\mathcal{F}_0 = \mathcal{F}_0(d,\delta)$. Assume that f satisfies $|f| \leq 1$ and $\|f\|_{U(d)} \geq \delta$. Then there exists g with $\|g\|^*_{U(d)} \leq 1$ and $\langle g; f\rangle \geq \delta$. By the Dual Form, there exists $h \in \mathcal{F}'$ and ψ with $\|\psi\|_1 < \delta/2$ such that $g = h + \psi$. Since

$$\delta \leq \langle g; f\rangle = \langle h + \psi; f\rangle = \langle h; f\rangle + \langle \psi; f\rangle$$

and $\langle \psi; f\rangle < \delta/2$, we have that $\langle h; f\rangle < \delta/2$. Since $h \in \mathcal{F}'$, there exists $h' \in \mathcal{F}_0$ with $\langle h'; f\rangle > \delta/2$ and we have the statement of the Inverse Theorem. □

3.2. **Dual functions and anti-uniform spaces.**

Lemma 3.4. *Let $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in \widetilde{V}_d$, belong to $L^{2^{d-1}}(\mu)$. Then*

$$\|\mathcal{D}_d(f_{\vec{\epsilon}} \colon \vec{\epsilon} \in \widetilde{V}_d)\|^*_{U(d)} \leq \prod_{\vec{\epsilon}\in\widetilde{V}_d} \|f_{\vec{\epsilon}}\|_{2^{d-1}}.$$

Proof. For every $h \in L^{2^{d-1}}(\mu)$, we have that

$$\begin{aligned}\big|\langle h; g\rangle\big| =& \Big|\mathbb{E}_{x\in Z,\, \vec{t}\in Z^d} h(x + \vec{0}\cdot\vec{t}) \prod_{\vec{\epsilon}\in\widetilde{V}_d} f_{\vec{\epsilon}}(x + \vec{\epsilon}\cdot\vec{t})\Big| \\ \leq& \|h\|_{U(d)} \cdot \prod_{\vec{\epsilon}\in\widetilde{V}_d} \|f_{\vec{\epsilon}}\| \leq \|h\|_{U(d)} \cdot \prod_{\vec{\epsilon}\in\widetilde{V}_d} \|f_{\vec{\epsilon}}\|_{2^{d-1}}\end{aligned}$$

by the Cauchy-Schwarz-Gowers Inequality (CSG) and Inequality (11). □

In particular, for $f \in L^{2^{d-1}}(\mu)$, we have that $\|\mathcal{D}_d f\|^*_{U(d)} \leq \|f\|^{2^d-1}_{U(d)}$. On the other hand, by (6),

$$\|f\|^{2^d}_{U(d)} = \langle \mathcal{D}_d f; f\rangle \leq \|\mathcal{D}_d f\|^*_{U(d)} \cdot \|f\|_{U(d)}$$

and thus $\|\mathcal{D}_d f\|^*_{U(d)} \geq \|f\|^{2^d-1}_{U(d)}$. We conclude:

Proposition 3.5. *For every $f \in L^{2^{d-1}}(\mu)$, $\|\mathcal{D}_d f\|^*_{U(d)} = \|f\|^{2^d-1}_{U(d)}$.*

While the following proposition is not used in the sequel, it gives a helpful description of the anti-uniform space:

Proposition 3.6. *The unit ball of the anti-uniform space of level d is the closed convex hull in $L^{2^{d-1}/(2^{d-1}-1)}(\mu)$ of the set*

$$\big\{\mathcal{D}_d f\colon f\in L^{2^{d-1}}(\mu),\ \|f\|_{U(d)}\leq 1\big\}.$$

Proof. The proof is a simple application of duality.

Let $B\subset L^{2^{d-1}/(2^{d-1}-1)}(\mu)$ be the unit ball of the anti-uniform norm $\|\cdot\|^*_{U(d)}$. Let K be the convex hull of the set in the statement and let $\overline{K}$ be its closure in $L^{2^{d-1}/(2^{d-1}-1)}(\mu)$.

By Proposition 3.5, for every f with $\|f\|_{U(d)}\leq 1$, we have $\mathcal{D}_d f\in B$. Since B is convex, $K\subset B$. Furthermore, B is contained in the unit ball of $L^{2^{d-1}/(2^{d-1}-1)}(\mu)$ and is a weak* compact subset of this space. Therefore, B is closed in $L^{2^{d-1}/(2^{d-1}-1)}(\mu)$ and $\overline{K}\subset B$.

We check that $\overline{K}\supset B$. If this does not hold, there exists $g\in L^{2^{d-1}/(2^{d-1}-1)}(\mu)$ satisfying $\|g\|^*_{U(d)}\leq 1$ and $g\notin\overline{K}$. By the Hahn-Banach Theorem, there exists $f\in L^{2^{d-1}}(\mu)$ with $\langle f;h\rangle\leq 1$ for every $h\in K$ and $\langle f;g\rangle>1$. This last property implies that $\|f\|_{U(d)}>1$. Taking $\phi=\|f\|^{-1}_{U(d)}\cdot f$, we have that $\|\phi\|_{U(d)}=1$ and $\mathcal{D}_d\phi\in K$. Thus by the first property of f, $\langle\mathcal{D}_d\phi;f\rangle\leq 1$. But

$$\langle\mathcal{D}_d\phi;f\rangle=\|f\|^{-2^d+1}_{U(d)}\langle\mathcal{D}_d f;f\rangle=\|f\|_{U(d)}$$

and we have a contradiction. □

It can be shown that when Z is finite, the set appearing in Proposition 3.6 is already closed and convex:

Proposition 3.7. *Assume Z is finite. Then the set*

$$\big\{\mathcal{D}_d f\colon \|f\|_{U(d)}\leq 1\big\}$$

is the unit ball of the anti-uniform norm.

We omit the proof of this result, as the proof (for finite Z) is similar to that of Theorem 3.8 below, which seems more useful. For the general case, the analogous statement is not as clear because the "uniform space" does not consist only of functions.

3.3. **Approximation results for anti-uniform functions.** We now use the vocabulary reviewed and developed to prove the following decomposition theorem:

Theorem 3.8. *Assume $d \geq 1$ is an integer. For every anti-uniform function g with $\|g\|^*_{U(d)} = 1$, integer $k \geq d-1$, and $\delta > 0$, the function g can be written as*

$$g = \mathcal{D}_d f + h,$$

where

$$\|f\|_{2^k} \leq 1/\delta; \quad \|f\|_{U(d)} \leq 1;$$
$$\|h\|_{2^k/(2^k-1)} \leq \delta.$$

As in the Dual Form of the Inverse Theorem, we do not assume that the function g is bounded.

Proof. Fix $k \geq d-1$ and $\delta > 0$. For $f \in L^{2^k}(\mu)$, define

$$\|\!|f|\!\| = \begin{cases} \left(\|f\|^{2^k}_{U(d)} + \delta^{2^k}\|f\|^{2^k}_{2^k}\right)^{1/2^k} & \text{if } k \geq d; \\ \left(\|f\|^{2^d}_{U(d)} + \delta^{2^d}\|f\|^{2^d}_{2^{d-1}}\right)^{1/2^d} & \text{if } k = d-1. \end{cases} \tag{13}$$

Since $\|f\|_{U(d)} \leq \|f\|_{2^{d-1}} \leq \|f\|_{2^k}$ for every $f \in L^{2^k}(\mu)$, $\|\!|f|\!\|$ is well defined on $L^{2^k}(\mu)$ and $\|\!|\cdot|\!\|$ is a norm on this space, equivalent to the norm $\|\cdot\|_{2^k}$.

Let $\|\!|\cdot|\!\|^*$ be the dual norm of $\|\!|\cdot|\!\|$: for $g \in L^{2^k/(2^k-1)}(\mu)$,

$$\|\!|g|\!\|^* = \sup\Big\{\big|\langle f; g\rangle\big| \colon f \in L^{2^k}(\mu),\ \|\!|f|\!\| \leq 1\Big\}.$$

This dual norm is equivalent to the norm $\|\cdot\|_{2^k/(2^k-1)}$. Since $\|\!|f|\!\| \geq \|f\|_{U(d)}$ for every $f \in L^{2^k}(\mu)$, we have that

$$\|\!|g|\!\|^* \leq \|g\|^*_{U(d)} \text{ for every } g \in L^{2^k/(2^k-1)}(\mu).$$

Fix an anti-uniform function g with $\|g\|^*_{U(d)} \leq 1$. Since $\|\!|f|\!\| \geq \|f\|_{U(d)}$ for any f, we have that

$$c := \|\!|g|\!\|^* \leq \|g\|^*_{U(d)} \leq 1.$$

Set $g' = c^{-1}g$, and so $\|\!|g'|\!\|^* = 1$.

Since the norm $\|\!|\cdot|\!\|$ is equivalent to the norm $\|\cdot\|_{2^k}$ and the Banach space $(L^{2^k}(\mu), \|\cdot\|_{2^k})$ is reflexive, the Banach space $(L^{2^k}(\mu), \|\!|\cdot|\!\|)$ is also reflexive. This means that $(L^{2^k}(\mu), \|\!|\cdot|\!\|)$ is the dual of the Banach space $(L^{2^k/(2^k-1)}(\mu), \|\!|\cdot|\!\|^*)$. Therefore, there exists $f' \in L^{2^k}(\mu)$ with

$$\|\!|f'|\!\| = 1 \text{ and } \langle g'; f'\rangle = 1.$$

By definition (13) of $\|\!|f'|\!\|$,

$$\|f'\|_{U(d)} \leq 1 \text{ and } \|f'\|_{2^k} \leq 1/\delta. \tag{14}$$

Assume first that $k \geq d$. (At the end, we explain the modifications needed to cover the case $k = d-1$.)

By (3), (6), and the symmetries of the measure μ_d, for every $\phi \in L^{2^k}(\mu)$ and every $t \in \mathbb{R}$,

$$\|f' + t\phi\|_{U(d)}^{2^d} = \|f'\|_{U(d)}^{2^d} + 2^d t \langle \mathcal{D}_d f'; \phi \rangle + o(t), \tag{15}$$

where by $o(t)$ we mean any function such that $o(t)/t \to 0$ as $t \to 0$.

Raising this to the power 2^{k-d}, we have that

$$\|f' + t\phi\|_{U(d)}^{2^k} = \|f'\|_{U(d)}^{2^k} + 2^k t \|f'\|_{U(d)}^{2^k - 2^d} \langle \mathcal{D}_d f'; \phi \rangle + o(t).$$

On the other hand,

$$\|f' + t\phi\|_{2^k}^{2^k} = \|f'\|_{2^k}^{2^k} + 2^k t \langle f'^{2^k - 1}; \phi \rangle + o(t).$$

Combining these expressions and using the definition (13) of $|\!|\!| f' + t\phi |\!|\!|$ and of $|\!|\!| f' |\!|\!|$, we have that

$$\begin{aligned}
|\!|\!| f' + t\phi |\!|\!|^{2^k} =& \|f' + t\phi\|_{U(d)}^{2^k} + \delta^{2^k} \|f' + t\phi\|_{2^k}^{2^k} \\
=& |\!|\!| f' |\!|\!|^{2^k} + 2^k t \|f'\|_{U(d)}^{2^k - 2^d} \langle \mathcal{D}_d f'; \phi \rangle + \delta^{2^k} 2^k t \langle f'^{2^k - 1}; \phi \rangle + o(t) \\
=& 1 + 2^k t \|f'\|_{U(d)}^{2^k - 2^d} \langle \mathcal{D}_d f'; \phi \rangle + \delta^{2^k} 2^k t \langle f'^{2^k - 1}; \phi \rangle + o(t).
\end{aligned}$$

Raising this to the power $1/2^k$, we have that

$$|\!|\!| f' + t\phi |\!|\!| = 1 + t \|f'\|_{U(d)}^{2^k - 2^d} \langle \mathcal{D}_d f'; \phi \rangle + \delta^{2^k} t \langle f'^{2^k - 1}; \phi \rangle + o(t).$$

Since for every $\phi \in L^{2^k}(\mu)$ and every $t \in \mathbb{R}$ we have

$$1 + t\langle g'; \phi \rangle = \langle g'; f' + t\phi \rangle \leq |\!|\!| f' + t\phi |\!|\!|,$$

it follows that

$$1 + t\langle g'; \phi \rangle \leq 1 + t \|f'\|_{U(d)}^{2^k - 2^d} \langle \mathcal{D}_d f'; \phi \rangle + \delta^{2^k} t \langle f'^{2^k - 1}; \phi \rangle + o(t) \ .$$

Since this holds for every t, we have

$$\langle g'; \phi \rangle = \|f'\|_{U(d)}^{2^k - 2^d} \langle \mathcal{D}_d f'; \phi \rangle + \delta^{2^k} \langle f'^{2^k - 1}; \phi \rangle.$$

Since this holds for every ϕ, we conclude that

$$g' = \|f'\|_{U(d)}^{2^k - 2^d} \mathcal{D}_d f' + \delta^{2^k} f'^{2^k - 1}.$$

Thus

$$g = c \|f'\|_{U(d)}^{2^k - 2^d} \mathcal{D}_d f' + c\delta^{2^k} f'^{2^k - 1}.$$

Set

$$f = \left(c \|f'\|_{U(d)}^{2^k - 2^d}\right)^{1/(2^d - 1)} f' \text{ and } h = c\delta^{2^k} f'^{2^k - 1}.$$

Then

$$g = \mathcal{D}_d f + h$$

and by (14),

$$\|f\|_{U(d)} \le 1 \ ; \ \|f\|_{2^k} \le 1/\delta$$
$$\|h\|_{2^k/(2^k-1)} = c\delta^{2^k}\|f'\|_{2^k}^{2^k-1} \le \delta.$$

Finally, if $k = d-1$, for every $\phi \in L^{2^k}(\mu)$ and every $t \in \mathbb{R}$, we still have that (15) holds, and

$$\|f' + t\phi\|_{2^{d-1}}^{2^d} = \|f'\|_{2^{d-1}}^{2^d} + 2^d t\|f'\|_{2^{d-1}}^{2^{d-1}}\langle f'^{2^{d-1}-1};\phi\rangle + o(t).$$

Thus

$$\|\!|f' + t\phi|\!\| = 1 + t\langle \mathcal{D}_d f';\phi\rangle + \delta^{2^d}\|f'\|_{2^{d-1}}^{2^{d-1}}\langle f'^{2^{d-1}-1};\phi\rangle + o(t).$$

As above, we deduce that

$$g' = \mathcal{D}_d f' + \delta^{2^d}\|f'\|_{2^{d-1}}^{2^{d-1}} f'^{2^{d-1}-1}.$$

Taking

$$f = c^{1/(2^d-1)} f' \text{ and } h = c\delta^{2^d}\|f'\|_{2^{d-1}}^{2^{d-1}} f'^{2^{d-1}-1},$$

we have the statement. □

When Z is finite, we can say more:

Theorem 3.9. *Assume that Z is finite. Given a function g with $\|g\|_{U(d)}^* = 1$ and $\delta > 0$, the function g can be written as*

$$g = \mathcal{D}_d f + h,$$

where

$$\|f\|_\infty \le 1/\delta; \quad \|f\|_{U(d)} \le 1;$$
$$\|h\|_1 \le \delta.$$

Proof. By Theorem 3.8, for every $k \ge d-1$ we can write

$$g = \mathcal{D}_d f_k + h_k,$$

where

$$\|f_k\|_{2^k} \le 1/\delta; \|h_k\|_{2^k/(2^k-1)} \le \delta; \|f_k\|_{U(d)} \le 1.$$

Let $N = |Z|$. Since $\|f_k\|_{2^k} \le 1/\delta$, it follows that $\|f_k\|_\infty \le N/\delta$. Similarly, $\|h_k\|_\infty \le N\delta$. By passing to a subsequence, since the functions are uniformly bounded we can therefore assume that $f_k \to f$ and that $h_k \to h$ pointwise as $k \to +\infty$. Thus $\mathcal{D}_d f_k \to \mathcal{D}_d f$ pointwise and so

$$g = \mathcal{D}_d f + h.$$

Since $\|f_k\|_{U(d)} \to \|f\|_{U(d)}$, it follows that $\|f\|_{U(d)} \le 1$. For every $k \ge d-1$, we have that $\|h_k\|_1 \le \|h_k\|_{2^k/(2^k-1)} \le \delta$. Since $\|h_k\|_1 \to \|h\|_1$, it follows that $\|h\|_1 \le \delta$. For $\ell \ge k \ge d-1$,

$$\|f_\ell\|_{2^k} \le \|f_\ell\|_{2^\ell} \le 1/\delta.$$

Taking the limit as $\ell \to +\infty$, we have that $\|f\|_{2^k} \leq 1/\delta$ for every $k \geq d-1$ and so $\|f\|_\infty \leq 1/\delta$. $\square$

Question 3.10. *Does Theorem 3.9 also hold when Z is infinite?*

We conjecture that the answer is positive, but the proof given does not cover this case.

3.4. **Applications.** Theorems 3.8 and 3.9 give insight into the $U(d)$ norm, connecting it to the L^p norms. For example, we have:

Corollary 3.11. *Let ϕ be a function with $\|\phi\| \leq 1$ and $\|\phi\|_{U(d)} = \theta > 0$. Then for every $p \geq 2^{d-1}$, there exists a function f such that $\|f\|_p \leq 1$ and $\langle \mathcal{D}_d f; \phi\rangle > (\theta/2)^{2^d}$.*

Furthermore, if Z is finite, there exists a function f with $\|f\|_\infty \leq 1$ and $\langle \mathcal{D}_d f; \phi\rangle > (\theta/2)^{2^d}$.

Note that for finite Z, this last statement is reminiscent of the Inverse Theorem.

Proof. It suffices to prove the result when $p = 2^k$ for some integer $k \geq d-1$. There exists g with $\|g\|^*_{U(d)} = 1$ and $\langle g; \phi\rangle = \theta$. Taking $\delta = \theta/2$ in Theorem 3.8, we have the first statement. For the second statement, we apply Theorem 3.9. $\square$

Theorem 3.9 leads to another reformulation of the Inverse Theorem, without any explicit reference to the Gowers norms. Once again, we mean the determination of a family $\mathcal{F}''(d,\delta)$, for each $d \geq 1$ and $\delta > 0$, satisfying:

Inverse Theorem, Reformulated Version. *For every $\delta > 0$ and every function ϕ on $\mathbb{Z}_N$ with $\|\phi\|_\infty \leq 1$, the function $\mathcal{D}_d\phi$ can be written as $\mathcal{D}_d\phi = g + h$ with $g \in \mathcal{F}''(d,\delta)$ and $\|h\|_1 \leq \delta$.*

We show that the statement is equivalent to the Dual Form of the Inverse Theorem, with simple and explicit relations between the families $\mathcal{F}'(d,\delta)$ and $\mathcal{F}''(d,\delta)$, again with different relations in each direction.

Proof. First assume the Dual Form. Given ϕ with $\|\phi\|_\infty \leq 1$, we have that $\|\phi\|_{U(d)} \leq 1$ and thus $\|\mathcal{D}_d\phi\|^*_{U(d)} \leq 1$. By the Dual Form, $\mathcal{D}_d\phi = h + \psi$, where $h \in \mathcal{F}'(d,\delta)$ and $\|\psi\|_1 \leq \delta$, which is exactly the Reformulated Version.

Conversely, assume the Reformulated Version. Let $g \in B_{U(d)^*}(1)$, where $B_{U(d)^*}(1)$ denotes the ball of radius 1 in anti-uniform norm of level d. Then by Theorem 3.8, $g = \mathcal{D}_d h + \psi$, where $\|h\|_\infty \leq 2/\delta$ and $\|\psi\|_1 \leq \delta/2$. Define $\mathcal{F}' = \mathcal{F}'(d,\delta)$ to be equal to $(2/\delta)^{2^d-1}\mathcal{F}''(d,\eta)$, where η is a positive constant to be defined later and $\mathcal{F}''(d,\eta)$ is as in the Reformulated Version. By the Reformulated Version, $\mathcal{D}_d h = f+\psi$, with

$$f \in \mathcal{F}' \text{ and } \|\psi\|_1 \leq (2/\delta)^{2^d-1}\eta.$$

Then $g = f+\phi+\psi$ with $f \in \mathcal{F}'$ and $\|\phi+\psi\|_1 \leq \delta/2 + (2/\delta)^{2^d-1}\eta$. Taking $\eta = (\delta/2)^{2^d}$, we have the result. □

3.5. **Anti-uniformity norms and embeddings.** This section is a conjectural, and somewhat optimistic, exploration of the possible uses of the theory of anti-uniform norms developed here. The main interest is not the sketches of proofs included, but rather the questions posed and the directions that we conjecture may be approached using these methods.

Definition 3.12. If G is a $(d-1)$-step nilpotent Lie group and Γ is a discrete, cocompact subgroup of G, the compact manifold $X = G/\Gamma$ is $(d-1)$-*step nilmanifold.* The natural action of G on X by left translations is written as $(g,x) \mapsto g.x$ for $g \in G$ and $x \in X$.

We recall the following "direct" result (a converse to the Inverse Theorem), proved along the lines of arguments in [8]:

Proposition 3.13 (Green and Tao [4], Proposition 12.6)**.** *Let $X = G/\Gamma$ be a $(d-1)$-step nilmanifold, $x \in X$, $g \in G$, F be a continuous function on X, and $N \geq 2$ be an integer. Let f be a function on $\mathbb{Z}_N$ with $|f| \leq 1$. Assume that for some $\eta > 0$,*

$$\left|\mathbb{E}_{0\leq n<N} f(n)F(g^n \cdot x)\right| \geq \eta.$$

Then there exists a constant $c = c(X,F,\eta) > 0$ such that

$$\|f\|_{U(d)} \geq c.$$

The key point is that the constant c depends only on X, F, and η, and not on f, N, g or x.

Remark 3.14. In [4], the average is taken over the interval $[-N/2, N/2]$ instead of $[0,N)$, but the proof of Proposition 3.13 is the same for the modified choice of interval.

A similar result is given in Appendix G of [6], and proved using simpler methods, but there the conclusion is about the norm $\|f\|_{U(d,\mathbb{Z}_{N'})}$, where N' is sufficiently large with respect to N.

By duality, Proposition 3.13 can be rewritten as

Proposition 3.15. *Let $X = G/\Gamma, x, g, F$ be as in Proposition 3.13. Let $N \geq 2$ be an integer and let h denote the function $n \mapsto F(g^n \cdot x)$*

restricted to $[0, N)$ *and considered as a function on* $\mathbb{Z}_N$*. Then for every* $\eta > 0$*, we can write*

$$h = \phi + \psi$$

where ϕ *and* ψ *are functions on* $\mathbb{Z}_N$ *with* $\|\phi\|^*_{U(d)} \leq c(X, F, \eta)$ *and* $\|\psi\|_1 \leq \eta$*.*

Proposition 3.13 does not imply that $\|h\|^*_{U(d)}$ is bounded independent of N, and using (12), one can easily construct a counterexample for $d = 2$ and $X = \mathbb{T}$. On the other hand, for $d = 2$ we do have that $\|h\|^*_{U(d)}$ is bounded independent of N when the function F is sufficiently smooth. Recalling that the Fourier series of a continuously differentiable function on $\mathbb{T}$ is absolutely convergent and directly computing using Fourier coefficients, we have:

Proposition 3.16. *Let* F *be a continuously differentiable function on* $\mathbb{T}$ *and let* $\alpha \in \mathbb{T}$*. Let* $N \geq 2$ *be an integer and let* h *denote the restriction of the function* $n \mapsto F(\alpha^n)$ *to* $[0, N)$*, considered as a function on* $\mathbb{Z}_N$*. Then*

$$\|h\|^*_{U(2)} \leq c\|\widehat{F}\|_{\ell^1(\mathbb{Z})},$$

where c *is a universal constant.*

A similar result holds for functions on $\mathbb{T}^k$.

It is natural to ask whether a similar result holds for $d > 2$. For the remained of this section, we assume that every nilmanifold X is endowed with a smooth Riemannian metric. For $k \geq 1$, we let $\mathcal{C}^k(X)$ denote the space of k-times continuously differentiable functions on X, endowed with the usual norm $\|\cdot\|_{\mathcal{C}^k(X)}$. We ask if the dual norm is bounded independent of N:

Question 3.17. *Let* $X = G/\Gamma$ *be a* $(d-1)$*-step nilmanifold. Does there exist an integer* $k \geq 1$ *and a positive constant* c *such that for all choices of a function* $F \in \mathcal{C}^k(X)$*,* $g \in G$*,* $x \in X$ *and integer* $N \geq 2$*, writing* h *for the restriction to* $[0, N)$ *of the function* $n \mapsto F(g^n \cdot x)$*, considered as a function on* $\mathbb{Z}_N$*, we have*

$$\|h\|^*_{U(d)} \leq c\|F\|_{\mathcal{C}^k(X)}?$$

Definition 3.18. If $g \in G$ and $x \in X$ are such that $g^N \cdot x = x$, we say that the map $n \mapsto g^n \cdot x$ is an *embedding* of $\mathbb{Z}_N$ in X.

Proposition 3.19. *The answer to Question 3.17 is positive under the additional hypothesis that* $n \mapsto g^n \cdot x$ *is an embedding of* $\mathbb{Z}_N$ *in* X*, meaning that* $g^N \cdot x = x$*.*

The proof of this proposition is similar to that of Proposition 5.6 in [9] and so we omit it.

More generally, we can phrase these results and the resulting question for groups other than $\mathbb{Z}_N$. We restrict ourselves to the case of $\mathbb{T}$, as the extension to $\mathbb{T}^k$ is clear. By the same argument used for Proposition 3.13, we have:

Proposition 3.20. *Let $X = G/\Gamma$ be a $(d-1)$-step nilmanifold, $x \in X$, u be an element in the Lie algebra of G, and F be a continuous function on X. Let f be a function on $\mathbb{T}$ with $|f| \leq 1$. Assume that for some $\eta > 0$ we have*

$$\Big|\int f(t)F\big(\exp(tu)\cdot x\big)\,dt\Big| \geq \eta,$$

where we identify $\mathbb{T}$ with $[0,1)$ in this integral. Then there exists a constant $c = c(X, F, \eta) > 0$ such that

$$\|f\|_{U(d)} \geq c.$$

By duality, Proposition 3.20 can be rewritten as

Proposition 3.21. *Let $X = G/\Gamma, x, u, F$, and $c = c(X, F, \eta)$ be as in Proposition 3.20. Let h denote the restriction of the function $t \mapsto F\big(\exp(tu)\cdot x\big)$ to $[0,1)$, considered as a function on $\mathbb{T}$. Then for every $\eta > 0$, we can write*

$$h = \phi + \psi,$$

where ϕ and ψ are functions on $\mathbb{T}$ with $\|\phi\|^*_{U(d)} \leq c$ and $\|\psi\|_1 \leq \eta$.

We also pose the analog of Question 3.17 for the group $\mathbb{T}$:

Question 3.22. *Let $X = G/\Gamma$ be a $(d-1)$-step nilmanifold. Does there exist an integer $k \geq 1$ and a positive constant c such that for all choices of a function $F \in \mathcal{C}^k(X)$, u in the Lie algebra of G, and $x \in X$, writing h for the restriction of the function $t \mapsto F\big(\exp(tu)\cdot x\big)$ to $[0,1)$, considered as a function on $\mathbb{T}$, we have*

$$\|h\|^*_{U(d)} \leq c\|F\|_{\mathcal{C}^k(X)}?$$

Analogous to Proposition 3.19, the answer to this question is positive under the additional hypothesis that $t \mapsto \exp(tu)\cdot x$ is an *embedding* of $\mathbb{T}$ in X, meaning that $\exp(u)\cdot x = x$.

4. Multiplicative structure

4.1. **Higher order Fourier Algebras.** In light of Theorem 3.8, the family of functions g on Z of the form $g = \mathcal{D}_d f$ for $f \in L^{2^k}(\mu)$ for some $k \geq d-1$ is an interesting one for further study. More generally, we also consider cubic convolution products for functions $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in \widetilde{V}_d$, belonging to $L^{2^k}(\mu)$ for some $k \geq d-1$. We restrict ourselves to the case $k = d-1$, as it gives rise to interesting algebras.

Definition 4.1. For an integer $d \geq 1$, define $K(d)$ to be the convex hull of

$$\Big\{\mathcal{D}_d(f_{\vec{\epsilon}}\colon \vec{\epsilon} \in \widetilde{V}_d)\colon f_\epsilon \in L^{2^{d-1}}(\mu) \text{ for every } \vec{\epsilon} \in \widetilde{V}_d \text{ and } \prod_{\vec{\epsilon} \in \widetilde{V}_d} \|f_{\vec{\epsilon}}\|_{2^{d-1}} \leq 1\Big\}.$$

By Lemma 2.6, $K(d)$ is included in the algebra $\mathcal{C}(Z)$ of continuous functions on Z and, by (8), it is included in the unit ball of $\mathcal{C}(Z)$ for the uniform norm.

Define $\overline{K}(d)$ to be the closure of $K(d)$ in $\mathcal{C}(Z)$ under the uniform norm.

Define $A(d)$ to be the linear subspace of $\mathcal{C}(Z)$ spanned by $\overline{K}(d)$, endowed with the norm $\|\cdot\|_{A(d)}$ such that $\overline{K}(d)$ is its unit ball.

We call $A(d)$ the *Fourier algebra of order d*.

We begin with some simple remarks. Clearly, for $d \geq 2$, if Z is a finite group, then $K(d) = \overline{K}(d)$, and every function on Z belongs to $A(d)$.

It follows from the definitions that $A(1)$ consists of the constant functions with the norm $\|\cdot\|_{A(1)}$ being absolute value.

For every $g \in A(d)$, again by using (8) we have that

$$\|g\|_\infty \leq \|g\|_{A(d)}. \tag{16}$$

By Lemma 3.4, we have that g belongs to that anti-uniform space of level d and that

$$\|g\|^*_{U(d)} \leq \|g\|_{A(d)}.$$

Lemma 4.2. *Let $K_c(d)$ to be the convex hull of*

$$\Big\{\mathcal{D}_d(f_{\vec{\epsilon}}\colon \vec{\epsilon} \in \widetilde{V}_d)\colon f_\epsilon \in \mathcal{C}(Z) \text{ for every } \vec{\epsilon} \in \widetilde{V}_d \text{ and } \prod_{\vec{\epsilon} \in \widetilde{V}_d} \|f_{\vec{\epsilon}}\|_{2^{d-1}} \leq 1\Big\}.$$

Then $K(d)$ is the closure of $K_c(d)$ in $\mathcal{C}(Z)$ under the uniform norm.

Let (g_n) be a sequence in $A(d)$ such that $\|g_n\|_{A(d)}$ is bounded. If g_n converges uniformly to some function g, then $g \in A(d)$ and $\|g\|_{A(d)} \leq \sup_n \|g_n\|_{A(d)}$.

The space $A(d)$ endowed with the norm $\|\cdot\|_{A(d)}$ is a Banach space.

Proof. By density, the first statement follows immediately from Inequality (16). The second statement follows immediately from the definition.

Let h_n be a sequence in $A(d)$ with $\sum_n \|h_n\|_{A(d)} < +\infty$. We have to show that the series $\sum_n h_n$ converges in $A(d)$. By (16), the series converges uniformly. Set h to be the sum of the series. By the second part of the lemma, h belongs to $A(d)$ and $\|h\|_{A(d)} \leq \sum_n \|g_n\|_{A(d)}$. Applying the same argument again, we obtain that for every N,

$$\big\|h - \sum_{n=1}^{N} h_n\big\|_{A(d)} = \big\| \sum_{n=N+1}^{\infty} h_n\big\|_{A(d)} \leq \sum_{n=N+1}^{\infty} \|h_n\|_{A(d)}$$

and the result follows. □

4.2. **The case $d = 2$.** We give a further description for $d = 2$, relating these notions to classical objects of Fourier analysis.

By definition, $\widetilde{V}_2 = \{01, 10, 11\}$. Every function g defined as a cubic convolution product of $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in \widetilde{V}_2$, satisfies

$$\begin{aligned} \sum_{\xi \in \widehat{Z}} |\widehat{g}(\xi)|^{2/3} &= \sum_{\xi \in \widehat{Z}} \prod_{\vec{\epsilon} \in \widetilde{V}_2} |\widehat{f}_{\vec{\epsilon}}(\xi)|^{2/3} \\ &\leq \prod_{\vec{\epsilon} \in \widetilde{V}_2} \big(\sum_{\xi \in \widehat{Z}} |\widehat{f}_{\vec{\epsilon}}(\xi)|^2 \big)^{1/3} = \prod_{\vec{\epsilon} \in \widetilde{V}_2} \|f_{\vec{\epsilon}}\|_2^{2/3}. \end{aligned} \tag{17}$$

Thus

$$\sum_{\xi \in \widehat{Z}} |\widehat{g}(\xi)| \leq \prod_{\vec{\epsilon} \in \widetilde{V}_2} \|f_{\vec{\epsilon}}\|_2.$$

It follows that for $g \in A(2)$, we have that

$$\sum_{\xi \in \widehat{Z}} |\widehat{g}(\xi)| \leq \|g\|_{A(2)}.$$

On the other hand, let g be a continuous function on Z with $\sum_{\xi \in \widehat{Z}} |\widehat{g}(\xi)| < +\infty$. This function can be written as (in this example, we make an exception to our convention that all functions are real-valued)

$$g(x) = \sum_{\xi \in \widehat{Z}} \widehat{g}(\xi)\, \xi(x) = \sum_{\xi \in \widehat{Z}} \widehat{g}(\xi) \mathbb{E}_{t_1, t_2 \in Z} \xi(x + t_1) \xi(x + t_2) \overline{\xi}(x + t_1 + t_2).$$

By Lemma 4.2, we have that $g \in A(2)$ and $\|g\|_{A(2)} \leq \sum_{\xi \in \widehat{Z}} |\widehat{g}(\xi)|$.

We summarize these calculations:

Proposition 4.3. *The space $A(2)$ coincides with the Fourier algebra $A(Z)$ of Z:*

$$A(Z) := \big\{ g \in \mathcal{C}(Z) \colon \sum_{\xi \in \widehat{Z}} |\widehat{g}(\xi)| < +\infty \big\}$$

and, for $g \in A(Z)$, $\|g\|_{A(2)} = \|g\|_{A(Z)}$, which is equal by definition to the sum of the series.

4.3. $A(d)$ **is an algebra of functions.**

Theorem 4.4. *The Banach space $A(d)$ is invariant under pointwise multiplication and $\|\cdot\|_{A(d)}$ is an algebra norm, meaning that for all $g, g' \in A(d)$,*

$$\|gg'\|_{A(d)} \leq \|g\|_{A(d)}\,\|g'\|_{A(d)}. \tag{18}$$

Proof. By the first part of Lemma 4.2 and density, it suffices to show that when

$$g(x) = \mathcal{D}_d(f_{\vec{\epsilon}}\colon \vec{\epsilon} \in \widetilde{V}_d)(x) \quad \text{and} \quad g'(x) = \mathcal{D}_d(f'_{\vec{\epsilon}}\colon \vec{\epsilon} \in \widetilde{V}_d)(x),$$

with $f_{\vec{\epsilon}}$ and $f'_{\vec{\epsilon}} \in \mathcal{C}(Z)$ for every $\vec{\epsilon} \in \widetilde{V}_d$ and with

$$\prod_{\epsilon \in \widetilde{V}_d} \|f_{\vec{\epsilon}}\|_{2^{d-1}} \leq 1 \text{ and } \prod_{\epsilon \in \widetilde{V}_d} \|f'_{\vec{\epsilon}}\|_{2^{d-1}} \leq 1,$$

we have $gg' \in A(d)$ and $\|gg'\|_{A(d)} \leq 1$.

Rewriting, we have that

$$g(x)g'(x) = \mathbb{E}_{\vec{s} \in Z^d}\Big(\mathbb{E}_{\vec{t} \in Z^d} \prod_{\vec{\epsilon} \in \widetilde{V}_d} f_{\vec{\epsilon}}(x + \vec{\epsilon} \cdot \vec{t})\; f'_{\vec{\epsilon}}(x + \vec{\epsilon} \cdot \vec{s})\Big)$$

Letting $\vec{u} = \vec{s} - \vec{t}$, this becomes

$$\begin{aligned} g(x)g'(x) &= \mathbb{E}_{\vec{u} \in \mathbb{Z}^d}\Big(\mathbb{E}_{\vec{t} \in Z^d} \prod_{\vec{\epsilon} \in \widetilde{V}_d} f_{\vec{\epsilon}}(x + \vec{\epsilon} \cdot \vec{t})\; f'_{\vec{\epsilon}}(x + \vec{\epsilon} \cdot \vec{u} + \vec{\epsilon} \cdot \vec{t})\Big) \\ &= \mathbb{E}_{\vec{u} \in Z^d}\Big(\mathbb{E}_{\vec{t} \in Z^d} \prod_{\vec{\epsilon} \in \widetilde{V}_d} \big(f_{\vec{\epsilon}} \cdot f'_{\vec{\epsilon}, \vec{\epsilon} \cdot \vec{u}}\big)(x + \vec{\epsilon} \cdot \vec{t})\Big) = \mathbb{E}_{\vec{u} \in Z^d} g^{(\vec{u})}(x), \end{aligned}$$

where

$$\begin{aligned} g^{(\vec{u})}(x) &:= \mathcal{D}_d\big(f_{\vec{\epsilon}} \cdot f'_{\vec{\epsilon}, \vec{\epsilon} \cdot \vec{u}}\colon \vec{\epsilon} \in \widetilde{V}_d\big)(x) \\ &= \mathbb{E}_{\vec{t} \in Z^d} \prod_{\vec{\epsilon} \in \widetilde{V}_d} \big(f_{\vec{\epsilon}} \cdot f'_{\vec{\epsilon}, \vec{\epsilon} \cdot \vec{u}}\big)(x + \vec{\epsilon} \cdot \vec{t}). \end{aligned} \tag{19}$$

We claim that

$$\mathbb{E}_{\vec{u} \in Z^d} \prod_{\vec{\epsilon} \in \widetilde{V}_d} \|f_{\vec{\epsilon}} \cdot f'_{\vec{\epsilon}, \vec{\epsilon} \cdot \vec{u}}\|_{2^{d-1}} \leq \prod_{\vec{\epsilon} \in V_d} \|f_{\vec{\epsilon}}\|_{2^{d-1}}\, \|f'_{\vec{\epsilon}}\|_{2^{d-1}} \leq 1. \tag{20}$$

Namely,

$$\begin{aligned}
&\mathbb{E}_{\vec{u}\in Z^d}\prod_{\vec{\epsilon}\in\widetilde{V}_d}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\vec{\epsilon}\cdot\vec{u}}\|_{2^{d-1}}\\
&=\mathbb{E}_{u_1,\dots,u_{d-1}\in Z}\Big(\prod_{\substack{\vec{\epsilon}\in\widetilde{V}_d\\ \epsilon_d=0}}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\epsilon_1u_1+\dots+\epsilon_{d-1}u_{d-1}}\|_{2^{d-1}}\\
&\qquad\qquad\mathbb{E}_{u_d\in Z}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\epsilon_1u_1+\dots+\epsilon_{d-1}u_{d-1}+u_d}\|_{2^{d-1}}\Big)\\
&\leq\mathbb{E}_{u_1,\dots,u_{d-1}\in Z}\Big(\prod_{\substack{\vec{\epsilon}\in\widetilde{V}_d\\ \epsilon_d=0}}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\epsilon_1u_1+\dots+\epsilon_{d-1}u_{d-1}}\|_{2^{d-1}}\\
&\qquad\qquad\prod_{\substack{\vec{\epsilon}\in\widetilde{V}_d\\ \epsilon_d=1}}\Big(\mathbb{E}_{u_d\in Z}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\epsilon_1u_1+\dots+\epsilon_{d-1}u_{d-1}+u_d}\|_{2^{d-1}}^{2^{d-1}}\Big)^{1/2^{d-1}}\Big).
\end{aligned}$$

But, for all $u_1,u_2,\dots,u_{d-1}\in Z$ and every $\vec{\epsilon}\in\widetilde{V}_d$ with $\epsilon_d=1$,

$$\begin{aligned}
&\mathbb{E}_{u_d\in Z}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\epsilon_1u_1+\dots+\epsilon_{d-1}u_{d-1}+u_d}\|_{2^{d-1}}^{2^{d-1}}\\
&\qquad=\mathbb{E}_{v\in Z}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},v}\|_{2^{d-1}}^{2^{d-1}}=\|f_{\vec{\epsilon}}\|_{2^{d-1}}^{2^{d-1}}\,\|f'_{\vec{\epsilon}}\|_{2^{d-1}}^{2^{d-1}}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\mathbb{E}_{u_1,\dots,u_{d-1}\in Z}\prod_{\substack{\vec{\epsilon}\in\widetilde{V}_d\\ \epsilon_d=0}}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\epsilon_1u_1+\dots+\epsilon_{d-1}u_{d-1}}\|_{2^{d-1}}\\
&\leq\prod_{\substack{\vec{\epsilon}\in\widetilde{V}_d\\ \epsilon_d=0}}\Big(\mathbb{E}_{u_1,\dots,u_{d-1}\in Z}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\epsilon_1u_1+\dots+\epsilon_{d-1}u_{d-1}}\|_{2^{d-1}}^{2^{d-1}}\Big)^{1/2^{d-1}}.
\end{aligned}$$

But, for $\vec{\epsilon}\in\widetilde{V}_d$ with $\epsilon_d=0$, we have that $\epsilon_1,\dots,\epsilon_{d-1}$ are not all equal to 0 and

$$\begin{aligned}
\mathbb{E}_{u_1,\dots,u_{d-1}\in Z}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\epsilon_1u_1+\dots+\epsilon_{d-1}u_{d-1}}\|_{2^{d-1}}^{2^{d-1}}&=\mathbb{E}_{w\in Z}\|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},w}\|_{2^{d-1}}^{2^{d-1}}\\
&=\|f_{\vec{\epsilon}}\|_{2^{d-1}}^{2^{d-1}}\,\|f'_{\vec{\epsilon}}\|_{2^{d-1}}^{2^{d-1}}.
\end{aligned}$$

Combining these relations, we have proved claim (20).

On the other hand, since by assumption each of the functions $(\vec{u},x)\mapsto f_{\vec{\epsilon}}(x)f'_{\vec{\epsilon},\vec{\epsilon}\cdot\vec{u}}(x)$ is continuous on $Z^d\times Z$, we have that each of the functions $\vec{u}\mapsto\|f_{\vec{\epsilon}}\,.\,f'_{\vec{\epsilon},\vec{\epsilon}\cdot\vec{u}}\|_{2^{d-1}}$ is continuous on Z^d. Moreover, by the definition (19) of the functions $g^{(\vec{u})}$, it follows that the function $(\vec{u},x)\mapsto g^{(\vec{u})}(x)$ is also continuous on $Z^d\times Z$ (see also Corollary 2.4 and Lemma 2.6).

Therefore, for every $\delta > 0$, there exists a finite subset $\{\vec{u}_1, \dots, \vec{u}_n\}$ of Z^d and non-negative numbers $\lambda_1, \dots, \lambda_n$ with $\sum_{j=1}^n \lambda_j = 1$ such that

$$\Big\|gg' - \sum_{j=1}^n \lambda_j g^{(\vec{u}_j)}\Big\|_\infty = \sup_{x\in Z}\Big|\mathbb{E}_{\vec{u}\in\mathbb{Z}^d} g^{(\vec{u})}(x) - \sum_{j=1}^n \lambda_j g^{(\vec{u}_j)}(x)\Big| < \delta;$$

$$\Big|\mathbb{E}_{\vec{u}\in Z^d} \prod_{\vec{\epsilon}\in\widetilde{V}_d} \|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\vec{\epsilon}\cdot\vec{u}}\|_{2^{d-1}} - \sum_{j=1}^n \lambda_j \prod_{\vec{\epsilon}\in\widetilde{V}_d} \|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\vec{\epsilon}\cdot\vec{u}_j}\|_{2^{d-1}}\Big| < \delta,$$

and combining this with (20), we now have that

$$\sum_{j=1}^n \lambda_j \prod_{\vec{\epsilon}\in\widetilde{V}_d} \|f_{\vec{\epsilon}}\cdot f'_{\vec{\epsilon},\vec{\epsilon}\cdot\vec{u}_j}\|_{2^{d-1}} < 1+\delta.$$

Therefore, the function $\sum_{j=1}^n \lambda_j g^{(\vec{u}_j)}$ belongs to the set $(1+\delta)K(d)$. Letting δ tend to 0, we conclude that gg' belongs to $\overline{K}(d)$ and in particular, $gg' \in A(d)$ and $\|gg'\|_{A(d)} \le 1$. □

4.4. **Tao's uniform almost periodicity norms.** In [13], Tao introduced a sequence of norms, the uniform almost periodicity norms, that also play a dual role to the Gowers uniformity norms:

Definition 4.5 (Tao [13])**.** For $f\colon Z \to \mathbb{C}$, define $\|f\|_{\mathrm{UAP}^0(Z)}$ to be equal to $|c|$ if f is equal to the constant c, and to be infinite otherwise. For $d \ge 1$, define $\|f\|_{\mathrm{UAP}^{d+1}(Z)}$ to be the infimum of all constants $M > 0$ such that for all $n \in \mathbb{Z}$,

$$T^n f = M\mathbb{E}_{h\in H}(c_{n,h} g_h),$$

for some finite nonempty set H, collection of functions $(g_h)_{h\in H}$ from Z to $\mathbb{C}$ satisfying $\|g_h\|_{L^\infty(Z)} \le 1$, collection of functions $(c_{n,h})_{n\in Z, h\in H}$ from Z to $\mathbb{C}$ satisfying $\|c_{n,h}\|_{\mathrm{UAP}^d(Z)} \le 1$, and a random variable h taking values in H.

When the underlying group is clear, we omit it from the notation and write $\|f\|_{\mathrm{UAP}^d(Z)} = \|f\|_{\mathrm{UAP}^d}$.

Remark 4.6. The definition given in [13] implicitly assumes that Z is finite; to extend to the case that Z is infinite, take H to be an arbitrary probability space and view the functions g_h and $c_{n,h}$ as random variables.

Tao shows that this defines finite norms UAP^d for $d \ge 1$ and that for finite Z, the uniformly almost periodic functions of order d (meaning functions for which the UAP^d norm is bounded) form a Banach algebra:

$$\|fg\|_{\mathrm{UAP}^d} \le \|f\|_{\mathrm{UAP}^d}\|g\|_{\mathrm{UAP}^d}.$$

The UAP^{d-1} and $A(d)$ norms are related: both are algebra norms and they satisfy similar properties, such as

$$\|f\|_{\mathrm{UAP}^{d-1}} \geq \|f\|^*_{U(d)}$$

and

$$\|f\|_{A(d)} \geq \|f\|^*_{U(d)}.$$

For $d = 2$, the two norms are in fact the same (an exercise in [14] due to Green and shown in Section 4.2 below). However, in general we do not know if they are equal:

Question 4.7. *For a function $f\colon Z \to \mathbb{C}$, is*

$$\|f\|_{A(d)} = \|f\|_{\mathrm{UAP}^{d-1}}$$

for all $d \geq 2$?

In particular, while the UAP norms satisfy

$$\|f\|_{\mathrm{UAP}(d-1)} \geq \|f\|_{\mathrm{UAP}(d)}$$

for all $d \geq 2$, we do not know if the same inequality holds for the norms $A(d)$.

4.5. **Decomposable functions on Z_d.** Recall that Z_d is the subgroup of Z^{2^d} defined in (2) and μ_d is its Haar measure. Elements $\mathbf{x} \in Z_d$ are written as $\mathbf{x} = (x_{\vec{\epsilon}}\colon \vec{\epsilon} \in V_d)$.

Definition 4.8. Let $M(d)$ be the convex hull of the family of functions on Z_d that can be written as

(21)
$$F(\mathbf{x}) = \prod_{\vec{\epsilon}\in V_d} f_{\vec{\epsilon}}(x_{\vec{\epsilon}}) \text{ with } f_{\vec{\epsilon}} \in L^{2^d}(\mu) \text{ for every } \vec{\epsilon} \text{ and } \prod_{\vec{\epsilon}\in V_d} \|f_{\vec{\epsilon}}\|_{2^d} \leq 1.$$

By the remark following (11), $M(d)$ is included in the unit ball of $L^2(\mu_d)$. Define $\overline{M}(d)$ to be the closure of $M(d)$ in $L^2(\mu_d)$.

Define the space $D(d)$ of *decomposable functions* to be the linear span of $\overline{M}(d)$, endowed with the norm $\|\cdot\|_{D(d)}$ such that $\overline{M}(d)$ is the unit ball.

We have that $D(d) \subset L^2(\mu_d)$ and that $\|F\|_{L^2(\mu_d)} \leq \|F\|_{D(d)}$ for every $F \in D(d)$. If Z is a finite group, then every function on Z_d belongs to $D(d)$ and $\overline{M}(d) = M(d)$.

Lemma 4.9. *Let $M_c(d)$ to be the convex hull of the family of functions on Z_d that can be written as*

(22)
$$F(\mathbf{x}) = \prod_{\vec{\epsilon}\in V_d} f_{\vec{\epsilon}}(x_{\vec{\epsilon}}) \textit{ with } f_{\vec{\epsilon}} \in \mathcal{C}(Z) \textit{ for every } \vec{\epsilon} \textit{ and } \prod_{\vec{\epsilon}\in V_d} \|f_{\vec{\epsilon}}\|_{2^d} \leq 1.$$

Then $\overline{M}(d)$ *is the closure of* $M_c(d)$ *in* $L^2(\mu_d)$.

Let (F_n) *be a sequence in* $D(d)$ *such that* $\|F_n\|_{D(d)}$ *is bounded. If* F_n *converges to some function* F *in* $L^2(\mu_d)$, *then* $F \in D(d)$ *and* $\|F\|_{D(d)} \leq \sup_n \|F_n\|_{D(d)}$.

The space $D(d)$ *endowed with the norm* $\|\cdot\|_{D(d)}$ *is a Banach space.*

The proof of this lemma is nearly identical to that of Lemma 4.2 and so we omit it.

4.6. **Diagonal translations.**

Definition 4.10. For $t \in Z$, we write $t^\Delta = (t, t, \dots, t) \in Z_d$. The map $\mathbf{x} \mapsto \mathbf{x} + t^\Delta$ is called the *diagonal translation by* t.

Let $\mathcal{I}(d)$ denote the subspace of $L^2(\mu_d)$ consisting of functions invariant under all diagonal translations. The orthogonal projection π on $\mathcal{I}(d)$ is given by

$$\pi F(\mathbf{x}) = \mathbb{E}_{t \in Z} F(\mathbf{x} + t^\Delta).$$

Proposition 4.11. *If* $F \in D(d)$, *then* $\pi F \in D(d)$ *and* $\|\pi F\|_{D(d)} \leq \|F\|_{D(d)}$.

Furthermore, $\pi F \in L^\infty(\mu_d)$ *and* $\|\pi F\|_{L^\infty(\mu_d)} \leq \|F\|_{D(d)}$.

In particular, functions $F \in D(d) \cap \mathcal{I}(d)$ *are bounded on* Z_d *and satisfy* $\|F\|_\infty \leq \|F\|_{D(d)}$.

Proof. We start by proving the second statement. We begin with the case that F is defined as in (21). Rewriting the definition of $\pi F(x)$, we have that for every $\mathbf{x} \in Z_d$,

$$|\pi F(\mathbf{x})| = \Big| \mathbb{E}_{t \in Z} \prod_{\vec{\epsilon} \in V_d} f_{\vec{\epsilon}, x_{\vec{\epsilon}}}(t) \Big| \leq \prod_{\vec{\epsilon} \in V_d} \|f_{\vec{\epsilon}, x_{\vec{\epsilon}}}\|_{2^d} \leq 1$$

and thus $\|\pi F\|_{L^\infty(\mu_d)} \leq 1$. This bound remains valid if F belongs to the convex hull $M(d)$ of functions of this type. Assume now that F belongs to $\overline{M}(d)$. Let (F_n) be a sequence in $M(d)$ converging to F in $L^2(\mu_d)$. We have that $\|\pi F_n\|_{L^\infty(\mu_d)} \leq 1$ for every n, and since the map π is continuous in the $L^2(\mu_d)$-norm, $\pi F_n \to \pi F$ in $L^2(\mu_d)$. It follows that $\|\pi F\|_{L^\infty(\mu_d)} \leq 1$. This proves the announced result.

We now prove the first statement of the proposition, assuming first that the function F has the form given in (22). We rewrite the definition of πF as

$$\pi F(\mathbf{x}) = \mathbb{E}_{t \in Z} \prod_{\vec{\epsilon} \in V_d} f_{\vec{\epsilon}, t}(x_{\vec{\epsilon}}).$$

Proceeding as in the proof of Theorem 4.4, we approximate the integral $\mathbb{E}_t$ in the uniform norm by a finite average that belongs to $M(d)$. We deduce that $\pi F \in \overline{M}(d)$. By density and since π is continuous in the

L^2-norm, the same result holds for every $F \in \overline{M}(d)$, completing the proof of the statement. The last statement follows immediately from the previous two. $\square$

Theorem 4.12. *For $F \in D(d)$ and $G \in D(d) \cap \mathcal{I}(d)$, we have that FG belongs to $D(d)$ and that $\|FG\|_{D(d)} \leq \|F\|_{D(d)}\|G\|_{D(d)}$.*

In particular, $D(d)\cap\mathcal{I}(d)$, endowed with pointwise multiplication and the norm $\|\cdot\|_{B(d)}$, is a Banach algebra.

Proof. Since $\pi G = G$ when $G \in D(d) \cap \mathcal{I}(d)$, it suffices to show that for all $F, G \in D(d)$, we have $F.\pi(G) \in D(d)$ and

$$\|F.\pi G\|_{D(d)} \leq \|F\|_{D(d)}\|G\|_{D(d)}. \tag{23}$$

First consider the case that F and G are product of continuous functions. More precisely, assume that

$$F(\mathbf{x}) = \prod_{\vec{\epsilon}\in V_d} f_{\vec{\epsilon}}(x_{\vec{\epsilon}}),\ G(\mathbf{x}) = \prod_{\vec{\epsilon}\in V_d} g_{\vec{\epsilon}}(x_{\vec{\epsilon}}),$$

where $f_{\vec{\epsilon}}$ and $g_{\vec{\epsilon}}$ belong to $\mathcal{C}(Z)$ for every $\vec{\epsilon} \in V_d$ and

$$\prod_{\vec{\epsilon}\in V_d} \|f_{\vec{\epsilon}}\|_{2^d} \leq 1 \text{ and } \prod_{\vec{\epsilon}\in V_d} \|g_{\vec{\epsilon}}\|_{2^d} \leq 1.$$

Then

$$(F.\pi G)(\mathbf{x}) = \mathbb{E}_{t\in Z} \prod_{\vec{\epsilon}\in V_d} (f_{\vec{\epsilon}}.g_{\vec{\epsilon},t})(x_{\vec{\epsilon}}) = \mathbb{E}_{t\in Z} H^{(t)}(\mathbf{x}),$$

where

$$H^{(t)}(\mathbf{x}) := \prod_{\vec{\epsilon}\in V_d} (f_{\vec{\epsilon}}.g_{\vec{\epsilon},t})(x_{\vec{\epsilon}}).$$

Furthermore,

$$\mathbb{E}_{t\in Z} \prod_{\vec{\epsilon}\in V_d} \|f_{\vec{\epsilon}}.g_{\vec{\epsilon},t}\|_{2^d} \leq \prod_{\vec{\epsilon}\in V_d} \Big(\mathbb{E}_{t\in Z}\|f_{\vec{\epsilon}}.g_{\vec{\epsilon},t}\|_{2^d}^{2^d}\Big)^{1/2^d} = \prod_{\vec{\epsilon}\in V_d} \|f_{\vec{\epsilon}}\|_{2^d}\|g_{\vec{\epsilon}}\|_{2^d} \leq 1.$$

We now proceed as in the end of the proof of Theorem 4.4. For every $\delta > 0$, choose an integer $n \geq 1$, a finite subset $\{t_1, \dots, t_n\}$ of Z, and non-negative $\lambda_1, \dots, \lambda_n$ with $\sum_{j=1}^n \lambda_j = 1$ such that

$$\Big|(F \cdot \pi G)(\mathbf{x}) - \sum_{j=1}^n \lambda_j H^{(t_j)}(\mathbf{x})\Big| = \Big|(\mathbb{E}_{t\in Z} H^{(t)}(\mathbf{x}) - \sum_{j=1}^n \lambda_j H^{(t_j)}(\mathbf{x})\Big| < \delta$$

for every $\mathbf{x} \in \mathbb{Z}^d$ and

$$\sum_{j=1}^n \lambda_j \prod_{\vec{\epsilon}\in V_d} \|f_{\vec{\epsilon}} \cdot g_{\vec{\epsilon},t_j}\|_{2^d} < \delta + \mathbb{E}_{t\in Z} \prod_{\vec{\epsilon}\in V_d} \|f_{\vec{\epsilon}} \cdot g_{\vec{\epsilon},t}\|_{2^d} \leq 1 + \delta.$$

This last bound implies that the function $sum_{j=1}^n \lambda_j H^{(t_j)}$ belongs to $(1+\delta)M(d)$ and by the preceding bound, we have that the function $(F \cdot \pi G) \in \overline{M}(d)$.

The same results holds when F and G belong to $M_c(d)$. Assume now that F and G belong to $\overline{M}(d)$. By the first part of Lemma 4.9, there exist sequences (F_n) and (G_n) in $M_c(d)$ with $F_n \to F$ and $G_n \to G$ in $L^2(\mu_d)$. By the second part of Proposition 4.11, $\|\pi G_n\|_{L^\infty(\mu_d)} \leq 1$ for every n and $\|\pi G\|_{L^\infty(\mu_d)} \leq 1$.

Since $F_n \to F$ in $L^2(\mu_d)$, we have that $(F_n - F) \cdot \pi G \to 0$ in $L^2(\mu_d)$. Since $\pi G_n - \pi G \to 0$ in $L^2(\mu_d)$, by passing to a subsequence we can assume that $\pi G_n - \pi G \to 0$ μ_d-almost everywhere. For every $n \in \mathbb{N}$, we have that $|F(\pi G_n - \pi G)|^2 \leq 4|F|^2$ and thus by the Lebesgue Dominated Convergence Theorem, $\|F(\pi G_n - \pi G)\|_{L^2(\mu_d)} \to 0$. Finally, $F_n \cdot \pi G_n$ converges to $F \cdot \pi G$ in $L^2(\mu_d)$ and $F \cdot \pi G \in \overline{M}(d)$, completing the proof of Theorem 4.12. □

5. A result of finite approximation

5.1. **A decomposition theorem.** For a probability space (X, μ), we assume throughout that it belongs to one of the two following classes:

- μ is nonatomic. We refer to this case as the *infinite case*.
- X is finite and μ is the uniform probability measure on X. We refer to this case as the *finite case*.

This is not a restrictive assumption: Haar measure on a compact abelian group always falls into one of these two categories.

As usual, all subsets or partitions of X are implicitly assumed to be measurable.

Definition 5.1. Let $m \geq 2$ be an integer and let $(X_1, \dots, X_m)$ be a partition of the probability space (X, μ). The partition is *almost uniform* if:

- In the infinite case, $\mu(X_i) = 1/m$ for every i.
- In the finite case, $|X_i| = \lfloor |X|/m \rfloor$ or $\lceil |X|/m \rceil$ for every i.

The main result of this paper is:

Theorem 5.2. *Let $d \geq 1$ be an integer and let $\delta > 0$. There exists an integer $M = M(d, \delta) \geq 2$ and a constant $C = C(d, \delta) > 0$ such that the following holds: if $f_{\vec{\epsilon}}$, $\vec{\epsilon} \in \widetilde{V}_{d+1}$, are $2^{d+1} - 1$ functions belonging to $L^{2^d}(\mu)$ with $\|f_{\vec{\epsilon}}\|_{L^{2^d}(\mu)} \leq 1$ and*

$$\phi(x) = \mathcal{D}_{d+1}(f_{\vec{\epsilon}} \colon \vec{\epsilon} \in \widetilde{V}_{d+1})(x),$$

then for every $\delta > 0$ there exist an almost uniform partition $(X_1, \dots, X_m)$ of Z with $m \leq M$ sets, a nonnegative function ρ on Z, and for $1 \leq i \leq m$ and every $t \in Z$, a function $\phi_i^{(t)}$ on Z such that

(i) $\|\rho\|_{L^2(\mu)} \leq \delta$;

(ii) $\|\phi_i^{(t)}\|_\infty \leq 1$ *and* $\|\phi_i^{(t)}\|_{A(d)} \leq C$ *for every i and every t;*

(iii)
$$\Bigl|\phi(x+t) - \sum_{i=1}^{m} 1_{X_i}(x)\phi_i^{(t)}(x)\Bigr| \leq \rho(x) \text{ for all } x, t \in Z.$$

Combining this theorem with an approximation result, this leads to a deeper understanding of properties of the dual norm.

Remark 5.3. In fact we show a bit more: each function $\phi_i^{(t)}$ is the sum of a bounded number of functions that are cubic convolution products of functions whose $L^{2^{d-1}}(\mu)$ norms are bounded by 1.

Remark 5.4. The function ϕ in the statement of Theorem 5.2 satisfies $|\phi| \leq 1$ and thus $0 \leq \rho \leq 2$.

Remark 5.5. Theorem 5.2 holds for $d = 1$, keeping in mind that $A(1)$ consists of constant functions and that $\|\cdot\|_{A(1)}$ is the absolute value. In this case, the results can be proven directly and we sketch this approach. In Section 4.2, we showed that the Fourier coefficients of the function ϕ satisfy
$$\sum_{\xi \in \widehat{Z}} |\widehat{\phi}(\xi)|^{2/3} \leq 1.$$
Let ψ be the trigonometric polynomial obtained by removing the Fourier coefficients in ϕ that are less than δ^3. The error term satisfies $\|\phi - \psi\|_\infty \leq \delta$ and so the function ρ in the theorem can be taken to be the constant δ. There are at most $1/\delta^2$ characters ξ such that $\widehat{\psi}(\xi) \neq 0$. Taking a finite partition such that each of these characters is essentially constant on each set in the partition, we have that for every t, the function ϕ_t is essentially constant on each piece of the partition.

A counterexample. The function ϕ belongs to $A(d+1)$, with $\|\phi\|_{A(d+1)} \leq 1$. But Theorem 5.2 can not be extended to all functions belonging to $A(d+1)$, even for $d = 1$. Meaning, the conclusion of the theorem does not remain valid for the family of functions belonging to the unit ball $A(2)$. We explain this point further.

Assume instead that the conclusion remains valid for functions in the unit ball $A(2)$. Take $Z = \mathbb{T}$, $\delta = 1/10$, let $M = M(1, \delta)$ be associated to δ as in the statement, and let $\theta = 1/2M$. Define ϕ to be the triangular function over the base $[-\theta, \theta]$:

$$\phi(x) = \begin{cases} 1 - \dfrac{|x|}{\theta} & \text{if } |x| < \theta; \\ 0 & \text{otherwise.} \end{cases}$$

Then the Fourier coefficients of ϕ are non-negative and

$$\|\phi\|_{A(2)} = \sum_{n \in \mathbb{Z}} \widehat{\phi}(n) = \phi(0) = 1.$$

Let $m \leq M$, $(X_i \colon 1 \leq i \leq M)$, $\phi_i^{(t)}$ and ρ be associated to ϕ and δ as in the statement. Recalling that $A(1)$ consists only of constant functions and using part (iii), we have that for every $1 \leq i \leq m$ and $x, x' \in X_i$,

$$|\phi(x+t) - \phi(x'+t)| \leq \rho(x) + \rho(x') \text{ for every } t \in \mathbb{T}.$$

Taking $t = -x$ and using the definition of ϕ, it follows that for all $x, x' \in X_i$ with $|x - x'| \geq \theta$, the sum $\rho(x) + \rho(x') \geq 1$. Therefore, defining $F_i = \{x \in X_i \colon \rho(x) < 1/2\}$, for all $x, x' \in F_i$, $|x - x'| < \theta$. This implies that F_i is included in an interval of length θ, and we conclude that

$$\mu(\{x \in \mathbb{T} \colon \rho(x) \geq 1/2\}) \geq m(1/m - \theta) \geq 1 - M\theta = 1/2.$$

This contradicts $\|\rho\|_2 \leq \delta = 1/10$.

Remark 5.6. On the other hand, for $A(2)$, we have a weaker form of the Theorem 5.2. Maintaining the same notation, for $\|\phi\|_{A(2)} \leq 1$, part (iii) of the conclusion becomes that for every t, the left hand side of the equation has $L^2(\mu)$ norm bounded by δ.

Before turning to the proof of Theorem 5.2, we need some definitions, notation, and further results. Throughout the remainder of this section, we assume that an integer $d \geq 1$ is fixed, and the dependence of all constants on d is implicit in all statements. For notational convenience, we study functions belonging to $A(d+1)$ instead of $A(d)$.

5.2. **Regularity Lemma.**

Definition 5.7. Fix an integer $D \geq 2$. Let (Z, μ) be a probability space of one of the two types considered in Definition 5.1.

Let ν be a measure on Z^D such that each of its projections on Z is equal to μ.

Let $\mathcal{P}$ be a partition of Z. An atom of the product partition $\mathcal{P} \times \ldots \times \mathcal{P}$ (D times) of Z^D is called a *rectangle* of $\mathcal{P}$.

A $\mathcal{P}$-*function* on Z^D is a function f that is constant on each rectangle of $\mathcal{P}$.

For a function F on Z^D, we define $F_{\mathcal{P}}$ to be the $\mathcal{P}$-function obtained by averaging over each rectangle with respect to the measure ν: for every $x \in Z^D$, if R is the rectangle containing x, then

$$F_{\mathcal{P}}(x) = \begin{cases} \dfrac{1}{\nu(R)} \displaystyle\int F\, d\nu & \text{if } \nu(R) \neq 0; \\ 0 & \text{if } \nu(R) = 0. \end{cases}$$

An m-step function is a $\mathcal{P}$-function for some partition $\mathcal{P}$ into at most m sets.

As with d, we assume that the integer D is fixed throughout and omit the explicit dependencies of the statements and constants on D.

We make use of the following version of the Regularity Lemma, a modification of the analytic version of Szemerédi's Regularity Lemma in [10]:

Theorem 5.8 (Regularity Lemma, revisited)**.** *For every D and $\delta > 0$, there exists $M = M(D, \delta)$ such that if (X, μ) and ν are as in Definition 5.7, then for every function F on Z^D with $|F| \leq 1$, there is an almost uniform partition $\mathcal{P}$ of Z into $m \leq M$ sets such that for every m-step function U on Z^D with $|U| \leq 1$,*

$$\Big| \int U(F - F_{\mathcal{P}})\, d\nu \Big| \leq \delta\ .$$

We defer the proof to Appendix A. In the remainder of this section, we carry out the proof of Theorem 5.2.

5.3. **An approximation result for decomposable functions.** We return to our usual definitions and notation. We fix $d \geq 1$ and apply the Regularity Lemma to the probability space (Z, μ), $D = 2^d$ and the probability measure μ_d on Z^{2^d}.

In this section, we show an approximation result that allows to pass from weak to strong approximations:

Proposition 5.9. *Let F be a function on Z_d belonging to $D(d)$ with $\|F\|_{D(d)} \leq 1$ and $\|F\|_{L^\infty(\mu_d)} \leq 1$. Let $\theta > 0$ and $\mathcal{P}$ be the partition of Z associated to F and θ by the Regularity Lemma (Theorem 5.8). Then there exist constants $C = C(d) > 0$ and $c = c(d) > 0$ such that*

$$\|F - F_{\mathcal{P}}\|_{L^2(\mu_d)} \leq (C\theta^c + \theta)^{1/2}.$$

We first prove a result that allows us to pass from sets to functions:

Lemma 5.10. *Assume that F is a function on Z_d with $\|F\|_{L^\infty(\mu_d)} \leq 1$. Let $\theta > 0$ and let $\mathcal{P}$ be the partition of Z associated to F and θ by the*

Regularity Lemma (Theorem 5.8). If $f_{\vec\epsilon}$, $\vec\epsilon \in V_d$, are functions on Z satisfying $\|f_{\vec\epsilon}\|_{2^d} \le 1$ for every $\vec\epsilon$, then

$$\Big|\mathbb{E}_{\mathbf{x}\in Z_d}(F - F_{\mathcal{P}})(\mathbf{x}) \prod_{\vec\epsilon\in V_d} f_{\vec\epsilon}(x_{\vec\epsilon})\Big| \le C\theta^c, \tag{24}$$

where $c = c(d)$ and $C = C(d)$ are positive constants.

In other words, writing $\|\cdot\|^*_{D(d)}$ for the dual norm of the norm $\|\cdot\|_{D(d)}$, we have that

$$\|F - F_{\mathcal{P}}\|^*_{D(d)} \le C\theta^c.$$

Proof. Let $\eta > 0$ be a parameter, with its value to be determined. By construction, $\mathcal{P}$ is an almost uniform partition of Z into $m < M(\eta)$ pieces and the function $F = F_{\mathcal{P}}$ satisfies

$$|\mathbb{E}_{Z_d} U(F - F_{\mathcal{P}})| \le \eta \tag{25}$$

for every m-step function U on Z_d with $|U| \le 1$. We show (24).

By possibly changing the constant C, we can further assume that the functions $f_{\vec\epsilon}$ are all non-negative. For $\vec\epsilon \in \{0,1\}^d$, set

$$f'_{\vec\epsilon}(x) = \min\big(f_{\vec\epsilon}(x), \eta\big) \text{ and } f''_{\vec\epsilon}(x) = f_{\vec\epsilon} - f'_{\vec\epsilon}(x).$$

Thus the average of (24) can be written as a sum of 2^d averages, which we deal with separately.

a) We first show that

$$\Big|\mathbb{E}_{\mathbf{x}\in Z_d}(F - F_{\mathcal{P}})(\mathbf{x}) \prod_{\vec\epsilon\in V_d} f'_{\vec\epsilon}(x_{\vec\epsilon})\Big| \le \eta^{2^d}\theta. \tag{26}$$

For $u \in \mathbb{R}_+$, write

$$A(\vec\epsilon, u) = \{x \in Z \colon f_{\vec\epsilon}(x) \le u\}.$$

For each $\vec\epsilon \in \{0,1\}^d$, we have that

$$f'_{\vec\epsilon}(x) = \int_0^\eta \mathbf{1}_{A(\vec\epsilon,u)}(x)\, du$$

and so the average of the left hand side of (26) is the integral over $\mathbf{u} = (u_{\vec\epsilon} \colon \vec\epsilon \in V_d) \in [0,\eta]^{2^d}$ of

$$\mathbb{E}_{\mathbf{x}\in Z_d}(F - F_{\mathcal{P}})(\mathbf{x}) \prod_{\vec\epsilon\in V_d} \mathbf{1}_{A(\vec\epsilon,u_{\vec\epsilon})}(x_{\vec\epsilon}).$$

By (25), for each $\mathbf{u} \in [0,\eta]^{2^d}$, the absolute value of this average is bounded by θ. Integrating, we obtain the bound (26).

b) Assume now that for each $\vec{\epsilon} \in \{0,1\}^d$, the function $g_{\vec{\epsilon}}$ is equal either to $f'_{\vec{\epsilon}}$ or to $f''_{\vec{\epsilon}}$, and that there exists $\vec{\alpha} \in \{0,1\}^d$ with $g_{\vec{\alpha}} = f''_{\vec{\epsilon}}$. We show that

$$\Big|\mathbb{E}_{\mathbf{x}\in Z_d}(F - F_{\mathcal{P}})(\mathbf{x}) \prod_{\vec{\epsilon}\in V_d} g_{\vec{\epsilon}}(x_\epsilon)\Big| \leq 2\eta^{-2^d+1}.$$

Since $|F - F_{\mathcal{P}}| \leq 2$ and the functions $g_{\vec{\epsilon}}$ are nonnegative, it suffices to show that

$$\mathbb{E}_{\mathbf{x}\in Z_d} \prod_{\vec{\epsilon}\in\{0,1\}^d} g_{\vec{\epsilon}}(x_{\vec{\epsilon}}) \leq \eta^{-2^d+1}.$$

By Corollary 2.5, the left hand side is bounded by

$$\begin{aligned}\prod_{\substack{\vec{\epsilon}\in V_d\\ \vec{\epsilon}\neq\vec{\alpha}}} \|g_{\vec{\epsilon}}\|_{2^{d-1}} \cdot \|g_{\vec{\alpha}}\|_1 \leq \|g_{\vec{\alpha}}\|_1 = \int 1_{f_{\vec{\alpha}}>\eta}(x) f_{\vec{\alpha}}(x)& \\ \leq \|f_{\vec{\alpha}}\|_{2^d}\, \mu\{x\in Z\colon f_{\vec{\alpha}}(x)\geq\eta\}^{(2^d-1)/2^d} \leq \eta^{-2^d+1},&\end{aligned}$$

and we have the statement.

c) The left hand side of (24) is thus bounded by

$$\eta^{2^d}\theta + 2(2^d-1)\eta^{-2^d+1}.$$

Taking $\eta = \theta^{-1/(2^{d+1}-1)}$, we obtain the bound (24). $\square$

We now use this to prove the proposition:

Proof of Proposition 5.9. Since F belongs to $D(d)$ with $\|F\|_{D(d)} \leq 1$, it follows from the definition of this norm and from Lemma 5.10 that $|\mathbb{E}_{\mathbf{x}\in Z_d}(F - F_{\mathcal{P}})(\mathbf{x})F(\mathbf{x})| \leq C\theta^c$.

On the other hand, $F_{\mathcal{P}}$ is an m-step function and by the property of the partition $\mathcal{P}$ given in the Regularity Lemma(Theorem 5.8), we have that $|\mathbb{E}_{\mathbf{x}\in Z_d}(F - F_{\mathcal{P}})(\mathbf{x})F_{\mathcal{P}}(\mathbf{x})| \leq \theta$. Finally, $\mathbb{E}_{\mathbf{x}\in Z_d}\big((F - F_{\mathcal{P}})(\mathbf{x})^2\big) \leq C\theta^c + \theta$. $\square$

5.4. **Proof of Theorem 5.2.** We maintain the notation and hypotheses from the statement of Theorem 5.2.

a) A decomposition. Define $\mathrm{P} : L^1(\mu_d) \to L^1(\mu)$ to be the operator of conditional expectation. The most convenient definition of this operator is by duality: for $h \in L^\infty(\mu)$ and $H \in L^1(\mu_d)$,

$$\int_Z h(x)\ \mathrm{P}\, H(x)\, d\mu(x) = \int_{Z_d} h(x_{\vec{0}})H(\mathbf{x})\, d\mu_d(\mathbf{x}).$$

Recall that $\|\mathrm{P}\, H\|_{L^1(\mu_d)} \leq \|H\|_{L^1(\mu_d)}$.

By definition, when

$$H(x) = \prod_{\vec{\epsilon} \in V_d} f_{\vec{\epsilon}}(x_{\vec{\epsilon}}),$$

where the functions $f_{\vec{\epsilon}}$ belong to $L^{2^{d-1}(\mu)}$, then

$$\mathrm{P}\, H(x) = \mathbb{E}_{\vec{t} \in Z_d} \prod_{\vec{\epsilon} \in V_d} f_{\vec{\epsilon}}(x + \vec{\epsilon} \cdot \vec{t}). \tag{27}$$

For $\mathbf{x} \in Z_d$, define

$$G(\mathbf{x}) = \bigotimes_{\vec{\epsilon} \in \widetilde{V}_d} f_{\vec{\epsilon}0}(\mathbf{x}) = \prod_{\vec{\epsilon} \in \widetilde{V}_d} f_{\vec{\epsilon}0}(x_{\vec{\epsilon}})$$

and

$$F(\mathbf{x}) = \Big(\pi \bigotimes_{\vec{\epsilon} \in V_d} f_{\vec{\epsilon}1}\Big)(\mathbf{x}) = \mathbb{E}_{u \in Z} \prod_{\vec{\epsilon} \in V_d} f_{\vec{\epsilon}1}(x_{\vec{\epsilon}} + u).$$

For $x \in Z$, we have

$$\phi(x) = \mathbb{E}_{\vec{s} \in Z_d} \prod_{\vec{\epsilon} \in \widetilde{V}_d} \Big(f_{\vec{\epsilon}0}(x + \vec{\epsilon} \cdot \vec{s})\, \mathbb{E}_{u \in Z} \prod_{\vec{\epsilon} \in V_d} f_{\vec{\epsilon}1}(x + \vec{\epsilon} \cdot \vec{s} + u) \Big) = \mathrm{P}(G \cdot F).$$

Recall that for $t \in Z$, ϕ_t is the function on Z defined by $\phi_t(x) = \phi(x+t)$.

For $t \in Z$ and $\mathbf{x} \in Z_d$, define

$$G_{t^\Delta}(\mathbf{x}) = G(x + t^\Delta) = \prod_{\vec{\epsilon} \in \widetilde{V}_d} f_{\vec{\epsilon}0}(x_{\vec{\epsilon}} + t).$$

Since the function F is invariant under diagonal translations, for $x, t \in Z$ we have that

$$\phi_t(x) = \mathrm{P}(G_{t^\Delta} \cdot F)(x).$$

By Proposition 4.11, the function $F \in D(d)$ and $\|F\|_{D(d)} \leq 1$. Thus $\|F\|_{L^\infty(\mu_d)} \leq 1$.

Let $\delta > 0$. Let c and C be as in Proposition 5.9 and let $\theta > 0$ be such that $(C\theta^c + \theta)^{1/2} < \delta$. Let $\mathcal{P}$ and $F_{\mathcal{P}}$ be associated to F and θ as in the Regularity Lemma. Let $\mathcal{P} = (A_1, \dots, A_m)$.

For $x, t \in Z$, we have that

$$\phi_t(x) = \mathrm{P}(G_{t^\Delta} \cdot (F - F_{\mathcal{P}})) + \mathrm{P}(G_{t^\Delta} \cdot F_{\mathcal{P}})$$

and we study the two parts of this sum separately.

b) Bounding the rest. Define

$$\rho(x) = \big(\mathrm{P}(F - F_{\mathcal{P}})^2\big)^{1/2}.$$

We have that

$$\|\rho\|_2 = \|\mathrm{P}(F - F_{\mathcal{P}})^2\|_{L^1(\mu_d)}^{1/2} \le \|(F - F_{\mathcal{P}})^2\|_{L^1(\mu_d)}^{1/2} = \|F - F_{\mathcal{P}}\|_{L^2(\mu_d)} \le \delta,$$

where the last inequality follows from Proposition 5.9.

Moreover,

$$\big|\mathrm{P}(G_{t^\vartriangle} \cdot (F - F_{\mathcal{P}}))\big| \le \big(\mathrm{P}(G_{t^\vartriangle}^2)\big)^{1/2} \cdot \big(\mathrm{P}(F - F_{\mathcal{P}})^2\big)^{1/2} \le \rho(x)$$

by (27) and Lemma 2.3.

c) The main term. We write elements of $\{1, \dots, m\}^{2^d}$ as

$$\mathbf{j} = (j_{\vec{\epsilon}} \colon \vec{\epsilon} \in V_d).$$

For $\mathbf{j} = (j_{\vec{\epsilon}} \colon \vec{\epsilon} \in V_d) \in \{1, \dots, m\}^{2^d}$, write

$$R_{\mathbf{j}} = \prod_{\vec{\epsilon} \in V_d} A_{j_{\vec{\epsilon}}}.$$

The function $F_{\mathcal{P}}$ is equal to a constant on each rectangle $R_{\mathbf{j}}$. Let $c_{\mathbf{j}}$ denote this constant. We have that $|c_{\mathbf{j}}| \le 1$.

For $1 \le i \le m$ and $t, x \in Z$, define

$$\phi_i^{(t)}(x) := \mathbb{E}_{\vec{s} \in Z^d} \sum_{\substack{\mathbf{j} \in \{1,\dots,m\}^{2^d} \\ j_{\vec{0}} = i}} c_{\mathbf{j}} \prod_{\vec{\epsilon} \in \widetilde{V}_d} 1_{A_{j_{\vec{\epsilon}}}}(x + \vec{\epsilon} \cdot \vec{s}).\phi_{\vec{\epsilon}0}(x + \vec{\epsilon} \cdot \vec{s}).$$

Since distinct rectangles are disjoint, it follows that

$$\Big| \sum_{\substack{\mathbf{j} \in \{1,\dots,m\}^{2^d} \\ j_{\vec{0}} = i}} c_{\mathbf{j}} \prod_{\vec{\epsilon} \in \widetilde{V}_d} 1_{A_{j_{\vec{\epsilon}}}}(x + \vec{\epsilon} \cdot \vec{s}).\phi_{\vec{\epsilon}0}(x + \vec{\epsilon} \cdot \vec{s}) \Big| \le \prod_{\vec{\epsilon} \in \widetilde{V}_d} |\phi_{\vec{\epsilon}0}(x + \vec{\epsilon} \cdot \vec{s})|.$$

Thus

$$|\phi_i^{(t)}(x)| \le 1.$$

On the other hand, the function $\phi_i^{(t)}$ is the sum of $m^{2^d - 1}$ functions belonging to $A(d)$ with norm ≤ 1 and thus

$$\|\phi_i^{(t)}\|_{A(d)} \le C = M^{2^d - 1}.$$

We claim that

$$P(G_{t^\vartriangle} \cdot F_{\mathcal{P}}) = \sum_{i=1}^{m} 1_{A_i}(x) \phi_i^{(t)}(x). \tag{28}$$

From the definitions, we have that

$$(G_{t^\Delta} \cdot F_{\mathcal{P}})(\mathbf{x}) = \sum_{\mathbf{j}\in\{1,\dots,m\}^{2^d}} c_{\mathbf{j}} \prod_{\vec{\epsilon}\in\vec{t}V_d} f_{\vec{\epsilon}0}(x_{\vec{\epsilon}}) \prod_{\vec{\epsilon}\in V_d} 1_{A_{j_{\vec{\epsilon}}}}(x_{\vec{\epsilon}}).$$

Grouping together all terms of the sum with $j_{\vec{0}} = i$ and using (27), we obtain (28). This completes the proof of Theorem 5.2. □

6. Further directions

We have carried out this study of Gowers norms and associated dual norms in the setting of compact abelian groups. This leads to a natural question: what is the analog of the Inverse Theorem for groups other than $\mathbb{Z}_N$? What would be the generalization for other finite groups or for infinite groups such as the torus, or perhaps even for totally disconnected (compact abelian) groups?

In Section 3.5, we give examples of functions with small dual norm, obtained by embedding in a nilmanifold. One can ask if this process is general: does one obtain all functions with small dual norm, up to a small error in L^1 in this way? In particular, for $\mathbb{Z}_N$ this would mean that in the Inverse Theorem we can replace the family $\mathcal{F}(d,\delta)$ by a family of nilsequences with "bounded complexity" that are periodic, with period N, meaning that they all come from embeddings of $\mathbb{Z}_N$ in a nilmanifold.

By the computations in Section 4.2, we see a difference between $A(2)$ and the dual functions: the cubic convolution product f of functions belonging to $L^2(\mu)$ satisfies $\sum |\widehat{f}|^{2/3} < \infty$, while $A(2)$ is the family of functions f such that $\sum |\widehat{f}(\xi)| < +\infty$. It is natural to ask what analogous distinctions are for $d > 2$.

Appendix A. Proof of the regularity lemma

We make use of the following version of the Regularity Lemma in a Hilbert space introduced in [10]:

Lemma A.1 (Lovasz and Szegedy [10])**.** *Let $K_1, K_2, \dots$ be arbitrary nonempty subsets of a Hilbert space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|_{\mathcal{H}}$. Then for every $\varepsilon > 0$ and $f \in \mathcal{H}$, there exists $k \le \lceil 1/\varepsilon^2\rceil$ and $f_i \in K_i$, $i = 1,\dots,k$ and $\gamma_1,\dots,\gamma_k \in \mathbb{R}$ such that for every $g \in K_{k+1}$,*

$$|\langle g, f - (\gamma_1 f_1 + \ldots + \gamma_k f_k)\rangle| \le \varepsilon \cdot \|g\|_{\mathcal{H}} \cdot \|f\|_{\mathcal{H}}.$$

For the proof of Theorem 5.8, we follow the proof of the strong form of the Regularity Lemma in [10].

Proof of Theorem 5.8. We only consider the infinite case, as the proof in the finite case is similar.

Choose a sequence of integers $s(1) < s(2) < \dots$ such that

$$(s(1)s(2)\dots s(i))^2 < s(i+1)$$

for each $i \in \mathbb{N}$ and such that $D/\varepsilon < s(1)$.

For every i, let K_i consist of $s(i)$-step functions.

By Lemma A.1, there exists $k \leq \lceil 1/\varepsilon^2 \rceil$ and there exists an $s(1)\dots s(k)$-step function F^* such that

$$\Big| \int U(F - F^*)\, d\nu \Big| \leq \varepsilon \tag{29}$$

for any $s(k+1)$-step function U. Choose m with $D/\varepsilon < m < s(k+1)$ and refine the partition defining F^* into a partition $\mathcal{S} = \{S_1, \dots, S_m\}$ into m sets. Then F^* is an $\mathcal{S}$-function and the bound (29) remains valid for every m-step function U.

Partition each set S_i into subsets of measure $1/m^2$ and a remainder set of measure less than $1/m^2$. Take the union of all these remainder sets and partition this union into sets of measure $1/m^2$. Thus we obtain a partition $\mathcal{P} = \{A_1, \dots, A_{m^2}\}$ of Z into m^2 sets of equal measure.

At least $m^2 - m$ of these m^2 sets are *good*, meaning that the set is included in some set of the partition $\mathcal{S}$. Let G denote the union of these good sets and call it the *good part* of Z. We have that

$$\nu\big(Z^D \setminus G^D\big) \leq D/m \leq \varepsilon.$$

We claim that if U is an m-step function with $|U| \leq 1$, then

$$\Big| \int U(F - F_{\mathcal{P}})\, d\nu \Big| \leq 4\varepsilon.$$

To show this, set $U' = \mathbf{1}_G \cdot U$. Then

$$\Big| \int (U - U')(F - F_{\mathcal{P}})\, d\nu \Big| \leq 2 \int |U - U'|\, d\nu \leq 2\varepsilon.$$

Moreover, U' is an m-step function with $|U'| \leq 1$ and by hypothesis,

$$\Big| \int U'(F - F^*)\, d\nu \Big| \leq \varepsilon.$$

Thuswe are reduced to showing that

$$\Big| \int U'(F^* - F_{\mathcal{P}})\, d\nu \Big| \leq \varepsilon.$$

Instead, we assume that

$$\int U'(F^* - F_{\mathcal{P}})\, d\nu > \varepsilon$$

and derive a contradiction (the other bound is proved similarly).

Define a new function U'' on Z^D. Set $U'' = 0 = U'$ outside G^D. Let R be a product of good sets. The functions F^* and $F_{\mathcal{P}}$ are constant on R and thus the function $F^* - F_{\mathcal{P}}$ is constant on R. Define U'' on R to be equal to 1 if this constant is positive and to be -1 if this constant is negative. Then $U''(F^* - F_{\mathcal{P}}) \geq U'(F - F_{\mathcal{P}})$ on R and so

$$\int U''(F^* - F_{\mathcal{P}})\, d\nu \geq \int U'(F^* - F_{\mathcal{P}})\, d\nu > \varepsilon.$$

On the other hand, U'' is a $\mathcal{P}$-function and so by definition of $F_{\mathcal{P}}$, $\int U''(F - F_{\mathcal{P}})\, d\nu = 0$ and

$$\int U''(F^* - F)\, d\nu > \varepsilon.$$

But U'' is an m-step function with $|U''| \leq 1$ and by (29) this integral is $< \varepsilon$, leading to a contradiction.

□

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